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Relativistic perihelion shift of Mercury revisited

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Abstract

In 1915, Albert Einstein derived a formula for an additional relativistic perihelion shift of Mercury. We give several critical comments of how this formula is treated, since the perihelion shift of Mercury is an ill-conditioned problem. We also present various sources of nonnegligible errors, which should be taken into account.

KEYWORDS

heliocentric system, line of apsides, perihelion advance, planets, Solar system

1 | INTRODUCTION

The perihelion shift (advance) of Mercury's orbit is considered to be one of the fundamental tests of the validity of the general theory of relativity, see Figure 1 and Foster & Nightingale (2006); Misner et al. (1997); Roseveare (1982); Rydin (2011). This paper is a continuation of our previous paper Křížek (2017), where we present the main drawbacks in the methodology of how this test is handled.

In 1915, Albert Einstein published (1915, p. 839) a formula for the relativistic perihelion shift, for one period, of

$$\varepsilon = 24\pi^3 \frac{a^2}{T^2 c^2 \left(1 - e^2\right)} = 5.012 \times 10^{-7}\ \text{rad} \approx 0.1'', \quad (1)$$

where according to contemporary data.

$T = 7.6005 \times 10^6$ s is the orbital period of Mercury,

$e = 0.2056$ is the numerical eccentricity of its (almost) elliptical orbit,

$a = 57.909 \times 10^9$ m is the length of its corresponding semimajor axis,

$c = 299{,}792{,}458$ m/s is the speed of light in vacuum.

Substituting these values into Equation (1), we obtain a value of

$$E = \varepsilon \frac{\tau}{T} \frac{180}{\pi} 3600'' = 43'' \quad \text{per century}, \quad (2)$$

where $\tau = 3{,}155{,}814{,}954$ s is the number of seconds in one century. We have to treat the angle $\varepsilon \approx 0.1''$ from (1) or $E \approx 0.01^\circ\ \text{cy}^{-1}$ from Equation (2) very carefully, since it is extremely small. The value (2) is in a good agreement with calculations made by Le Verrier (1859) and later also by Newcomb, namely,

$$E = 43.37'' \quad \text{per century}, \quad (3)$$

see Newcomb (Newcomb 1895, chap. IX, p. 184).

In astrophysical community, it is believed that the small difference between (2) and (3) is not just a coincidence, and thus the simple formula (Equation [1]) is applied to test the validity of the general theory of relativity without any doubts how it was derived and what are its properties.

2 | A METHOD OF ALBERT EINSTEIN

Einstein did not solve his equations of the general theory of relativity for the Solar system analytically, since they are represented by a very complicated nonlinear system of hyperbolic partial differential equations. Their exact solution is not even known for two bodies with positive masses. Therefore, Einstein had to make a whole series

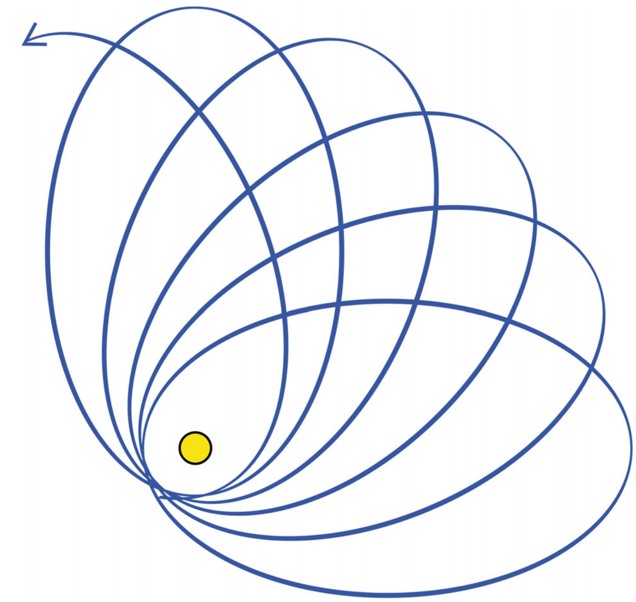


FIGURE 1 Idealized uniform aphelion and perihelion shifts of Mercury in the direction of circulation

of simplifications to get some value of the perihelion shift of Mercury (see [2]). In other words, Formula (1) has not been derived as a consequence of Einstein's equations in terms of mathematical implications.

For instance, Einstein restricts himself only to the equatorial plane. Mercury is substituted by a massless test particle and the gravitational influence of other planets is not taken into account to get a pure relativistic effect. Einstein considers the zero right-hand side of his field equations outside the Sun. He neglects higher-order nonlinear terms when calculating the Christoffel symbols corresponding to space variables. After five pages of other approximations, Einstein (1915, pp. 833–837) obtains (without any error estimates) an ordinary differential equation for angle ϕ whose solution leads to the elliptic integral (see Figure 2)

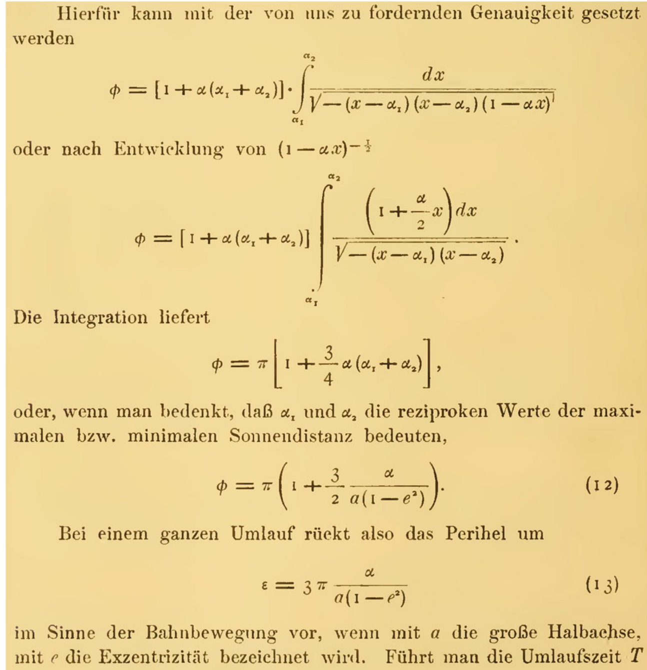
$$\phi = [1 + \alpha\,(\alpha_1 + \alpha_2)] \int_{\alpha_1}^{\alpha_2} \frac{\mathrm{d}x}{\sqrt{-\,(x - \alpha_1)\,(x - \alpha_2)\,(1 - \alpha x)}}, \tag{4}$$

where $\phi > 180°$ is the angle between the radius-vector of perihelion and the radius-vector of aphelion of Mercury's orbit (see Figure 3),

$$\alpha_1 = \frac{1}{a(1+e)} = 1.432 \cdot 10^{-11}\ \mathrm{m}^{-1},$$
$$\alpha_2 = \frac{1}{a(1-e)} = 2.174 \cdot 10^{-11}\ \mathrm{m}^{-1}, \tag{5}$$

where

$$\alpha = \frac{2GM_\odot}{c^2} = 2953\ \mathrm{m} \tag{6}$$



Hierfür kann mit der von uns zu fordernden Genauigkeit gesetzt werden

$$\phi = [1+\alpha(\alpha_1+\alpha_2)] \cdot \int_{\alpha_1}^{\alpha_2} \frac{dx}{\sqrt{-(x-\alpha_1)(x-\alpha_2)(1-\alpha x)}}$$

oder nach Entwicklung von $(1-\alpha x)^{-\frac{1}{2}}$

$$\phi = [1+\alpha(\alpha_1+\alpha_2)] \int_{\alpha_1}^{\alpha_2} \frac{\left(1+\frac{\alpha}{2}x\right)dx}{\sqrt{-(x-\alpha_1)(x-\alpha_2)}}.$$

Die Integration liefert

$$\phi = \pi\left[1+\frac{3}{4}\alpha(\alpha_1+\alpha_2)\right],$$

oder, wenn man bedenkt, daß α_1 und α_2 die reziproken Werte der maximalen bzw. minimalen Sonnendistanz bedeuten,

$$\phi = \pi\left(1+\frac{3}{2}\frac{\alpha}{a(1-e^2)}\right). \tag{12}$$

Bei einem ganzen Umlauf rückt also das Perihel um

$$\varepsilon = 3\pi\frac{\alpha}{a(1-e^2)} \tag{13}$$

im Sinne der Bahnbewegung vor, wenn mit a die große Halbachse, mit e die Exzentrizität bezeichnet wird. Führt man die Umlaufszeit T

FIGURE 2 Page 838 from the original paper (Einstein 1915) which contains an erroneous integration over the interval $[\alpha_1, \alpha_2]$ due to Proposition 1, see Appendix A

is the Schwarzschild radius of the Sun and G is the gravitational constant.

Because the integral in Equation (4) has no known analytical solution, Einstein employs the linear part of the following Taylor expansion

$$\frac{1}{\sqrt{1-\alpha x}} = 1 + \frac{1}{2}\alpha x + \frac{3}{8}\alpha^2 x^2 + \cdots$$

which is a fairly good approximation on the interval $[\alpha_1, \alpha_2]$, since $\alpha\alpha_i \ll 1$ for $i = 1, 2$. Hence, the angle ϕ from Equation (4) is expressed as follows:

$$\phi = [1 + \alpha\,(\alpha_1 + \alpha_2)] \int_{\alpha_1}^{\alpha_2} \frac{\left(1 + \frac{1}{2}\alpha x\right)\mathrm{d}x}{\sqrt{-\,(x - \alpha_1)\,(x - \alpha_2)}}$$
$$\stackrel{?}{=} \pi\left[1 + \frac{3}{4}\alpha\,(\alpha_1 + \alpha_2)\right] = \pi\left[1 + \frac{3\alpha}{2a\left(1 - e^2\right)}\right], \tag{7}$$

where the first equality should be replaced by $\approx$, as explained above, and the last equality holds due to (5). However, Einstein does not present any details as to how the above integration denoted by $\stackrel{?}{=}$ was performed. We will postpone this question to Appendix A. Note that the singularities near the end points α_1 and α_2 in (7) are integrable.

Finally, from Equations (7) and (6), and Kepler's third law $a^3/T^2 = GM_\odot/\left(4\pi^2\right)$, it follows that after one period

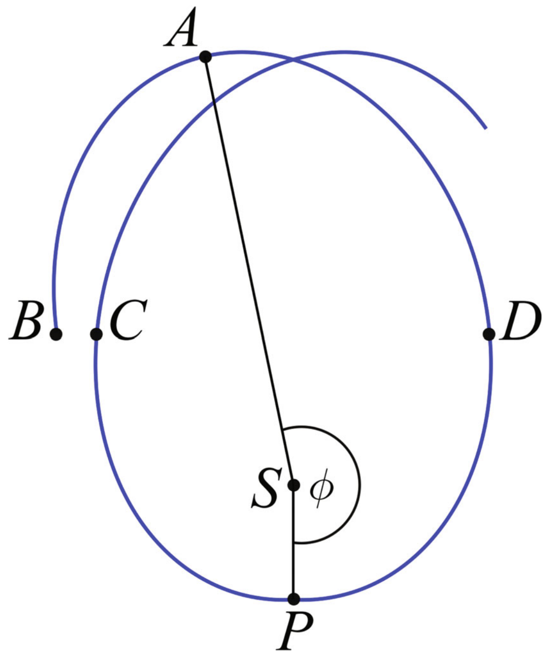


FIGURE 3 The letters S, P, and A stand for the Sun, the perihelion, and the aphelion of Mercury's orbit, respectively. The angle $\angle PSA$ between the perihelion radius-vector and the aphelion radius-vector is denoted by ϕ. The lengths of these vectors are $|PS| = a(1-e)$ and $|AS| = a(1+e)$

(more precisely, between two successive perihelion passages) the perihelion shifts about the angle

$$\varepsilon = 2(\phi - \pi) = 3\pi \frac{\alpha}{a\left(1-e^2\right)} = 3\pi \frac{2GM_\odot}{ac^2\left(1-e^2\right)}$$
$$= 24\pi^3 \frac{a^2}{T^2c^2\left(1-e^2\right)}, \tag{8}$$

that is, the relationship (1), which according to (2) yields the idealized value of the relativistic perihelion shift of Mercury 43″ per century. Here, one subtracts two numbers of almost the same size, namely $\phi - \pi$, which is a delicate numerical operation, see Brandts et al. (2016).

3 | SEVERAL CRITICAL REMARKS ON FORMULA (1)

Remark 1. Einstein derived Formula (1) for weak gravitational fields and small velocities $v \ll c$. Anyway, it is often applied also to other systems, for example, to the star S2 orbiting around the supermassive black hole Sgr A* and reaching the maximum velocity 7,650 km/s. Its orbit is seen in the projection to the celestial sphere and the first orbit of S2 was measured only very roughly. Despite that, Abuter et al. (2020) claim that the measured pericenter shift corresponds exactly to the value $\varepsilon = 11'$, which is obtained from Equation (1) for $T = 16$ yr, $a \approx 970$ au, and $e = 0.885$ (note that it would be more precise to measure apocenter shift). Furthermore, note that the rotation of the central black hole was not taken into account. There are, of course, many other errors coming from various sources like the pixel structure of CCD detectors, calibration errors, errors in the determination of physical constants, systematic errors, tidal forces, influence of neighboring stars, and so forth. Anyway, Abuter et al. (2020) claim that their measurements represent another proof that the general relativity is valid (see also Genzel (2020)).

Remark 2. Notice that the input parameters a, T, and e to Formula (1) are squared. Therefore, this formula is very sensitive to their precise measurements. For instance, we have

$$(a + \Delta a)^2 = a^2 + 2a\Delta a + (\Delta a)^2.$$

Hence, if the relative error in determining the semimajor axis a is, for example, 1%, then the error in a^2 is about 2% (we usually have $|\Delta a| \ll a$).

Remark 3. It is curious that Formula (1) always yields a positive perihelion shift $\varepsilon > 0$ even for the zero eccentricity $e = 0$. In this case, there are infinitely many perihelia, namely, the perihelion is at each point of a circular trajectory. In fact, by Equation (1) we always have

$$\varepsilon \geq 24\pi^3 a^2 T^{-2} c^{-2} > 0.$$

However, this does not allow us to consider a retrograde perihelion shift with negative ε, i.e., when $\phi < \pi$ in Figure 3. For instance, Andrea Ghez claims that her team observed that the movement of the pericenter of the star S2 around the black hole Sgr A* is retrograde (see Ghez (2020, 45 min)).

Remark 4. For Newtonian elliptic orbits, the eccentricity is defined as follows

$$e = \frac{\sqrt{a^2 - b^2}}{a},$$

where b is the length of the corresponding semiminor axis. However, what is the definition of e for non-elliptic orbits? (cf. Figure 5.) How do we define the semiminor axis b for the situation sketched in Figure 3? Is $b = |BD|/2$ or $b = |CD|/2$? How to define B, C, and D at all? Should we define a as $a = |AP|/2$ or $a = (|AS| + |PS|)/2$ or by another manner? Formula (1) thus contains not well-defined quantities a and e. Moreover, the static space around the Sun is curved. So are α_1 and α_2 from Equation (5) well defined?

Remark 5. Also, T in Equation (1) is not well defined. Note that all three of Kepler's laws describe the reality only approximately, since there is always a modeling error. Nevertheless, a natural question arises: *Are we justified to use Kepler's third law in deriving the relativistic formula* (8) *which leads to (1)*? The period T appearing in Kepler's third law corresponds to angle 2π exactly. Hence, we should replace the last equality in Equation (8) by the $\approx$ sign, since the radius-vector of Mercury makes a larger angle than 2π between two passages of neighboring aphelia (or perihelia), see Figure 1.

Remark 6. Elliptical orbits of test particles can only be obtained for the central force field, which is proportional to the gravitational potential $1/r$. Nevertheless, this does not hold in the Solar system. According to Rydin (2011), the solar oblateness (quadrupole moment) contributes to the overall perihelion shift of Mercury only $0.0254''$ per century (see also Pireaux & Rozelot (2003)). On the other hand, a somewhat much larger value of $3.4''$ per century is presented in the study by Weinberg (1972, p. 200). He claims that the most probable reason for this effect is that the length of the rotational axis of the Sun oscillates. Also, a differential rotation of the inner Sun may have a nonnegligible effect, see Dicke & Goldberg (1967). If the central core is an axially symmetric ellipsoid that rotates more rapidly than the surface of the Sun, then a part of the Mercury's perihelion shift (1) could be explained in another way.

Remark 7. Many small errors can produce a total error larger than the tiny relativistic effect (1). For instance, Mercury is tidally locked, which surely has an influence on its orbit, see also the previous Remark 1 to 6.

Remark 8. In 1898, Paul Gerber derived the following formula for the speed of light by means of retarded potentials (see Gerber (1898))

$$c^2 = 24\pi^3 \frac{a^2}{T^2\left(1-e^2\right)\Phi}, \qquad (9)$$

where Φ is the perihelion shift of Mercury during one orbital period. We see that this formula is the same as Equation (1) for

$$\varepsilon = \Phi.$$

Therefore, the corresponding tests of the validity of general theory of relativity by means of Formula (1) have to yield exactly the same value as tests of Gerber's theory of retarded potentials. So which theory do we test?

Remark 9. According to Janssen & Renn (2022); Weinstein (2022), in 1913, Einstein wrote a manuscript on the motion of the perihelion of Mercury together with his friend Michele Besso. Although Besso corrected many errors in their common calculations, Einstein does not mention him in the final version, Einstein (1915).

So Einstein worked on the problem of Mercury's perihelion shift for at least 2 years before the paper (Einstein 1915) appeared. Nonetheless, after that Einstein claimed: *I have not mentioned the work by Gerber originally, because I did not know it when I wrote my work on the perihelion motion of Mercury.*[1] However, it is interesting that Einstein used a similar notation as Gerber (1898), namely, the angle ϕ from Figure 3 satisfies $\phi = \Phi/2$, where Φ appears in Gerber's formula (9).

Remark 10. Although Formula (1) was derived for a massless test particle representing Mercury, it is often applied to close binaries whose components have positive masses M_1 and M_2 (see, e.g., Avdeev et al. 2020; Bowler 2010; Guinan & Maloney 1985; Hilditch 2001; Stovall et al. 2018; Susobhanan et al. 2018; Weisberg & Taylor 2005). Note that PSR J0737–3039 is the only known double pulsar. By Lyne (2006), it has a large relativistic periastron advance. The mass $M_\odot$ is usually replaced by the sum $M_1 + M_2$, see, e.g., Will (2014, p. 47). However, the exact solution of Einstein's equations for two bodies with positive masses is not known. Hence, we cannot reliably verify if such an approximation is correct.

4 | WHY THIS PROBLEM IS ILL-CONDITIONED?

The eccentricity of Mercury's orbit is a relatively large number $e = 0.2056$, but since the semiminor axis of Mercury has the length

$$b = a\sqrt{1-e^2} \approx 0.98a,$$

the orbit is almost circular, with the Sun being at one of the foci (see Figure 4). The problem of finding its perihelion is thus ill-conditioned, since for a circular orbit, the perihelion occurs at each point.

Consider now a rectangular heliocentric system whose position is unchanged with respect to the fixed distant stars. Denote by the letter O the observed value of Mercury's perihelion shift per century with respect to the Sun, and by C the calculated value per century using Newtonian mechanics. Since the vertex of O lies in the Sun, this angle seems always to be much smaller when it is observed from

[1] Ich habe die Gerbersche Arbeit ursprünglich schon deshalb nicht erwähnt, weil ich sie nicht kannte, als ich meine Arbeit über die Perihelbewegung des Merkur schrieb.

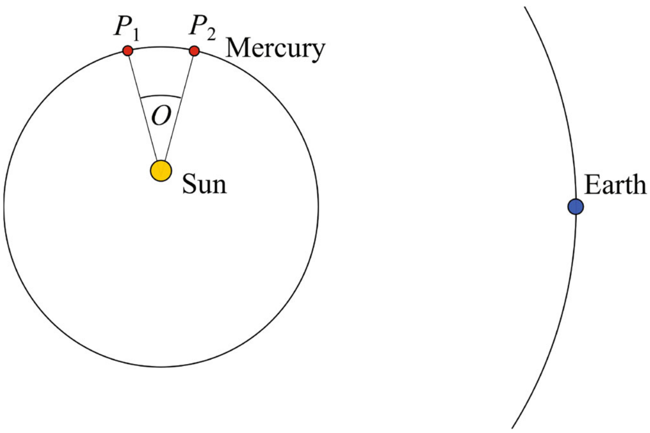


FIGURE 4 The angle O corresponds to the observed Mercury's perihelion shift per century. For better visualization, the angular distance O between two perihelia P_1 and P_2 in figure is magnified 200 times. The observational data from the Earth must be recalculated to find the value of angle O in the heliocentric coordinates

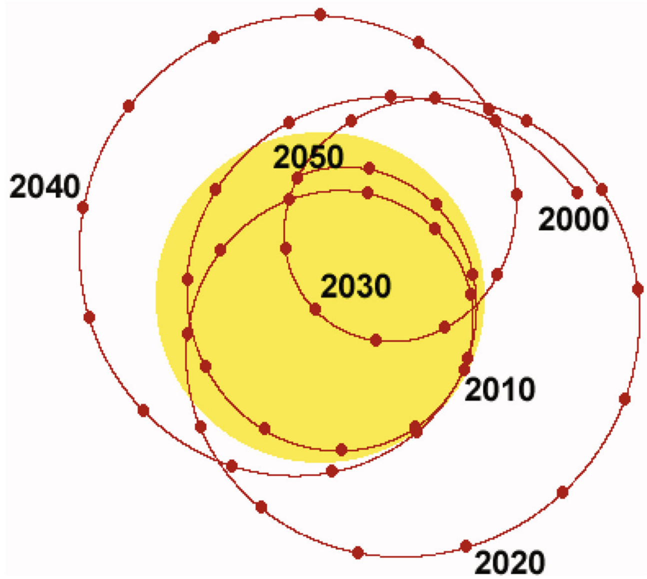


FIGURE 5 The trajectory of the Newtonian barycenter of the Solar system for the period 2000–2050. It causes a tortuous path of Mercury. The barycenter shifts each day by about 1,000 km, whereas the additional relativistic shift is on average only 96 km per year

the Earth (see Figure 4). Therefore, it must be recalculated to the heliocentric coordinates.

In the current astrophysical community, it is generally accepted that

$$O - C = E,$$

where E is the value (2) predicted by Einstein. However, this is again an ill-conditioned problem due to the subtraction of two quite inexact numbers of almost equal magnitude, see Brandts et al. (2016). An appropriate mixed polynomial-trigonometric interpolation of observed values leads to O, see Bretagnon & Francou (1988) (also Folkner et al. 2014), whereas C is obtained by numerical solution of the classical N-body problem (see, e.g., Narlikar & Rana 1985; Rana 1987). However, these values are subjected to many errors, which should not be ignored. For example, they include inexact input data, the modeling error of the Newtonian mechanics, numerical integration errors, and rounding errors. Also, note that the Newtonian barycenter of the Solar system shifts by about 1,000 km every day (cf. Figure 5 and Remark 4), whereas the idealized relativistic perihelion shift of Mercury according to Equation (2) is only $2\pi a(1 - e)E/100 = 96$ km per year and the average speed of Mercury is about 50 km/s.

The quantities $O \sim 575''$ cy^{-1} and $C \sim 532''$ cy^{-1} are not uniquely established in the literature. For instance, according to Amelkin (2019), the average perihelion shift of the orbit of Mercury, calculated in the framework of the planar limited problem, is $C = 556.5''$ cy^{-1}. By Rana (1987), the perihelion shift may increase about $24''$ or decrease about $11''$ within less than 1 year, which surely has a non-negligible influence on the average shift per century. Thus, $E = O - C$ is highly imprecise, since it is influenced by a large amount of various errors. Nevertheless, most authors claim that Equation (2) is exact, although they present different values of C, see, for example, Křížek (2017, p. 50). It is clear that if at least one term in relation $E = O - C$ is not correctly established, then this proclaimed equality cannot be properly used and the value E given by Equation (2) may differ from reality.

Hundreds of publications claim that the value $E = 43''$ per century from Equation (2) is very precise, that it nicely fits astronomical observations, and thus it perfectly confirms the general theory of relativity, see, for example, Clemence (1947); Duncombe (1956); Foster & Nightingale (2006); Kraniotis & Whitehouse (2003); Misner et al. (1997); Narlikar & Rana (1985); Nobili & Will (1986); Park et al. (2017); Pireaux & Rozelot (2003); Rana (1987); Ridao et al. (2014); Roseveare (1982). However, when applying Formula (1), we should keep in mind the way it was derived and what are its properties.

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APPENDIX A

In 2010, Anatoli Andrei Vankov pointed out without any proof that there is a computational error in Einstein's evaluation of the integral appearing in (7), see Vankov (2010, p. 21). Therefore, we introduce a detailed calculation, so that everyone can check that Vankov is right. The next statement is rather an example, but since it yields a different value of ϕ than Einstein's value from (7), see also Figure 2, we will call it a proposition.

Proposition 1. *We have*

$$[1+\alpha(\alpha_1+\alpha_2)]\int_{\alpha_1}^{\alpha_2}\frac{\left(1+\frac{1}{2}\alpha x\right)\mathrm{d}x}{\sqrt{-(x-\alpha_1)(x-\alpha_2)}} = \pi\left[1+\frac{5\alpha}{4}(\alpha_1+\alpha_2)+\frac{\alpha^2}{4}(\alpha_1+\alpha_2)^2\right].$$

Proof Define the linear transformation ℓ: $[\alpha_1, \alpha_2] \to [-1, 1]$ by

$$\ell(x) = \frac{x-s}{h}, \tag{A1}$$

where

$$s = \frac{\alpha_1+\alpha_2}{2} \tag{A2}$$

is the midpoint of $[\alpha_1, \alpha_2]$ and $h = \alpha_2 - s$ half its width. Then $\ell(\alpha_1) = -1$ and $\ell(\alpha_2) = 1$, but also

$$\sqrt{-(x-\alpha_1)(x-\alpha_2)} = \sqrt{-(hy+h)(hy-h)} = h\sqrt{1-y^2},$$

where the substitution $y = \ell(x)$ with inverse

$$x = hy + s$$

was used. Hence, $\mathrm{d}x = h\mathrm{d}y$ and by (A1) we obtain

$$\int_{\alpha_1}^{\alpha_2} \frac{\left(1+\frac{1}{2}\alpha x\right)\mathrm{d}x}{\sqrt{-(x-\alpha_1)(x-\alpha_2)}} = \left(1+\frac{\alpha s}{2}\right)\int_{-1}^{1}\frac{\mathrm{d}y}{\sqrt{1-y^2}} + \frac{\alpha h}{2}\int_{-1}^{1}\frac{y\,\mathrm{d}y}{\sqrt{1-y^2}} = \left(1+\frac{\alpha s}{2}\right)\pi, \tag{A3}$$

because the right-most integral exists over [0, 1] and cancels the one over [−1, 0], whereas the middle integral has arcsin y as antiderivative and thus evaluates to π.

Multiplying (A3) by $[1+\alpha(\alpha_1+\alpha_2)]$ and applying (A2), we find that

$$\begin{aligned}
&[1+\alpha(\alpha_1+\alpha_2)]\int_{\alpha_1}^{\alpha_2}\frac{\left(1+\frac{1}{2}\alpha x\right)\mathrm{d}x}{\sqrt{-(x-\alpha_1)(x-\alpha_2)}}\\
&= [1+\alpha(\alpha_1+\alpha_2)]\,\pi\left[1+\frac{\alpha}{4}(\alpha_1+\alpha_2)\right]\\
&= \pi\left[1+\frac{5\alpha}{4}(\alpha_1+\alpha_2)+\frac{\alpha^2}{4}(\alpha_1+\alpha_2)^2\right].
\end{aligned}$$

The proof is completed. ∎

By Proposition 1 the missing integration denoted by $\stackrel{?}{=}$ in (7) we get

$$\begin{aligned}
\phi &= \pi\left[1+\frac{5\alpha}{4}(\alpha_1+\alpha_2)+\frac{\alpha^2}{4}(\alpha_1+\alpha_2)^2\right]\\
&\approx \pi\left[1+\frac{5}{4}\alpha(\alpha_1+\alpha_2)\right]
\end{aligned} \tag{A4}$$

neglecting the quadratic term (of dimensionless order 10^{-15}). However, this differs from Equation (7), see also the middle of Figure 2. Hence, by Equations (5) and (6), we have

$$\varepsilon = 2(\phi-\pi) = \pi\frac{5\alpha}{2}(\alpha_1+\alpha_2) = 8.364\cdot 10^{-7}\ \mathrm{rad}$$

for one period which is a larger value than that in Equation (1). This is 5/3 times the value that Einstein computed. Therefore, Vankov in Einstein (1915, p. 11) proposes to replace the first factor $1+\alpha(\alpha_1+\alpha_2)$ in Equation (7) by $1+1/2\alpha(\alpha_1+\alpha_2)$, which then leads to the right-hand side of Equation (7), see also Janssen & Renn (2022, p. 69). Then Formula (1) is obtained from Equation (8). At present, there are several other independent ways of deriving (1), see, for example, Brumberg (1991); Kopeikin et al. (2011); Straumann (2013).