The geometry of analytic structures

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17 Aug 2021
Finite-infinite workshop
Wolfgang Pauli Institute, Vienna

A Canonical Example: Complex Structure

- $\mathfrak{C}$ complex holomorphic atlas on a C^∞ manifold M: local charts $z\colon \mathbb{C}^n \to M$, with holomorphic transitions.
- Holonomic Frames: Jacobians of local charts $T\mathfrak{C} \ni Tz = Z\colon M \to FM \subset T^{\oplus 2n}M$ of the frame bundle. Holonomic sections are described by a (1st order) PDE $\mathcal{H} \subset J^\infty FM$.
- G-structure (order-1): Pointwise, $T\mathfrak{C} = J^1\mathfrak{C}/\mathfrak{C}$ defines a sub-bundle $\mathcal{C} \subset FM = J^0 FM$, a principal $[GL(2n,\mathbb{R}) \supset GL(n,\mathbb{C}) = G]$-bundle.
- Strict Integrability $\implies$ Formal Integrability: Existence of holonomic sections $Z\colon M \to \mathcal{C}$, implies non-empty $(\mathcal{H} \cap J^\infty\mathcal{C}) \subset J^\infty FM$ (equivalently, the intrinsic torsion $\tau_\mathcal{C} = 0$).
- Geometric Objects: Principal G-bundle $\mathcal{C} \iff$ adapted $J_x \in \mathrm{End}(T_x M)$, $J_x^2 = -\mathrm{id}$. Vanishing $\tau_\mathcal{C} = 0 \iff$ vanishing *Nijenhuis tensor* $N_J = 0$.
- Formal integrability $\implies$ Strict Integrability (Newlander-Nirenberg'57): $N_J = 0$ implies existence of adapted holonomic local frames $Z\colon M \to \mathcal{C}$.

Analytic Structure

Structure Group: $An(n, \mathbb{R})$ analytic local diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ fixing 0.

- $\mathfrak{A}$ real analytic atlas on a C^∞ manifold M: local charts $x \colon \mathbb{R}^n \to M$, with real analytic transitions.
- Holonomic Frames: Jacobians of local charts $T\mathfrak{A} \ni Tx = X \colon M \to FM \subset T^{\oplus n}M$ of the frame bundle.
- G-structure (order-∞): Pointwise, $J^\infty \mathfrak{A}/\mathfrak{A}$ defines a sub-bundle* $\mathcal{A} \subset J^\infty FM$, a principal $[An(n, \mathbb{R}) = G]$-bundle.
- Formal Integrability: Analog of $\tau_{\mathcal{A}} = 0$**?**
- Strict Integrability: if defined, does $\tau_{\mathcal{A}} = 0$ imply existence of adapted holonomic frames**?**
- Geometric Objects: Analogs of $J^2 = -\mathrm{id}$ and $N_J = 0$**?**

* Germ vs Jet subtlety!

Strict Integrability: Street's Theorem

Definition (Nelson'59)

Let a frame $(X_i)_i$ and a function u be C^∞ on a C^∞ manifold M. The function u is X-analytic when $|X_{i_1} \cdots X_{i_N}(u)| < N! r^N$ locally uniformly on M.

Theorem (Street 2018, arXiv:1808.04635)

Let $(X_i)_i$ be a C^∞ frame on a C^∞ manifold M. If the structure functions c_{ij}^k in $[X_i, X_j] = c_{ij}^k X_k$ are X-analytic, then there exists a C^ω sub-atlas on M making $(X_i)_i$ a C^ω frame.

Conclusion: given a local solution frame $j^\infty X(M) \subset \mathcal{A} \subset J^\infty FM$, there exists also a local adapted holonomic frame $j^\infty \tilde{X}(M) \subset \mathcal{A} \cap \mathcal{H}$. In other words, $\mathcal{A}$ is strictly integrable!

Q: (IK 2014) $\exists$*"Newlander-Nirenberg" for real analytic structures?* MO172729
A: (Street 2018)

Formal Integrability $\implies$ Strict Integrability

- Formal integrability: $\mathcal{A} \subset J^\infty FM$ is formally integrable (as a PDE) when the Cartan connection leaves $\mathcal{A}$ invariant.
- Torsion freeness: The derivatives $X_{i_1} \cdots X_{i_N}(c_{ij}^k)$ in Street's condition can be computed pointwise from $j^\infty X$ and the condition is $An(n, \mathbb{R})$-invariant. Hence, it is a property of $\mathcal{A} \subset J^\infty FM$ and is a good candidate for torsion freeness (analog of $\tau_{\mathcal{A}} = 0$)!

Theorem

If $\mathcal{A} \subset J^\infty FM$ is formally integrable and torsion free, then there locally exist $j^\infty X(M) \subset \mathcal{A}$.

Proof: The fibers of $\mathcal{A}$ are not so big, so we can rescue Frobenius's theorem for the Cartan connection. Let $X^\infty \colon M \to \mathcal{A}$ be any non-holonomic local section of $\mathcal{A}$ and $X \colon M \to FM$ its lowest projection. Identify $J^\infty FM \cong (J^\infty \mathbb{R}_M)^{\oplus n^2}$ via $\tilde{X}_i = v_i^j X_j$.

Cartan connection transport equation for scalars:

$$\partial_\nu u_{\mu_1 \cdots \mu_N} = u_{\nu \mu_1 \cdots \mu_N}. \quad \ldots$$

Formal Integrability $\implies$ Strict Integrability

Proof: ... Change the fiber coordinates pointwise to $u_I = u_{i_1 \cdots i_N} = X^\infty_{(i_1} \cdots X^\infty_{i_N)}(u)$.
Equivalent Cartan transport equation:

$$X^\nu_j \partial_\nu u_I = u_{jI} + \sum_{|H| \le |I|} P^H_{j:I} u_H = (\Delta \cdot u^\infty + P \cdot u^\infty)_{j:I} .$$

Define the analytic norms $\|u\| = \sum_{N=0}^\infty \frac{r^N}{N!} \sum_{|H|=N} |u_H|$. Street's pointwise condition implies the estimates

$$\|\Delta\|_{r,s}, \|P\|_{r,s} \le \frac{Ce^{-1}}{\log s - \log r} .$$

Finish by invoking the following

Theorem (Ovsyannikov'65, Trèves'68)

Let $(V_\alpha)_\alpha$ be a scale of Banach spaces, $V_\alpha \subset V_\beta$ and $\|-\|_\beta < \|-\|_\alpha$, $\alpha > \beta$. The equation $\dot v = \mathcal{Q}(t) v$, $v(0) = v_0 \in \bigcup_\alpha V_\alpha$ will have a unique C^0 in time t local solution when $\|\mathcal{Q}(t)\|_{\beta,\alpha} \le \frac{Ce^{-1}}{\alpha-\beta}$. Moreover, the solution satisfies $\|v(t)\|_\beta \le \|v_0\|_\alpha \left(1 - \frac{C|t|}{\alpha-\beta}\right)^{-1}$. Also, when $\mathcal{Q}(t) = \mathcal{Q}(t,p)$ is C^1 in parameters $p \in \mathbb{R}^k$ with $\|\partial_p Q(t,p)\|_{\beta,\alpha} \le \frac{Ce^{-1}}{\alpha-\beta}$, then the solution $v(t,p)$ is C^1 in (t,p).

Geometric Structures: Analytic Structures in the Wild?

Theorem (Nelson'59, Kotake-Narasimhan'62)

On $\mathbb{R}^n$, let $B = \sum_{N=0}^{k} B^{\mu_1\cdots\mu_N}\partial_{\mu_1}\cdots\partial_{\mu_N}$ be elliptic with analytic coefficients. Then analytic is equivalent to B-analytic,

$$u|_\Omega \text{ analytic} \iff \sup_{x\in\Omega} |B^N u(x)| \le (kN)!\, r^{kN}.$$

- Hypothesis: B elliptic on M + (some condition):
  B-analytic $\iff$ analytic w.r.t unique analytic atlas.
- Counterexample: $B = h\partial_x h^{-1}$, where $h = h(x)$ is non-analytic on $\mathbb{R}$.
  Then u is B-analytic $\iff$ $u = hv$ for v analytic.
- Observation (DeTurk-Kazdan'81): A smooth Riemannian Einstein metric, $R[g] = cg$, on a compact manifold M is analytic in Riemann normal coordinates.
  Q: Does the Einstein equation define a "natural" analytic structure?

Discussion

- Analytic structure can be thought of geometrically!
- Interesting interplay of new and classic results from analysis and geometry (Nelson'59, Kotake-Narasimhan'62, Ovsyannikov'65, Street 2018).
- Work in progress:
  - Elliptic operator + (what condition?) $\implies$ analytic structure
  - Analytic vector bundle structure?
  - * Germ vs Jet: $X = \partial_x$ and $\tilde{X} = (1 + \varepsilon e^{-x^{-2}})\partial_x$ have $\mathcal{A} = \tilde{\mathcal{A}} \subset J^\infty F\mathbb{R}$!
    But $\mathrm{germ}_0(An(1,\mathbb{R}) \cdot X) \neq \mathrm{germ}_0(An(1,\mathbb{R}) \cdot \tilde{X})$.
    What is a better way to define $\tilde{\mathcal{A}}$?
- More examples/counter-examples?

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Thank you for your attention!

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