A look at the geometry of the 5-dimensional charged rotating black hole

[arXiv:2112.13266]

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05 Apr 2022
GR Seminar
Charles University, Prague

Black Hole Geometry Ansatz

- Ansatz for 5d charged rotating BH with equal angular momenta. [Kunz, Navarro-Lérida *et al.* (2005, ...)]:

$$
\begin{aligned}
\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu &= -f\,\mathrm{d}t^2 + \frac{m}{f}\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2\right) \\
&\quad + \frac{n}{f}r^2\left[\sin^2\theta\left(\mathrm{d}\phi - \frac{\omega}{r}\,\mathrm{d}t\right)^2 + \cos^2\theta\left(\mathrm{d}\psi - \frac{\omega}{r}\,\mathrm{d}t\right)^2\right] \\
&\quad + \frac{m-n}{f}r^2\sin^2\theta\cos^2\theta\,(\mathrm{d}\phi - \mathrm{d}\psi)^2
\end{aligned}
$$

$$
A = A_\mu\,\mathrm{d}x^\mu = a_0\,\mathrm{d}t + a_\phi\left(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi\right)
$$

  t – time, r – (Kunz) radial, θ – polar, ϕ, ψ – azimuthal coordinates. f, m, n, ω, a_0, a_ϕ depend only on $r \rightsquigarrow$ numerical solution

- Naive boundary conditions:
  - asymptotic flatness ($r = \infty$): $f, m, n \sim 1$, $\omega, a_0, a_\phi \sim 0$
  - regular horizon ($r = r_\mathrm{H}$): $|\omega|, |a_0|, |a_\phi| < 0$, $f, m, n, a'_\phi = 0$

- **Q:** Algebraic type (bulk, horizon)? Other geometric properties?

Problem With Radial Coordinate

Naive attempts at constructing horizon-penetrating coordinates revealed problems.
Same problem occurred when matching known solutions.

charged $\hat{Q} \neq 0$, non-rotating $\hat{J} = 0$	uncharged $\hat{Q} = 0$, rotating $\hat{J} \neq 0$
$f = \frac{(\mathrm{r}^2 - \mathrm{r}_+^2)(\mathrm{r}^2 - \mathrm{r}_-^2)}{\mathrm{r}^4}$	$f = \frac{\Sigma^2 - \hat{M}\mathrm{r}^2}{\Sigma^2 + a^2\hat{M}}$
$r^2 m = \mathrm{r}^2 f$	$r^2 m = \Sigma f$
$r^2 n = \mathrm{r}^2 f$	$r^2 n = \frac{\Sigma^2 - \hat{M}\mathrm{r}^2}{\Sigma}$
$\omega = 0$	$\frac{\omega}{r} = \frac{a\hat{M}}{\Sigma^2 + a^2\hat{M}}$
$a_0 = \frac{\hat{Q}}{\mathrm{r}^2}$	$a_0 = 0$
$a_\phi = 0$	$a_\phi = 0$
$\hat{M} = \mathrm{r}_+^2 + \mathrm{r}_-^2, \quad \hat{Q} = \frac{\sqrt{3}}{2}\mathrm{r}_+\mathrm{r}_-$	$\Sigma = \mathrm{r}^2 + a^2$
	$\hat{M} = (\mathrm{r}_+ + \mathrm{r}_-)^2, \quad a := \hat{J}/\hat{M} = \sqrt{\mathrm{r}_+\mathrm{r}_-}$

r – known regular radial coordinate. The Kunz radial coordinate

$$r = r_\mathrm{H}\sqrt{\frac{\sqrt{\mathrm{r}^2 - \mathrm{r}_-^2} + \sqrt{\mathrm{r}^2 - \mathrm{r}_+^2}}{\sqrt{\mathrm{r}^2 - \mathrm{r}_-^2} - \sqrt{\mathrm{r}^2 - \mathrm{r}_+^2}}} = r_\mathrm{H} + \mathcal{O}\left(\sqrt{\mathrm{r} - \mathrm{r}_+}\right) \quad \text{is } \textbf{singular!}$$

Reparametrized Ansatz

- The singular Kunz coordinate r explicitly appears in the Einstein-Maxwell equations.
  - **Q:** Can near-horizon expansions be trusted? Are the BH equations well-posed?
- Reparametrize ansatz by absorbing all explicit radial factors:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -f\,\mathrm{d}t^2 + \frac{\mathrm{m}}{f}\left(\frac{(\mathrm{r}\,\mathrm{dr})^2}{\mathrm{N}} + \mathrm{d}\theta^2\right) + \frac{\mathrm{N}}{\mathrm{m}}\left[\sin^2\theta\,(\mathrm{d}\phi - \varpi\,\mathrm{d}t)^2 + \cos^2\theta\,(\mathrm{d}\psi - \varpi\,\mathrm{d}t)^2\right] + \frac{\mathrm{m}^2 - f\mathrm{N}}{\mathrm{m}f}\sin^2\theta\cos^2\theta\,(\mathrm{d}\phi - \mathrm{d}\psi)^2$$

$$\mathrm{m} = mr^2\,, \quad \varpi = \frac{\omega}{r}\,, \quad \mathrm{N} = \frac{mnr^4}{f}\,, \quad \frac{(\mathrm{r}\,\mathrm{dr})^2}{\mathrm{N}} = \frac{(\mathrm{d}r)^2}{r^2}$$

From now on set $\mathrm{R} = \mathrm{r}^2$ and use $(-)' = \frac{\mathrm{d}}{\mathrm{dR}}(-)$.

Reduced Einstein-Maxwell Equations (EME)

R-autonomous conservation laws ($\hat{Q}$ – charge, $\hat{J}$ – angular momentum, $r_- < r_+$ – horizons):

$$N'' = 2 \implies \qquad N = (R - r_+^2)(R - r_-^2)$$

$$\left(\frac{N}{f}(a_0' + \varpi a_\phi')\right)' = 0 \implies \frac{N}{f}(a_0' + \varpi a_\phi') = -\hat{Q}$$

$$\left(\frac{N^2}{fm}\varpi' + 4\hat{Q}a_\phi\right)' = 0 \implies \frac{N^2}{fm}\varpi' + 4\hat{Q}a_\phi = -2\hat{J}$$

R-autonomous 2nd order BVP (between infinity $R = \infty$ and outer horizon $R = r_H^2 = r_+^2$):

$$f^2 m \left(\frac{m}{f^2}f'\right)' - mm'\left(\frac{m'}{m} - 2\frac{N'}{N}\right) f - \left(4\frac{m^2}{N} - f\right) f = \frac{8}{3}\frac{fm}{N}\left(2m^2(a_\phi')^2 - fa_\phi^2\right) - \frac{4\hat{Q}^2}{3}\frac{f^2m^2}{N^2}$$

$$fm\left(\frac{N}{m}m'\right)' + \left(\frac{f^2N}{m^2} - 2f\right)m = \frac{4}{3}f\left(2m^2(a_\phi')^2 - fa_\phi^2\right) + \frac{4\hat{Q}^2}{3}\frac{f^2m}{N} + 4\frac{f^2m^2}{N^2}(\hat{J} + 2\hat{Q}a_\phi)^2$$

$$m\,(ma_\phi')' - fa_\phi = 2\hat{Q}\frac{fm^2}{N^2}(\hat{J} + 2\hat{Q}a_\phi)$$

R-autonomous 1st order constraint:

$$C := -m^2(a_\phi')^2 + fa_\phi^2 - m + \frac{fN}{4m} + \hat{Q}^2\frac{fm}{N} + \frac{fm^2}{N^2}(\hat{J} + 2\hat{Q}a_\phi)^2 + \frac{Nm'}{8}\left(\frac{f'}{f} - 2\frac{m'}{m} + 4\frac{N'}{N}\right) - \frac{Nmf'}{8f}\left(2\frac{f'}{f} - \frac{m'}{m} + 2\frac{N'}{N}\right) = 0$$

with compatibility condition $(C/f^2)' = 0 \mod \mathrm{BVP}$

Asymptotic Flatness

Find Bondi coordinates $(u_\pm, \mathrm{r}, \theta, \phi_\pm, \psi_\pm)$ at null infinity such that [Satishchandran-Wald 2019]

$$g_{\mu\nu} = \begin{pmatrix} -1 & \pm 1 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{r}^2 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{r}^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & \mathrm{r}^2\cos^2\theta \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\mathrm{r}^{-2}) & \mathcal{O}(\mathrm{r}^{-2}) & \mathcal{O}(\mathrm{r}^{-1}) & \mathcal{O}(\mathrm{r}^{-1}) & \mathcal{O}(\mathrm{r}^{-1}) \\ \mathcal{O}(\mathrm{r}^{-2}) & 0 & 0 & 0 & 0 \\ \mathcal{O}(\mathrm{r}^{-1}) & 0 & \mathcal{O}(\mathrm{r}^{0}) & \mathcal{O}(\mathrm{r}^{0}) & \mathcal{O}(\mathrm{r}^{0}) \\ \mathcal{O}(\mathrm{r}^{-1}) & 0 & \mathcal{O}(\mathrm{r}^{0}) & \mathcal{O}(\mathrm{r}^{0}) & \mathcal{O}(\mathrm{r}^{0}) \\ \mathcal{O}(\mathrm{r}^{-1}) & 0 & \mathcal{O}(\mathrm{r}^{0}) & \mathcal{O}(\mathrm{r}^{0}) & \mathcal{O}(\mathrm{r}^{0}) \end{pmatrix}$$

$$A_\mu = \begin{pmatrix} \mathcal{O}(\mathrm{r}^{-2}) & 0 & \mathcal{O}(\mathrm{r}^{-1}) & \mathcal{O}(\mathrm{r}^{-1}) & \mathcal{O}(\mathrm{r}^{-1}) \end{pmatrix}$$

Combine asymptotic flatness with Frobenius method at singular point $\mathrm{R} = \infty$:

Bondi coordinates	asymptotic flatness	w/ EME analysis
$\mathrm{d}u_\pm = \mathrm{d}t \pm \sqrt{\frac{\mathrm{m}}{\mathrm{N}}}\frac{\mathrm{dR}}{2f}$	$f = 1 + \mathcal{O}(\mathrm{R}^{-1})$	$f = 1 - \frac{\hat{M}}{\mathrm{R}} + \mathcal{O}(\mathrm{R}^{-2})$
$\mathrm{r} = \sqrt{\mathrm{R}}$	$\mathrm{m} = \mathrm{R} + \mathcal{O}(\mathrm{R}^0)$	$\mathrm{m} = \mathrm{R} - \frac{1}{2}(\hat{M} + \mathrm{r}_+^2 + \mathrm{r}_-^2) + \mathcal{O}(\mathrm{R}^{-1})$
$\theta = \theta$	$a_\phi = \mathcal{O}(\mathrm{R}^{-1/2})$	$a_\phi = \frac{a_\phi^{(-1)}}{\mathrm{R}} + \mathcal{O}(\mathrm{R}^{-2})$
$\mathrm{d}\phi_\pm = \mathrm{d}\phi \pm \varpi\sqrt{\frac{\mathrm{m}}{\mathrm{N}}}\frac{\mathrm{dR}}{2f}$	$\mathrm{N} = \mathrm{R}^2 + \mathcal{O}(\mathrm{R}^1)$	$\mathrm{N} = \mathrm{R}^2 - (\mathrm{r}_+^2 + \mathrm{r}_-^2)\mathrm{R} + \mathrm{r}_+^2\mathrm{r}_-^2$
$\mathrm{d}\psi_\pm = \mathrm{d}\psi \pm \varpi\sqrt{\frac{\mathrm{m}}{\mathrm{N}}}\frac{\mathrm{dR}}{2f}$	$\varpi = \mathcal{O}(\mathrm{R}^{-3/2})$	$\varpi = \frac{\hat{J}}{\mathrm{R}^2} + \mathcal{O}(\mathrm{R}^{-3})$
	$a_0 = \mathcal{O}(\mathrm{R}^{-1})$	$a_0 = \frac{\hat{Q}}{\mathrm{R}} + \mathcal{O}(\mathrm{R}^{-2})$

2 free BVP integration constants + external parameters

Horizon Regularity

Find Kruskal-like coordinates $(U, V, \theta, \Phi, \Psi)$ covering the past/future horizons and the bifurcation sphere. Both $g_{\mu\nu}$ and A_μ must be smoot at the horizon $\mathrm{R} = \mathrm{r}_\mathrm{H}^2$ ($\mathrm{r}_\mathrm{H} = \mathrm{r}_+ > \mathrm{r}_-$).

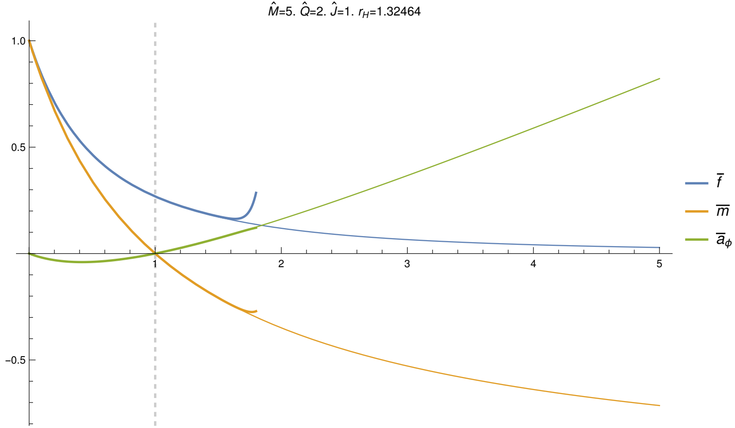
Combine horizon regularity with Frobenius method at singular point $\mathrm{R} = \mathrm{r}_\mathrm{H}^2$ ($\rho = \mathrm{R} - \mathrm{r}_\mathrm{H}^2$):

Kruskal-like coordinates	horizon regularity	w/ EME analysis
$U = \sqrt{\mathrm{R}/\mathrm{r}_\mathrm{H}^2 - 1}\, e^{\kappa t/2}\mathcal{O}(1)$	$f = \mathcal{O}(\rho)$	$f = f^{(1)}\rho + f^{(2)}\rho^2 + f^{(3)}\rho^3 + \mathcal{O}\left(\rho^4\right)$
$V = \sqrt{\mathrm{R}/\mathrm{r}_\mathrm{H}^2 - 1}\, e^{-\kappa t/2}\mathcal{O}(1)$	$\mathrm{m} = \mathcal{O}(\rho)$	$\mathrm{m} = \mathrm{m}^{(1)}\rho + \mathcal{O}\left(\rho^2\right)$
$\theta = \theta$	$a_\phi = \mathcal{O}(1)$	$a_\phi = a_\phi^{(0)} + \mathcal{O}(\rho)$
$\Phi = \phi - \varpi(\mathrm{r}_\mathrm{H}^2)t$	$\mathrm{N} = \mathcal{O}(\rho)$	$\mathrm{N} = (\mathrm{r}_+^2 - \mathrm{r}_-^2)\rho + \rho^2$
$\Psi = \psi - \varpi(\mathrm{r}_\mathrm{H}^2)t$	$\varpi = \mathcal{O}(1)$	$\varpi = \varpi^{(0)} + \mathcal{O}(\rho)$
	$a_0 = \mathcal{O}(1)$	$a_0 = a_0^{(0)} + \mathcal{O}(\rho)$

4 free BVP integration constants + external parameters

Well-Posedness

- Unique solution of 2nd order 3×3 BVP: generic intersection of 2-param. family with 4-param. family in 6-param. solution space.
- External charges $\hat{M}, \hat{J}, \hat{Q}$ determine solution: shift $r_-^2 \mapsto 0$ by R-autonomy, then invert $r_H^2 := r_+^2 \leftrightsquigarrow \hat{M}$.



Normalized f, m, a_ϕ vs r_H^2/R; extrapolated **outer BVP** to seed *inner IVP*.

Algebraic Type

- Past/future horizon ($\mathsf{R} = \mathsf{r}_\mathsf{H}^2$): **Type II** — one of the Kruskal-like $\mathrm{d}U$ or $\mathrm{d}V$ is a Weyl aligned null direction (WAND)
- Bifurcation sphere: **Type D** — both $\mathrm{d}U$ and $\mathrm{d}V$ are WANDs
- Bulk, outside the horizon: **not Type II** —
  off-shell case-by-case analysis of 5d Bel-Debever criteria; [Ortaggio 2009]
  compatibility with EME ruled out by checking against the general $\mathcal{O}(\mathsf{R}^{-5})$ asymptotic solution, unless $\hat{J} = 0$ or $\hat{Q} = 0$ both of which are Type D.
- Ricci $R_{\mu\nu}$ and Maxwell $F_{\mu\nu}$: **Type D**
- $g_{\mu\nu} \neq \eta_{\mu\nu} - 2Hk_\mu k_\nu$: The spacetime is **not Kerr-Schild** (w.r.t Minkowski metric) with null **geodesic** k_μ. [Ortaggio-Pravda-Pravdová 2009]

Discussion

- The original works on the 5d charged rotating black hole (inadvertently?) used a **singular** r coordinate.
- We have confirmed the well-posedness of the reduced Einstein-Maxwell equations w.r.t a **regular** r coordinate, with regularity at the **horizon** and asymptotic flatness at **null infinity**.
- Frobenius analysis of the singular points at $\mathrm{r} = \infty$ and $\mathrm{r} = \mathrm{r}_\mathrm{H}$ opens door to **more robust** numerical methods.
- We have confirmed the **geometric horizon conjecture** (algebraically special Weyl tensor on the horizon).
  [Coley-McNutt-Shoom 2017]
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Thank you for your attention!