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Regular Tessellations of Maximally Symmetric Hyperbolic Manifolds

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Abstract: We first briefly summarize several well-known properties of regular tessellations of the three two-dimensional maximally symmetric manifolds, $\mathbb{E}^2$, $\mathbb{S}^2$, and $\mathbb{H}^2$, by bounded regular tiles. For instance, there exist infinitely many regular tessellations of the hyperbolic plane $\mathbb{H}^2$ by curved hyperbolic equilateral triangles whose vertex angles are $2\pi/d$ for $d = 7, 8, 9, \ldots$ On the other hand, we prove that there is no curved hyperbolic regular tetrahedron which tessellates the three-dimensional hyperbolic space $\mathbb{H}^3$. We also show that a regular tessellation of $\mathbb{H}^3$ can only consist of the hyperbolic cubes, hyperbolic regular icosahedra, or two types of hyperbolic regular dodecahedra. There exist only two regular hyperbolic space-fillers of $\mathbb{H}^4$. If $n > 4$, then there exists no regular tessellation of $\mathbb{H}^n$.

Keywords: Euclidean space; spherical and hyperbolic geometry; hypersphere; n-simplex; n-cube; icosahedron; 120-cell; 600-cell

MSC: 51M20

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1. Introduction

There exist three three-dimensional maximally symmetric manifolds: the Euclidean space $\mathbb{E}^3$, the hypersphere $\mathbb{S}^3$, and the hyperbolic space $\mathbb{H}^3$, see e.g., ([1], Chapt. 13, [2], p. 721). There are no other maximally symmetric manifolds up to scaling. One way to imagine these manifolds is to tile them with congruent bounded regular polyhedral cells. For $\mathbb{S}^3$ and $\mathbb{H}^3$, these cells are obviously curved. Their edges are geodesics and their faces are formed by families of geodesics.

For simplicity, the term "regular" will be often omitted from now on. We shall deal not only with the three-dimensional case but, for any integer $n \geq 1$, we shall look for all regular face-to-face tessellations of the n-dimensional maximally symmetric manifolds, $\mathbb{E}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$, by congruent bounded regular polytopic cells (Euclidean, spherical, or hyperbolic). The unit *hypersphere* $\mathbb{S}^n$ is defined as follows:

$$\mathbb{S}^n = \{x \in \mathbb{E}^{n+1} \mid |x| = 1\},$$

where $|\cdot|$ denotes the standard Euclidean norm. Note that the Gaussian sectional curvature of $\mathbb{S}^n$ is equal to 1 at each point. The *hyperbolic space* $\mathbb{H}^n$ is the maximally symmetric n-dimensional Riemannian manifold, whose sectional curvature is equal to (-1) at each point and each pair of principal directions. This manifold is simply connected.

Now, we briefly summarize some well-known facts about regular polytopes in Euclidean space E^n that will be used later. In $\mathbb{E}^2$, there exist infinitely many regular polygons: equilateral triangles, squares, pentagons, etc. Nevertheless, in $\mathbb{E}^3$ there exist only five regular polyhedra (called *Platonic bodies*) whose faces are congruent regular polygons, and the same number of faces meet at every vertex. Denote by v, e, and f the number of their vertices, edges, and faces, respectively. Then the famous Euler formula,

$$f + v = e + 2, \tag{1}$$

is clearly valid. The Platonic bodies can be arranged into dual pairs with faces and vertices interchanged (see Table 1). Note that the regular tetrahedron is self-dual.

The number d of edges coming together at every vertex is said to be the *vertex degree*. Thus, we obtain

$$dv = 2e. \tag{2}$$

It is easy to find that the dihedral angles between two adjacent faces of the tetrahedron and octahedron in $\mathbb{E}^3$ are (see [3])

$$\alpha_1 = \arccos\tfrac{1}{3} \approx 70.529^\circ \quad \text{and} \quad \alpha_3 = 180^\circ - \alpha_1 \approx 109.471^\circ, \tag{3}$$

respectively. The dihedral angle of the dodecahedron is

$$\alpha_4 = 180^\circ - \arctan 2 \approx 116.565^\circ \tag{4}$$

and of the icosahedron

$$\alpha_5 = \arccos\Big(-\frac{\sqrt{5}}{3}\Big) \approx 138.19^\circ. \tag{5}$$

Table 1. Five Platonic bodies in $\mathbb{E}^3$. Here, f is the number of their faces, e is the number of their edges, v is the number of vertices, d is the degree of each vertex, α is the dihedral angle (rounded to integers) between two adjacent faces, S_i denotes for the symmetric group of all permutations and A_i the alternating group of all even permutations of i elements (see [4], p. 86).

Name	f	e	v	d	α	Point Group
tetrahedron	4 triangles	6	4	3	71°	A_4
cube	6 squares	12	8	3	90°	S_4
octahedron	8 triangles	12	6	4	109°	S_4
dodecahedron	12 pentagons	30	20	3	117°	A_5
icosahedron	20 triangles	30	12	5	138°	A_5

For $n > 3$, we may define a *regular polytope* in $\mathbb{E}^n$ by induction: all its $(n-1)$-dimensional facets are congruent regular polytopes and all of its vertices have the same vertex degree.

If $n = 4$, then there exist exactly six regular polytopes, see Table 2. In this case, relation (2) holds again, and the following Euler–Poincaré formula,

$$f + v = e + c, \tag{6}$$

is valid. Here, the symbol c stands for the number of congruent polyhedral cells on their surface. They can only be from Table 1. The other symbols have the same meaning as for $n = 3$.

Theorem 1. *For every $n \geq 5$, there exist exactly three regular polytopes in $\mathbb{E}^n$, namely, the n-simplex, the n-cube, and the n-orthoplex.*

See [5], and for their volumes see ([6], pp. 452–453).

Table 2. Regular polytopes in $\mathbb{E}^4$ discovered by Ludwig Schläfli. The symbol c stands for the number of polyhedral cells on their surfaces. The other symbols have the same meaning as in Table 1.

Name	c	f	e	v	d	Duality
4-simplex	5 tetrahedra	10	10	5	4	self-dual
4-cube	8 cubes	24	32	16	4	dual to 4-orthoplex
4-orthoplex	16 tetrahedra	32	24	8	6	dual to 4-cube
24-cell	24 octahedra	96	96	24	8	self-dual
120-cell	120 dodecahedra	720	1200	600	4	dual to 600-cell
600-cell	600 tetrahedra	1200	720	120	12	dual to 120-cell

2. Two-Dimensional Regular Tessellations

Below we present several well-known results for the dimension $n = 2$, which will be used in Section 3 to derive local properties of regular tessellations for $n = 3$. A *regular polygon* in $\mathbb{E}^2$, $\mathbb{S}^2$, and $\mathbb{H}^2$ is a bounded (possibly curved) polygon all of whose sides are geodesics of the same length and all its vertex angles have the same size. In Euclidean plane $\mathbb{E}^2$, there exist, up to translation, rotation, reflection, and scaling, only three regular tessellations; namely, by equilateral triangles with vertex degree $d = 6$, by squares with $d = 4$, and by regular hexagons with $d = 3$ (see Figure 1).

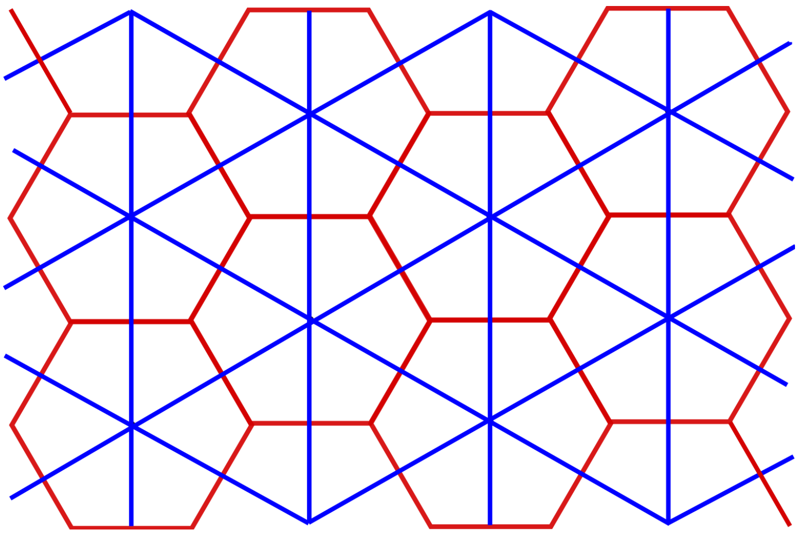

Figure 1. The hexagonal tessellation of $\mathbb{E}^2$ is dual to the triangular tessellation and vice versa.

Consider now a Platonic body, all of whose vertices belong to $\mathbb{S}^2$. The radial projection (sometimes also called the central orthographic projection) from the center of $\mathbb{S}^2$ maps all edges onto $\mathbb{S}^2$. This generates five regular tessellations of $\mathbb{S}^2$ by spherical polygons with at least three sides. Their vertices are consequently connected by geodesics, and the sum of all angles around every vertex equals 360°. These regular tessellations are called the *projected tetrahedron*, the *projected octahedron*, the *projected icosahedron*, the *projected cube*, and the *projected dodecahedron*. There exist, however, other regular tessellations of $\mathbb{S}^2$ (see Figure 2).

Theorem 2. *Every regular tessellation of $\mathbb{S}^2$ is obtained by the radial projection of all five Platonic bodies onto $\mathbb{S}^2$ and by connecting two antipodal points by $d \geq 2$ great semicircles, ending at these vertices with mutual angle $2\pi/d$.*

Proof. Formulas (1) and (2) are evidently also valid for each regular tessellation of the sphere $\mathbb{S}^2$. Furthermore, for p-gonal spherical tiles, we find that

$$2e = fp. \tag{7}$$

Multiplying (1) by $2d$ and substituting (2) and (7), we get the following necessary condition:

$$pf(d-2) = 2d(f-2). \tag{8}$$

Hence, every regular tessellation of $\mathbb{S}^2$ has to satisfy the Diophantine Equation (8).

First, let $p \geq 6$. Then, we find that (8) has no solution for $d \geq 3$, since the following relation,

$$0 = pf(d-2) - 2d(f-2) \geq 6fd - 12f - 2df + 4d = 4d(f+1) - 12f \geq 12(f+1) - 12f = 12,$$

is impossible.

The remaining cases can be investigated by inspection. By equality (8) for $p = 5$, we obtain $3df = 10f - 4d < 10f$, i.e., $d \leq 3$, which yields only one solution, $d = 3$ and $f = 12$. Similarly, for $p = 4$ we also get only one solution $d = 3$ and $f = 6$. For $p = 3$ there exist three solutions, $d \in \{3, 4, 5\}$ and $f \in \{4, 8, 20\}$, respectively. We find that the above solutions can be obtained by the radial projection of all five Platonic bodies onto the sphere $\mathbb{S}^2$ (cf. Figure 2).

Finally, for $p = 2$, the above Diophantine Equation (8) has infinitely many solutions for any $d = f \geq 2$. Here, the case $d = f = 2$ corresponds to two hemispheres. $\square$

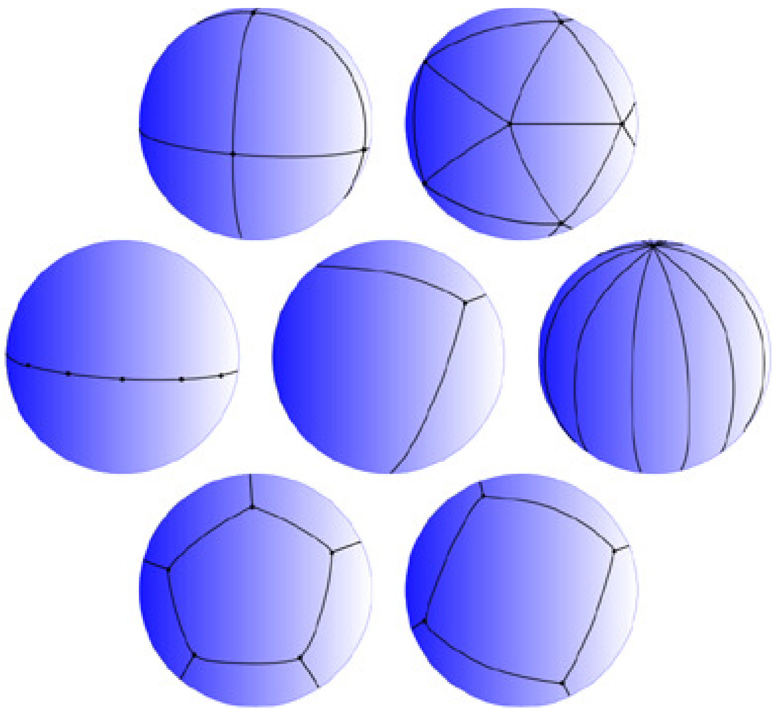

Figure 2. Regular tessellations of $\mathbb{S}^2$. The projected tetrahedron in the center is self-dual. The projected cube is dual to the projected octahedron and the projected dodecahedron is dual to the projected icosahedron. The hosohedron tessellation depicted on the right consists of $d \geq 2$ congruent lune tiles (2-gons). Its dual is on the left. It is formed by two hemispheres.

To summarize the above result, there are only the following regular tessellations of the sphere $\mathbb{S}^2$ (see Figure 2):

- By 12 spherical regular pentagons for $d = 3$;
- By 6 spherical squares for $d = 3$;
- By 4, 8, and 20 spherical equilateral triangles for the vertex degree $d \in \{3, 4, 5\}$, respectively;
- By $d = f > 2$ spherical 2-gons with mutual vertex angle $2\pi/d$;
- By $f = 2$ hemispheres.

Remark 1. *The size of the foregoing spherical space-fillers of* $\mathbb{S}^2$ *cannot be chosen arbitrarily (as in* $\mathbb{E}^2$*), because they must fit to the unit sphere. For instance, there are three different equilateral triangular space-fillers of* $\mathbb{S}^2$ *whose vertex angles are* $120°$*,* $90°$*, and* $72°$ *(see Figure 2). Obviously, their edges have uniquely determined lengths,* $\pi - \arccos(1/3)$*,* $\pi/2$*, and* $2\arctan(1/\tau)$*, respectively, where* $\tau = (\sqrt{5}+1)/2$ *is the golden section. In higher dimensional tessellations of* $\mathbb{S}^n$ *and* $\mathbb{H}^n$*, for any* $n \geq 2$*, all edges also have uniquely defined lengths.*

The *dual tessellation* of a regular tessellation is formed by taking the center of each polygon (possibly curved) as a vertex and joining the centers of neighboring polygons by geodesics (see e.g., Figure 1). Thus, the existence of a dual tessellation is obvious. The projected dodecahedron and the projected icosahedron posses the global five-fold symmetry in every vertex. On the other hand, no uniform crystal mesh in $\mathbb{E}^n$ for any $n \leq 3$ exists with a global five-fold symmetry, but for $n \geq 4$, a global five-fold symmetry can be achieved (see [7]). According to ([8], p. 135), the regular tessellations of the hyperbolic plane $\mathbb{H}^2$ may have a global seven-fold symmetry, eight-point symmetry, etc., in every vertex.

The hyperbolic plane $\mathbb{H}^2$ has somewhat non-intuitive geometry. In particular, Hilbert proved (see [9]) that $\mathbb{H}^2$ cannot be isometrically imbedded into $\mathbb{E}^3$. In 1955, Blanuša [10] proved that $\mathbb{H}^2$ can be isometrically imbedded into $\mathbb{E}^6$. Nevertheless, it is not known whether the dimension 6 can be reduced.

Therefore, we should somehow deform the hyperbolic plane to get some idea of what its regular tessellations look like (see [11]). For instance, the hyperbolic plane can be represented by the interior of the unit circle k (the so-called *Poincaré disk*) in $\mathbb{E}^2$. Let us point out that its geodesics are circular arcs that are perpendicular to k (cf. Figure 3). Note that these arcs may degenerate to a straight line passing through the center of k. One can derive that there exists exactly one circular arc passing through two different arbitrary points, A and B, that is perpendicular to the boundary circle k at its endpoints $P \in k$ and $Q \in k$. The circular arc has its center outside k (if it is not a straight line). The distance between the points A and B is defined, e.g., in ([3], p. 163).

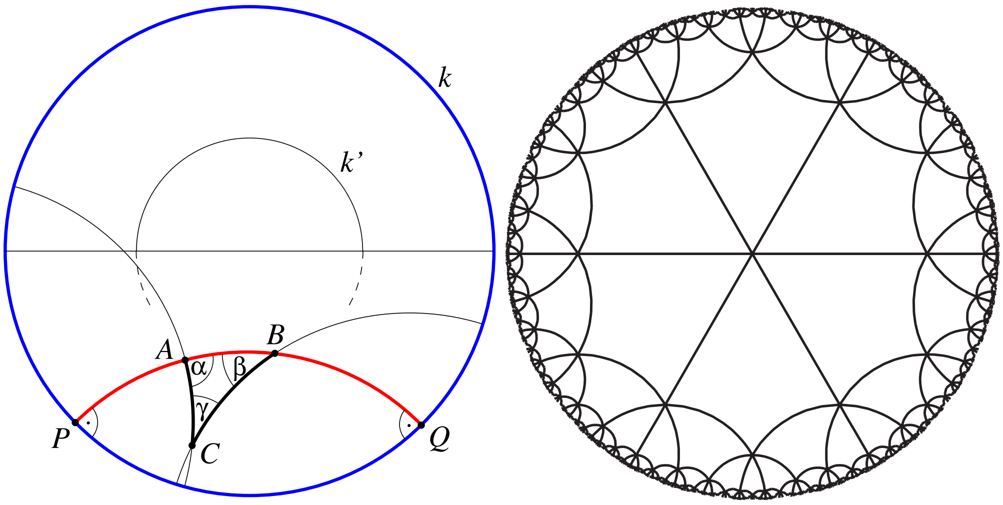

Figure 3. (**Left**): The shortestconnections are represented by circular arcs (or straight lines passing through the center). They are orthogonal to k at their endpoints. Lambert's formula states that $\alpha + \beta + \gamma < 180°$ in the curved triangle ABC. (**Right**): Regular tessellation of the Poincaré disk by hyperbolic squares. The angle at each of their four vertices is only $60°$.

Note that the length of a circle k' (with radius r) concentric to k is larger than $2\pi r$, see the left part of Figure 3. This is similar to measuring the length of a circle around a saddle point on some curved surface.

There exist infinitely many different tessellations by regular curved hyperbolic polygons of $\mathbb{H}^2$; namely:

- by hyperbolic equilateral triangles with vertex degree $d \geq 7$; more precisely, for every $d \geq 7$ there exists one equilateral hyperbolic triangle with vertex angles $2\pi/d$ that tile $\mathbb{H}^2$,
- By hyperbolic squares for any $d \geq 5$ (see the right part of Figure 3);
- By hyperbolic pentagons for any $d \geq 4$;
- By hyperbolic hexagons for any $d \geq 4$; and
- By hyperbolic heptagons, octagons, etc., for any $d \geq 3$.

Note that d stands also for the number of hyperbolic polygons meeting at every vertex.

An analogue of the hosohedron tessellation from Figure 2 in two-dimensional Euclidean space $\mathbb{E}^2$ consists of congruent parallel strips of four quadrants, etc. Since such individual tiles are unbounded, we shall not take them into account. For the same reason, we shall not consider an analogue of the hosohedron tessellation in the hyperbolic plane $\mathbb{H}^2$.

3. Three-Dimensional Regular Tessellations

David Brander [12] proved that, for any $n > 1$, the hyperbolic space $\mathbb{H}^n$ can be isometrically imbedded into $\mathbb{E}^{6n-6}$. An open question is whether the huge dimension $6n - 6$ is optimal or if it can be reduced. Note that there is a local isometric imbedding from $\mathbb{H}^n$ to $\mathbb{E}^{2n-1}$, see [13].

Regular face-to-face tessellations of $\mathbb{E}^n$, $\mathbb{S}^n$, and $\mathbb{H}^3$ for $n > 1$ are formed by congruent bounded regular polytopic cells that are possibly curved. They can again be defined by induction, namely, its $(n-1)$-dimensional facets are regular cells (possibly curved) and all their vertices have the same vertex degree.

The dual tessellation of a given three-dimensional congruent regular tessellation is defined as in Section 2.

Lemma 1. *For any regular tessellation there exists its dual.*

Lemma 2. *Let d be the vertex degree of a three-dimensional regular tessellation and let each edge be surrounded by p cells. Then, the faces of the dual space-filler cell are p-gons (possibly curved) and their number is d.*

Proofs of Lemmas 1 and 2 immediately follow from the assumed geometric regularity.

The Euclidean space $\mathbb{E}^3$ can be tessellated only by cubes with vertex degree $d = 6$ and with $m = 8$ cubes meeting at each vertex.

Suppose now that all vertices of each Schläfli regular polytope in $\mathbb{E}^4$ lie on the unit hypersphere $\mathbb{S}^3$. Then $\mathbb{S}^3$ can be tessellated by means of the radial projection of every polytope from Table 2 into $\mathbb{S}^3$. Therefore, $v \in \{5, 8, 16, 24, 120, 600\}$ vertices can be uniformly distributed on $\mathbb{S}^3$ and, in this way, we obtain the following regular tessellations of $\mathbb{S}^3$ by regular spherical cells:

- Five, sixteen, and six-hundred spherical tetrahedral cells with vertex degree $d \in \{4, 6, 12\}$, respectively. From this, one can derive that $m \in \{4, 8, 20\}$ cells meet at each vertex and each edge is surrounded by $p \in \{3, 4, 5\}$ cells, respectively. Hence, the corresponding dihedral angles are 120°, 90°, and 72°. The projected five-cell regular tessellation is self-dual;
- Eight spherical cubes with $d = 4$, where $m = 4$ cubes meet at each vertex and $p = 3$ cubes surround each edge. This tessellation is dual to the projected tetrahedral 16-cell;
- Twenty-four spherical octahedral cells with $d = 8$ and with $m = 6$ cells meeting at each vertex and $p = 3$ octahedral cells around each edge. This tessellation is also self-dual.
- One-hundred-and-twenty spherical dodecahedral cells with $d = 4$. Furthermore, $m = 4$ cells meet at each vertex. There are $p = 3$ dodecahedral cells around each edge. The dihedral angle α_4 given by (4) is very close to 120°. Therefore, the radial projection onto $\mathbb{S}^3$ only slightly deforms the usual 120-cell. This tessellation is dual to the projected tetrahedral 600-cell.

There are, however, other regular tessellations of $\mathbb{S}^3$ having only two antipodal points (as in the two-dimensional case sketched on the right part of Figure 2). Their tiles resemble slices (we shall again not consider analogues these tessellations in $\mathbb{E}^3$ and $\mathbb{H}^3$, since the corresponding tiles are unbounded, see [14]). For instance, the equator of $\mathbb{S}^3$ represented by the sphere $\mathbb{S}^2$ is tessellated as shown in Figure 2.

Obviously, the Euler–Poincaré relation (6) holds again for the above regular spherical tessellations of $\mathbb{S}^3$. They can be visualized in $\mathbb{E}^3$ by means of the stereographic projection. Furthermore, notice that the curved icosahedral cell does not appear in the above list of all regular tessellations of $\mathbb{S}^3$, since the corresponding angle α_5 given by (5) is greater than 120°. Therefore, it is a natural candidate for a space-filler of $\mathbb{H}^3$. We will again deal with a hyperbolic geometry as in the two-dimensional case (see [15]).

The hyperbolic space $\mathbb{H}^3$ can be visualized similarly to $\mathbb{H}^2$ in Figure 3. Instead of the Poincaré disk, we have to consider the *Poincaré ball* whose boundary is a two-dimensional sphere that does not belong to $\mathbb{H}^3$. In such a visualization, faces of curved cells are parts of spherical surfaces. In the next theorem, we show that the manifold $\mathbb{H}^3$ can be tessellated, for instance, by curved icosahedral cells with $d = 20$ and $m = 12$ cells meeting at every vertex. Every edge is surrounded by $p = 3$ cells. This tessellation is self-dual, since each edge is surrounded by three cells; i.e., the dual space-filler has triangular faces by Lemma 2 (see the right cell of Figure 4).

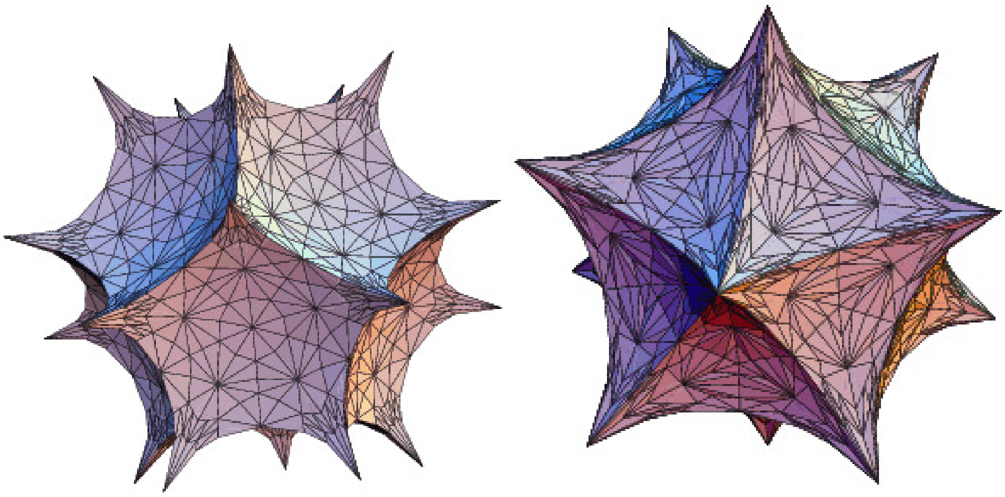

Figure 4. Schematic visualizations of curved hyperbolic dodecahedral and icosahedral space-filler cells of $\mathbb{H}^3$. (From Wolfram MathWorld).

Furthermore, let us point out that the quantities d, m, p in Table 3 below attain the same values as v, f, d from Table 1, respectively, i.e., formulae similar to (1) and (2) hold again for d, m, p. The proof of the following theorem differs from that presented in ([14], p. 157).

Theorem 3. *There are at most four regular tessellations of $\mathbb{H}^3$, namely, by a hyperbolic cube, by one of the two types of hyperbolic dodecahedra, and by a hyperbolic icosahedron.*

Proof. We shall investigate regular tessellations of all three manifolds—$\mathbb{E}^3$, $\mathbb{S}^3$, and $\mathbb{H}^3$—using properties stated in Sections 1 and 2. A necessary condition for the existence of a regular tessellation of a three-dimensional maximally symmetric manifold is that local symmetries represented by the symmetric group S_4 or alternating groups A_4 and A_5 (see Table 1 and [16]) have to be satisfied at each vertex. The reason is that these manifolds are locally almost Euclidean at each point. In other words, the intersection of an arbitrarily small sphere centered at each vertex with a maximally symmetric manifold should allow for the production of regular tessellations on this sphere (see the first column of Table 3 and Figure 2).

Obviously, we have only a finite number of different face-to-face regular tessellations for $n = 3$ (up to translation, rotation, reflection, and scaling) contrary to the case $n = 2$. All admissible cases are summarized in Table 3 except for regular tessellations of $\mathbb{S}^3$ that have only two antipodal points. These exceptional cases can be excluded, since they do not contribute to the number of regular tessellations of $\mathbb{H}^3$ by bounded tiles.

The first row of Table 3 contains all possible types of tiles that can be used, i.e., three-dimensional cells that can possibly be curved. In this way, we obtain $5 \times 5 = 25$ possibilities to inspect. They include the six regular tessellations of $\mathbb{S}^3$ mentioned above and one regular tessellation of $\mathbb{E}^3$ by ordinary cubes.

Some other cases can be excluded a priori. Consider, for instance, the local projected dodecahedron in the left lower corner of Table 3. Since its faces are spherical pentagons (see Figure 2), we immediately find that the corresponding space-filler cannot be a tetrahedral, cubic, octahedral, or dodecahedral cell. This is indicated by the symbol y in Table 3. Similarly, we can exclude four cases corresponding to the local projected cube, which are indicated by x. Consequently, the symbols x and y mean that the cross-section of the cell with an arbitrarily small sphere $\mathbb{S}^2$ centered in an arbitrary vertex is not a spherical square nor a spherical pentagon, i.e., these cases are impossible.

Table 3. Possible regular tessellations of $\mathbb{E}^3$, $\mathbb{S}^3$, and $\mathbb{H}^3$. The corresponding projected cells are in parentheses. The first column shows the local two-dimensional spherical tessellation about every vertex. In the first row, the symbol d stands for the vertex degree, m is the number of cells meeting at each vertex, and p is the number of cells surrounding each edge. The first row then continues with the type of three-dimensional tiles that can be used (i.e., cells possibly curved). The symbols x, y, z indicate cases that cannot happen (see the proof of Theorem 3).

Projected	d	m	p	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
tetrahedron	4	4	3	$\mathbb{S}^3$ (5-cell)	$\mathbb{S}^3$ (8-cell)	z	$\mathbb{S}^3$ (120-cell)	z
octahedron	6	8	4	$\mathbb{S}^3$ (16-cell)	$\mathbb{E}^3$	z	$\mathbb{H}^3$	z
cube	8	6	3	x	x	$\mathbb{S}^3$ (24-cell)	x	x
icosahedron	12	20	5	$\mathbb{S}^3$ (600-cell)	$\mathbb{H}^3$	z	$\mathbb{H}^3$	z
dodecahedron	20	12	3	y	y	y	y	$\mathbb{H}^3$

Notice that the regular tessellations on the main diagonal of Table 3 are self-dual. The other dual tessellations are placed symmetrically with respect to the main diagonal. Therefore, we can exclude other possibilities by the following duality argument indicated by the symbol z in Table 3. If such a tessellation were to exist, then by Lemma 1 its dual tessellation would also exist, which is a contradiction. Assume, for example, that there exists an icosahedral tessellation such that its intersection with a small ball centered at any vertex yields a spherical tetrahedron (see the first z in the last column of Table 3). Then, by Lemma 1 there should exist a dual tessellation, which contradicts the nonexistence of the regular tessellation indicated by the first y of the last row in Table 3.

The remaining four cases in Table 3 are indicated by $\mathbb{H}^3$. Two of them on the main diagonal are self-dual. □

Corollary 1. *There is no regular hyperbolic tetrahedral and also no octahedral space-filler of* $\mathbb{H}^3$.

The proof follows directly from Table 3, which contains all admissible cases. So there is no analogy with triangular tessellations of the hyperbolic plane $\mathbb{H}^2$ in the three-dimensional case.

Now let us introduce the four hyperbolic space-filler cells predicted by Theorem 3. First, consider the regular tessellation of $\mathbb{H}^3$ made by hyperbolic cubes. From Table 3, we can see that there are $p = 5$ hyperbolic cubes around any edge. Hence, their dihedral angles are 72° (see Figure 5). Moreover, each vertex is surrounded by $m = 20$ cells. Hence, the intersection of the corresponding regular tessellation with an arbitrarily small two-dimensional sphere centered at each vertex leads to the regular tessellation (projected icosahedron sketched in Figure 2).

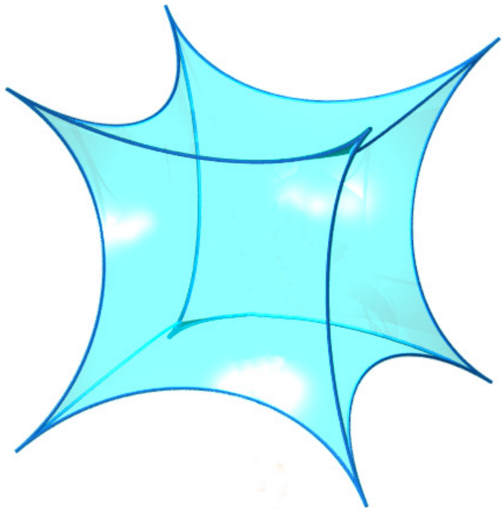

Figure 5. A simple visualization of the curved cubic space-filler cell of $\mathbb{H}^3$. Its dihedral angles are equal to 72°.

Corollary 2. *In the regular tessellations by a cubic space-filler of $\mathbb{S}^3$, $\mathbb{E}^3$, and $\mathbb{H}^3$, every edge is surrounded by $p \in \{3, 4, 5\}$ cubic cells, and every vertex is surrounded by $m \in \{4, 8, 20\}$ cubic cells, respectively. There are no other cubic space-fillers for the dimension $n = 3$.*

The proof also follows from Table 3. The corresponding dihedral angles of these cubes are 120°, 90°, and 72°.

By Lemma 1, there exists the dual regular tessellation to the regular hyperbolic tessellation, where each edge is surrounded by $p = 5$ cubic cells with vertex degree $d = 12$. Hence, by Lemma 2, the corresponding dual space-filler cell has 12 pentagonal faces. The dihedral angle $\alpha_4 \approx 117°$ of the usual Euclidean dodecahedron changes to 90° in the hyperbolic case, since $p = 4$ hyperbolic dodecahedra surround each edge (see Table 3). Each vertex is surrounded by $m = 8$ dodecahedral hyperbolic cells.

Nevertheless, there are two different regular tessellations of $\mathbb{H}^3$ formed by hyperbolic dodecahedra. For the second type, $p = 5$, the corresponding dihedral angle is 72°, and each vertex is surrounded by $m = 20$ cells. Hence, the intersection of the corresponding regular tessellation with an arbitrarily small two-dimensional sphere centered at each vertex leads to the projected icosahedron (see Figure 2). This tessellation is self-dual.

Finally, the dihedral angle $\alpha_5 \approx 138°$ of the usual Euclidean icosahedron changes to 120° in the hyperbolic case, since there is a place for $p = 3$ hyperbolic icosahedra around any edge, see Table 3. Each vertex is surrounded by $m = 12$ icosahedral cells. The intersection of the corresponding regular tessellation with an arbitrarily small two-dimensional sphere $\mathbb{S}^2$ centered at each vertex leads to the projected dodecahedron (see Figure 2).

4. Four-Dimensional Tessellations

The Euclidean space $\mathbb{E}^n$ can be tessellated by n-cubes for any positive integer n. By [5] for $n = 4$, the 4-orthoplex and the 24-cell are space-fillers as well (see Table 4). Their dihedral angles between adjacent cells are 120° in both cases.

Using the radial projection of the regular polytopes in $\mathbb{E}^{n+1}$ into the hypersphere $\mathbb{S}^n$, we get from Theorem 1 that $\mathbb{S}^4$ for each $n \geq 4$ can be tessellated by hyperspherical n-simplices, hyperspherical n-cubes, and hyperspherical n-orthoplexes. Let us note that the regular tessellations on the main diagonal of Table 4 are self-dual, and there exist only two dual tessellations of $\mathbb{S}^4$ that are placed symmetrically with respect to the main diagonal.

There are, again, other regular tessellations of $\mathbb{S}^4$ that have only two antipodal points (like in the two-dimensional case). The equator of $\mathbb{S}^4$ represented by the hypersphere $\mathbb{S}^3$ can be tessellated by the spherical cells of Section 3. Nevertheless, we shall again not consider these special cases, since they do not lead to regular tessellations of $\mathbb{E}^4$ and $\mathbb{H}^4$ by bounded tiles.

Table 4. Possible regular tessellations of the three four-dimensional maximally symmetric manifolds $\mathbb{E}^4$, $\mathbb{S}^4$, and $\mathbb{H}^4$. The first column shows local three-dimensional spherical tessellation about every vertex. In the first row, the symbol d stands for the vertex degree and m is the number of hypercells meeting at each vertex. The first row then continues with the type of four-dimensional tiles that are used. The symbols u, w, x, y, and z again indicate cases that cannot happen.

Projected	d	m	4-Simplex	4-Cube	4-Orthoplex	24-Cell	120-Cell	600-Cell
4-simplex	5	5	$\mathbb{S}^4$	$\mathbb{S}^4$	z	z	z	z
4-orthoplex	8	16	$\mathbb{S}^4$	$\mathbb{E}^4$	z	z	z	z
4-cube	16	8	x	x	$\mathbb{E}^4$	x	x	x
24-cell	24	24	y	y	y	$\mathbb{E}^4$	y	y
600-cell	120	600	u	u	u	u	$\mathbb{H}^4$	u
120-cell	600	120	w	w	w	w	w	$\mathbb{H}^4$

Theorem 4. *There exist exactly two bounded regular hyperbolic space-filler cells of $\mathbb{H}^4$, namely, the hyperbolic* 120*-cell and* 600*-cell.*

The proof is similar to that for the manifold $\mathbb{H}^3$ from the previous section.

5. Higher-Dimensional Tessellations

The Euclidean space $\mathbb{E}^n$ can be tessellated by n-cubes for any $n \geq 1$. Using Theorem 1 and the radial projection, we find that the spherical n-cubes tessellate $\mathbb{S}^n$ for any $n \geq 1$. By Theorems 3 and 5, hyperbolic n-cubes tessellate $\mathbb{H}^n$ only for $n \leq 3$.

According to Section 2, there exist infinitely many regular space-fillers of the hyperbolic plane $\mathbb{H}^2$. On the other hand, the following statement holds:

Theorem 5. *There is no regular tessellation of $\mathbb{H}^n$ for $n \geq 5$.*

The proof follows from Theorems 1 and 4. The main results of this paper are summarized in Table 5.

Table 5. The number of regular tessellations (which do not have only two antipodal points) of the maximally symmetric manifolds.

n	$\mathbb{E}^n$	$\mathbb{S}^n$	$\mathbb{H}^n$
1	1	1	not def.
2	3	5	∞
3	1	6	4
4	3	3	2
5	1	3	0
6	1	3	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

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