On Extreme Computational Complexity of the Einstein Equations

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Abstract We show how to explicitly express the first of the 10 Einstein partial differential equations to demonstrate their extremely large general complexity. Consequently, it is very difficult to use them, for example, to realistically model the evolution of the Solar system, since their analytical solution even for at least two massive bodies is not known. Significant computational problems associated with their numerical solution are illustrated as well. Thus, we cannot verify whether the Einstein equations describe the motion of two or more bodies sufficiently accurately.

Keywords Einstein equations · Schwarzschild solution · Finite difference method · n-body simulations

Mathematics subject classification: 65M06 · 35L70

1 Historical Facts of Importance

Karl Schwarzschild was probably the first scientist who ever realized that our universe at any fixed time might be non-Euclidean and that it can be modeled by the three-dimensional hypersphere $\mathbb{S}^3$ or the three-dimensional hyperbolic pseudosphere $\mathbb{H}^3$, see his paper Schwarzschild and Über das zulässige Krümmungsmaaß des Raumes (1900) from 1900. In 1915, he became famous, since he calculated the first nontrivial solution of the Einstein vacuum equations (see (6) below).

On November 18, 1915, Albert Einstein submitted his famous paper Einstein (1915b) about Mercury's perihelion shift. Here the gravitational field is described by the following equations using the present standard notation and the Einstein summation convention:

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$$\Gamma^{\varkappa}{}_{\mu\nu,\varkappa} - \Gamma^{\lambda}{}_{\mu\varkappa}\Gamma^{\varkappa}{}_{\lambda\nu} = 0, \quad \varkappa, \lambda, \mu, \nu = 0, 1, 2, 3, \tag{1}$$

where

$$\Gamma^{\varkappa}{}_{\mu\nu} := \frac{1}{2} g^{\varkappa\lambda}(g_{\nu\lambda,\mu} + g_{\lambda\mu,\nu} - g_{\mu\nu,\lambda}) \tag{2}$$

are the *Christoffel symbols of the second kind* (sometimes also called the connection coefficients), $g_{\mu\nu} = g_{\mu\nu}(x^0, x^1, x^2, x^3)$ are components of the unknown 4×4 twice differentiable symmetric metric tensor of the spacetime of one time variable x^0 and three space variables x^1, x^2, x^3, and

$$\det(g_{\mu\nu}) = -1. \tag{3}$$

Furthermore, $g^{\mu\nu}$ is the 4×4 symmetric inverse tensor to $g_{\mu\nu}$. For brevity, the first classical derivatives of a function $f = f(x^0, x^1, x^2, x^3)$ are denoted as $f_{,\varkappa} := \partial f / \partial x^{\varkappa}$. For simplicity, the dependence of all functions on the spacetime coordinates will be often not indicated. Note that the infinitesimally small spacetime distance $\mathrm{d}s$ is usually expressed by physicists as follows: $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}$.

On November 20, 1915, David Hilbert submitted the paper Hilbert (1915) which was published on March 31, 1916. He did not require the validity of the restrictive algebraic constraint (3). Using a variational principle, he derived the following complete equations for the gravitational field (see Fig. 1):

$$R_{\mu\nu} := \Gamma^{\varkappa}{}_{\mu\nu,\varkappa} \underline{\underline{- \Gamma^{\varkappa}{}_{\mu\varkappa,\nu} + \Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\varkappa}{}_{\lambda\varkappa}}} - \Gamma^{\lambda}{}_{\mu\varkappa}\Gamma^{\varkappa}{}_{\lambda\nu} = 0. \tag{4}$$

The doubly underlined terms do not appear in (1). The number of the Christoffel symbols in (1) is $10(4 + 4 \times 4) = 200$. From (2) we observe that (4) is a system of nonlinear second-order partial differential equations. Its left-hand side $R_{\mu\nu}$ is at present called the *Ricci tensor* and the equations $R_{\mu\nu} = 0$ are called the *Einstein vacuum equations*.

Gravitationsgleichungen nur zweite Ableitungen der Potentiale $g^{\mu\nu}$ enthalten, so muß H die Gestalt haben

$$H = K + L$$

wo K die aus dem Riemannschen Tensor entspringende Invariante (Krümmung der vierdimensionalen Mannigfaltigkeit)

$$K = \sum_{\mu,\nu} g^{\mu\nu} K_{\mu\nu},$$

$$K_{\mu\nu} = \sum_{\varkappa}\left(\frac{\partial}{\partial w_\nu}\begin{Bmatrix}\mu\varkappa\\ \varkappa\end{Bmatrix} - \frac{\partial}{\partial w_\varkappa}\begin{Bmatrix}\mu\nu\\ \varkappa\end{Bmatrix}\right) + \sum_{\varkappa,\lambda}\left(\begin{Bmatrix}\mu\varkappa\\ \lambda\end{Bmatrix}\begin{Bmatrix}\lambda\nu\\ \varkappa\end{Bmatrix} - \begin{Bmatrix}\mu\nu\\ \lambda\end{Bmatrix}\begin{Bmatrix}\lambda\varkappa\\ \varkappa\end{Bmatrix}\right)$$

Fig. 1 Hilbert's original paper Hilbert (1915, p. 402)

By Sauer (1999, p. 569), Hilbert's knowledge and understanding of the calculus of variation and of invariant theory readily put him into a position to fully grasp the Einstein gravitational theory. Hilbert's original paper (see Fig. 1) contains the Ricci tensor $R_{\mu\nu}$ from (4) denoted by $K_{\mu\nu}$ (K is the Ricci scalar) and the Christoffel symbol $\Gamma^{\varkappa}{}_{\mu\nu}$ denoted by $-\begin{Bmatrix}\mu\nu\\ \varkappa\end{Bmatrix}$. Hilbert's paper was submitted for publication five days earlier than Einstein's note Einstein (1915a).

On November 25, 1915, Albert Einstein submitted a three and a half page note Einstein (1915a) which contains the same expression of the Ricci tensor as in (4). The way in which the Einstein equations (see (16) below) were derived is not described there. Einstein's short note Einstein (1915a) was published already on December 2, 1915, i.e. within one week. One can easily see that the classical Minkowski metric

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1), \tag{5}$$

where all non-diagonal entries are zeros, is a trivial solution to the Einstein vacuum equations (4) since all the Christoffel symbols (2) are zeros.

On January 13, 1916, Karl Schwarzschild submitted the paper Schwarzschild (1916b) containing the first nontrivial static spherically symmetric solution of the Einstein vacuum equations. At present his solution is usually written as follows:

$$g_{\mu\nu} = \mathrm{diag}\left(-\frac{r-S}{r}, \frac{r}{r-S}, r^2, r^2\sin^2\theta\right), \tag{6}$$

where all non-diagonal entries are zeros, $S \geq 0$ is a fixed real constant, and the standard spherical coordinates

$$x^1 = r\cos\varphi\cos\theta, \quad x^2 = r\sin\varphi\cos\theta, \quad x^3 = r\sin\theta,$$

are employed, $r > S$, $\varphi \in [0, 2\pi)$, and $\theta \in [0, \pi]$. The metric tensor (6) is called the *exterior Schwarzschild solution* of the Einstein equations (4). We see that for $S = 0$ the formula (6) reduces to the Minkowski metric in the spherical coordinates. Note that already on December 22, 1915, Karl Schwarzschild wrote a letter to Einstein announcing that he was reading Einstein's paper on Mercury and found a solution to field equations (see Vankov 2011 for this letter). This solution is very similar to (6).

Finally, on March 20, 1916, Albert Einstein submitted his fundamental work Einstein (1916), where the general theory of relativity was established. One year later, Einstein in (1952) added the term $\Lambda g_{\mu\nu}$ to the right-hand side of his field equations (see (25) below) and Willem de Sitter (1917) found their vacuum solution which describes the behavior of the expansion function of the entire universe.

The outline of this paper is as follows. In Sect. 2, we prove that the Schwarzschild solution (6) satisfies (4), but does not satisfy (1). In Sect. 3, we introduce the Einstein equations and in Sect. 4, we show that they have an extremely complicated explicit expression. In Sect. 5, we investigate their enormous computational complexity when they are solved numerically. Finally, in Sect. 6, we present some concluding remarks.

2 Exterior and Interior Schwarzschild Solution

Classical relativistic tests McCausland (1999), Misner et al. (1997), Will (2014) (such as bending of light, Mercury's perihelion shift, gravitational redshift, Shapiro's fourth test of general relativity) are based on verification of very simple algebraic formulae derived by various simplifications and approximations of the Schwarzschild solution (6) which is very special and corresponds only to one spherically symmetric nonrotating body. Therefore, in this section we present several theorems on the Schwarzschild solution.

Einstein and also Schwarzschild assumed that the gravitational field has the following properties:

1. It is static, i.e., all components $g_{\mu\nu}$ are independent of the time variable x^0.
2. It is spherically symmetric with respect to the coordinate origin, i.e., the same solutions will be obtained after a linear orthogonal transformation.
3. The equations $g_{\rho 0} = g_{0\rho} = 0$ hold for any $\rho = 1, 2, 3$.
4. The spacetime is asymptotically flat, i.e., the metric tensor $g_{\mu\nu}$ tends to the Minkowski metric in infinity.

Theorem 1 *The exterior Schwarzschild solution* (6) *with* $S > 0$ *satisfies Eqs.* (4)*, but does not satisfy* (1).

Proof We will proceed in three steps:

1. At first, we recall the following definition of the Christoffel symbols of the first kind:

$$\Gamma_{\lambda\mu\nu} := \frac{1}{2}(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}). \tag{7}$$

From this and the relation $g_{\mu\nu} = g_{\nu\mu}$ we find that the Christoffel symbols are, in general, symmetric with respect to the second and third subscript

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2}(g_{\lambda\nu,\mu} + g_{\mu\lambda,\nu} - g_{\nu\mu,\lambda}) = \Gamma_{\lambda\nu\mu}. \tag{8}$$

Using (7) and (6), we obtain

$$\begin{aligned}
\Gamma_{001} &= \frac{1}{2} g_{00,1} = -\frac{r-(r-S)}{2r^2} = -\frac{S}{2r^2},\\
\Gamma_{100} &= -\frac{1}{2} g_{00,1} = \frac{S}{2r^2},\\
\Gamma_{111} &= \frac{1}{2} g_{11,1} = -\frac{S}{2(r-S)^2},
\end{aligned}$$

$$\Gamma_{122} = -\frac{1}{2}g_{22,1} = -r,$$
$$\Gamma_{133} = -\frac{1}{2}g_{33,1} = -r\sin^2\theta,$$
$$\Gamma_{221} = \frac{1}{2}g_{22,1} = r,$$
$$\Gamma_{233} = \frac{1}{2}g_{33,2} = r^2\sin\theta\cos\theta,$$
$$\Gamma_{331} = \frac{1}{2}g_{33,1} = r\sin^2\theta,$$
$$\Gamma_{332} = \frac{1}{2}g_{33,2} = r^2\sin\theta\cos\theta$$

and the other components $\Gamma_{\lambda\mu\nu} = \Gamma_{\lambda\nu\mu}$ are zeros.

2. Similarly to (8) we can find that the Christoffel symbols of the second kind are symmetric with respect to the second and third (lower) indices and by (2) and (7) we have

$$\Gamma^{\varkappa}{}_{\mu\nu} = g^{\varkappa\lambda}\Gamma_{\lambda\mu\nu}.$$

Since the metric tensor (6) is diagonal, its inverse reads

$$g^{\mu\nu} = \mathrm{diag}\left(-\frac{r}{r-S}, \frac{r-S}{r}, \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right) \tag{9}$$

and all non-diagonal entries are zeros. From this and Step 1 we get

$$\Gamma^0{}_{01} = g^{00}\Gamma_{001} = \frac{S}{2r(r-S)},$$
$$\Gamma^1{}_{00} = g^{11}\Gamma_{100} = \frac{S(r-S)}{2r^3},$$
$$\Gamma^1{}_{11} = g^{11}\Gamma_{111} = -\frac{S}{2r(r-S)},$$
$$\Gamma^1{}_{22} = g^{11}\Gamma_{122} = -(r-S),$$
$$\Gamma^1{}_{33} = g^{11}\Gamma_{133} = -(r-S)\sin^2\theta,$$
$$\Gamma^2{}_{12} = g^{22}\Gamma_{221} = \frac{1}{r^2}r = \frac{1}{r},$$
$$\Gamma^2{}_{33} = g^{22}\Gamma_{233} = \sin\theta\cos\theta,$$
$$\Gamma^3{}_{31} = g^{33}\Gamma_{331} = \frac{1}{r^2\sin^2\theta}r\sin^2\theta = \frac{1}{r},$$
$$\Gamma^3{}_{32} = g^{33}\Gamma_{332} = \frac{1}{r^2\sin^2\theta}r^2\sin\theta\cos\theta = \mathrm{cotan}\,\theta$$

and the other components $\Gamma^{\varkappa}{}_{\mu\nu} = \Gamma^{\varkappa}_{\nu\mu}$ are zeros.

3. Finally, we will evaluate all entries of the Ricci tensor defined in (4). In particular, by (9) and Step 2 we obtain

$$
\begin{aligned}
R_{00} &= \Gamma^{\varkappa}{}_{00,\varkappa} \underline{\underline{- \Gamma^{\varkappa}{}_{0\varkappa,0} + \Gamma^{\lambda}{}_{00}\Gamma^{\varkappa}{}_{\lambda\varkappa}}} - \Gamma^{\lambda}{}_{0\varkappa}\Gamma^{\varkappa}{}_{\lambda 0} \\
&= \Gamma^{1}{}_{00,1} \underline{\underline{+ \Gamma^{1}{}_{00}(\Gamma^{2}{}_{12} + \Gamma^{3}{}_{13})}} - 2\Gamma^{1}{}_{00}\Gamma^{0}{}_{01} \\
&= \frac{S2r^3 - S(r-S)6r^2}{4r^6} \underline{\underline{+ \frac{S(r-S)}{2r^3}\left(\frac{1}{r} + \frac{1}{r}\right)}} - 2\frac{S(r-S)}{2r^3}\frac{S}{2r(r-S)} \\
&= \frac{rS}{2r^4} - \frac{3S(r-S)}{2r^4} \underline{\underline{+ \frac{S(r-S)}{2r^3}\frac{2}{r}}} - \frac{S^2}{2r^4} = 0,
\end{aligned}
$$

where the doubly underlined terms correspond to the doubly underlined terms in (4). From this it is obvious that the Schwarzschild solution (6) does not satisfy (1), since $S(r-S)/r^4 \neq 0$ for $S > 0$. (Note that (3) is not valid.)
Similarly, we find that

$$
\begin{aligned}
R_{11} &= \Gamma^{\varkappa}{}_{11,\varkappa} \underline{\underline{- \Gamma^{\varkappa}{}_{1\varkappa,1} + \Gamma^{\lambda}{}_{11}\Gamma^{\varkappa}{}_{\lambda\varkappa}}} - \Gamma^{\lambda}{}_{1\varkappa}\Gamma^{\varkappa}{}_{\lambda 1} \\
&= \Gamma^{1}{}_{11,1} \underline{\underline{- \Gamma^{0}{}_{10,1} - \Gamma^{1}{}_{11,1} - \Gamma^{2}{}_{12,1} - \Gamma^{3}{}_{13,1} + \Gamma^{1}{}_{11}(\Gamma^{0}{}_{01} + \Gamma^{2}{}_{12} + \Gamma^{3}{}_{13})}} \\
&\quad - \Gamma^{0}{}_{01}\Gamma^{0}{}_{01} - \Gamma^{2}{}_{12}\Gamma^{2}{}_{12} - \Gamma^{3}{}_{13}\Gamma^{3}{}_{13} \\
&= \underline{\underline{\frac{S(2r-S)}{2r^2(r-S)^2} + \frac{2}{r} - \frac{S}{2r(r-S)}\left(\frac{S}{2r(r-S)} + \frac{2}{r}\right)}} - \frac{S^2}{4r^2(r-S)^2} - \frac{2}{r^2} = 0, \\
R_{22} &= \Gamma^{\varkappa}{}_{22,\varkappa} \underline{\underline{- \Gamma^{\varkappa}{}_{2\varkappa,2} + \Gamma^{\lambda}{}_{22}\Gamma^{\varkappa}{}_{\lambda\varkappa}}} - \Gamma^{\lambda}{}_{2\varkappa}\Gamma^{\varkappa}{}_{\lambda 2} \\
&= \Gamma^{1}{}_{22,1} \underline{\underline{- \Gamma^{3}{}_{23,2} + \Gamma^{1}{}_{22}(\Gamma^{2}{}_{12} + \Gamma^{3}{}_{13})}} - 2\Gamma^{2}{}_{21}\Gamma^{1}{}_{22} - \Gamma^{3}{}_{23}\Gamma^{3}{}_{23} \\
&= -1 + \underline{\underline{\frac{1}{\sin^2\theta} - (r-S)\frac{2}{r}}} + \frac{2}{r}(r-S) - \cot\text{an}^2\theta = 0, \\
R_{33} &= \Gamma^{\varkappa}{}_{33,\varkappa} \underline{\underline{- \Gamma^{\varkappa}{}_{3\varkappa,3} + \Gamma^{\lambda}{}_{33}\Gamma^{\varkappa}{}_{\lambda\varkappa}}} - \Gamma^{\lambda}{}_{3\varkappa}\Gamma^{\varkappa}{}_{\lambda 3} \\
&= \underline{\underline{\Gamma^{1}{}_{33}\Gamma^{3}{}_{13} + \Gamma^{2}{}_{33}\Gamma^{3}{}_{23}}} - \Gamma^{3}{}_{31}\Gamma^{1}{}_{33} - \Gamma^{3}{}_{32}\Gamma^{2}{}_{33} = 0,
\end{aligned}
$$

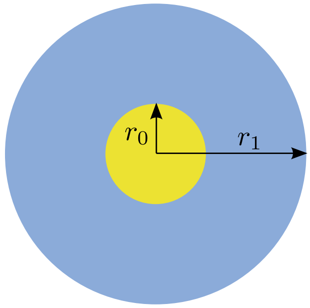
and the other non-diagonal entries of $R_{\mu\nu}$ are also zeros.

Therefore, (4) is satisfied. □

Theorem 1 nicely demonstrates Schwarzschild's ingenuity to find a nontrivial solution to a very complicated system of partial differential equations (4) for an arbitrary constant $S > 0$ (see also Křížek 2019a, Appendix). For a spherically symmetric object of mass $M > 0$, the constant

$$
S = \frac{2MG}{c^2} \tag{10}
$$

Fig. 2 Spherical shell between two concentric spheres



is called its *Schwarzschild gravitational radius*, where G is the gravitational constant and c is the speed of light in a vacuum.

Denote by

$$S_{\mu\nu} = -\Gamma^{\varkappa}{}_{\mu\varkappa,\nu} + \Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\varkappa}{}_{\lambda\varkappa}$$

the doubly underlined terms in (4). By Einstein (1915a) one additional algebraic condition (3) surprisingly guarantees that 10 differential operators $S_{\mu\nu} = S_{\nu\mu}$ simultaneously vanish when (4) is valid. This implies (1).

For positive numbers $r_0 < r_1$ consider the spherical shell $\{(x^1, x^2, x^3) \in \mathbb{E}^3 \mid r_0^2 \leq (x^1)^2 + (x^2)^2 + (x^3)^2 \leq r_1^2\}$ with interior radius r_0 and exterior radius r_1 (see Fig. 2). It is a region between two concentric spheres. Its volume in the Euclidean space $\mathbb{E}^3$ around the mass M is equal to

$$V = \frac{4}{3}\pi\left(r_1^3 - r_0^3\right).$$

However, the spacetime around the mass M is curved. Therefore, we have to consider the *proper* (*relativistic*) *volume* defined as

$$\tilde{V} := \int_{r_0}^{r_1} r^2\sqrt{\frac{r}{r-S}}\,\mathrm{d}r \times \int_0^{\pi}\left(\int_0^{2\pi}\sin\theta\,\mathrm{d}\varphi\right)\mathrm{d}\theta = 4\pi\int_{r_0}^{r_1} r^2\sqrt{\frac{r}{r-S}}\,\mathrm{d}r.$$

In Křížek and Křížek (2018), we prove the following astonishing theorem.

Theorem 2 *If $M > 0$ and $r_0 > S$ are any fixed numbers satisfying* (10)*, then*

$$\tilde{V} - V \to \infty \quad \text{as } r_1 \to \infty.$$

We observe that the difference of volumes $\tilde{V} - V$ increases over all limits for $r_1 \to \infty$, which is quite surprising property. Namely, Theorem 2 can be applied, for instance, to a small imperceptible pinhead, since the fixed mass $M > 0$ can be arbitrarily small. Consequently, a natural question arises whether the exterior Schwarzschild solution (6) approximates reality well.

In 1916, Karl Schwarzschild (1916a) found the first nonvacuum solution of the Einstein equations, cf. (12) below. He assumed that the ball with coordinate radius $r_0 > 0$ is formed by an ideal incompressible nonrotating fluid with constant density to avoid a possible internal mechanical stress in the solid that may have a nonnegligible influence on the resulting gravitational field. Then by Ellis (2012) (see also Florides 1974, p. 529; Stephani 2004, p. 213; Interior 2020) the corresponding metric is given by

$$g_{\mu\nu} = \mathrm{diag}\left(-\frac{1}{4}\left(3\sqrt{1-\frac{S}{r_0}} - \sqrt{1-\frac{Sr^2}{r_0^3}}\right)^2, \frac{r_0^3}{r_0^3 - Sr^2}, r^2\sin^2\theta, r^2\right), \quad (11)$$

where $r \in [0, r_0]$. The metric tensor (11) is called the *interior Schwarzschild solution*, see Stephani (2004, p. 213). It is again a static solution, meaning that it is independent of time. To avoid the division by zero in the component g_{11}, we require

$$\frac{r_0^3}{r_0^3 - Sr^2} = \left(1 - \frac{Sr^2}{r_0^3}\right)^{-1} > 0 \quad \text{for all } r \in [0, r_0]$$

which leads to the inequality

$$r_0 > S.$$

Hence, we can define the *composite Schwarzschild metric* $g_{\mu\nu}$ by (6) for $r > r_0$ and by (11) for $r \in [0, r_0]$. It is easy to check that $g_{\mu\nu}$ is continuous everywhere.

Theorem 3 *The composite Schwarzschild metric $g_{\mu\nu}$ is not differentiable for $r = r_0$.*

Proof We will show that the component g_{11} is not differentiable. From (6) and (11) we observe that the first classical derivative does not exist at $r = r_0$ (see Fig. 3). Namely, the component $g_{11}(r)$ of the interior solution (11) is an increasing function on $[0, r_0]$, whereas from (6) we see that the one-sided limit of the first derivative of the component $g_{11}(r) = r/(r - S)$ of the exterior solution is negative

$$\lim_{r \to r_0^+} \frac{\partial g_{11}}{\partial r}(r) < 0.$$

All Riemannian spacetime manifolds have to be locally flat which is not true in this case. □

The piecewise rational function g_{11} cannot be smoothed near r_0, since then the Einstein equations would not be valid in a close neighborhood of r_0. Thus we observe that the composite Schwarzschild solution (6)+(11) is not a global solution of an idealized spherically symmetric star and its close neighborhood (see Fig. 2).

3 Einstein Equations of General Relativity

The Einstein field equations consist of 10 equations (cf. Einstein 1916) for 10 components of the unknown twice differentiable symmetric metric tensor $g_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3, \tag{12}$$

where $R_{\mu\nu}$ is the symmetric Ricci tensor defined by (4), the contraction

$$R = g^{\mu\nu} R_{\mu\nu} \tag{13}$$

is the *Ricci scalar* (i.e. the scalar curvature), and $T_{\mu\nu}$ is the symmetric *tensor of density of energy and momentum*. Let us emphasize that the 10 Einstein equations (12) are not independent, since the covariant divergence of the right-hand side is supposed to be zero (see, e.g., Misner et al. 1997, p. 146), i.e.,

$$T^{\mu\nu}{}_{;\nu} := T^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}{}_{\lambda\nu} T^{\lambda\nu} + \Gamma^{\nu}{}_{\lambda\nu} T^{\mu\lambda} = 0, \tag{14}$$

where $T^{\mu\nu} = g^{\varkappa\mu} g^{\lambda\nu} T_{\varkappa\lambda}$. The covariant divergence of the Ricci tensor is nonzero, in general, but the covariant divergence of the whole left-hand side of (12) is zero automatically for $g_{\mu\nu}$ smooth enough, e.g., if the third derivatives of $g_{\mu\nu}$ are continuous (which is not the case sketched in Fig. 3). Therefore, we have only six independent equations in (12). The number of independent components of the metric tensor is also six, since we have four possibilities of choosing four coordinates.

Finally, the contravariant symmetric 4×4 metric tensor $g^{\mu\nu}$ which is inverse to the covariant metric tensor $g_{\mu\nu}$ satisfies

$$g^{\mu\nu} = \frac{g^*_{\mu\nu}}{\det(g_{\mu\nu})}, \quad \det(g_{\mu\nu}) := \sum_{\pi \in S_4} (-1)^{\operatorname{sgn} \pi} g_{0\nu_0} g_{1\nu_1} g_{2\nu_2} g_{3\nu_3}, \tag{15}$$

where the entries $g^*_{\mu\nu}$ form the 4×4 matrix of 3×3 algebraic adjoints of $g_{\mu\nu}$, S_4 is the symmetric group of 24 permutations π of indices $(\nu_0, \nu_1, \nu_2, \nu_3)$, $\operatorname{sgn} \pi = 0$ for

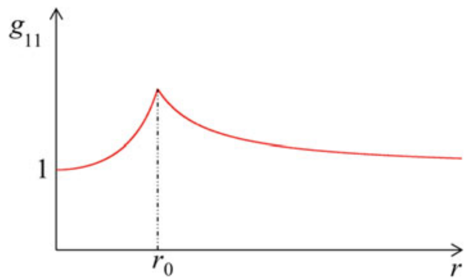


Fig. 3 Behavior of the non-differentiable component $g_{11} = g_{11}(r)$ of the composite metric tensor from (6) and (11)

an even permutation and sgn $\pi = 1$ for an odd permutation. Notice that the constant on the right-hand side of (12) is very small in the SI base units implying that the Ricci curvature tensor $R_{\mu\nu}$ is also very small (if components of $T_{\mu\nu}$ are not too large).

Einstein (1915a) presented the field equations (12) in an equivalent form which is at present written as follows:

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right), \tag{16}$$

where $T := T_{\mu\nu}g^{\mu\nu} = T^{\mu}{}_{\mu}$ denotes the trace of $T_{\mu\nu}$. To see that (16) is equivalent with (12), we multiply (12) by $g^{\mu\nu}$. Then the traces of the corresponding tensors satisfy

$$-R = R - 2R = \frac{8\pi G}{c^4}T.$$

Theorem 4 *If $g_{\mu\nu}$ is a solution to* (12)*, then* $(-g_{\mu\nu})$ *also solves* (12)*.*

Proof From (2) we find that the Christoffel symbols remain the same if we replace $g_{\mu\nu}$ by $(-g_{\mu\nu})$, namely,

$$\Gamma^{\varkappa}{}_{\mu\nu} = \frac{1}{2}(-g^{\varkappa\lambda})(-g_{\nu\lambda,\mu} - g_{\lambda\mu,\nu} + g_{\mu\nu,\lambda}).$$

Using (4), we find that the Ricci tensor $R_{\mu\nu}$ in (12) does not change as well. Concerning the second term on the left-hand side of (12), we observe from (13) that $\left(-\frac{1}{2}Rg_{\mu\nu}\right)$ also remains unchanged if we replace $g_{\mu\nu}$ by $(-g_{\mu\nu})$. □

Example 1 For comparison, we also note that the first-order classical derivatives of the Newton potential u for the situation sketched in Fig. 2 are continuous. It is described by the Poisson equation

$$\Delta u = 4\pi G\rho, \tag{17}$$

where ρ is the mass density. Let the right-hand side $f = 4\pi G\rho$ be spherically symmetric and such that $f(r) = 1$ for $r \in [0, 1]$ and $f(r) = 0$ otherwise. The Laplace operator in spherical coordinates reads

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \operatorname{cotan}\theta\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}\right).$$

The term in parenthesis on the right-hand side is zero for the spherically symmetric case. By the well-known method of variations of constants, we find the following solution to the Poisson equation (17):

$$u(r,\varphi,\theta)=\begin{cases}\dfrac{1}{6}r^2-\dfrac{1}{2} & \text{for } r\in[0,1],\\ -\dfrac{1}{3r} & \text{otherwise.}\end{cases}$$

Hence, both u and $\partial u/\partial r$ are continuous at $r_0 = 1$.

4 Explicit Form of the First Einstein Equation

In this section, we want to point out the extreme complexity of the Einstein equations. In (12), the dependence of the Ricci scalar R and the Ricci tensor $R_{\mu\nu}$ on the metric tensor $g_{\mu\nu}$ is not indicated. Therefore, the Einstein equations (12) seem to be quite simple (see Misner et al. 1997, p. 42). To avoid this deceptive opinion, we will show now how to derive an explicit form of the first Einstein equation.

First, we shall consider only the case when $T_{\mu\nu} = 0$ (and without the cosmological constant Einstein 1952). Multiplying (12) by $g^{\mu\nu}$ and summing over all μ and ν, we obtain by (13) that

$$0 = g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} = R - \frac{1}{2}R\delta^{\mu}{}_{\mu} = R - \frac{1}{2}4R,$$

where $\delta^{\mu}{}_{\nu}$ is the Kronecker delta. Thus, $R = 0$ and the Einstein vacuum equations can be rewritten in the well-known form

$$R_{\mu\nu} = 0. \tag{18}$$

The unknown metric tensor $g_{\mu\nu}$ is not indicated. Concerning nonuniqueness expressed by Theorem 4, we observe from (2) that we can add any constant to any component $g_{\mu\nu} = g_{\nu\mu}$ and the Einstein equations $R_{\mu\nu} = 0$ will still be valid.

Equation (18) looks seemingly very simple, since the unknown metric tensor $g_{\mu\nu}$ is not indicated there. So now we will rewrite it so that this metric tensor appears explicitly there. Using (4), we can express the first Einstein equation of (18) as follows:

$$\begin{aligned}
0 = R_{00} &= \Gamma^{\varkappa}{}_{00,\varkappa} - \Gamma^{\varkappa}{}_{0\varkappa,0} + \Gamma^{\lambda}{}_{00}\Gamma^{\varkappa}{}_{\lambda\varkappa} - \Gamma^{\lambda}{}_{0\varkappa}\Gamma^{\varkappa}{}_{0\lambda}\\
&= \underline{\Gamma^0{}_{00,0}} + \Gamma^1{}_{00,1} + \Gamma^2{}_{00,2} + \Gamma^3{}_{00,3} - \underline{\Gamma^0{}_{00,0}} - \Gamma^1{}_{01,0} - \Gamma^2{}_{02,0} - \Gamma^3{}_{03,0}\\
&\quad + \underline{\Gamma^0{}_{00}(\Gamma^0{}_{00}} + \Gamma^1{}_{01} + \Gamma^2{}_{02} + \Gamma^3{}_{03}) + \underline{\Gamma^1{}_{00}(\Gamma^0{}_{10}} + \Gamma^1{}_{11} + \Gamma^2{}_{12} + \Gamma^3{}_{13})\\
&\quad + \underline{\Gamma^2{}_{00}(\Gamma^0{}_{20}} + \Gamma^1{}_{21} + \Gamma^2{}_{22} + \Gamma^3{}_{23}) + \underline{\Gamma^3{}_{00}(\Gamma^0{}_{30}} + \Gamma^1{}_{31} + \Gamma^2{}_{32} + \Gamma^3{}_{33})\\
&\quad - \underline{\Gamma^0{}_{00}\Gamma^0{}_{00}} - \Gamma^0{}_{01}\Gamma^1{}_{00} - \Gamma^0{}_{02}\Gamma^2{}_{00} - \Gamma^0{}_{03}\Gamma^3{}_{00} - \underline{\Gamma^1{}_{00}\Gamma^0{}_{01}} - \Gamma^1{}_{01}\Gamma^1{}_{01}\\
&\quad - \Gamma^1{}_{02}\Gamma^2{}_{01} - \Gamma^1{}_{03}\Gamma^3{}_{01} - \underline{\Gamma^2{}_{00}\Gamma^0{}_{02}} - \Gamma^2{}_{01}\Gamma^1{}_{02} - \Gamma^2{}_{02}\Gamma^2{}_{02} - \Gamma^2{}_{03}\Gamma^3{}_{02}\\
&\quad - \underline{\Gamma^3{}_{00}\Gamma^0{}_{03}} - \Gamma^3{}_{01}\Gamma^1{}_{03} - \Gamma^3{}_{02}\Gamma^2{}_{03} - \Gamma^3{}_{03}\Gamma^3{}_{03},
\end{aligned}$$

where the underlined terms cancel. Hence, the first Einstein equation can be rewritten by means of the Christoffel symbols in the following way:

$$\begin{aligned} 0 = R_{00} &= \Gamma^1{}_{00,1} + \Gamma^2{}_{00,2} + \Gamma^3{}_{00,3} - \Gamma^1{}_{01,0} - \Gamma^2{}_{02,0} - \Gamma^3{}_{03,0} \\ &+ \Gamma^0{}_{00}(\Gamma^1{}_{01} + \Gamma^2{}_{02} + \Gamma^3{}_{03}) + \Gamma^1{}_{00}(-\Gamma^0{}_{10} + \Gamma^1{}_{11} + \Gamma^2{}_{12} + \Gamma^3{}_{13}) \\ &+ \Gamma^2{}_{00}(-\Gamma^0{}_{20} + \Gamma^1{}_{21} + \Gamma^2{}_{22} + \Gamma^3{}_{23}) + \Gamma^3{}_{00}(-\Gamma^0{}_{30} + \Gamma^1{}_{31} + \Gamma^2{}_{32} + \Gamma^3{}_{33}) \\ &- 2\Gamma^1{}_{02}\Gamma^2{}_{01} - 2\Gamma^1{}_{03}\Gamma^3{}_{01} - 2\Gamma^2{}_{03}\Gamma^3{}_{02} - (\Gamma^1{}_{01})^2 - (\Gamma^2{}_{02})^2 - (\Gamma^3{}_{03})^2. \qquad (19) \end{aligned}$$

Using (2) and the symmetry of $g_{\mu\nu}$, we obtain

$$\begin{aligned} 2\Gamma^\varkappa{}_{\mu\nu} &= g^{\varkappa 0}(g_{\mu 0,\nu} + g_{\nu 0,\mu} - g_{\mu\nu,0}) + g^{\varkappa 1}(g_{\mu 1,\nu} + g_{\nu 1,\mu} - g_{\mu\nu,1}) \\ &+ g^{\varkappa 2}(g_{\mu 2,\nu} + g_{\nu 2,\mu} - g_{\mu\nu,2}) + g^{\varkappa 3}(g_{\mu 3,\nu} + g_{\nu 3,\mu} - g_{\mu\nu,3}). \end{aligned}$$

Thus, by (18), we can express the first Einstein equation $R_{00} = 0$ by means of the metric coefficients and their first- and second-order derivatives as follows:

$$\begin{aligned} 0 = 4R_{00} = 2\Big[& g^{10}_{,1} g_{00,0} + g^{11}_{,1}(2g_{01,0} - g_{00,1}) + g^{12}_{,1}(2g_{02,0} - g_{00,2}) + g^{13}_{,1}(2g_{03,0} - g_{00,3}) \\ &+ g^{10} g_{00,01} + g^{11}(2g_{01,01} - g_{00,11}) + g^{12}(2g_{02,01} - g_{00,21}) + g^{13}(2g_{03,01} - g_{00,31}) \\ &+ g^{20}_{,2} g_{00,0} + g^{21}_{,2}(2g_{01,0} - g_{00,1}) + g^{22}_{,2}(2g_{02,0} - g_{00,2}) + g^{23}_{,2}(2g_{03,0} - g_{00,3}) \\ &+ g^{20} g_{00,02} + g^{21}(2g_{01,02} - g_{00,12}) + g^{22}(2g_{02,02} - g_{00,22}) + g^{23}(2g_{03,02} - g_{00,32}) \\ &+ g^{30}_{,3} g_{00,0} + g^{31}_{,3}(2g_{01,0} - g_{00,1}) + g^{32}_{,3}(2g_{02,0} - g_{00,2}) + g^{33}_{,3}(2g_{03,0} - g_{00,3}) \\ &+ g^{30} g_{00,03} + g^{31}(2g_{01,03} - g_{00,13}) + g^{32}(2g_{02,03} - g_{00,23}) + g^{33}(2g_{03,03} - g_{00,33}) \\ &- g^{10}_{,0} g_{00,1} - g^{11}_{,0} g_{11,0} - g^{12}_{,0}(g_{02,1} + g_{12,0} - g_{01,2}) - g^{13}_{,0}(g_{03,1} + g_{13,0} - g_{01,3}) \\ &- g^{10} g_{00,10} - g^{11} g_{11,00} - g^{12}(g_{02,10} + g_{12,00} - g_{01,20}) - g^{13}(g_{03,10} + g_{13,00} - g_{01,30}) \\ &- g^{20}_{,0} g_{00,2} - g^{21}_{,0}(g_{01,2} + g_{21,0} - g_{02,1}) - g^{22}_{,0} g_{22,0} - g^{23}_{,0}(g_{03,2} + g_{23,0} - g_{02,3}) \\ &- g^{20} g_{00,20} - g^{21}(g_{01,20} + g_{21,00} - g_{02,10}) - g^{22} g_{22,00} - g^{23}(g_{03,20} + g_{23,00} - g_{02,30}) \\ &- g^{30}_{,0} g_{00,3} - g^{31}_{,0}(g_{01,3} + g_{31,0} - g_{03,1}) - g^{32}_{,0}(g_{02,3} + g_{32,0} - g_{03,2}) - g^{33}_{,0} g_{33,0} \\ &- g^{30} g_{00,30} - g^{31}(g_{01,30} + g_{31,00} - g_{03,10}) - g^{32}(g_{02,30} + g_{32,00} - g_{03,20}) - g^{33} g_{33,00}\Big] \\ &+ (g^{00} g_{00,0} - g^{01} g_{00,1} - g^{02} g_{00,2} - g^{03} g_{00,3}) \\ &\times \Big[g^{10}(2g_{10,1} - g_{11,0}) + g^{11} g_{11,1} + g^{12}(2g_{12,1} - g_{11,2}) + g^{13}(2g_{13,1} - g_{11,3}) \\ &+ g^{20}(g_{10,2} + g_{20,1} - g_{12,0}) + g^{21} g_{11,2} + g^{22} g_{22,1} + g^{23}(g_{13,2} + g_{23,1} - g_{12,3}) \\ &+ g^{30}(g_{10,3} + g_{30,1} - g_{13,0}) + g^{31} g_{11,3} + g^{32}(g_{12,3} + g_{32,1} - g_{13,2}) + g^{33} g_{33,1}\Big] \\ &+ (g^{10} g_{00,0} + g^{11} g_{11,1} - g^{12} g_{11,2} - g^{13} g_{11,3}) \\ &\times \Big[-g^{00} g_{00,1} - g^{01} g_{11,0} - g^{02}(g_{12,0} + g_{02,1} - g_{10,2}) - g^{03}(g_{13,0} + g_{03,1} - g_{10,3}) \\ &+ g^{10}(2g_{10,1} - g_{11,0}) + g^{11} g_{11,1} + g^{12}(2g_{12,1} - g_{11,2}) + g^{13}(2g_{13,1} - g_{11,3}) \\ &+ g^{20}(g_{10,2} + g_{20,1} - g_{12,0}) + g^{21} g_{11,2} + g^{22} g_{22,1} + g^{23}(g_{13,2} + g_{23,1} - g_{12,3}) \\ &+ g^{30}(g_{10,3} + g_{30,1} - g_{13,0}) + g^{31} g_{11,3} + g^{32}(g_{12,3} + g_{32,1} - g_{13,2}) + g^{33} g_{33,1}\Big] \end{aligned}$$

$$
\begin{aligned}
&+\left[g^{20}g_{00,0}+g^{21}(2g_{01,0}-g_{00,1})+g^{22}(2g_{02,0}-g_{00,2})+g^{23}(2g_{03,0}-g_{00,3})\right]\\
&\times\Big[-g^{00}g_{00,2}-g^{01}(g_{21,0}+g_{01,2}-g_{20,1})-g^{02}g_{22,0}-g^{03}(g_{23,0}+g_{03,2}-g_{20,3})\\
&+g^{10}(g_{20,1}+g_{10,2}-g_{21,0})+g^{11}g_{11,2}+g^{12}g_{22,1}+g^{13}(g_{23,1}+g_{13,2}-g_{21,3})\\
&+g^{20}(2g_{20,2}-g_{22,0})+g^{21}(2g_{21,2}-g_{22,1})+g^{22}g_{22,2}+g^{23}(2g_{23,2}-g_{22,3})\\
&+g^{30}(g_{20,3}+g_{30,2}-g_{23,0})+g^{31}(g_{21,3}+g_{31,2}-g_{23,1})+g^{32}g_{22,3}+g^{33}g_{33,2}\Big]\\
&+\left[g^{30}g_{00,0}+g^{31}(2g_{01,0}-g_{00,1})+g^{32}(2g_{02,0}-g_{00,2})+g^{33}(2g_{03,0}-g_{00,3})\right]\\
&\times\Big[-g^{00}g_{00,3}-g^{01}g_{01,3}-g^{02}(g_{32,0}+g_{02,3}-g_{30,2})-g^{03}g_{33,0}\\
&+g^{10}(g_{30,1}+g_{10,3}-g_{31,0})+g^{11}g_{11,3}+g^{12}(g_{32,1}+g_{12,3}-g_{31,2})+g^{13}g_{33,1}\\
&+g^{20}(g_{30,2}+g_{20,3}-g_{32,0})+g^{21}(g_{31,2}+g_{21,3}-g_{32,1})+g^{22}g_{22,3}+g^{23}g_{33,2}\\
&+g^{30}(2g_{30,3}-g_{33,0})+g^{31}(2g_{31,3}-g_{33,1})+g^{32}(2g_{32,3}-g_{33,2})+g^{33}g_{33,3}\Big]\\
&-2\left[g^{10}g_{00,2}+g^{11}(g_{01,2}+g_{21,0}-g_{02,1})+g^{12}g_{22,0}+g^{13}(g_{03,2}+g_{23,0}-g_{02,3})\right]\\
&\times\left[g^{20}g_{00,1}+g^{21}g_{11,0}+g^{22}(g_{02,1}+g_{12,0}-g_{01,2})+g^{23}(g_{03,1}+g_{13,0}-g_{01,3})\right]\\
&-2\left[g^{10}g_{00,3}+g^{11}(g_{01,3}+g_{31,0}-g_{03,1})+g^{12}(g_{02,3}+g_{32,0}-g_{03,2})+g^{13}g_{33,0}\right]\\
&\times\left[g^{30}g_{00,1}+g^{31}g_{11,0}+g^{32}(g_{02,1}+g_{12,0}-g_{01,2})+g^{33}(g_{03,1}+g_{13,0}-g_{01,3})\right]\\
&-2\left[g^{20}g_{00,3}+g^{21}(g_{01,3}+g_{31,0}-g_{03,1})+g^{22}(g_{02,3}+g_{32,0}-g_{03,2})+g^{23}g_{33,0}\right]\\
&\times\left[g^{30}g_{00,2}+g^{31}(g_{01,2}+g_{21,0}-g_{02,1})+g^{32}g_{22,0}+g^{33}(g_{03,2}+g_{23,0}-g_{02,3})\right]\\
&-\left[g^{10}g_{00,1}+g^{11}g_{11,0}+g^{12}(g_{02,1}+g_{12,0}-g_{01,2})+g^{13}(g_{03,1}+g_{13,0}-g_{01,3})\right]^2\\
&-\left[g^{20}g_{00,2}+g^{21}(g_{01,2}+g_{21,0}-g_{02,1})+g^{22}g_{22,0}+g^{23}(g_{03,2}+g_{23,0}-g_{02,3})\right]^2\\
&-\left[g^{30}g_{00,3}+g^{31}(g_{01,3}+g_{31,0}-g_{03,1})+g^{32}(g_{02,3}+g_{32,0}-g_{03,2})+g^{33}g_{33,0}\right]^2. \qquad (20)
\end{aligned}
$$

Now we should substitute all entries of (20) with double upper indices for (15). For instance, the entry g^{11} in the second line of (20) could be rewritten by means of the Sarrus rule for 3×3 symmetric matrices g^*_{11} by

$$
\begin{aligned}
g^{11} &= \frac{g^*_{11}}{\det(g_{\mu\nu})}\\
&= \frac{g_{00}g_{22}g_{33}+2g_{02}g_{03}g_{23}-g_{00}(g_{23})^2-g_{22}(g_{03})^2-g_{33}(g_{02})^2}{\sum_{\pi\in S_4}(-1)^{\operatorname{sgn}\pi}\, g_{0\nu_0}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3}}, \qquad (21)
\end{aligned}
$$

where the sum in the denominator contains $4! = 24$ terms. Note that the optimal expression for the minimum number of arithmetic operations to calculate the inverse of a 4×4 matrix is not known, yet. The other nine entries g^{00}, g^{01}, g^{02}, g^{03}, g^{12}, g^{13}, g^{22}, g^{23}, and g^{33} can be expressed similarly.

However, we have to evaluate also the first derivatives of $g^{\mu\nu}$. Consider for instance the entry $g^{11}_{,1}$ in the first line of (20). Then by (21) we get

$$
\begin{aligned}
g^{11}_{,1} &= \frac{\partial}{\partial x_1}\left(\frac{g^*_{11}}{\det(g_{\mu\nu})}\right) \\
&= \Bigg(\frac{1}{\sum_{\pi\in S_4}(-1)^{\operatorname{sgn}\pi} g_{0\nu_0}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3}}\Big(g_{00}g_{22}g_{33} + 2g_{02}g_{03}g_{23} \\
&\quad - g_{00}(g_{23})^2 - g_{22}(g_{03})^2 - g_{33}(g_{02})^2\Big)\Bigg)_{,1} \\
&= \Bigg[\Big(g_{00,1}g_{22}g_{33} + 2g_{02,1}g_{03}g_{23} - g_{00,1}(g_{23})^2 - g_{22,1}(g_{03})^2 - g_{33,1}(g_{02})^2 \\
&\quad + g_{00}g_{22,1}g_{33} + 2g_{02}g_{03,1}g_{23} + g_{00}g_{22}g_{33,1} + 2g_{02}g_{03}g_{23,1} - 2g_{00}g_{23,1} \\
&\quad - 2g_{22}g_{03,1} - 2g_{33}g_{02,1}\Big)\left(\sum_{\pi\in S_4}(-1)^{\operatorname{sgn}\pi} g_{0\nu_0}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3}\right) \\
&\quad - \Big(g_{00}g_{22}g_{33} + 2g_{02}g_{03}g_{23} - g_{00}(g_{23})^2 - g_{22}(g_{03})^2 - g_{33}(g_{02})^2\Big) \\
&\quad \times \sum_{\pi\in S_4}(-1)^{\operatorname{sgn}\pi}\Big(g_{0\nu_0,1}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3} + g_{0\nu_0}g_{1\nu_1,1}g_{2\nu_2}g_{3\nu_3} \\
&\quad + g_{0\nu_0}g_{1\nu_1}g_{2\nu_2,1}g_{3\nu_3} + g_{0\nu_0}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3,1}\Big)\Bigg] \\
&\quad \times \left(\sum_{\pi\in S_4}(-1)^{\operatorname{sgn}\pi} g_{0\nu_0}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3}\right)^{-2}.
\end{aligned}
\tag{22}
$$

Substituting all $g^{\mu\nu}$ and also its first derivatives into (20), we get the explicit form of the first Einstein equation $R_{00} = 0$ of the second order for 10 unknowns $g_{00}, g_{01}, g_{02}, \dots, g_{33}$. It is evident that such an equation is extremely complicated. Relation (19) takes only four lines, relation (20) takes 40 lines and after substitution of all entries with determinants given by (21), (22), etc., into (20), the Eq. (18) for the component R_{00} of the Ricci tensor will occupy at least 10 pages. The other nine equations $R_{\mu\nu} = 0$ can be expressed similarly.

The explicit expression of the left-hand side of (12) for a given covariant divergence-free $T_{\mu\nu} \neq 0$ in terms of the unknown components of $g_{\mu\nu}$ is even more complicated. Using (13) and (20)–(22), we still have to express the term $-\frac{1}{2}Rg_{\mu\nu}$ similarly. Up to now, nobody has calculated how many terms the Einstein equations really contain, in general.

5 Computational Complexity of the Einstein Equations

According to (20)–(22), we observe that the Einstein equations are highly nonlinear. From the end of Sect. 4, we find that the explicit form of all 10 equations (12) will occupy at least one hundred pages. For comparison note that the Laplace equation $\Delta u = 0$ has only three terms $\partial^2 u/\partial x_i^2$, $i = 1, 2, 3$, on its left-hand side and the famous Navier-Stokes equations have 24 terms.

Let n denote the number of mass bodies. If $n = 0$, then the simplest solution to the Einstein equation is the Minkowski metric (5). If $n = 1$, then there are several other simple solutions to (12) that use spherical or axial symmetry of one body, e.g., the Schwarzschild metrics (6) and (11), or the Kerr metric Misner et al. (1997, p. 878). However, these solutions are local, not global (cf. Theorem 3). Moreover, the analytical solution of (12) is not known for two or more mass bodies. Thus, we have a serious problem to verify whether the Einstein equations describe well the n-body problem for $n > 1$ (e.g., in the Solar system).

There are many numerical methods for solving partial differential equations such as the finite difference method, the finite volume method, the boundary element method, the finite element method Brandts et al. (2020), etc. For the numerical solution of the Einstein equations, we have to include back all arguments of the functions

$$g_{\mu\nu} = g_{\mu\nu}(x^0, x^1, x^2, x^3),$$
$$g_{\mu\nu,\varkappa} = g_{\mu\nu,\varkappa}(x^0, x^1, x^2, x^3),$$
$$g_{\mu\nu,\varkappa\lambda} = g_{\mu\nu,\varkappa\lambda}(x^0, x^1, x^2, x^3)$$

appearing in (20)–(22) for all $\mu, \nu, \varkappa, \lambda = 0, 1, 2, 3$.

For example, in the simplest setting of the finite difference method one has to establish a four-dimensional regular space-time mesh, e.g., with N^4 mesh points

$$(x_i^0, x_j^1, x_k^2, x_l^3) \quad \text{for } i, j, k, l = 1, 2, \ldots, N.$$

Then the 10 values $g_{\mu\nu}(x^0, x^1, x^2, x^3)$, their $40 = 10 \times 4$ first derivatives and $100 = 10 \times (1 + 2 + 3 + 4)$ second derivatives (of the Hessian) appearing in (20)–(22) have to be replaced by finite differences at all mesh points. For instance, the second derivative $g_{00,11}$ appearing in the second line of (20) can be approximated by the standard central difference as

$$g_{00,11}(x_i^0, x_j^1, x_k^2, x_l^3) \approx \frac{g_{00}(x_i^0, x_j^1 + h, x_k^2, x_l^3) - 2g_{00}(x_i^0, x_j^1, x_k^2, x_l^3) + g_{00}(x_i^0, x_j^1 - h, x_k^2, x_l^3)}{h^2},$$

where $h = N^{-1}$ is the discretization parameter. The huge system of nonlinear partial differential equations described in Sect. 4 would then be replaced by a much larger system of nonlinear algebraic equations for approximate values of the metric tensor at all mesh points. In particular, at each mesh point, the corresponding discrete Einstein equations will be much longer than the Einstein equations themselves written explicitly. Hence, for example, if $N \approx 100$, the discrete system on each time level will occupy millions pages of extremely complicated and highly nonlinear algebraic equations.

It is well known that explicit numerical methods for solving evolution problems are unstable. Therefore, one should apply implicit methods. Nevertheless, up to now, we do not know any convergent and stable method that would yield a realistic numerical solution of the above system with guaranteed error bounds of discretization, iteration, and rounding errors.

Moreover, there are large problems with initial conditions. Since (12) is a second-order hyperbolic system of equations, one should prescribe initial conditions for all 10 components $g_{\mu\nu}$ and all their 40 first derivatives. However, this is almost impossible if all data are not spherically symmetric. The main reason is that spacetime tells matter how to move and matter tells spacetime how to curve Misner et al. (1997). So the initial space manifold is a priori not known, in general. Thus we also have serious problems to prove the existence and uniqueness of the solution of the Einstein equations and compare their solution with reality. There are similar large problems with boundary conditions and the divergence-free right-hand side (14) of the Einstein equations (12).

Another non-negligible problem lies in the nonuniqueness of topology. The reason is that the knowledge of the metric tensor $g_{\mu\nu}$ does not determine uniquely the topology of the corresponding space-time manifold. For instance, the Euclidean space $\mathbb{E}^3$ has obviously the same metric $g_{\mu\nu} = \delta_{\mu\nu}$, $\mu, \nu = 1, 2, 3$, as $\mathbb{S}^1 \times \mathbb{E}^2$ but different topology for a time-independent case with $T_{\mu\nu} = 0$ in (12). Here $\mathbb{S}^1$ stands for the unit circle. Hence, solving the Einstein equations does not mean that we obtain the shape of the associated space-time manifold. Other examples can be found in Misner et al. (1997, p. 725).

6 Concluding Remarks

Validation and verification of problems of mathematical physics and their computer implementation is a very important part of numerical analysis. We always encounter two basic types of errors: modeling error and numerical errors (such as discretization error, iteration error, round-off errors, and also undiscovered programing bugs). Validation tries to estimate the modeling error and to answer the question: *Do we solve the correct equations?* On the other hand, verification tries to quantify the numerical errors and to answer the question: *Do we solve the equations correctly?*

There is a general belief in the current astrophysical community that the Einstein equations best describe gravity Křížek (2019a). However, their extreme complexity prevents from verifying whether they model, for instance, the Solar system better than Newtonian n-body simulations with $n > 1$. Hence, in this case the Einstein equations are, in fact, non-computable by present computer facilities, and thus nontestable in their general form. Moreover, from the previous exposition it is obvious that by (12) we are unable to calculate trajectories of the Jupiter-Sun system, even 1 mm of Jupiter's trajectory, for example. The reason is that the mass of Jupiter is not negligible with respect to the Sun's mass. On the other hand, such trajectories can be calculated numerically very precisely by the n-body simulations (e.g., with $n = 8$ planets) even though their analytical solution is not known. Thus the modeling error e_0, which is the difference between observed trajectories and the analytical solution, is also not known. However, the modeling error e_0 can be easily estimated by the triangle inequality

$$|e_0| \leq |e_1| + |e_2|,$$

where e_1 is the numerical error and e_2 is the total error which is the difference between observed and numerically calculated trajectories.

Classical relativistic tests are based on verification of very simple algebraic formulae (see, e.g., (23) below) derived by various simplifications and approximations of the Schwarzschild solution (6) of the Einstein equations (12) which is very special and corresponds only to the exterior of one spherically symmetric body, that is $n = 1$. However, we cannot test good approximation properties of the Einstein equations (12) by means of one particular exterior Schwarzschild solution. Such an approach could be used only to disprove their good modeling properties of reality. Analogously, good modeling properties of the Laplace equation $\Delta u = 0$ cannot be verified by testing some of its trivial linear solutions, since there exist infinitely many other nontrivial solutions.

In Einstein (1915b), Einstein replaced Mercury by a massless point, the position of the Sun was fixed, and the other planets were not taken into account (see Křížek (2017) for many other simplifications that were done). To express the gravitational field, Einstein used Eqs. (1) instead of (4) for $\mu = \nu = 0$. This important fact is suppressed (cf. Theorem 1). In this way, Einstein derived under various further approximations the following formula for the relativistic perihelion shift of Mercury Einstein (1915b, p. 839):

$$\varepsilon = 24\pi^3 \frac{a^2}{T^2 c^2 (1 - e^2)} = 5.012 \times 10^{-7} \, \text{rad}, \tag{23}$$

where $T = 7.6005 \times 10^6$ s is the orbital period, $e = 0.2056$ is the eccentricity of its elliptic orbit, and $a = 57.909 \times 10^9$ m is the length of its semimajor axis. From this he got an idealized value of the perihelion shift 43″ per century. However, this number does imply that the system (12) describes trajectories of planets better than Newtonian mechanics as demonstrated in Sects. 4 and 5.

Note that Paul Gerber in 1898 derived the following formula for the speed of light by means of retarded potentials (see Gerber 1898):

$$c^2 = 24\pi^3 \frac{a^2}{T^2 (1 - e^2) \Phi},$$

where Φ is the perihelion shift of Mercury during one orbital period. We see that this formula is the same as (23). So the corresponding tests of the general theory of relativity based on (4) or by the Einstein system (1) yield the same values as tests of the Gerber theory of retarded potentials. So which theory is correct?

A shift (advance) of the line of apsides of binary pulsars cannot be derived similarly to Mercury's perihelion shift, since the analytical solution of the corresponding two-body problem with nonzero masses is not known. Thus, only some heuristic formulae can be employed to this highly nonlinear problem. Recent observations of gravitational waves also do not confirm that (12) models reality well, since these waves are described by a simplified linearized equations with the D'Alembert operator. For a collision of two black holes a post-Newtonian approach was employed (see, e.g., Mroué et al. 2013). Moreover, a large gravitational redshift was not taken into account Křížek and Somer (2018). Abbott et al. (2016) considered only the cosmological gravitational redshift $z = 0.09$ of a binary black hole merger, but they forgot that the redshift of any black hole is $z = \infty$. In Křížek and Somer (2018), we demonstrate that the resulting black hole masses were overestimated approximately twice.

Note also that Fig. 4, which should illustrate the propagation of gravitational waves, contradicts the general theory of relativity. To see this, denote by d the coordinate distance of two black holes and by T their orbital period. Multiply the inequality

$$\pi > 2$$

by d/T. Then we immediately get a contradiction

$$v = \frac{\pi d}{T} > \frac{2d}{T} = \frac{|AB|}{T} = c, \tag{24}$$

where v is the orbital velocity, c the speed of gravitational waves (equal to the speed of light), and $|AB|$ is the distance of two consecutive maximum amplitudes of the right black hole as indicated in Fig. 4. However, $v \leq \frac{1}{3}c$ by Abbott et al. (2016). Figure 4 shows only a dipole character of gravitational waves and not their proclaimed quadrupole character. Furthermore, the double Archimedean spiral illustrating gravitational waves has by definition a different shape near the center.

Finally, we would like to emphasize that no equation of mathematical physics describes reality absolutely exactly on any scale. Therefore, each mathematical model has only a limited scope of its application. In particular, also the Einstein equations with cosmological constant $\Lambda \neq 0$ (see Einstein 1952)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{25}$$

should not be applied to the entire universe as it is often done, since they are nonlinear and thus not scale invariant. Note that the observable universe is at least 15 orders of magnitude larger than one astronomical unit.

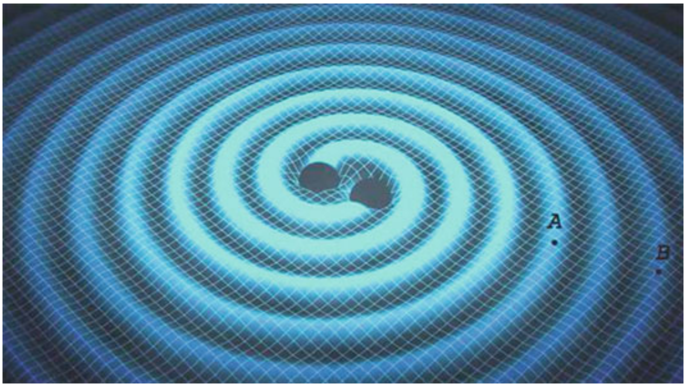


Fig. 4 Popular illustration implying that the orbital velocity v of binary black holes is larger than the speed of light c, see (24)

There are only three maximally symmetric three-dimensional manifolds $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$ that are used to model a homogeneous and isotropic universe for a fixed time. In this case, the Einstein equations lead to the famous Friedmann ordinary differential equation Křížek and Somer (2016), i.e.

$$\text{Einstein equations} + \text{maximum symmetry} \Longrightarrow \text{Friedmann equation.} \qquad (26)$$

The Friedmann equation is applied to calculate luminosity distances of type Ia supernovae. On the basis of these distances, it is claimed that the Einstein equations well describe the entire universe. This is a typical circular argument.

The current cosmological model, which is based on the Friedman equation, possesses over 20 paradoxes (see, e.g., Křížek 2019a; Křížek and Somer 2016; Vavryčuk 2018). From this and implication (26), it is evident that the Einstein equations should not be applied to modeling the entire universe. During its expansion, the topology cannot change. The most probable model is $\mathbb{S}^3$ whose present radius is very roughly $R = 10^{26}$ m, the volume is $2\pi^2 R^3$ and the total mass is estimated to 2×10^{53} kg. However, radius increases with time. So during the Big Bang, the topology of the universe should also be $\mathbb{S}^3$. By Křížek (2019b) the maximum mass density is about 10^{18} kg/m^3 which would correspond to the radius $R = 10^9$ m.

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