Quantitative properties of the Schwarzschild metric

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(Submitted on 22.02.2018; Accepted on 27.02.2018)

Abstract. In this paper we show that the difference between the Euclidean geometry and Schwarzschild geometry curved by a tiny mass ball can be quite large on galactic and cosmological scales. We also provide formulas for the proper (relativistic) radius and volume of a homogeneous mass ball. For instance, the homogeneous ball, whose mass and radius is the same as that of the Earth, has relativistic volume about 457 km^3 larger than its Euclidean volume. Similarly, the Euclidean circumference of the Sun is about 3 km shorter than its relativistic circumference, provided the Sun would be homogeneous. Finally, we give some cosmological applications. In particular, the most probable model of a homogeneous and isotropic universe for a fixed time is a three-dimensional hypersphere, since a homogeneous distribution of mass yields a positive curvature.

Key words: Exterior and interior Schwarzschild metric, proper radius, coordinate radius, proper volume, Earth, Sun

Количествени свойства на метриката на Шварцшилд

Михал Крижек, Филип Крижек

В тази статия се показва, че разликата между Евклидовото пространство и геометрията на Шварцшилд, изкривена от малка масова топка, може да бъде доста голяма при галактически и космологични мащаби.

Dedicated to Prof. Lawrence Somer on the occasion of his 70th birthday

1 Introduction

Consider a fixed nonrotating ball in vacuum with mass $M > 0$ and with a spherically symmetric mass distribution. Denote by

$$S = \frac{2MG}{c^2} \tag{1}$$

its *Schwarzschild gravitational radius*, where $G = 6.674 \cdot 10^{-11}$ m^{3}kg^{-1}s^{-2} is the gravitational constant and $c = 299\,792\,458$ m/s is the speed of light in vacuum. Let $R > S$ be the coordinate radius of the ball defined by

$$R = \frac{o}{2\pi},$$

where o is its circumference.

According to Birkhoff's theorem (see Birkhoff 1923), the space outside the mass ball is described by the *exterior Schwarzschild metric* (see e.g. Schwarzschild 1916a, Landau & Lifshitz 1975, Misner, Thorne & Wheeler 1997, Moore 2013)

$$\mathrm{d}l^2 = \frac{r}{r-S}\,\mathrm{d}r^2 + r^2 \sin^2\theta\,\mathrm{d}\varphi^2 + r^2\mathrm{d}\theta^2 \tag{2}$$

for a fixed time, where $r > R$, $\varphi \in [0, 2\pi)$, and $\theta \in [0, \pi]$ are the standard spherical coordinates (see Fig. 1). The corresponding metric tensor is the exact vacuum solution of Einstein's equations. If $M \to 0$, then by (1) the Schwarzschild metric (2) changes into the standard Euclidean metric.

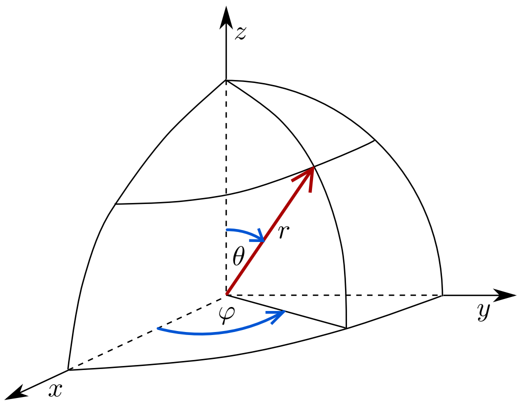

Fig. 1. The point (x, y, z) in the standard spherical coordinates (r, φ, θ)

In Section 2, we investigate the exterior Schwarzschild metric corresponding to a spherical shell outside a fixed ball with spherical mass distribution. We prove that the difference between its relativistic volume and the Euclidean volume is unbounded if the outer radius of the spherical shell tends to infinity. In Section 3, we concentrate on the interior Schwarzschild metric of a homogeneous mass ball and present several useful formulae for the relativistic radius and volume. Finally, in Section 4 we give an application of the interior Schwarzschild metric in cosmology.

2 An unexpected property of the exterior Schwarzschild metric

For positive numbers $R < Q$ consider a spherical shell with interior radius R and exterior radius Q (see Fig. 2). Its volume in the Euclidean space $\mathbb{E}^3$ is equal to

$$V = \frac{4}{3}\pi(Q^3 - R^3). \tag{3}$$

Now we will derive a formula for the proper volume $\tilde{V}$ of the spherical shell with coordinate radii $R < Q$ in a curved space around the mass ball with coordinate radius R. Here the tilde indicates a curved space. By (2) we find that the exterior volume element is equal to

$$\mathrm{d}\tilde{V} = \sqrt{\frac{r}{r-S}}\,\mathrm{d}r \cdot (r \sin\theta \,\mathrm{d}\varphi) \cdot (r\,\mathrm{d}\theta).$$

Therefore, by Fubini's theorem the *proper (relativistic) volume* is defined as

$$\tilde{V} = \int_R^Q r^2 \sqrt{\frac{r}{r-S}}\,\mathrm{d}r \cdot \int_0^\pi \Big(\int_0^{2\pi} \sin\theta \,\mathrm{d}\varphi \Big) \mathrm{d}\theta = 4\pi \int_R^Q r^2 \sqrt{\frac{r}{r-S}}\,\mathrm{d}r. \qquad (4)$$

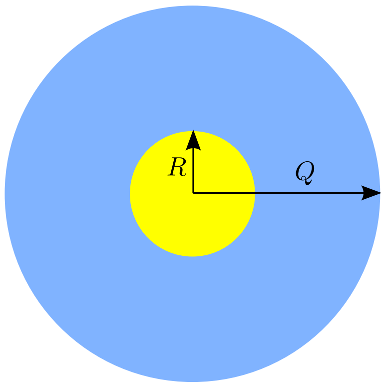

Fig. 2. Spherical shell $\{(x,y,z) \in \mathbb{E}^3 \mid R^2 \le x^2+y^2+z^2 \le Q^2\}$ is the region between two concentric spheres.

Theorem 1. *If $M > 0$ and $R > S$ are any fixed numbers satisfying* (1), *then*

$$\tilde{V} - V \to \infty \quad \text{as } Q \to \infty.$$

Proof. By differentiation, we can verify that (cf. Gradštejn & Ryžik 1971, p. 97)

$$\int r^2 \sqrt{\frac{r}{r-S}}\,\mathrm{d}r = \Big(\frac{r^2}{3} + \frac{5Sr}{12} + \frac{5S^2}{8} \Big) \sqrt{r(r-S)} + \frac{5S^3}{16} \ln(2\sqrt{r(r-S)} + 2r - S).$$

From this, (4), and (3) we get

$$\tilde{V} - V = 4\pi \int_R^Q r^2 \sqrt{\frac{r}{r-S}}\,\mathrm{d}r - \frac{4}{3}\pi(Q^3 - R^3) = \qquad (5)$$

$$= \frac{4\pi}{3} \Big[\Big(Q^2 + \frac{5SQ}{4} + \frac{15S^2}{8} \Big) \sqrt{Q(Q-S)} + \frac{15S^3}{16} \ln(2\sqrt{Q(Q-S)} + 2Q - S)$$

$$- \Big(R^2 + \frac{5SR}{4} + \frac{15S^2}{8} \Big) \sqrt{R(R-S)} - \frac{15S^3}{16} \ln(2\sqrt{R(R-S)} + 2R - S)$$

$$- Q^3 + R^3 \Big].$$

The relation (5) was derived from the Schwarzschild metric (2) exactly without any approximations. Since

$$Q > R > S$$

and since the logarithmic function is increasing, the difference of the two terms containing ln in (5) is positive.[3] Thus from the inequality

$$\sqrt{Q(Q-S)} > Q - S \tag{6}$$

we get the following lower bound

$$\tilde{V} - V > \Big(Q^2 + \frac{5SQ}{4} + \frac{15S^2}{8}\Big)(Q-S) - Q^3 + C = \frac{SQ^2}{4} + \frac{5S^2Q}{8} + \overline{C},$$

where C contains all remaining terms not depending on Q and where $\overline{C} = C - 15S^3/8$. Letting $Q \to \infty$, we obtain the statement of the theorem. $\square$

We observe that the difference of volumes $\tilde{V} - V$ increases over all limits for $Q \to \infty$, which is quite surprising property. Namely, Theorem 1 can be applied for instance to a billiard ball or a small steel ball from a bearing (see Example 1 below) or an imperceptible pinhead, since the mass $M > 0$ can be arbitrarily small. Consequently, a natural question arises: How large can Q be so that the relativistic relation (4) approximates reality well.

Example 1. Setting $M = 0.033$ kg, $R = 0.01$ m, and $Q = 5 \cdot 10^{20}$ m, which is the radius of our Galaxy, we find that $S = 5 \cdot 10^{-29}$ m and by (5) the difference

$$\tilde{V} - V \approx 10\,000 \text{ km}^3.$$

This is about 10^{19} times more than the volume of the ball itself. From this we see that the use of Einstein's equations to galactic distances is questionable. Their application to cosmological distances may yield, by Theorem 1, surprising results, which may have nothing to do with reality (see also Křížek & Somer 2016).

Remark 1. If $M > 0$ and $Q > S$ are any fixed numbers satisfying (1), then by (5) we get $\tilde{V} < \infty$ for $R \to S^+$, i.e., there is no singularity in volume near the Schwarzschild radius.

Remark 2. According to (2), the curved Schwarzschild space is in the radial direction described by the metric

$$\mathrm{d}\rho = \sqrt{\frac{r}{r-S}}\mathrm{d}r.$$

By differentiation, we can check that

$$\int \sqrt{\frac{r}{r-S}}\mathrm{d}r = \sqrt{r(r-S)} + S\ln(\sqrt{r} + \sqrt{r-S}).$$

[3] Since $\ln a - \ln b = \ln(a/b)$, the argument in parenthesis is dimensionless.

Replacing the volume V by $\hat{V} = \frac{4}{3}\pi((\rho(Q))^3 - (\rho(R))^3)$ in Theorem 1, we can also derive that $\hat{V} - \tilde{V} \to \infty$ as $Q \to \infty$. To see this we first introduce the following lower estimate

$$\begin{aligned}(\rho(Q))^3 &= (\sqrt{Q(Q-S)} + S\ln(\sqrt{Q} + \sqrt{Q-S}))^3\\ &> Q(Q-S)\sqrt{Q(Q-S)} + 3SQ(Q-S)\ln(\sqrt{Q} + \sqrt{Q-S}\,),\\ &> Q(Q-S)^2 + 3SQ(Q-S)\ln\sqrt{Q},\end{aligned}$$

where the last inequality is due to (6). From this and (5) we get

$$\begin{aligned}\hat{V} - \tilde{V} &> Q(Q-S)^2 + 3SQ(Q-S)\ln\sqrt{Q} - C\\ &\quad - \Big(Q^2 + \frac{5SQ}{4} + \frac{15S^2}{8}\Big)Q - \frac{15S^3}{16}\ln(2\sqrt{Q(Q-S)} + 2Q - S)\\ &> SQ^2\Big(3\ln\sqrt{Q} - \frac{13}{4}\Big) - S^2Q\Big(3\ln\sqrt{Q} + \frac{7}{8}\Big) - S^3\ln(4Q) - C \to \infty\end{aligned}$$

as $Q \to \infty$, where C contains all remaining terms not depending on Q.

Remark 3. The classical tests of the validity of the General Theory of Relativity (bending of light near the Sun's surface and the Mercury's perihelion shift (see Einstein 1915, Křížek 2017) are based just on the Schwarzschild metric (2). The bending angle $\Phi = 1.75''$ is obtained from the relation $\Phi = 4GM/(c^2R)$ which is derived from (2) under several simplifications (see Stephani 2004).

3 The interior Schwarzschild metric

In 1916 Karl Schwarzschild (see Schwarzschild 1916b) found the first nonvacuum solution of Einstein's equations. He assumed that the ball with coordinate radius $R > 0$ is formed by an ideal incompressible fluid to avoid a possible internal mechanical stress in the solid that may have a nonnegligible influence on the resulting gravitational field. Then by Ellis 2012 (see also Stephani 2004, p. 213; Florides 1974, p. 529; wikipedia) the corresponding time independent metric (i.e. $\mathrm{d}t = 0$) is given by

$$\mathrm{d}l^2 = \frac{R^3}{R^3 - Sr^2}\,\mathrm{d}r^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2 + r^2\mathrm{d}\theta^2, \tag{7}$$

where $r \in [0, R]$. The relation (7) is called the *interior Schwarzschild solution*, see Stephani 2004, p. 213. To avoid the division by zero in the first coefficient on the right-hand side of (7), we require

$$\frac{R^3}{R^3 - Sr^2} = \Big(1 - \frac{Sr^2}{R^3}\Big)^{-1} > 0 \quad \text{for all } r \in [0, R],$$

that is

$$R > S.$$

Moreover, we observe that for $r = R$ the metric (7) for the interior of the ball continuously matches the Schwarzschild metric (2) defined outside the ball.

Now let us calculate the proper radius of the ball. For $\mathrm{d}\varphi = 0$ and $\mathrm{d}\theta = 0$ the equality (7) obviously reduces to

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{1-\alpha^2 r^2}, \tag{8}$$

where

$$\alpha = \sqrt{\frac{S}{R^3}}. \tag{9}$$

For $r \in [0, \alpha^{-1})$ we can easily verify that

$$F(r) = \frac{1}{\alpha}\arcsin(\alpha r) \tag{10}$$

is a primitive function of

$$f(r) = \frac{1}{\sqrt{1-\alpha^2 r^2}}. \tag{11}$$

From the inequality $R > S$ and (9) we see that

$$R < R\sqrt{\frac{R}{S}} = \alpha^{-1}.$$

Given (8), (10), and (11), we now define the *proper (relativistic) radius* of the homogeneous mass ball as

$$\tilde{R} = \int_0^R \frac{\mathrm{d}r}{\sqrt{1-\alpha^2 r^2}} = \frac{1}{\alpha}\arcsin(\alpha R). \tag{12}$$

From this we find by l'Hospital's rule for fixed $R > 0$ that

$$\tilde{R} \to R \quad \text{for } \alpha \to 0,$$

i.e., when $M \to 0$, which follows from (1) and (9).

For a given circumference o of the ball there exists exactly one coordinate radius R, whereas its proper radius $\tilde{R}$ is not uniquely determined. It depends on the mass M. The larger M is, the larger is $\tilde{R}$.

Example 2. Consider a homogeneous ball with mass $M = M_\odot = 2 \cdot 10^{30}$ kg and coordinate radius $R = R_\odot = 695\,700$ km corresponding to the Sun. Then by (12) we have $\tilde{R}_\odot - R_\odot = 492$ m. Hence, the associated Euclidean circumference is about 3 km shorter than $2\pi\tilde{R}_\odot$, where $\tilde{R}_\odot$ is the proper radius.

Theorem 2. *If $M > 0$ is fixed, then the function $R \mapsto \tilde{R} - R$ is decreasing and strictly convex on the interval (S, ∞).*

The proof follows easily from (9) and (10) under the condition $R > S$ (cf. Fig. 3).

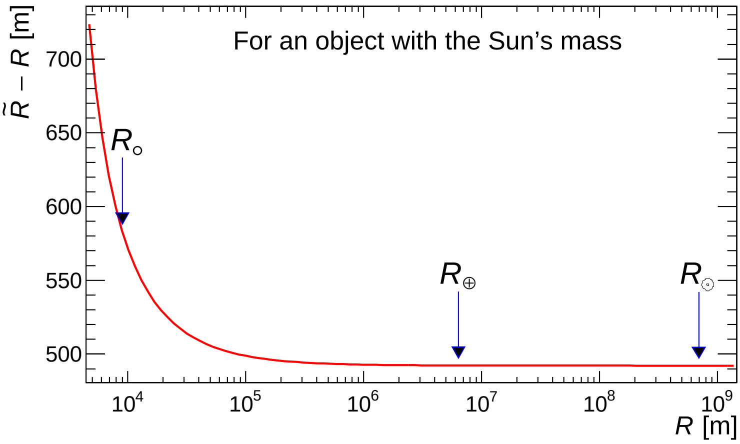

Fig. 3. Dependence of the difference $\tilde{R} - R$ on the coordinate radius R for the fixed mass $M = M_\odot = 2 \cdot 10^{30}$ kg (see Theorem 2). Here $R_\odot = 695\,700$ km is the coordinate radius of the Sun. The radius $R_\oplus = 6378$ km corresponds approximately to a white dwarf and $R_\circ = 9$ km to a hypothetical neutron star.

Further we derive a formula for the proper volume of a homogeneous mass ball. By (7) and (8) the volume element reads

$$\mathrm{d}\tilde{V} = \frac{\mathrm{d}r}{\sqrt{1 - \alpha^2 r^2}} \cdot (r \sin\theta \,\mathrm{d}\varphi) \cdot (r \,\mathrm{d}\theta).$$

We can easily check that

$$H(r) = \frac{1}{2\alpha^3} \arcsin \alpha r - \frac{r}{2\alpha^2} \sqrt{1 - \alpha^2 r^2}$$

is a primitive function of

$$h(r) = \frac{r^2}{\sqrt{1 - \alpha^2 r^2}}$$

on the interval $[0, \alpha^{-1})$, cf. Peebles 1993, p. 298.

By Fubini's theorem for the *proper (relativistic) volume* of the homogeneous ball is defined as

$$\tilde{V} = \int_0^R \frac{r^2 \,\mathrm{d}r}{\sqrt{1 - \alpha^2 r^2}} \cdot \int_0^\pi \Big(\int_0^{2\pi} \sin\theta \,\mathrm{d}\varphi \Big) \mathrm{d}\theta = H(R) \cdot 4\pi$$
$$= \frac{2\pi}{\alpha^2} \Big(\frac{\arcsin \alpha R}{\alpha} - R\sqrt{1 - \alpha^2 R^2} \Big). \qquad (13)$$

For fixed $R > 0$ it can be shown by l'Hospital's rule and the relation (9) that

$$\tilde{V} \to V := \frac{4}{3}\pi R^3 \quad \text{for } \alpha \to 0. \tag{14}$$

Example 3. Consider a homogeneous ball with mass $M = M_\oplus = 5.97219 \cdot 10^{24}$ kg and coordinate radius $R = R_\oplus = 6378$ km corresponding to the Earth. Then by (13) we get a surprisingly large relativistic effect, namely,

$$\tilde{V}_\oplus - V_\oplus = 457 \text{ km}^3,$$

where

$$V_\oplus \approx 10^{12} \text{ km}^3. \tag{15}$$

Theorem 3. *If $M > 0$ is fixed, then the function $R \mapsto \tilde{V}/V$ is decreasing and strictly convex on the interval (S, ∞).*

The proof follows from relations (9), (13), and the inequality $R > S$.

Remark 4. The convergence in (14) is monotonically decreasing. Hence, the proper volume $\tilde{V}$ is larger than the coordinate volume V for $M > 0$. Moreover, the matter defends to gravitational compression in such a way that it increases its proper volume $\tilde{V}$ when R is fixed and M increases. In other words, the larger M is, the larger is $\tilde{V}$ for a fixed circumference and fixed V.

Example 4. For the Sun with coordinate radius $R_\odot = 695\,700$ km we obtain by (13) that (cf. (15))

$$\tilde{V}_\odot - V_\odot = 1.796 \cdot 10^{12} \text{ km}^3, \quad \overline{V}_\odot - \tilde{V}_\odot = 1.197 \cdot 10^{12} \text{ km}^3,$$

where $V_\odot = \frac{4}{3}\pi R_\odot^3$ and $\overline{V}_\odot = \frac{4}{3}\pi \tilde{R}_\odot^3$ (cf. Example 2). For comparison, by (5) we get

$$\tilde{V} - V = 4.49 \cdot 10^{12} \text{ km}^3$$

for $Q = R_\odot$ and $R \searrow S_\odot$, where $S_\odot = 2.953$ km is the Schwarzschild radius.

4 Applications in cosmology

According to the *Einstein cosmological principle*, the "universe" is homogeneous and isotropic for large spatial scales and fixed time. The *homogeneity* is expressed by a translation symmetry (i.e. the space has at any point the same density, temperature, pressure, etc.), while *isotropy* is expressed by rotational symmetry (i.e. there are no preferred directions at any point and an observer is not able to distinguish a given direction from another direction by means of local physical measurements).

First, let us present an argument that favours the bounded three-dimensional hypersphere (see (16) below) as a model of our universe for a fixed time. Assume, to the contrary, that the universe is unbounded, i.e. infinite. Then it would have everywhere on large scales the same density, temperature, pressure,[4] and so on, at a given time instant after the Big Bang as required by the Einstein cosmological principle. In this case, information would have to be transmitted at infinite speed which is impossible. The popular theory of inflation cannot explain such a homogeneity and isotropy of an infinite universe. Moreover, the actual space could not first be finite after its origin and then change to infinite.

Using (8), the metric (7) can be rewritten as

$$\mathrm{d}l^2 = \frac{1}{1-\alpha^2 r^2}\,\mathrm{d}r^2 + r^2 \sin^2\theta\,\mathrm{d}\varphi^2 + r^2\mathrm{d}\theta^2,$$

where $r \in [0, \alpha^{-1})$, $\varphi \in [0, 2\pi)$, $\theta \in [0, \pi]$, and $\alpha > 0$ given by (9) is fixed. We observe that it is similar to the well-known metric of the stationary homogeneous and isotropic universe

$$\mathrm{d}l^2 = \frac{1}{1-r^2}\,\mathrm{d}r^2 + r^2 \sin^2\theta\,\mathrm{d}\varphi^2 + r^2\mathrm{d}\theta^2, \quad r \in [0, 1),$$

for a fixed time and the three-dimensional hypersphere

$$\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{E}^4 \,|\, x^2 + y^2 + z^2 + w^2 = a^2\} \tag{16}$$

with radius $a = 1$, see Stephani 2004, p. 214; Peebles 1993, p. 297; Friedman 1922; Weinberg 1972, p. 344. Such a maximally symmetric manifold has in all its points a constant positive curvature that is equal to 1. The shortest paths on $\mathbb{S}^3$ are arcs of the great circles. The sum of angles in a curved triangle, whose sides are arcs of great circles, is greater than 180°. Hence, the sum of angles of a curved triangle, whose sides are geodesics, inside a homogeneous mass ball will also be greater than 180°. Thus, a homogeneous mass distribution causes a positive curvature. (Note that Alexander Friedmann assumes a negative mass density distribution to get a hyperbolic geometry with a negative curvature, see Friedmann 1924.)

When the mass distribution of our universe was homogeneous and isotropic at some fixed time instant t, its curvature had to be positive due to the above similarity. The hypersphere (16) with radius $a = a(t)$ is the only mathematical model of such a homogeneous universe, see Křížek & Mészáros 2016. Its radius $a = a(t)$ is an increasing function satisfying the Friedmann equation (see Friedman 1922). Later, when the matter started to collapse locally due to many gravitational perturbations, the global curvature had to remain positive. It could not jump to zero or negative curvature, since the corresponding manifolds are infinite and unbounded, whereas $\mathbb{S}^3$ is bounded.

There is also another argument against $\mathbb{E}^3$ being the correct model of our universe. By Einstein's General theory of relativity, matter curves space. Nevertheless, the zero curvature of $\mathbb{E}^3$ is independent of the decreasing mean mass density $\rho = \rho(t) > 0$ with time.

[4] Furthermore, these quantities should attain arbitrarily large values at all points just after the Big Bang.

Acknowledgements. We are indebted to Yurii V. Dumin, Attila Mészáros, Vojtěch Pravda, and Vladimír Wagner for fruitful discussions and to Hana Bílková for drawing the figures. This paper was supported by RVO 67985840 and LG 15052 of the Czech Republic.

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