HARMONIC AND PLURIHARMONIC BEREZIN TRANSFORMS

Miroslav Engliš

Abstract. We show that, perhaps surprisingly, the asymptotic behaviour of the Berezin transform as well as some properties of Toeplitz operators on a variety of weighted harmonic and pluriharmonic Bergman spaces seem to be the same as in the holomorphic case.

Let Ω be a bounded domain in $\mathbf{C}^n$, $L^2_{\text{hol}}(\Omega) \subset L^2(\Omega)$ the Bergman space of all square-integrable holomorphic functions on Ω, and $K(x,y)$ its reproducing kernel, i.e. the Bergman kernel. Thus

$$f(x) = \int_\Omega f(y)\ K(x,y)\ dy = \langle f, K_x\rangle, \qquad K_x := K(\cdot, x),$$

for all $f \in L^2_{\text{hol}}$ and $x \in \Omega$. Recall that for $\phi \in L^\infty(\Omega)$, the Toeplitz operator T_ϕ with symbol ϕ is defined by

$$T_\phi :\ L^2_{\text{hol}} \to L^2_{\text{hol}}, \qquad T_\phi f := P(\phi f),$$

where $P : L^2 \to L^2_{\text{hol}}$ is the orthogonal projection (the Bergman projection). The Berezin symbol of a (bounded linear) operator T on L^2_{hol} is, by definition, the function $\widetilde{T}$ on Ω defined by

$$\widetilde{T}(x) := \frac{\langle TK_x, K_x\rangle}{K(x,x)} = \Big\langle T\frac{K_x}{\|K_x\|}, \frac{K_x}{\|K_x\|}\Big\rangle.$$

Finally, the Berezin transform of $f \in L^\infty$ is, by definition, the Berezin symbol of the Toeplitz operator T_f:

$$Bf(x) = \widetilde{T_f}(x) = K(x,x)^{-1} \int_\Omega f(y)\ |K(x,y)|^2\ dy.$$

It is immediate that the mapping $T \mapsto \widetilde{T}$ is linear, $\widetilde{I} = \mathbf{1}$, $(T^*)^\sim = \overline{\widetilde{T}}$, $\|\widetilde{T}\|_\infty \le \|T\|$, and $\widetilde{T}$ is a real-analytic function on Ω; similarly for $f \mapsto Bf$. Since the function $\langle TK_y, K_x\rangle$, being holomorphic in x and $\overline{y}$, is uniquely determined by its restriction to the diagonal $x = y$, it also follows that both mappings $T \mapsto \widetilde{T}$ and $f \mapsto Bf$ are one-to-one — a fact which is of crucial importance for some applications.

Research supported by GA AV ČR grant no. A1019304 and by AV ČR IRP no. AV0Z10190503.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

There are also the weighted analogues of all the above objects: namely, for any continuous, positive weight function w on Ω, the subspace $L^2_{\text{hol}}(\Omega, w)$ of all holomorphic functions in $L^2(\Omega, w)$ is closed and possesses a reproducing kernel $K_w(x, y)$ — the weighted Bergman kernel; and one may define the Toeplitz operators $T^{(w)}_f$, Berezin symbols $\widetilde{T}^{(w)}$ and Berezin transform $B^{(w)}$ in the same way as before.

Consider now a strictly plurisubharmonic function Φ on Ω. Then $g_{i\bar{j}} = \dfrac{\partial^2\Phi}{\partial z_i \partial \overline{z}_j}$ defines a Kähler metric on Ω, with the associated volume element $d\mu(z) = \det[g_{i\bar{j}}]\, dz$ (dz being the Lebesgue measure). For any $h > 0$, we then have, in particular, the weighted Bergman spaces $L^2_{\text{hol}}(\Omega, e^{-\Phi/h}\, d\mu) =: L^2_{\text{hol},h}$, and the corresponding reproducing kernels $K_h(x, y)$, Toeplitz operators $T^{(h)}_f$, and Berezin transforms $B_h f$. It turns out that the following theorem holds.

Theorem. ([E1],[BMS]) *Assume that Ω is smoothly bounded and strictly pseudoconvex, and $e^{-\Phi}$ is a defining function for Ω. Then as $h \searrow 0$,*

$$K_h(x, x) \approx e^{\Phi(x)/h}\, h^{-n} \sum_{j=0}^{\infty} h^j\, b_j(x); \tag{1}$$

$$B_h f \approx \sum_{j=0}^{\infty} h^j\, Q_j f; \qquad \textit{and} \tag{2}$$

$$T^{(h)}_f T^{(h)}_g \approx \sum_{j=0}^{\infty} h^j\, T^{(h)}_{C_j(f,g)} \qquad (\textit{in operator norm}), \tag{3}$$

for some functions $b_j \in C^\infty(\Omega)$, some differential operators Q_j, with $Q_0 = I$ and $Q_1 = g^{\bar{j}i}\partial_i\overline{\partial}_j$, the Laplace-Beltrami operator with respect to the metric $g_{i\bar{j}}$; and some bidifferential operators C_j, where $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = \frac{i}{2\pi}\{f, g\}$ (the Poisson bracket of f and g).

The proof of the theorem makes use of the domain

$$\widetilde{\Omega} := \{(x, t) \in \Omega \times \mathbf{C} : \ |t|^2 < e^{-\Phi(x)}\}$$

which by the hypotheses is smoothly bounded and strictly pseudoconvex, and admits $r(x, t) := |t|^2 - e^{-\Phi(x)}$ as a defining function. Its boundary $\mathcal{X} = \partial\widetilde{\Omega}$ is a compact manifold, and $\alpha = \operatorname{Im}\partial r$ is a contact form on $\mathcal{X}$ (i.e. $\alpha \wedge (d\alpha)^{n-1}$ is a nonvanishing volume element). Let $H^2(\mathcal{X})$ be the Hardy subspace of all functions in $L^2(\mathcal{X})$ that extend holomorphically to $\widetilde{\Omega}$. According to a formula of Forelli, Rudin and Ligocka, the reproducing kernel $K_{\mathcal{X}}$ of $H^2(\mathcal{X})$ — the Szegö kernel — satisfies

$$K_{\mathcal{X}}((x, t), (y, s)) = \frac{1}{2\pi n!} \sum_{k=0}^{\infty} (t\overline{s})^k\, K_{1/(k+n+1)}(x, y).$$

On the other hand, by results of Fefferman, Boutet de Monvel and Sjöstrand,

$$K_{\mathcal{X}}\,|_{\text{diagonal}} = \frac{a}{r^{n+1}} + b\, \log r, \qquad a, b \in C^\infty(\overline{\widetilde{\Omega}}).$$

Employing the usual Cauchy estimates for the function $f_x(t\overline{s}) := K_{\mathcal{X}}((x,t),(x,s))$ of one complex variable on the disc $|t\overline{s}| < e^{-\Phi(x)}$, the expansion (1) follows (where $h = 1/(k+n+1)$, $k \to \infty$). In fact, this even gives a similar expansion for $K_h(x,y)$ for $(x,y) \in \Omega \times \Omega$ close to the diagonal, and (2) then follows by an application of the stationary phase method. Finally, (3) can be proved using the Boutet de Monvel-Guillemin theory of generalized Toeplitz operators (with pseudodifferential symbols).

A completely analogous result also holds for an arbitrary Kähler manifold Ω such that the second cohomology class $[\omega]$ of the Kähler form ω is integral: namely, there exists then an Hermitian line bundle $\mathcal{L}$ over Ω with compatible connection ∇ such that $\operatorname{curv} \nabla = \omega$. For $k = 1, 2, \dots$, consider, instead of the spaces $L^2_{\mathrm{hol}}(\Omega, e^{-k\Phi}\, d\mu)$, the subspaces of all holomorphic square-integrable sections of the k-th power $\mathcal{L}^{*\otimes k}$ of the dual bundle $\mathcal{L}^*$. Taking the unit disc bundle $\widetilde{\Omega} \subset \mathcal{L}^*$ in $\mathcal{L}^*$ in the place of the domain $\widetilde{\Omega}$ from the preceding paragraph, a totally parallel argument again shows that (1) and (2) hold, and the Guillemin-Boutet de Monvel theory of generalized Toeplitz operators again yields also (3) (cf. [BMS],[Zel]).

The last theorem has an elegant application to *quantization on Kähler manifolds*. Recall that the traditional problem of quantization consists in looking for a map $f \mapsto Q_f$ from $C^\infty(\Omega)$ into operators on some (fixed) Hilbert space which is linear, conjugation-preserving, $Q_{\mathbf{1}} = I$, and as the Planck constant $h \searrow 0$,

$$[Q_f, Q_g] \approx \frac{ih}{2\pi} Q_{\{f,g\}}. \tag{4}$$

(The spectrum of Q_f is then interpreted as the possible outcomes of measuring the observable f in an experiment; and (4) amounts to a correct semiclassical limit.) Our last theorem implies that (4) holds for $Q_f = T_f^{(h)}$, the Toeplitz operators on the Bergman spaces $L^2_{\mathrm{hol},h}$ (or on the spaces of holomorphic L^2-sections of the bundles $\mathcal{L}^{*\otimes 1/h}$). This is the so-called *Berezin-Toeplitz quantization*.

There is also another approach to quantization, discarding the operators Q_f but rather looking for a noncommutative associative product $*$ on $C^\infty(\Omega)$, depending on h, such that as $h \searrow 0$,

$$f * g \to fg, \qquad \frac{f*g - g*f}{h} \to \frac{i}{2\pi}\{f,g\}.$$

Such products are called a <u>star-products</u>, and are the subject of *deformation quantization*. The relationship to Bergman spaces is the following: in view of the injectivity of the map $T \mapsto \widetilde{T}$ from operators to their Berezin symbols, we can define for two bounded operators T, U on $L^2_{\mathrm{hol},h}$ a "product" of their symbols by

$$\widetilde{T} * \widetilde{U} := \widetilde{TU}.$$

This gives a noncommutative associative product on

$$\{\widetilde{T} : \ T \text{ a bounded operator on } L^2_{\mathrm{hol},h}\} \subset C^\omega(\Omega).$$

It can be shown from part (2) of the last theorem (i.e. from the asymptotics of B_h) that if h is made to vary, these products can be glued into a star-product on $C^\infty(\Omega)$. This is the so-called *Berezin quantization*.

From the point of view of these applications, the weighted Bergman spaces $L^2_{\mathrm{hol},h}$ (or their analogues $L^2_{\mathrm{hol}}(\mathcal{L}^{*\otimes k})$ for manifolds) have an obvious disadvantage in that their very definition requires a holomorphic structure (hence, in particular, they can make sense only on Kähler manifolds). On the other hand, the other ingredients — the operator symbols, the Toeplitz operators and the Berezin transform — make sense not only for L^2_{hol}, but for any subspace of L^2 with a reproducing kernel. Hence it seems very natural to investigate whether any such spaces other than weighted Bergman spaces can be used for quantization.

For instance, one such candidate might be the harmonic Bergman spaces L^2_{harm} of all harmonic functions in L^2. As in the holomorphic case, these possess a reproducing kernel, the *harmonic Bergman kernel* $H(x,y)$; in contrast to the usual Bergman kernel, $H(x,y)$ is real-valued and symmetric, $H(x,y)=H(y,x)\in\mathbf{R}$. Similarly, one has *pluriharmonic Bergman spaces* L^2_{ph} (and pluriharmonic Bergman kernels).

Still another candidate are *Sobolev spaces of holomorphic functions* (Sobolev-Bergman spaces), i.e. the subspaces W^s_{hol} of all holomorphic functions in the (possibly weighted) Sobolev spaces W^s, $s\in\mathbf{R}$. In fact, one can show that in the situation from the last theorem (i.e. when $e^{-\Phi}$ is a defining function), the weighted Bergman spaces $L^2_{\mathrm{hol},h}$, for $h=1/m$, coincide (as sets) with $W^s_{\mathrm{hol}}(\Omega)$ where $s=\frac{n+1-m}{2}\le 0$.

It is also possible to combine these two approaches and look at Sobolev spaces of (pluri)harmonic functions.

In this talk, we discuss in more detail the situation for the harmonic and pluriharmonic Bergman spaces.

Unfortunately, it turns out that — from the point of view of the quantization applications at least — bad things happen. First of all, recall that for the Berezin-Toeplitz quantization we needed that the Toeplitz operators satisfy

$$\frac{1}{h}[T^{(h)}_f,T^{(h)}_g]\approx\frac{i}{2\pi}T^{(h)}_{\{f,g\}}\qquad\text{as } h\searrow 0.$$

However, for Toeplitz operators on L^2_{harm}, this fails even on $\Omega=\mathbf{D}$, the unit disc in $\mathbf{C}$, with the hyperbolic metric (given by Kähler potential $\Phi(z)=\log\frac{1}{1-|z|^2}$) and $f(z)=z$, $g(z)=\overline{z}$. Second, recall that the Berezin quantization (the star-products) was based on the fact that the correspondence $T\mapsto\widetilde{T}$ between operators and their symbols was one-to-one. However, this fails on any harmonic Bergman space: if f,g are any two linearly independent elements in L^2_{harm}, then the operator $T=\langle\cdot,\overline{f}\rangle g-\langle\cdot,\overline{g}\rangle f$ is easily seen to satisfy $\langle TH_x,H_x\rangle=f(x)g(x)-g(x)f(x)=0\ \forall x$; hence $\widetilde{T}\equiv 0$, while apparently $T\ne 0$. Thus, there is no hope to perform the quantization. (See [E2] for the details.)

In view of these failures, it would be only natural to expect that also the other assertions of our theorem (e.g. the asymptotics of the Berezin transform, or the injectivity of the map $f\mapsto Bf$) break down. The following results therefore came as some surprise for the author.

Recall that a domain $\Omega \subset \mathbf{C}^n$ is called complete Reinhardt if $x \in \Omega$ and $|y_j| \le |x_j|\ \forall j$ imply $y \in \Omega$. In particular, such domains are invariant under the rotations

$$z \mapsto (z_1 e^{i\theta_1}, z_2 e^{i\theta_2}, \dots, z_n e^{i\theta_n}), \qquad \forall \theta_1, \dots, \theta_n \in \mathbf{R}. \tag{5}$$

Theorem 1. *Let $\Omega \subset \mathbf{C}^n$ be complete Reinhardt and let ν be any finite measure on Ω invariant under the rotations (5). Then on $L^2_{\mathrm{ph}}(\Omega, d\nu)$,*

$$\widetilde{T}_f = 0 \implies T_f = 0 \qquad (\textit{i.e. } \widetilde{B}f = 0 \implies f = 0).$$

Thus, although the Berezin symbol map $T \mapsto \widetilde{T}$ is not injective on all operators, it is injective on Toeplitz operators.

Theorem 2. *Consider the following situations,*

$$L^2_{\mathrm{harm}}(\mathbf{D}, \tfrac{1+h}{\pi h}(1-|z|^2)^{1/h}),$$
$$L^2_{\mathrm{ph}}(\mathbf{C}^n, h^{-n} e^{-|z|^2/h})$$

(i.e. the harmonic Bergman spaces on the disc with respect to the usual weights and the pluriharmonic Fock spaces on $\mathbf{C}^n$), and also the pluriharmonic analogues of the standard weighted Bergman spaces on bounded symmetric domains in $\mathbf{C}^n$. Then the associated Berezin transforms possess the asymptotic expansion (2), i.e. there exist differential operators Q_j such that $\forall f \in C^\infty \cap L^\infty$,

$$B_h f(x) = \sum_{j=0}^{\infty} h^j\, Q_j f(x) \qquad \textit{as } h \searrow 0.$$

In fact, these are the same Q_j as in the holomorphic case.

Theorem 3. *The assertion of the last theorem also holds for*

$$L^2_{\mathrm{harm}}(\mathbf{R}^n, h^{-n/2} e^{-|x|^2/h})$$

(the harmonic Fock space on $\mathbf{R}^n$), with $Q_j = (\Delta/4)^j$.

The proofs of these theorems go by explicit calculations of the reproducing kernels in question (which are possible owing to the rotational symmetry of the domains and measures) and the method of stationary phase; see [E3]. (For Theorem 3, one also needs the properties of certain spherical harmonics [ABR], and an interesting special function — one of the hypergeometric functions of Horn — plays a role.)

In a way, these theorems raise more questions than they answer. First of all, it is not clear whether the results are anomalies whose validity stems from the abundant symmetries of the domains, or whether they hold in more general settings. For instance, does Theorem 1 hold for the Toeplitz operators on the pluriharmonic Bergman space on a general smoothly bounded strictly pseudoconvex domain in $\mathbf{C}^n$? Or does Theorem 2 hold for the pluriharmonic analogues of the spaces $L^2_{\mathrm{hol}}(\Omega, e^{-\Phi/h} d\mu)$ from the traditional Berezin and Berezin-Toeplitz quantizations? For Theorem 3, it even makes sense to study the problem not only for pseudoconvex domains in $\mathbf{C}^n$, which are the natural arena for holomorphic functions, but for any open set in $\mathbf{R}^n$. (Currently, it is even unknown whether an analogue of Theorem 3 holds for the unit ball of $\mathbf{R}^n$.)

We remark that in the holomorphic case, the asymptotics of the weighted Bergman kernels, of the Berezin transform and of the Toeplitz operators were derived from the boundary behaviour of the Szegö kernel of the "inflated" domain $\widetilde{\Omega} = \{(x,t) \in \Omega \times \mathbf{C} : |t|^2 < e^{-\Phi}\}$, using the formula of Forelli-Rudin-Ligocka and the Fefferman-Boutet de Monvel-Sjöstrand theorem. It should be noted that the Forelli-Rudin-Ligocka formula holds also in the pluriharmonic case: if we denote by $H^2_{\text{ph}}(\mathcal{X})$, $\mathcal{X} = \partial\widetilde{\Omega}$, the subspace in $L^2(\mathcal{X})$ of all functions that have a pluriharmonic extension inside $\widetilde{\Omega}$, then the reproducing kernel of $H^2_{\text{ph}}(\mathcal{X})$ is given by

$$K^{\text{ph}}_{\mathcal{X}}((x,t),(y,s)) = \frac{1}{2\pi n!} \sum_{j=-\infty}^{\infty} (s\bar{t})^{[j]} \; K^{\text{ph}}_{1/(|j|+n+1)}(x,y),$$

where $z^{[j]} = z^j$ or $\overline{z}^{-j}$ according as $j \geq 0$ or < 0, and $K^{\text{ph}}_{1/m}(x,y)$ is the reproducing kernel of $L^2_{\text{ph}}(\Omega, e^{-m\Phi}\, d\mu)$. Thus in principle we can again get the asymptotics of $K^{\text{ph}}_{1/m}$, and of the pluriharmonic Berezin transform, from the boundary singularity of $K^{\text{ph}}_{\mathcal{X}}$. Unfortunately, what is missing is the pluriharmonic analogue of the Fefferman-Boutet de Monvel-Sjöstrand theorem, i.e. the description of the boundary singularity of the pluriharmonic Szegö or Bergman kernels.

Similarly, it seems unknown what is the boundary singularity of the harmonic Bergman (or Szegö) kernel of a domain in $\mathbf{R}^n$. (There exist optimal estimates for the boundary growth, though; see [KK].) However, in this case there is no analogue of the Forelli-Rudin-Ligocka formula.

References

[ABR] S. Axler, P. Bourdon, W. Ramey, *Harmonic function theory*, Springer-Verlag, New York, 1992.

[BMS] M. Bordemann, E. Meinrenken, M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and $gl(n)$, $n \to \infty$ limits*, Comm. Math. Phys. **165** (1994), 281–296.

[E1] M. Engliš, *Weighted Bergman kernels and quantization*, Comm. Math. Phys. **227** (2002), 211–241.

[E2] M. Engliš, *Berezin and Berezin-Toeplitz quantizations for general function spaces*, preprint (2005).

[E3] M. Engliš, *Berezin transforms on harmonic and pluriharmonic Bergman spaces*, forthcoming.

[KK] H. Kang, H. Koo, *Estimates of the harmonic Bergman kernel on smooth domains*, J. Funct. Anal. **185** (2001), 220–239.

[Zel] S. Zelditch, *Szegö kernels and a theorem of Tian*, Int. Math. Res. Not. **6** (1998), 317–331.

Mathematics Institute, Žitná 25, 11567 Praha 1, Czech Republic
E-mail address: englis@math.cas.cz