Finite renormalizations in locally covariant perturbative algebraic QFT

[arXiv:1411.1302] w/ Valter Moretti
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A(lgebraic)QFT

- In a QFT on a manifold M, a field operator $\phi(f) = \int f(x)\phi(x)\,\mathrm{d}x$ smeared by a test function f is considered to be **localized** within $\operatorname{supp} f \subset M$. Typically, $\phi(f)$ is an unbounded self-adjoint operator on a Hilbert space of states $\mathcal{H}$. (**Ex:** free relativistic field)
- The operators $\mathcal{A}(U)$ localized within $U \subseteq M$ are closed under products. In more detail, $\mathcal{A}(U)$ is a **non-commutative $*$-algebra**.
- These algebras have special properties, like **monotonicity** $\mathcal{A}(U) \subset \mathcal{A}(V)$ when $U \subset V$ and **microcausality** $[\mathcal{A}(U), \mathcal{A}(V)] = 0$ when U and V are spacelike separated.
- AQFT takes the algebras $\mathcal{A}(U)$ ($U \subset M$) of **localized quantum observables** as fundamental, satisfying some axioms, and separates out finding their representations $\pi \colon \mathcal{A}(U) \to \mathrm{Op}(\mathcal{H})$.
- In general, there are many **inequivalent** representations of the same algebra of observables. Different physical **vacuum states** may belong to inequivalent representations (thermal states, spontaneously broken symmetries, non-equilibrium states).

Perturbative AQFT and (infinite) Renormalization

- Consider an interacting Lagrangian $\mathcal{L}[\phi] = \mathcal{L}_0[\phi] + \lambda\mathcal{L}_I[\phi]$, where $\mathcal{L}_0[\phi]$ is free and λ is a formal coupling parameter.
- Starting with free quantum fields $\phi(x)$, try to make sense of the interacting fields $\phi_I(x)$ via **Bogoliubov's formula** (Feynman diagrams)

$$\mathcal{T}_{\mathcal{L}_I}(\phi_I(x)\phi_I(y)\cdots) = \left(\mathcal{T} e^{\frac{i}{\hbar}\int_M \lambda\mathcal{L}_I[\phi]\,\mathrm{d}x}\right)^{-1} \mathcal{T}\left[(\phi(x)\phi(y)\cdots)\, e^{\frac{i}{\hbar}\int_M \lambda\mathcal{L}_I[\phi]\,\mathrm{d}x}\right]$$

  - Work over formal power series $\mathbb{C}[[\hbar,\lambda]]$. Ignores convergence.
  - Replace $\lambda \to \lambda(x)$ by test function. Separates UV and IR problems.
- **Want:** Time-ordered products $\mathcal{T}_{k+1}[\phi(x)\mathcal{L}_I[\phi](y_1)\cdots\mathcal{L}_I[\phi](y_k)]$ are well-defined (free field algebra)-valued distributions. Then $\phi_I(x)$ is well-defined order-by-order, to all orders.
- **Key observation:** $\mathcal{T}_k[A(x_1)\cdots B(x_k)] = A(x_1)\mathcal{T}_{k-1}[\cdots B(x_k)]$ if x_1 is chronologically later than $(x_2,\ldots,x_k)$.
  Epstein-Glaser: causality + $\mathcal{T}_{k-1} \implies \mathcal{T}_k[A(x_1)\cdots B(x_k)]$ outside $\Delta_k = \{x_1 = \cdots = x_k\}$!
  UV renormalization: extend distribution $\mathcal{T}_k[A(x_1)\cdots B(x_k)]$ from $M^k \setminus \Delta_k$ to M^k. Always possible, under reasonable hypotheses!
- **Elementary example:** "$\frac{1}{x}$" $\to \frac{1}{x+i0} + c\delta(x)$

Nonlinear Local Observables and Hadamard States

- **Warning:** typical integraction $\mathcal{L}_I[\phi] =$ "$\phi(x)^4$" $\neq \phi(x)\phi(x)\phi(x)\phi(x)$. OK if $x_1 \neq \cdots \neq x_4$, but UV divergence if $x_1, x_2, x_3, x_4 \to x$!
- To start the Epstein-Glaser induction, we still need a rule for

  $$\mathcal{T}_1(\text{local, nonlinear, classical}) \mapsto (\text{free quantum observable})$$

  Typical notation: $\mathcal{T}_1(A(x)) = {:}A(x){:}$ (**Wick ordering**)
- In QFT on Minkowski space, **Wick ordering**, aka **normal ordering**, aka **vacuum subtraction** has multiple equivalent definitions:
  - Momentum cutoff:

    $$\phi(x) \mapsto \phi(x)_\Lambda = \int_{|\mathbf{k}|<\Lambda} \sqrt{\frac{\hbar}{2\omega_\mathbf{k}}} (\hat{a}_k e^{-i\omega_\mathbf{k} t} + \hat{a}^\dagger_{-k} e^{+i\omega_\mathbf{k} t}) e^{i\mathbf{k}\cdot\mathbf{x}} \, \mathrm{d}\mathbf{k}$$

    $$\phi(x)^2 \mapsto {:}\phi(x)^2{:} = \lim_{\Lambda\to\infty} \phi(x)^2_\Lambda - \hbar F(\hbar, \Lambda).$$
  - Point splitting:

    $${:}\phi(x)\phi(y){:} = \phi(x)\phi(y) - \hbar \left(\tfrac{1}{\hbar}\langle\phi(x)\phi(y)\rangle_{\mathrm{Fock}}\right), \quad \text{then let } y \to x.$$
  - For higher powers of ϕ, must subtract **lower powers** of ϕ with **singular coefficients**.
- **Point splitting:** Generalizes to curved spacetimes $(M, \mathbf{g})$, but there is no preferred vacuum state $\langle - \rangle_{(M,\mathbf{g})}$!
- **Hadamard states:** Preferred **class** of states $\langle - \rangle_\Omega$ such that $\langle\phi(x)\phi(y)\rangle_\Omega \sim \hbar H_{(M,\mathbf{g})}(x,y) + \text{l.o.t}_\Omega$. $H_{(M,\mathbf{g})}(x,y)$ depends only on local geometry.
  Wick ordering: ${:}\phi(x)^2{:} = \lim_{y\to x} \phi(x)\phi(y) - \hbar H_{(M,g)}(x,y)$.

Short summary on pAQFT

Theorem (Main theorem of perturbative renormalization)

Given a free QFT, there always exists a renormalized $\mathcal{T}_{k\geq 1}$. Given two renormalized time-ordered products, $\mathcal{T}_{k\geq 1}$ and $\mathcal{T}'_{k\geq 1}$ and an interaction $\mathcal{L}_I[\phi]$, the difference can be absorbed by a ***finite renormalization****:*

$$\mathcal{T}'_{\mathcal{L}_I}[A_I(x)B_I(y)\cdots] = \mathcal{T}_{\mathcal{L}_I+O(\hbar)}[(A_I(x)+O(\hbar))(B_I(y)+O(\hbar))\cdots]$$

- **Special features:**
  - No path inegral.
  - No Euclidean Wick rotation.
  - Mathematically precise framework for textbook QFT.
- **Surveys and summaries:**
  - Hollands, *Renormalized quantum Yang-Mills fields in curved spacetime* RMP (2009) **20** 1033 0705.3340
  - Brunetti et al., *Advances in Algebraic Quantum Field Theory* Springer (2015)
  - Fröb, *Anomalies in Time-Ordered Products and Applications to the BV-BRST Formulation of Quantum Gauge Theories* CMP (2019) **372** 281 1803.10235

Finite Renormalization vs Anomalies

- If O is any classical local observable, then any **quantization prescription** $O \mapsto {:}O{:}$ suffers from **ambiguities.** Why not use ${:}O{:}' = {:}O{:} + O(\hbar)$? These are **finite renormalizations**!
- This is a manifestation of the well-known **operator ordering ambiguity** in quantum mechanics. Quantization is **not unique**!
- An unlucky quantization can result in **anomalies**:
  - Internal or gauge symmetries not preserved.
  - Conservation laws violated (e.g., $\nabla^a T_{ab} \neq 0$).
- Can **anomalies** be cancelled by exploiting **ambiguities**? A precise **classification** of the ambiguities is necessary to answer the question.
- Renormalization ambiguities on **curved spacetime**:
  - How much does the definition depend on the vacuum state?
  - Is the definition local?
  - Is the definition covariant?
  - How much more ambiguity compared to Minkowski spacetime?

Finite Renormalization on Minkowski Spacetime

- Sub-singular Wick ordering subtractions are not unique, changing them, generally produces

$$:\phi^k:' = :\phi^k: + \sum_{i<k} Z_i :\phi^i:$$

  where the $Z_i:\phi^i:$ are the **finite renormalization counter-terms.**
- On Minkowski space, there are many ways to constrain them:
  - Poincaré invariance.
  - Uniqueness of Fock vacuum.
  - Scaling dimensions.
  - Internal symmetries (e.g., $\phi \mapsto -\phi$). Etc.
- Examples for the free scalar field, $\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$:
  - $\phi^4 \mapsto :\phi^4: + Z_1 m^2 :\phi^2: + Z_2 m^4$
  - $(\partial\phi)^2 \mapsto :(\partial\phi)^2: + Z_1 m^2 :\phi^2: + Z_2 m^4$
  - $\partial_a\phi\partial_b\phi \mapsto :\partial_a\phi\partial_b\phi: + Z_1 \eta_{ab} m^2 :\phi^2: + Z_2 \eta_{ab} m^4$

Locally Covariant Fields on Curved Spacetime

- Our work is in the framework of Locally Covariant QFT on Curved Spacetimes (Hollands-Wald, Brunetti-Fredenhagen-Verch, …).
- A QFT is an assignment of a $*$-algebra of observables to a spacetime, $(M, \mathbf{g}) \to \mathcal{A}(M, \mathbf{g})$. It is **locally covariant** if
  - a causal isometric embedding $(M, \mathbf{g}) \to (M', \mathbf{g}')$ induces an injective homomorphism $\mathcal{A}(M, \mathbf{g}) \to \mathcal{A}(M', \mathbf{g}')$;
  - these homomorphisms respect spacelike commutativity, time slice property.
- A local field $(M, \mathbf{g}) \mapsto \Phi_{(M,\mathbf{g})}$ is a distribution on M valued in $\mathcal{A}(M, \mathbf{g})$. It is **locally covariant** when $\Phi_{(M,g)}(f) \in \mathcal{A}(M, \mathbf{g})$ respects the inclusions and isomorphisms induced by isometries.
- In categorical language, $\mathcal{A}$ is a **covariant functor** from spacetimes to algebras and Φ is a **natural transformation** from the functor of test functions to the algebra functor $\mathcal{A}$.

Result of Hollands and Wald (2001) [arXiv:gr-qc/0103074]

- Consider a massive, curvature coupled scalar field

$$\mathcal{L} = -\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - \xi R\phi^2.$$

- To any polynomial $P(\phi)$, we can associate a locally covariant local field $:P(\phi):$ that essentially reduces to the corresponding Wick polynomial on Minkowski space.
- The assignment of the field is not unique. Under technical conditions, the **ambiguity** is precisely characterized as follows: Given two prescriptions $:\cdots:$ and $:\cdots:'$, there exists a sequence of coefficients C_k such that for each n:

$$:\phi^n:' - :\phi^n: = \sum_{k=0}^{n-1} \binom{n}{k} C_{n-k} :\phi^k: \quad \text{(setting } \hbar = 1\text{)},$$

with each $C_k = C_k[\mathbf{g}, m^2, \xi]$ a **scalar** diff-op. that depends **polynomially** on the local Riemann tensor $\mathbf{R}$ and its derivatives, depends **polynomially** on m^2 and depends **analytically** on ξ.

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Problems with Hollands & Wald

- The result of H-W is intuitive and appealing, reducing to the folklore result on Minkowski spacetime.
- **But:** no vectors B_μ or spinors ψ, no derivatives $\partial_\mu \phi$, no time ordered products $T(:\phi^2(x)::\bar{\psi}\gamma^\mu \nabla_\mu \psi(y):)$, no covariance for background gauge field transformations $(M, \mathbf{g}, \mathbf{A}) \mapsto (M, \mathbf{g}, \mathbf{A} + \partial \mathbf{u})$.
- H-W do **claim** a reasonable result that covers some of these cases, but for a proof they only say that it should be analogous to the scalar case.
- The technical conditions involve **analyticity** in an essential and technically cumbersome way. It is unnatural in smooth differential geometry.
- **Goal:** Eventually address all these issues.
  But for now, just generalize to Wick powers of bosonic vector-valued fields and eliminate the **analyticity** axiom.

Existence vs Classification

- In [arXiv:gr-qc/0103074] H-W **classified** the renormalization ambiguities, **conditional** on the **existence** of at least one construction consistent with their axioms.
- There is an obvious candidate construction scheme: point split Hadamard parametrix regularization scheme.
- In the later work [arXiv:gr-qc/0111108], the proved **existence**, by showing this method to be consistent with the axioms.
- In our work (with V. Moretti and/or A. Melati), we have restricted our attention to **classification**, while **existence** is left to future work. It is expected that the Hadamard regularization scheme will still work.

Our Axioms / Renormalization Conditions

- We can essentially reproduce the H-W result, with updated axioms:
  - **normalization**, $:\phi: = \phi$
  - **commutators**, $[:A(x):, \phi(y)] = i:\{A(x), \phi(y)\}:$
  - **completeness**, $\forall x\colon [A, \phi(x)] = 0 \iff A = \alpha 1$
  - **scaling**, $(\mathbf{g}, \phi, \mathbf{t}) \mapsto (\mu^{-2}\mathbf{g}, \mu^{d_\phi}\phi, \mu^{d_\mathbf{t}}\mathbf{t})$
    $$\implies :\phi^k: \mapsto \mu^{kd_\phi}(:\phi^k: + O(\log\mu))$$
  - **locality** and **covariance**
  - **smoothness**, $\omega(:A_{\mathbf{g},\mathbf{t}}(x):)$ is jointly smooth in (x, s) under smooth compactly supported variations of $(\mathbf{g}_s, \mathbf{t}_s)$, for some non-empty class of states ω (e.g., *Hadamard*).
  - **Leibniz rule**, **perturbative agreement** (not explicitly used)
- The technical **analyticity** requirement of H-W (*analyticity* upon restriction to *analytic* $(\mathbf{g}, m^2, \xi)$) has been replaced by our **smoothness** axiom with respect to $(\mathbf{g}, \mathbf{t})$.
- Also, $\phi = (\phi_i)$, $\mathbf{t} = (t_j)$ could be any **natural multi-component field**. We restrict to tensor fields.

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Conditions on the background fields

- The components of the dynamical fields may have different scaling degrees, $\mu^{d_\phi}\phi = (\mu^{d_i}\phi_i)$. We do not need to require any conditions on the weights d_i.
- The components of the background fields may also have different scaling degrees, $\mu^{d_\mathbf{t}}\mathbf{t} = (\mu^{s_j} t_j)$. Each t_j is a component of a covariant tensor of rank ℓ_j. A background field **t** is **admissible** if

$$\ell_j + s_j \geq 0 \quad \text{(for all } j\text{)}.$$

  When the equality $\ell_j + s_j = 0$ holds, the component t_j is said to be **marginal**. We denote by $\mathbf{z} = (t_j)_{\text{marginal}}$ the marginal components.
- **Example:** m^2 $(\ell = 0,\ s = 2)$, ξ $(\ell = 0,\ s = 0)$, g_{ab} $(\ell = 2,\ s = -2)$
- In the physics literature, the scaling weights d_i and s_j are sometimes called the **mass dimension**.

Theorem (Kh-Melati-Moretti)

*Let ϕ be a multicomponent **locally covariant** tensor field, coupled to **admissible** background tensor fields $\mathbf{t}$, with marginal components $\mathbf{z}$. Let $\{{:}\phi^n{:}\}_{n=1,2,\dots}$ and $\{{:}\phi^n{:}'\}_{n=1,2,\dots}$ be two families of Wick powers of ϕ. Then there exists a family of locally-covariant c-number fields $\{C_k\}_{k=1,2,\dots}$, such that $C_1 = 0$ and, for every $k = 1, 2, \dots$,*

(i) $$ {:}\phi_{i_1} \cdots \phi_{i_n}{:}' = {:}\phi_{i_1} \cdots \phi_{i_n}{:} + \sum_{k=0}^{n-1} \binom{n}{k} {:}\phi_{(i_1} \cdots \phi_{i_k}{:}\, C^{n-k}_{i_{k+1}\cdots i_n)}[\mathbf{g}, \mathbf{t}] , $$

(ii) *each $C^k_{i_1\cdots i_k}[\mathbf{g}, \mathbf{t}]$ is homogeneous of appropriate degree,*

(iii) *more precisely $C^k_{i_1\cdots i_k}[\mathbf{g}, \mathbf{t}] = \sum_{j=1}^{N_k} c^k_j[\mathbf{g}, \mathbf{t}](P^k_j)_{i_1\cdots i_k}[\mathbf{g}, \mathbf{t}]$ for **equivariant polynomials** $P^k_j[\mathbf{g}, \mathbf{t}] = P^k_j(\mathbf{g}^{-1}, \varepsilon, \mathbf{R}, \nabla\mathbf{R}, \mathbf{t}, \nabla\mathbf{t}, \cdots)$, with **smooth invariant** invariant scalar $c^k_j[\mathbf{g}, \mathbf{t}] = c^k_j(\mathbf{z})$ coefficients.*

N.B.: For mixed Bose-Fermi fields ϕ, it suffices to use **fermionic signs**, $X_{(i_1\cdots i_n)} = \sum_{\sigma\in S_n} (-)^\sigma X_{\sigma i_1 \cdots \sigma i_n}$. But **spin equivariance** needs more attention!

Notes on the proof (1 of 4)

We closely follow the structure of our previous work on scalars (which followed the original H-W proof, with greater attention to detail).

Starting from **normalization**, use induction on **commutators** and **completeness** to get

$$:\phi_{i_1} \cdots \phi_{i_n}:' = :\phi_{i_1} \cdots \phi_{i_n}: + \sum_{k=0}^{n-1} \binom{n}{k} :\phi_{(i_1} \cdots \phi_{i_k}: C^{n-k}_{i_{k+1} \cdots i_n)}[\mathbf{g}, \mathbf{t}] \, ,$$

with c-number coefficients $C^{n-k}_{i_{k+1} \cdots i_n}[\mathbf{g}, \mathbf{t}]$.

For scalar ϕ and $\mathbf{t} = (m^2, \xi)$, we get the H-W formula

$$:\phi^n:' - :\phi^n: = \sum_{k=0}^{n-1} \binom{n}{k} C_{n-k}[\mathbf{g}, m^2, \xi] :\phi^k: .$$

Notes on the proof (2 of 4)

Using **locality** and **smoothness**, we conclude that the coefficients $(\mathbf{g}, \mathbf{t}) \mapsto C^k[\mathbf{g}, \mathbf{t}]$ are *local* and *regular*, hence $C^k(x, \mathbf{g}, \partial\mathbf{g}, \ldots, \mathbf{t}, \partial\mathbf{t}, \ldots)$.

Theorem (Peetre-Slovák)

*A map $C^\infty \to C^\infty$ that is **local** (compatible with restriction to smaller domains) and **regular** (maps smooth families to smooth families) must be a smooth differential operator of locally bounded order.*

- Original result for linear maps, Peetre (1959, 1960).
- Extension to nonlinear maps, Slovák (1988).
- Great exposition, Navarro-Sancho [arXiv:1411.7499].

Key place where the **analyticity** was previously used by H-W.

Notes on the proof (3 of 4)

Theorem (Thomas Replacement)

A smooth homogeneous tensor function of $\mathbf{g}, \partial\mathbf{g}, \ldots \mathbf{T}, \partial\mathbf{T}, \ldots$ *is equivariant under diffeomorphisms iff it is a smooth homogeneous pointwise* $\mathbf{g}$*-isotropic function of* $\mathbf{R}, \nabla\mathbf{R}, \ldots \mathbf{T}, \nabla\mathbf{T}, \ldots$ *and* ε*.*

- Original, T.Y. Thomas (1920s). More modern, Slovák (1992).
- Concise, self-contained proof (our paper).

Using **covariance** (under diffeomorphisms) and **scaling**, the structure of the differential operators C^k can be refined to

$$\begin{aligned} u \cdot C^k[\mathbf{g}, \mathbf{t}] &= u \cdot C^k(x, \mathbf{g}, \partial\mathbf{g}, \ldots, \mathbf{t}, \partial\mathbf{t}, \ldots) \\ &= P^k_{\mathbf{g}}(\mathbf{R}, \nabla\mathbf{R}, \cdots, \mathbf{t}, \nabla\mathbf{t}; u), \end{aligned}$$

where $P^k_{\mathbf{g}}$ are **homogeneous g-isotropic scalar** functions, which is **linear** in u auxiliary tensors.

Notes on the proof (4 of 4)

Theorem (Luna, Richardson 1970s + incremental improvement)

*A **smooth equivariant** function on fin.dim. $O(\mathbf{g})$ or $SO(\mathbf{g})$ reps is a linear combination of **polynomial equivariants** with coefficients essentially **smooth** functions of **polynomial scalar invariants**.*

Theorem (FFT of Invariant Theory, Weyl 1930)

*Scalar $\mathbf{g}$-isotropic **polynomials** are generated by (a) outer products, (b) index contractions with $\mathbf{g}$, (c) index contractions with ε.*

Theorem (Folklore)

A positive weight, homogeneous function that is smooth around zero is a polynomial.

Thus, with only **admissible** background fields $\mathbf{t}$,

$$u \cdot C^k[\mathbf{g}, \mathbf{t}] = P^k_{\mathbf{g}}(\mathbf{R}, \nabla\mathbf{R}, \ldots, \mathbf{t}, \nabla\mathbf{t}, \ldots; u)$$

is a sum of **homogeneous invariant polynomials**, whose coefficients are (locally) **smooth functions** of (finitely many) **invariant scalar polynomials** in (marginal components) $\mathbf{z}$.

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Example: scalar Klein-Gordon, with derivative

Scalar Scalar Klein-Gordon in n-dimensions:

$$\Box_{\mathbf{g}}\phi - m^2\phi + \xi R\,\phi = 0\,, \quad \left(\Phi = (\phi, \nabla_a\phi),\ \Phi \mapsto \mu^{\frac{n-2}{2}}\Phi\right).$$

Admissible: m^2 ($\ell + s = 0 + 2$), ξ ($\ell + s = 0 + 0$); marginal: ξ.

$$\begin{bmatrix} :\phi^2:' \\ :\phi\nabla_a\phi:' \\ :\nabla_{(a}\phi\nabla_{b)}\phi:' \end{bmatrix} = \begin{bmatrix} :\phi^2: \\ :\phi\nabla_a\phi: \\ :\nabla_{(a}\phi\nabla_{b)}\phi: \end{bmatrix} + \begin{bmatrix} \alpha_1 m^2 + \alpha_2 R + A_{\xi,m^2} \\ \beta_1\nabla_a R + B_{\xi,m^2} \\ g_{ab}\left(\gamma_1 m^4 + \gamma_2 m^2 R + \gamma_3 R^2\right) + \left(\gamma_4 m^2 + \gamma_5\Box\right) R_{ab} + C_{\xi,m^2} \end{bmatrix}$$

with **smooth** $\{\alpha,\beta,\gamma\}_j = \{\alpha,\beta,\gamma\}_j(\xi)$, where also

$$A_{\xi,m^2} = \alpha_3\nabla^a\xi\nabla_a\xi + \alpha_4\Box\xi\,,$$

$$\begin{aligned} B_{\xi,m^2} &= \beta_2\nabla_a m^2 + \beta_3 m^2\nabla_a\xi \\ &+ \beta_4 R\nabla_a\xi + \beta_5 R_{ab}\nabla^b\xi \\ &+ \beta_6(\nabla^b\xi\nabla_b\xi)\nabla_a\xi + \beta_7\Box\xi\nabla_a\xi \\ &+ \beta_8\nabla^b\xi\nabla_{(b}\nabla_{a)}\xi + \beta_9\nabla_a\Box\xi\,, \end{aligned}$$

$$\begin{aligned} C_{\xi,m^2} &= \gamma_6\nabla_{(a}\xi\nabla_{b)}m^2 + \gamma_7 m^2\nabla_{(a}\xi\nabla_{b)}\xi + \gamma_8 R\nabla_a\xi\nabla_b\xi + \gamma_9 R_{ab}(\nabla\xi)^2 \\ &+ \gamma_{10}R_{c(a}\nabla_{b)}\xi\nabla^c\xi + \gamma_{11}g_{ab}\nabla^c\xi\nabla_c m^2 + \gamma_{12}g_{ab}m^2(\nabla\xi)^2 \\ &+ \gamma_{13}g_{ab}R(\nabla\xi)^2 + \gamma_{14}g_{ab}R^{bc}\nabla_b\xi\nabla_c\xi + \gamma_{15}\nabla_{(a}\nabla_{b)}m^2 \\ &+ \gamma_{16}m^2\nabla_{(a}\nabla_{b)}\xi + \gamma_{17}\Box\xi\nabla_{(a}\nabla_{b)}\xi + \gamma_{18}R\nabla_{(a}\nabla_{b)}\xi + \gamma_{19}R_{ab}\Box\xi \\ &+ \gamma_{20}g_{ab}\Box m^2 + \gamma_{21}g_{ab}m^2\Box\xi + \gamma_{22}g_{ab}(\Box\xi)^2 + \gamma_{23}g_{ab}R\Box\xi \\ &+ \gamma_{24}\nabla_{(a}\xi\nabla_{b)}\Box\xi + \gamma_{25}\nabla_{(a}\nabla_{b)}\Box\xi \\ &+ \gamma_{26}g_{ab}\nabla^c\xi\nabla_c\Box\xi + \gamma_{27}g_{ab}\Box^2\xi\,. \end{aligned}$$

Example: Vector Klein-Gordon

Vector Klein-Gordon in *n*-dimensions:

$$\Box_{\mathbf{g}} A_a - m^2 A_a + \xi_a^b R\, A_b = 0\,, \quad \left(A_b \mapsto \mu^{\frac{n-2}{4}} A_b\right)\,.$$

Admissible: m^2 ($\ell + s = 0 + 2$), ξ_a^b ($\ell + s = 2 - 2$); marginal: ξ_a^b.

$$:A_a A_b:' = :A_a A_b: + (y_1 m^2 + y_2 R) g_{ab} + y_3 R_{ab} + (y_4 m^2 + y_5 R)\xi_{ab} + B_\xi\,,$$

where

$$\begin{aligned} B_\xi &= y_6 g_{ab} \Box \xi_c^c + y_7 \nabla_{(a} \nabla_{b)} \xi_c^c + y_8 g_{ab} \nabla^c \xi_d^d \nabla_c \xi_d^d + y_9 g_{cd} \nabla_{(a} \xi^{cd} \nabla_{b)} \xi_c^c \\ &+ y_{10} \left(\nabla_{(a} \nabla_{b)} \xi_{cd}\right) \xi^{cd} + y_{11} \nabla_{(a} \xi^{cd} \nabla_{b)} \xi_{cd} + y_{12} g_{ab} \left(\Box \xi_{cd}\right) \xi^{cd} + y_{13} g_{ab} \nabla^c \xi_{de} \nabla_c \xi^{de} \\ &+ y_{14} \xi_{ab} \Box \xi_c^c + y_{15} \xi_{ab} \nabla^c \xi_d^d \nabla_c \xi_d^d + y_{16} \Box \xi_{ab} + y_{17} \xi_{ab} \left(\Box \xi_{cd}\right) \xi^{cd} + y_{18} \xi_{ab} \nabla^c \xi_{de} \nabla_c \xi^{de} \\ &+ y_{19} \xi_{cd} \nabla_{(a} \xi^{cd} \nabla_{b)} \xi_c^c + y_{20} \xi_{cd} \xi_{ef} \nabla_{(a} \xi^{ef} \nabla_{b)} \xi^{cd}\,, \end{aligned}$$

with **smooth** $y_j = y_j(\operatorname{tr}\boldsymbol{\xi} = \xi_a^a, \operatorname{tr}\boldsymbol{\xi}^2 = \xi_a^b \xi_b^a, \operatorname{tr}\boldsymbol{\xi}^3, \ldots, \operatorname{tr}\boldsymbol{\xi}^n)$ (locally).
Stable orbit types are separated by the **matrix discriminant**

$$p_0(\boldsymbol{\xi}) = \operatorname{disc}(\boldsymbol{\xi}) = \det\left(\operatorname{tr}\boldsymbol{\xi}^{i+j-2}\right)_{i,j=1}^{n}\,.$$

Discussion

- In **Kh-Moretti** (2016) and **Kh-Melati-Moretti** (2019) we have revisited the **classification** of finite renormalizations of **locally covariant bosonic fields**. We have replaced the H-W analyticity axiom by a smoothness axiom, and carefully generalized to dynamical and background **tensor fields**.
- **Reminder:** need to check that the smoothness axiom is verified!
- **Remark:** need incremental improvents in smooth invariant theory.
- It remains to generalize the results to ~~**tensor**~~ and **spinor fields**, background **gauge fields**, Wick products with **derivatives** and **time ordered products**.

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Thank you for your attention!