Local and gauge invariant observables in gravity

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The need for local observables

Consider a Classical or a Quantum Field Theory on an *n*-dim. spacetime M.

- In QFT, $\langle\hat{\phi}(x)\hat{\phi}(y)\rangle$ is singular for some pairs of (x, y).
- In classical FT, $\{\phi(x), \phi(y)\}$ is singular for some pairs of (x, y).
- Instead, use smearing

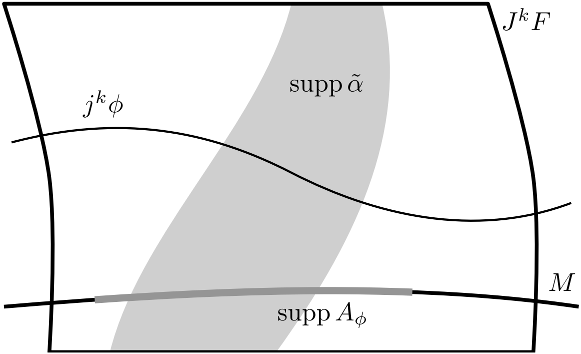
$$\phi(\tilde{\alpha}) = \int_M \phi(x)\alpha(x)\,\mathrm{d}\tilde{x}$$

  so that $\langle\hat{\phi}(\tilde{\alpha})\hat{\phi}(\tilde{\beta})\rangle$ and $\{\phi(\tilde{\alpha}), \phi(\tilde{\beta})\}$ are always finite, provided
  - $\tilde{\alpha}$, $\tilde{\beta}$ are **smooth** *n*-forms on M,
  - $\tilde{\alpha}$, $\tilde{\beta}$ have **compact** supports.
- **Smoothness** diffuses singularities.
  Compactness ensures convergence of all integrals.
- Support of a functional: $\operatorname{supp}\phi(\tilde{\alpha}) = \operatorname{supp}\tilde{\alpha} \subset M$.

Local observables

- Field ϕ is a section of some bundle $\pi\colon F \to M$ ($\pi^k\colon J^k F \to M$).
- Local observables may be **non-linear** and **depend on derivatives** (jets). An n-form $\tilde{\alpha} = \alpha(x, \phi(x), \partial\phi(x), \cdots)\,\mathrm{d}\tilde{x}$ on $J^k F$

defines a local observable $\quad A_\phi = \int_M (j^k\phi)^* \tilde{\alpha},$



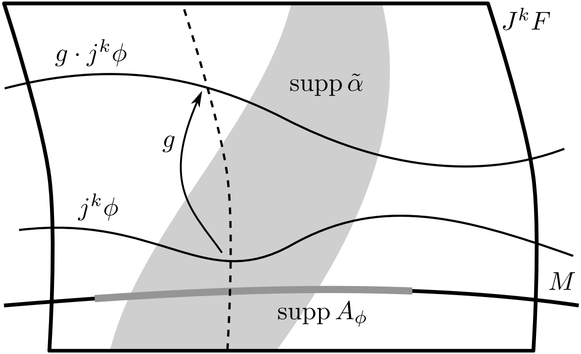
provided

supp $A_\phi = \pi^k$ supp $\tilde{\alpha}$

is compact!

Local observables in gauge theory (no gravity)

- Let $\mathscr{G}$ be the group of **gauge transformations.**
- Gauge transformations $g \in \mathscr{G}$ act on $J^k F$ (hence $j^k\phi \mapsto g \cdot j^k\phi$).
- **No gravity:** $\mathscr{G}$ fixes the fibers of $\pi^k \colon J^k F \to M$.



$$A_\phi = \int_M (j^k\phi)^* \tilde{\alpha}$$

is $\mathscr{G}$**-invariant** provided

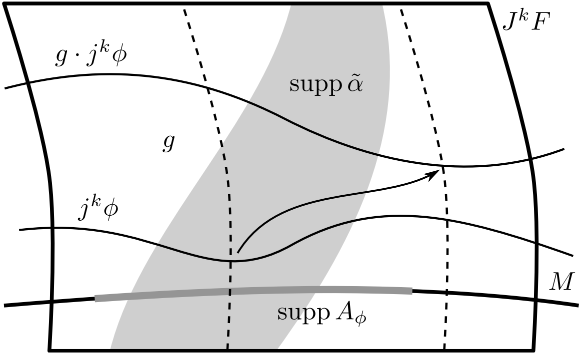
$$g^* \tilde{\alpha} = \tilde{\alpha} + \mathrm{d}(\cdots)$$

and

supp A_ϕ is compact!

No (such) local observables in gravity

- Gravity is General Relativity (GR), $F = S^2 T^* M$, $\mathscr{G} = \mathrm{Diff}(M)$.
- Diffeomorphisms **do not** fix the fibers of $\pi^k \colon J^k F \to M$. In fact, diffeomorphisms **act transitively** on these fibers.
- M is never compact, as needed by **global hyperbolicity**.



$\mathrm{supp}\, A_\phi = \mathrm{supp}\, \tilde{\alpha}$ compact

$\Downarrow$

$g^* \tilde{\alpha} \neq \tilde{\alpha} + \mathrm{d}(\cdots)$

$\Downarrow$

$A_\phi = \int_M (j^k \phi)^* \tilde{\alpha}$ is not $\mathscr{G}$**-invariant**!

Relaxing locality: an explicit example

- Take dim $M = 4$. Write the **dual Weyl tensor** as

$$\overset{*}{W}_{ab}{}^{cd} = W_{abc'd'}\varepsilon^{c'd'cd} = \varepsilon_{aba'b'}W^{a'b'cd}.$$

- Make use of curvature scalars (Komar-Bergmann 1960-61)

$$b^1 = W_{ab}{}^{cd}W_{cd}{}^{ab}, \qquad b^3 = W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab},$$
$$b^2 = W_{ab}{}^{cd}\overset{*}{W}_{cd}{}^{ab}, \qquad b^4 = W_{ab}{}^{cd}W_{cd}{}^{ef}\overset{*}{W}_{ef}{}^{ab}.$$

- Let φ be a **generic** metric ($\det|\partial b^i/\partial x^j| \neq 0$) and let $\beta = (b^1[\varphi](x), b^2[\varphi](x), b^3[\varphi](x), b^4[\varphi](x))$ for some $x \in M$.
- Take $a\colon \mathbb{R}^4 \to \mathbb{R}$, with sufficiently small compact support containing β, let $\tilde{\alpha} = a(b)\,\mathrm{d}b^1 \wedge \mathrm{d}b^2 \wedge \mathrm{d}b^3 \wedge \mathrm{d}b^4$ on $J^{k\geq 2}F$

$$\text{and} \quad A_\phi = \int_M (j^k\phi)^*\tilde{\alpha}.$$

- A_ϕ is well-defined on a Diff**-invariant** neighborhood $\mathcal{U} \ni \varphi$ among all metrics ϕ such that $R[\phi]_{ab} = 0$. A_ϕ is Diff**-invariant.**

Differential invariants of fields (algebra)

- In any gauge theory, the group $\mathscr{G}$ of gauge trans. acts on $J^k F$.
- **Differential invariants**: scalar $\mathscr{G}$-invariant functions on $J^k F$.
- **Theorem** (Lie-Tresse 1890s, Kruglikov-Lychagin 2011):
  - (generically) all differential invariants (all $k < \infty$) are generated by
  - a finite number of **invariants** and
  - a finite number of **differential operators** satisfying
  - a finitely generated set of **differential identities**.
- Examples
  - Non-gauge theory: every function on $J^k F$.
  - Yang-Mills theory: invariant polynomials of curvature $d_A A$.
  - Gravity: curvature scalars, built from Riemann R, ∇R, $\nabla\nabla R$, ...
- Gauge invariant observables: let $\tilde{\alpha} = a(b^1, \ldots, b^m)\, \mathrm{d}b^1 \wedge \cdots \wedge \mathrm{d}b^n$, for some $a\colon \mathbb{R}^m \to \mathbb{R}$ and differential invariants b^i, $i = 1, \ldots, m \geq n$,

  then $\quad A_\phi = \int_M (j^k \phi)^* \tilde{\alpha} \quad$ is **well-defined** and **gauge invariant**,

  provided $\operatorname{supp}[(j^k \phi)^* \tilde{\alpha}]$ is compact.

Moduli spaces of fields (geometry)

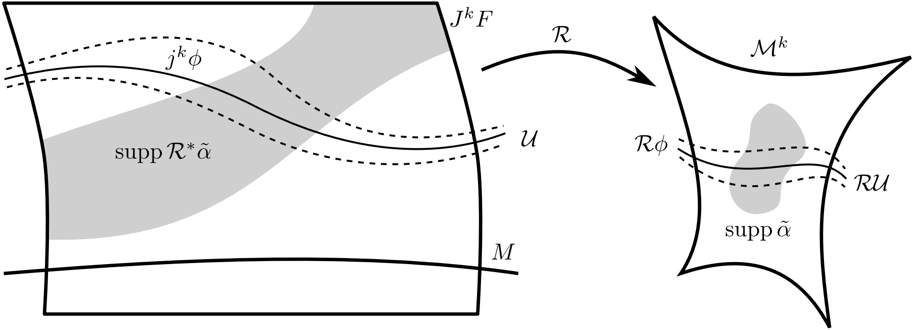
- In any gauge theory, the group $\mathscr{G}$ of gauge trans. acts on $J^k F$.
- **Moduli space**: quotient space $\mathcal{M}^k = (J^k F \setminus \Sigma^k)/\mathscr{G}$ (Σ^k is singular).
- Differential invariants are coordinates, separating points, on $\mathcal{M}^k$.
- Denote by $\mathcal{R} \colon J^k F \to \mathcal{M}^k$ the quotient map. Two (generic) field configurations ϕ and φ are gauge equivalent iff the images of $\mathcal{R}\phi(M)$ and $\mathcal{R}\varphi(M)$ coincide as submanifolds of $\mathcal{M}^k$ (for high k).
- Differential identities among differential invariants define a PDE $\mathcal{E}^k$ on n-dimensional submanifolds of $\mathcal{M}^k$, identifying submanifolds like $\mathcal{R}\phi(M)$.
- **Finite generation** means that there exists a k' such that all $\mathcal{M}^k$ and $\mathcal{E}^k$ ($k > k'$) can be recovered from $\mathcal{M}^{k'}$ and $\mathcal{E}^{k'}$.
- Choose **compactly supported** n-form $\tilde{\alpha}$ on $\mathcal{M}^k$ and $\mathcal{U}$ such that $\phi \in \mathcal{U}$ implies $\mathcal{R}\phi(M) \cap \operatorname{supp} \tilde{\alpha}$ is compact. Then $\mathcal{U}$ is $\mathscr{G}$-invariant,

  $$A_\phi = \int_M (j^k \phi)^* \mathcal{R}^* \tilde{\alpha} \quad \text{is **well-defined** and **gauge invariant**,}$$

  and the A_ϕ separate $\mathscr{G}$-orbits in $\mathcal{U}$.

New notion of local and gauge invariant observables

- A_ϕ may only be defined on an open subset $\mathcal{U} \subset \mathcal{S}$ of (covariant) phase space. Local charts!
- $A_\phi = \int_M (j^k\phi)^* \tilde{\alpha}$, with $j^k\phi(M) \cap \operatorname{supp} \tilde{\alpha}$ compact for every $\phi \in \mathcal{U}$.
- A_ϕ is gauge invariant if $\tilde{\alpha} = \mathcal{R}^* \tilde{\beta}$ of some n-form $\tilde{\beta}$ on $\mathcal{M}^k$.



- **NB:** Two metrics ϕ and ψ are Diff-equivalent iff $\mathcal{R}\phi = \mathcal{R}\psi$ in $\mathcal{M}^k$.

Poisson brackets

- Poisson brackets of **gauge invariant observables** are well-defined intrinsically, but require care to compute.
- **1st possibility.** Use hyperbolic gauge fixing to obtain Peierls bracket $E(-,-)$,

$$\{A_\phi, B_\phi\} = E(A'_\phi, B'_\phi) = \int_{M\times M} A'_\phi(x) \cdot E_\phi(x,y) \cdot B'_\phi(y),$$

  where $A_{\phi+t\psi} = A_\phi + tA'_\phi(\psi) + O(t^2)$ and $A'_\phi(\psi) = \int_M A'_\phi(x) \cdot \psi(x)$, with $A'_\phi(x)$ a compactly supported distribution.
- **2nd possibility.** Use the reduced equation $\mathcal{E}^k$ on $\mathcal{M}^k$ and apply the Peierls formalism (write as hyperbolic PDE + constraints, linearize, compute Green functions).
- For gravity: 1st possibility is well understood. 2nd possibility not yet explored.

Conclusion

- **Local gauge invariant** observables are important in both Classical (non-perturbative construction) and Quantum (perturbative or semi-classical renormalization) Field Theory.
- Usual restriction on “compact support” **excludes** gravitational gauge theories.
- Relaxing the support conditions opens the door to a large class of gauge invariant observables (even for gravitational theories), defined using **differential invariants** or **moduli spaces** of fields. They separate gauge orbits on open subsets of the phase space.
- The **Peierls formalism** computes their Poisson brackets.
- Limitations:
  - Observables may not be globally defined on all of phase space.
  - Naive approach separates only generic phase space points (e.g., metrics without isometries).
  - Need to connect with operational description of observables.

Thank you for your attention!