Paths in hyperspaces

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Abstract

We prove that the hyperspace of closed bounded sets with the Hausdorff topology, over an almost convex metric space, is an absolute retract. Dense subspaces of normed linear spaces are examples of, not necessarily connected, almost convex metric spaces. We give some necessary conditions for the path-wise connectedness of the Hausdorff metric topology on closed bounded sets. Finally, we describe properties of a separable metric space, under which its hyperspace with the Wijsman topology is path-wise connected.

Key words and phrases: hyperspace, Wijsman topology, Hausdorff metric, path-wise connectedness, absolute retract.

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1 Introduction

Given a metric space (X, d), let $CL(X)$ denote the hyperspace of closed nonempty subsets of X. We are interested in path-wise connectedness and related properties of hyperspace topologies on $CL(X)$, mainly the Hausdorff metric topology and the Wijsman topology. These two topologies come from identifying a closed set $A \subseteq X$ with its distance functional $x \mapsto \text{dist}(x, A)$, so that $CL(X)$ can be regarded as a subspace of $C(X, \mathbb{R})$, the space of all continuous real functions on X. Under this identification, the Hausdorff and Wijsman topologies are the topologies of uniform and pointwise convergence respectively. Both are different from the well known Vietoris topology (unless X is compact). The advantage of these topologies is metrizability: the Hausdorff topology on bounded sets is always metrizable and the Wijsman one is metrizable provided the base space is separable. The Vietoris topology on closed sets is metrizable only if the base space is compact. For a general reference concerning hyperspace topologies see Beer's book [3].

Global and local path-wise connectedness of the Vietoris topology on compact sets has been studied since 1930s. Borsuk and Mazurkiewicz [4] showed in 1931 that both $K(X)$, the hyperspace of compact subsets of X and $C(X)$, the hyperspace of subcontinua of X, are path-wise connected provided X is a metrizable continuum. The case of non-metrizable spaces was investigated by McWaters [14] and Ward [13]: they obtained results on generalized path-wise connectedness of the Vietoris hyperspace of compact sets. Local path-wise connectedness of $K(X)$ and $C(X)$, for

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a compact metric space X, were first characterized by Wojdysławski [15] in 1939. Namely, $K(X)$ is locally path-wise connected iff X is locally connected. The same is true for $C(X)$ and this property is equivalent to the fact that $K(X)$ (or $C(X)$) is an absolute neighborhood retract. The case of all metric spaces is due to Curtis [5]: $K(X)$ is locally path-wise connected (equivalently: $K(X) \in \mathrm{ANR}$) iff X is *locally continuum-wise connected*, i.e. for every $p \in X$ and its neighborhood V there is a neighborhood U of p such that any two points of U lie in a subcontinuum of V. A famous result of Curtis and Schori [7] says that $K(X)$ is homeomorphic to the Hilbert cube iff X is a locally connected, nondegenerate, metric continuum (for other results in this spirit see e.g. [5, 6]). Let us also mention a useful result of Curtis and Nguyen To Nhu [6]: the Vietoris hyperspace of finite sets over a locally path-wise connected metric space is an ANR. For a study of topological properties of compact Vietoris hyperspaces we refer to Nadler's book [11] or to a recent one by Illanes and Nadler [8].

Concerning other hyperspace topologies, not much is known. Antosiewicz and Cellina [1] showed that the hyperspace of closed bounded sets with the Hausdorff metric topology, over a convex subspace of a normed linear space, is an absolute retract. Sakai and Yang [12] proved that $CL(X)$ with the Fell topology is homeomorphic to the Hilbert cube minus a point iff X is a locally compact, locally connected, separable metrizable space with no compact components. Banakh, Kurihara and Sakai [2] showed that for a normed linear space X, $CL(X)$, $K(X)$ and some other subspaces of $CL(X)$, equipped with the Attouch-Wets topology, are absolute retracts; in case where X is a Banach space, $CL(X)$ is homeomorphic to a Hilbert space. Finally, Sakai, Yaguchi and the second author [9] gave general conditions for the ANR property of $CL(X)$ with the Wijsman topology. In [9] it is also proved that $CL(X)$ with the Wijsman topology is homeomorphic to the separable Hilbert space, provided X is a separable Banach space.

We give several results on path-wise connectedness and absolute neighborhood retract property for some hyperspace topologies. Using well known results for compact hyperspaces, we characterize path-wise connectednes of the Vietoris topology on closed sets over a metrizable space; we apply this result for the Wijsman topology. We note that for a noncompact metrizable space X, $CL(X)$ with the Vietoris topology is not locally connected. We prove that the hyperspace of closed bounded sets endowed with the Hausdorff topology is an absolute retract, provided the base space is almost convex (see the definitions below). This improves the result of Antosiewicz and Cellina [1] mentioned above. We give some necessary conditions for the path-wise connectedness of the Hausdorff topology on closed bounded sets. Finally, we discuss the path-wise connectedness of the Wijsman topology. We show, among others, that $CL(X)$ with the Wijsman topology is path-wise connected if (X, d) is separable and path-wise connected at infinity or continuum-wise connected.

Notation

For a given topological (metric) space X, we denote by $CL(X)$, $K(X)$, $C(X)$ and $CLB(X)$ the hyperspace of closed, compact, compact connected and closed bounded nonempty subsets of X respectively. For any set X, we denote by $\mathrm{Fin}(X)$ the collection of all nonempty finite subsets of X. ω denotes the set of all nonnegative integers.

Let (X, d) be a metric space. We will denote by $\mathrm{B}(A, r)$ and $\overline{\mathrm{B}}(A, r)$ the open and closed ball centered at $A \subseteq X$ and with radius $r \geqslant 0$, respectively. The *Hausdorff metric* on $CLB(X)$ is defined by

$$d_H(A,B) = \inf\{r > 0 \colon A \subseteq \mathrm{B}(B,r) \;\&\; B \subseteq \mathrm{B}(A,r)\}.$$

We can define $d_H(A,B)$ also for unbounded sets, setting $d_H(A,B) = +\infty$ if there is no $r > 0$ with $A \subseteq \mathrm{B}(B,r)$ and $B \subseteq \mathrm{B}(A,r)$. The topology induced by the Hausdorff metric is called the *Hausdorff topology*. It is reasonable to consider the Hausdorff topology on all closed subsets of X, because d_H is locally a metric. However, $CLB(X)$ is clopen in $CL(X)$ and hence $CL(X)$ is not connected for an unbounded metric space (X,d). The Hausdorff topology on $K(X)$ agrees with the Vietoris topology. The *Wijsman topology* on $CL(X)$ is the least topology $\mathcal{T}$ such that for every $x \in X$ the function $A \mapsto \mathrm{dist}(x,A)$ is continuous. By the formula

$$d_H(A,B) = \sup_{x\in X} \bigl|\mathrm{dist}(x,A) - \mathrm{dist}(x,B)\bigr|,$$

the Wijsman topology is weaker than the Hausdorff topology. For a noncompact metric space, the Wijsman topology is strictly weaker than the Vietoris one (even on finite sets). We will denote by $\mathcal{T}_V$, $\mathcal{T}_H$ and $\mathcal{T}_W$ the Vietoris, Hausdorff and Wijsman topology, respectively (the latter two depend on the metric).

A metric space (X,d) is *almost convex* if for every $x,y \in X$ and for every $s,t > 0$ with $d(x,y) < s+t$, there exists $z \in X$ such that $d(x,z) < s$ and $d(z,y) < t$. For example, a dense subspace of a normed linear space (or, more generally, of a convex metric space) is almost convex.

A *path* in a space X is a continuous map $\gamma \colon J \to X$ where $J \subseteq \mathbb{R}$ is a closed interval (usually $J = [0,1]$). We denote by B^{k+1} and S^k the standard $k+1$-dimensional closed ball and the standard k-dimensional sphere (which is the boundary of B^{k+1}), respectively. A topological space X is *k-connected* ($k \in \omega$) if every continuous map $f \colon S^k \to X$ has a continuous extension $F \colon B^{k+1} \to X$. In particular, "0-connected" means "path-wise connected". X is *homotopically trivial* if it is k-connected for every $k \in \omega$. Local versions of k-connectedness are defined in the obvious way. A metric space (X,d) is *path-wise connected at infinity* if for every $x \in X$ there exists a continuous map $f \colon [0,+\infty) \to X$ such that $f(0) = x$ and $\lim_{t\to+\infty} d(x,f(t)) = +\infty$. A topological space X is *continuum-wise connected* if every two points of X are contained in a subcontinuum of X (i.e. a compact connected subspace of X).

An *absolute neighborhood retract* (briefly ANR) is a metrizable space X such that for every metric space Y, every continuous map $f \colon A \to X$ defined on a closed set $A \subseteq Y$, has a continuous extension $F \colon U \to X$, for some open set U with $A \subseteq U \subseteq Y$. If, under these assumptions, $U = Y$ then X is an *absolute retract* (briefly AR). It is well known that an absolute neighborhood retract is locally k-connected for every $k \in \omega$ and a homotopically trivial ANR is an absolute retract.

A *(join) semilattice* is a commutative semigroup $(L,\vee)$ such that $a \vee a = a$ for every $a \in L$. A semilattice comes from a partially ordered set $(L,\leqslant)$ such that every two elements of L have a supremum. Specifically, setting $a \vee b = \sup\{a,b\}$, $(L,\leqslant)$ becomes a semilattice. Conversely, if $(L,\vee)$ is a semilattice then defining $a \leqslant b$ iff $a \vee b = b$, we get a partial order on L such that $a \vee b = \sup\{a,b\}$. A *Lawson semilattice* [10] is a topological semilattice $(L,\vee)$ (i.e. a semilattice equipped with the topology such that $\vee \colon L \times L \to L$ is continuous) which has a neighborhood base consisting of subsemilattices. Most of hyperspaces are Lawson semilattices with respect to $\cup$ (see Section 2.2).

2 General results

2.1 On path-wise connectedness

Fix a topological (metric) space X and let $\mathcal{T}$ be the Vietoris topology or the Wijsman topology on $CL(X)$. Observe that $(CL(X), \mathcal{T})$ has the following properties:

(i) If $\{A_n\}_{n\in\omega}$ converges to A then $\{C \cup A_n\}_{n\in\omega}$ converges to $C \cup A$ for each $C \in CL(X)$.

(ii) If $\{A_n\}_{n\in\omega}$ is increasing and such that $\bigcup_{n\in\omega} A_n$ is dense in X then $\{A_n\}_{n\in\omega}$ converges to X.

Most of the hyperspace topologies satisfy stronger condition than (i), namely the union operator $\cup\colon CL(X) \times CL(X) \to CL(X)$ is continuous. On the other hand, for a bounded metric space X the Hausdorff topology on $CL(X)$ does not necessarily satisfy (ii).

Proposition 2.1. *Let X be a separable topological space and let $\mathcal{T}$ be a topology on $CL(X)$ satisfying conditions (i), (ii) above. Then the following conditions are equivalent:*

(a) $(CL(X), \mathcal{T})$ is path-wise connected.

(b) For each $a, b \in X$ there is a continuous map $\gamma\colon [0, 1] \to (CL(X), \mathcal{T})$ such that $\gamma(0) = \{a\}$ and $b \in \gamma(1)$.

Proof. It is enough to show that (b) $\implies$ (a). Fix $C \in CL(X)$. We show that there exists a path joining C to X. Fix a countable dense set $\{d_n\colon n \in \omega\} \subseteq X$ with $d_0 \in C$. For each $n \in \omega$ choose a continuous map $\gamma_n\colon [0, 1] \to (CL(X), \mathcal{T})$ such that $\gamma(0) = \{d_n\}$ and $d_{n+1} \in \gamma(1)$. Define $\varphi_n\colon [n, n + 1] \to CL(X)$ by setting

$$\varphi_n(t) = C \cup \bigcup_{k<n} \gamma_k(1) \cup \gamma_n(t - n).$$

As $\mathcal{T}$ is nice (property (1)), we see that φ_n is continuous. Furthermore, $\varphi_n(n+1) = \varphi_{n+1}(n+1)$ and $\bigcup_{n\in\omega} \varphi_n(n)$ is dense. Thus we can define a map $\varphi\colon [0, +\infty] \to CL(X)$ by setting $\varphi \restriction [n, n+1] = \varphi_n$ and $\varphi(+\infty) = X$. Applying condition (2) for $\mathcal{T}$ we see that φ is continuous at $+\infty$. This completes the proof. □

2.2 Hyperspaces as Lawson semilattices

Let $(Y, \vee)$ be a Lawson semilattice and consider $\mathrm{Fin}(L)$ with the Vietoris topology. The formula

$$r(\{a_1, \dots, a_n\}) = a_1 \vee \dots \vee a_n$$

defines a map $r\colon \mathrm{Fin}(L) \to L$ which is easily seen to be continuous (because L has a basis consisting of subsemilattices). Identifying L with $\{\{x\}\colon x \in L\}$ we see that L is a retract of $\mathrm{Fin}(L)$. On the other hand, if L is metrizable and locally path-wise connected [and connected] then $\mathrm{Fin}(L)$ is an ANR [AR] (by the theorem of Curtis and Nguyen To Nhu [6]). Thus we obtain a useful result due to Banakh, Kurihara and Sakai [2]:

Theorem 2.2. *Let $(L, \vee)$ be a metrizable Lawson semilattice which is locally path-wise connected. Then L is an* ANR. *If, additionally, L is connected then L is an* AR.

Using similar arguments we can prove the following:

Proposition 2.3. *Let $(L, \vee)$ be a metrizable Lawson semilattice. Then L is k-connected for every $k > 0$.*

Proof. Let $f\colon S^k \to L$ be a continuous map. Then f extends naturally to a Vietoris continuous map $\overline{f}\colon \mathrm{Fin}(S^k) \to \mathrm{Fin}(L)$. As $k > 0$, $\mathrm{Fin}(S^k)$ is an AR, so there is a map $j\colon B^{k+1} \to \mathrm{Fin}(S^k)$ such that $j(x) = \{x\}$ for $x \in S^k$ (in fact, there is a straightforward formula for j, see [6]). Now setting $F = r\overline{f}j$, where $r\colon \mathrm{Fin}(L) \to L$ is the retraction defined above, we get a continuous extension of f. □

To apply the above results for hyperspaces we need to know that they are Lawson semilattices.

Proposition 2.4. *The Vietoris, Wijsman and Hausdorff hyperspaces are Lawson semilattices with respect to $\cup$.*

Proof. Let X be a topological (metric) space and let $\mathcal{T} \in \{\mathcal{T}_V, \mathcal{T}_H, \mathcal{T}_W\}$. Clearly, $\mathcal{T}$ has a base consisting of subsemilattices. We need to show the continuity of the union. First, let $\mathcal{T} = \mathcal{T}_V$ and fix $(A_0, B_0) \in CL(X) \times CL(X)$. If $A_0 \cup B_0 \in U^+$ then $(A_0, B_0) \in U^+ \times U^+$ and $\cup[U^+ \times U^+] \subseteq U^+$, where $\cup[M]$ is the image of $M \subseteq CL(X) \times CL(X)$ under $\cup\colon CL(X) \times CL(X) \to CL(X)$. If $A_0 \cup B_0 \in U^-$ then $(A_0, B_0) \in W$, where $W = (U^- \times CL(X)) \cup (CL(X) \times U^-)$, and we have $\cup[W] \subseteq U^-$. Thus, $\cup$ is continuous with respect to $\mathcal{T}_V$. For $\mathcal{T} = \mathcal{T}_W$ and $\mathcal{T} = \mathcal{T}_H$ the continuity of $\cup$ follows from the formulae:

$$\mathrm{dist}(x, A \cup B) = \min\{\mathrm{dist}(x, A), \mathrm{dist}(x, B)\},$$
$$d_H(A \cup B, A' \cup B') \leqslant \max\{d_H(A, A'), d_H(B, B')\}.$$

□

3 The Vietoris topology

In this section we note some results on path-wise connectedness of the Vietoris topology. Recall that the theorem of Borsuk and Mazurkiewicz [4] says that both $K(X)$ and $C(X)$ are path-wise connected whenever X is a metrizable continuum. Using this result, we are able to investigate the case of noncompact metric spaces. We use the following fact: if $\gamma\colon [0,1] \to K(X)$ is a path and $\gamma(0)$ is connected then $\bigcup \gamma[0,1]$ is a subcontinuum of X (see, e.g. Nadler [11]).

Theorem 3.1. *For a metrizable space X the following conditions are equivalent:*

(a) Every compact subset of X is contained in a continuum.

(b) $(K(X), \mathcal{T}_V)$ is path-wise connected.

Proof. (a) $\implies$ (b) Fix $A, B \in K(X)$. Let $C \subseteq X$ be a continuum such that $A \cup B \subseteq C$. Then $A, B \in K(C)$ and hence, by the theorem of Borsuk-Mazurkiewicz, there exists a path $\gamma\colon [0,1] \to K(C)$ such that $\gamma(0) = A$ and $\gamma(1) = B$.
(b) $\implies$ (a) Fix $A \in K(X)$ and $a \in X$. Let $\gamma\colon [0,1] \to K(X)$ be a path joining A and $\{a\}$. Then $D = \bigcup \gamma[0,1]$ is a continuum containing A. □

Example 3.2. An example of a path-wise connected subspace of the plane $\mathbb{R}^2$ which does not satisfy (a) above. Consider

$$X = (\{0\} \times [0, +\infty)) \cup S \cup T,$$

where

$$S = \{(x, |\sin(\pi/x)|/x)\colon x \in [0,1]\}$$

and $T = \{(x,y) \in \mathbb{R}^2\colon (x - 1/2)^2 + y^2 = 1/4 \text{ and } y < 0\}$. Observe that X is path-wise connected. Let $A = (\{0\} \cup \{1/n\colon n \in \omega\}) \times \{0\}$. Then $A \in K(X)$ but each closed connected subset of X containing A also contains $\{0\} \times [0, +\infty)$ and therefore is not compact.

Theorem 3.3. *For a metrizable space X the following conditions are equivalent:*

(a) $(C(X), \mathcal{T}_V)$ is path-wise connected.

(b) There exists $\mathcal{G} \subseteq K(X)$ containing all singletons of X, such that $(\mathcal{G}, \mathcal{T}_V)$ is path-wise connected.

(c) X is continuum-wise connected, i.e. each two points of X lie in a subcontinuum of X.

Proof. (a) $\implies$ (b) is obvious.
(b) $\implies$ (c) Fix $a, b \in X$ and let $\gamma\colon [0,1] \to \mathcal{G}$ be a path joining $\{a\}$ and $\{b\}$. Then $\bigcup \gamma[0,1]$ is a subcontinuum of X containing a, b.
(c) $\implies$ (a) Fix $A, B \in C(X)$. Let G be a subcontinuum of X intersecting both A and B. Then $A, B \in C(D)$, where $D = A \cup B \cup G$. By the theorem of Borsuk-Mazurkiewicz, there exists a path in $C(D)$ joining A and B. □

Using the above result and Proposition 2.1 we obtain the following.

Corollary 3.4. *Let X be a separable topological space. If X is path-wise connected or X is continuum-wise connected and metrizable then $(CL(X), \mathcal{T}_V)$ is path-wise connected.*

Let X be a metrizable space. A theorem of Curtis [5] says that $(K(X), \mathcal{T}_V)$ is locally path-wise connected (equivalently: $(K(X), \mathcal{T}_V) \in$ ANR) iff X is locally continuum-wise connected. Concerning $CL(X)$, we have the following negative result.

Theorem 3.5. *If X is a noncompact metrizable space then $(CL(X), \mathcal{T}_V)$ is not locally connected.*

Proof. Let $C = \{x_n\colon n \in \omega\}$ be a (one-to-one) sequence in X having no cluster point (by the noncompactness of X); then $C \in CL(X)$. Choose a disjoint family $\{U_n\}_{n\in\omega}$ of open sets such that $x_n \in U_n$ for $n \in \omega$. Let $U = \bigcup_{n\in\omega} U_n$. Then U^+ is a neighborhood of C. Let $\mathcal{V} \in \mathcal{T}_V$ be any nieghborhood of C such that $\mathcal{V} \subseteq U^+$. Then $\mathcal{V}$ contains a basic neighborhood $\mathcal{W} = V^+ \cap V_0^- \cap \dots \cap V_{m-1}^-$ of C, with $V_i \subseteq V$ and $V, V_0, \dots, V_{m-1}$ open in X. This implies, in particular, that $C \subseteq V$ and that for every $i < m$ there is an $n(i) \in \omega$ with $x_{n(i)} \in V_i$. Let $F = \{x_{n(0)}, \dots, x_{n(m-1)}\}$, so that $F \in \mathcal{W}$, and fix $k > \max\{n(i) \colon i < m\}$. We will prove that $\mathcal{S} = \mathcal{V} \cap U_k^-$ is clopen in $\mathcal{V}$, and this will imply that $\mathcal{V}$ is disconnected, because $C \in \mathcal{S}$ while $F \in \mathcal{W} \setminus \mathcal{S} \subseteq \mathcal{V} \setminus \mathcal{S}$.

Clearly, $\mathcal{S}$ is open in $\mathcal{V}$ because U_k^- is open in $CL(X)$. On the other hand, we may observe that $\mathcal{S} = \mathcal{V} \cap [\operatorname{cl} U_k]^-$: indeed, every element of $\mathcal{V}$ is a subset of U and hence it cannot contain any point of $(\operatorname{cl} U_k) \setminus U_k$ (because the sets U_i are pairwise disjoint). Therefore, $\mathcal{S}$ is also closed in $\mathcal{V}$. □

4 The Hausdorff metric topology

In this section we consider $CLB(X)$ endowed with the Hausdorff metric topology. The Hausdorff metric is actually defined on $CL(X)$ but one can easily observe that $CLB(X)$ is clopen in $CL(X)$ so $CL(X)$ is not connected unless X is bounded. If X is not compact then there is an unbounded metric on X; on the other hand if d is an unbounded metric on X then $\varrho(x, y) = \min\{1, d(x, y)\}$ defines a bounded, uniformly equivalent metric, so the Hausdorff metric induced by ϱ is equivalent to the one induced by d. It follows that a noncompact metrizable space admits a metric for which the Hausdorff hyperspace of closed bounded sets is disconnected.

Observe that if $\gamma \colon [0, 1] \to CLB(X)$ is a path then the map $\Gamma \colon [0, 1] \to CLB(X)$ defined by the formula

$$\Gamma(t) = \operatorname{cl}\Big(\bigcup_{s \leqslant t} \gamma(s)\Big)$$

is also a path in $CLB(X)$. Such a map will be called *an order arc* in $CLB(X)$.

Lemma 4.1. *For a metric space (X, d) the following conditions are equivalent:*

(a) $(CLB(X), \mathcal{T}_H)$ is path-wise connected.

(b) For each $p \in X$ and for each $n \in \omega$ there exists a path $\gamma \colon [0, 1] \to CLB(X)$ such that $\gamma(0) = \{p\}$ and $\mathrm{B}(p, n) \subseteq \bigcup_{t \leqslant 1} \gamma(t)$.

Proof. We need to show that (b) $\Longrightarrow$ (a). Fix $C, D \in CLB(X)$. Fix $n \in \omega$ with

$$n > \max\{d_H(\{c\}, D), d_H(C, \{d\})\},$$

where $c \in C$ and $d \in D$ are fixed arbitrarily. By (b) there exist paths $f, g \colon [0, 1] \to CLB(X)$ with $f(0) = \{c\}$, $g(0) = \{d\}$, $\mathrm{B}(c, n) \subseteq \bigcup_{t \leqslant 1} f(t)$ and $\mathrm{B}(d, n) \subseteq \bigcup_{t \leqslant 1} g(t)$. We may assume that f, g are order arcs, thus $\mathrm{B}(c, n) \subseteq f(1)$ and $\mathrm{B}(d, n) \subseteq g(1)$. Set $F = f(1) \cup g(1)$. Then $F \in CLB(X)$ and $C \cup D \subseteq \mathrm{B}(d, n) \cup \mathrm{B}(c, n) \subseteq F$. Define $\gamma \colon [0, 2] \to CLB(X)$ by setting

$$\gamma(t) = \begin{cases} C \cup f(t) & \text{if } t \in [0, 1], \\ C \cup f(1) \cup g(t-1) & \text{if } t \in [1, 2]. \end{cases}$$

Observe that γ is well-defined, continuous and $\gamma(0) = C$, $\gamma(2) = F$. It follows that C and F can be joined by a path. Similarly, there is a path joining D to F. □

4.1 Almost convex metric spaces

Recall that a metric space (X, d) is *almost convex* if for each $a, b \in X$ and for each $s, t > 0$ such that $d(a, b) < s + t$ there exists $x \in X$ with $d(a, x) < s$ and $d(x, b) < t$. Clearly, every convex metric space is almost convex and a dense subspace of an almost convex metric space is almost convex.

Lemma 4.2. *A metric space (X, d) is almost convex iff for each $A \subseteq X$ and for each $s, t > 0$ we have $\mathrm{B}(\mathrm{B}(A, s), t) = \mathrm{B}(A, s + t)$.*

Proof. If (X, d) satisfies the above condition then for $a, b \in X$ and $s, t > 0$ with $d(a, b) < s + t$ we have $b \in \mathrm{B}(a, s + t) = \mathrm{B}(\mathrm{B}(a, s), t)$ and hence there is $x \in \mathrm{B}(a, s)$ with $d(x, b) < t$. Thus (X, d) is almost convex.
Assume now that (X, d) is almost convex and fix $A \subseteq X$ and $s, t > 0$. Clearly $\mathrm{B}(\mathrm{B}(A, s), t) \subseteq \mathrm{B}(A, s + t)$. Fix $p \in \mathrm{B}(A, s + t)$. Let $a \in A$ be such that $d(a, p) < s + t$. There exists $x \in X$ with $d(a, x) < s$ and $d(x, p) < t$. Thus $p \in \mathrm{B}(\mathrm{B}(A, s), t)$. □

Lemma 4.3. *Let (X, d) be an almost convex metric space and let $C \in CL(X)$. Then the map $\gamma\colon [0, +\infty) \to CL(X)$ defined by the formula*

$$\gamma(t) = \overline{\mathrm{B}}(C, t)$$

is a constant 1 Lipschitz map with respect to the Hausdorff metric on $CL(X)$ and the standard metric on $[0, +\infty)$.

Proof. First observe that $\mathrm{cl}\,\mathrm{B}(A, r) = \overline{\mathrm{B}}(A, r)$ for every $A \subseteq X$ and $r > 0$. Thus, by Lemma 4.2 we have $\gamma(t + r) \subseteq \overline{\mathrm{B}}(\gamma(t), r)$ for every $t, r \geqslant 0$. It follows that $d_H(\gamma(t), \gamma(t + r)) \leqslant r$. □

Theorem 4.4. *Let (X, d) be an almost convex metric space. Then $(CLB(X), \mathcal{T}_H)$ is an absolute retract.*

Proof. $CLB(X)$ is path-wise connected by Lemma 4.1, but we need to show that it is locally path-wise connected. Fix $C, D \in CLB(X)$ and $r > d_H(C, D)$. Let $A = \overline{\mathrm{B}}(C, r) \cup \overline{\mathrm{B}}(D, r)$. Then $C \cup D \subseteq A$ and $A \in CLB(X)$. Define $\gamma\colon [0, 2r] \to CLB(X)$ by

$$\gamma(t) = \begin{cases} \overline{\mathrm{B}}(C, t), & \text{if } t \leqslant r, \\ \overline{\mathrm{B}}(C, r) \cup \overline{\mathrm{B}}(D, t - r), & \text{if } r \leqslant t \leqslant 2r. \end{cases}$$

By Lemma 4.3, γ is continuous with respect to the Hausdorff metric. Observe that $d_H(\gamma(t), C) \leqslant 2r$, Thus C and A can be joined by a path contained in the open ball centered at C and with radius $2r$. By symmetry, the same applies to D and A. This proves that $(CLB(X), d_H)$ is locally path-wise connected. As $CLB(X)$ is a Lawson semilattice, by Theorem 2.2, it is an ANR. On the other hand, $CLB(X)$ is homotopically trivial (Proposition 2.3), so it is an AR. □

Corollary 4.5. *Let X be a dense subset of a convex subset of a normed linear space, endowed with the metric induced by the norm. Then $(CLB(X), \mathcal{T}_H)$ is an absolute retract.*

The above result in the case of convex subsets of normed spaces was proved, using elementary although complicated methods, by Antosiewicz and Cellina [1].

4.2 C-connectedness

We investigate necessary conditions for the path-wise connectedness of $(CLB(X), d_H)$ and we present some counterexamples.

Let us call a metric space (X, d) *C-connected* (or *connected in Cantor's sense*) if for each $a, b \in X$ and for each $\varepsilon > 0$ there exist $x_0, \dots, x_n \in X$ such that $x_0 = a$, $x_n = b$ and $d(x_i, x_{i+1}) < \varepsilon$ for $i < n$. Clearly every connected space is C-connected and every compact C-connected space is connected. A sequence $(x_0, \dots, x_n)$ with $d(x_i, x_{i+1}) < \varepsilon$ for $i < n$ will be called an *ε-sequence of size n joining* x_0, x_n. Call a metric space (X, d) *uniformly C-connected* if for each bounded set $B \subseteq X$ and for each $\varepsilon > 0$ there exists $k \in \omega$ such that for each $x, y \in B$ there exists an ε-sequence in X of size at most k joining x, y. Observe that the closure of a uniformly C-connected subset of X is also uniformly C-connected.

Proposition 4.6. *Let (X, d) be a metric space. Then $(CLB(X), d_H)$ is C-connected if and only if (X, d) is uniformly C-connected.*

Proof. Suppose that $(CLB(X), d_H)$ is C-connected. Fix a closed bounded set $B \subseteq X$ and $p \in B$. Fix $\varepsilon > 0$ and let $(A_0, \dots, A_k)$ be an ε-sequence in $CLB(X)$ with $A_0 = \{p\}$ and $A_k = B$. Fix $x \in B$. We can find $x_{k-1} \in A_{k-1}$ such that $d(x_{k-1}, x) < \varepsilon$, because $d_H(A_{k-1}, B) < \varepsilon$. Inductively, we find $x_i \in A_i$ such that $d(x_i, x_{i+1}) < \varepsilon$. Then $x_0 = \{p\}$ and $(x_0, \dots, x_{k-1}, x)$ is an ε-sequence of size k joining p, x. It follows that every two points of B are joined by an ε-sequence of size at most $2k$. Thus (X, d) is uniformly C-connected.

Suppose now that (X, d) is uniformly C-connected and fix $B \in CLB(X)$. Fix $p \in B$. We show that $\{p\}$ and B can be joined be an ε-sequence for every $\varepsilon > 0$. As (X, d) is C-connected and is isometrically embedded in $CLB(X)$, it then follows that $CLB(X)$ is C-connected.

Fix $\varepsilon > 0$. Let k be such that for each $x \in B$ there exists an $\varepsilon/2$-sequence $(y_0(x), \dots, y_k(x))$ such that $y_0(x) = p$ and $y_k(x) = x$. Define $A_i = \operatorname{cl}\{y_i(x) \colon x \in B\}$. Observe that $A_i \in CLB(X)$ and $d_H(A_i, A_{i+1}) \leqslant \varepsilon/2 < \varepsilon$ for $i < k$. Thus $(A_0, \dots, A_k)$ is an ε-sequence in $CLB(X)$ joining $A_0 = \{p\}$ to $A_k = B$. □

Consider the following metric properties:

- C_1: Every bounded subset of X is contained in a uniformly C-connected bounded subset of X.
- C_2: For each $p \in X$ and for each $r > s > 0$ such that $\mathrm{B}(p, r) \setminus \mathrm{B}(p, s) \neq \emptyset$ there exists a uniformly C-connected set $S \subseteq \mathrm{B}(p, r)$ such that $p \in S$ and $S \cap \mathrm{B}(p, r) \setminus \mathrm{B}(p, s) \neq \emptyset$.

Proposition 4.7. *Let (X, d) be a metric space such that $(CLB(X), \mathcal{T}_H)$ is path-wise connected. Then (X, d) has properties C_1, C_2.*

Proof. Fix $p \in X$ and $r > 0$. Let $f \colon [0, 1] \to CLB(X)$ be a Hausdorff continuous order arc with $f(0) = \{p\}$ and $f(1) \supseteq \mathrm{B}(p, r)$. Fix $\varepsilon > 0$. Let $k \in \omega$ be such that $|t - s| < 1/k$ implies $d_H(f(t), f(s)) < \varepsilon$. Fix $x_0 \in \mathrm{B}(p, r)$. As $d_H(f(1), f(1 - 1/k)) < \varepsilon$, we can find $x_1 \in f(1 - 1/k)$ with $d(x_0, x_1) < \varepsilon$. Inductively, we find $x_i \in f(1 - i/k)$ with $d(x_{i-1}, x_i) < \varepsilon$. Finally $x_k = p$ which means that $(x_0, \dots, x_k)$ is an ε-sequence of size k joining p, x. This shows that $f(1)$ is uniformly C-connected and consequently (X, d) has property C_1. By the same argument, $f(t)$ is C-connected for each $t \in [0, 1]$. Now let $0 < s < r$ be such that $\mathrm{B}(p, r) \setminus \mathrm{B}(p, s) \neq \emptyset$. Then $d_H(\{p\}, f(1)) > s$.

Let $t_0 \in [0,1]$ be such that $d_H(\{p\}, f(t_0)) \in (s, r)$. Then $S = f(t_0)$ is a uniformly C-connected set with $p \in S$ and $S \cap \mathrm{B}(p,r) \setminus \mathrm{B}(p,s) \neq \emptyset$. This shows property C_2. $\square$

We now describe two examples of path-wise connected spaces with path-wise disconnected Hausdorff hyperspaces.

Example 4.8. Let (X, ϱ) be an unbounded metric space and define a bounded metric on X by the formula $d(x,y) = \min\{\varrho(x,y), 1\}$. Then $(CL(X), d_H)$ is not C-connected since (X, d) is not uniformly C-connected. Indeed, (X, d) is bounded, but for every k we can find two points of X which cannot be joined by a 1-sequence of size k. This shows that if a metrizable space X is not compact then there exists a metric d on X such that $(CLB(X), d_H)$ is not C-connected.

Example 4.9. Let $A = \bigcup_{n\in\omega}(3^{-n}, 2\cdot 3^{-n})$ and $B = \bigcup_{n\in\omega}[3^{-n}, 2\cdot 3^{-n}]$. Consider

$$X = \{(x,y) \in [0,1]^2 \colon \chi_A(x) \leqslant y \leqslant \chi_B(x)\}$$

with the topology inheritted from the plane. Then for any compatible metric d on X, $CLB(X)$ is not path-wise connected since (X, d) does not have property C_2. Indeed, if U is a neighborhood of $p = (0,0)$ contained in $[0,1] \times [0, 1/2)$ then the only C-connected subset of U containing p is $\{p\}$. On the other hand, if d is the Euclidean metric on X then (X, d) has property C_1.

PROBLEM: Do properties C_1, C_2 characterize path-wise connectedness of the Hausdorff topology?

5 The Wijsman topology

Let (X, d) be a metric space. Recall that the Wijsman topology is weaker than the Vietoris one; on $CLB(X)$ it is also weaker than the Hausdorff metric topology. $(CL(X), \mathcal{T}_W)$ is completely regular, it is metrizable iff X is separable. See Beer's book [3] for the details.
Applying Corollary 3.4 and Lemma 4.3 together with Proposition 2.1 we obtain the following.

Corollary 5.1. *If (X, d) is a separable continuum-wise connected metric space then $(CL(X), \mathcal{T}_W)$ is path-wise connected.*

Theorem 5.2. *If (X, d) is an almost convex metric space then $(CL(X), \mathcal{T}_W)$ is path-wise connected.*

Proof. If (X, d) is separable, this follows from Proposition 2.1 and Lemma 4.3. However in general, by Lemma 4.3, the formula

$$\gamma(t) = \begin{cases} \overline{\mathrm{B}}(A, t) & \text{if } t \in [0, +\infty), \\ X & \text{if } t = +\infty. \end{cases}$$

defines a Wijsman continuous path $\gamma \colon [0, +\infty] \to CL(X)$ joining A to X, for each $A \in CL(X)$. $\square$

The next result describes different situations. It appears that $(CL(X), \mathcal{T}_W)$ may be path-wise connected even if X is far from being connected.

Theorem 5.3. *Let (X,d) be a separable metric space such that for each $a,b \in X$ either there exists a uniformly continuous map $f\colon \mathbb{Q}\cap[0,1] \to X$ with $f(0)=a$ and $f(1)=b$, or else there exists a map $g\colon \mathbb{Q}\cap[0,+\infty) \to X$ such that $g(0)=b$, $\lim_{t\to+\infty} d(g(t),b)=+\infty$ and $g \restriction (\mathbb{Q}\cap[0,n])$ is uniformly continuous for every $n\in\omega$. Then $CL(X)$ with the Wijsman topology is path-wise connected.*

Proof. We use Proposition 2.1. Fix $a,b\in X$. Assume first that there exists a uniformly continuous map $f\colon \mathbb{Q}\cap[0,1]\to X$ with $f(0)=a$ and $f(1)=b$. Define a path $\gamma\colon [0,1]\to CL(X)$ by setting

$$\gamma(t) = \operatorname{cl}\{f(q)\colon q\in\mathbb{Q}\cap[0,t]\}.$$

Clearly, $\gamma(0)=\{a\}$ and $b\in\gamma(1)$. We need to show that γ is continuous. Fix $\varepsilon>0$. Let $\delta>0$ be such that $d(f(q_0),f(q_1))<\varepsilon$ whenever $|q_0-q_1|<\delta$, $q_0,q_1\in\mathbb{Q}\cap[0,1]$. Fix $t_0,t_1\in[0,1]$ with $|t_0-t_1|<\delta$. Consider $x=f(q_0)$, where $q_0\in\mathbb{Q}\cap[0,t_0]$. Choose $q_1\in\mathbb{Q}\cap[0,t_1]$ such that $|q_0-q_1|<\delta$. Then $d(x,f(q_1))<\varepsilon$ and $f(q_1)\in\gamma(t_1)$. It follows that $\gamma(t_0)\subseteq\overline{\mathrm{B}}(\gamma(t_1),\varepsilon)$. By symmetry we get $d_H(\gamma(t_0),\gamma(t_1))\leqslant\varepsilon$. It follows that γ is continuous with respect to the Hausdorff metric on $CL(X)$. Hence, γ is also continuous with respect to the Wijsman topology.

Now assume that there exists a map $g\colon \mathbb{Q}\cap[0,+\infty)\to X$ such that $g(0)=b$, $\lim_{t\to+\infty} d(g(t),b) = +\infty$ and $g\restriction(\mathbb{Q}\cap[0,n])$ is uniformly continuous for each $n\in\omega$. Define $\gamma\colon[0,+\infty]\to CL(X)$ by setting

$$\gamma(t)=\{a\}\cup\operatorname{cl}\{g(q)\colon q\in\mathbb{Q}\cap[t,+\infty)\}$$

for $t\in[0,+\infty)$ and $\gamma(+\infty)=\{a\}$. Clearly, $b\in\gamma(0)$. Observe that $\gamma\restriction[0,n]$ can be represented in the form

$$\gamma(t)=\eta(t)\cup B_n,$$

where $B_n=\{a\}\cup\operatorname{cl}\{g(q)\colon q\in[n,+\infty)\}$ and

$$\eta(t)=\operatorname{cl}\{g(q)\colon q\in\mathbb{Q}\cap[t,n]\}.$$

Thus, by the previous argument, $\gamma\restriction[0,n]$ is continuous with respect to the Wijsman topology. It remains to show that γ is continuous at $+\infty$. Fix $x\in X$. Then $\operatorname{dist}(x,\gamma(+\infty))=d(x,a)$. On the other hand, $d(g(q),x)\geqslant d(g(q),b)-d(x,b)$ so there exists $n_0\in\omega$ such that $d(g(q),x)\geqslant d(x,a)$ for $q\in\mathbb{Q}\cap[n_0,+\infty)$. Hence $\operatorname{dist}(x,\gamma(t))=d(x,a)=\operatorname{dist}(x,\gamma(+\infty))$ for $t\geqslant n_0$. □

Corollary 5.4. *Let (X,d) be a separable metric space which is path-wise connected at infinity. Then $(CL(X),\mathcal{T}_W)$ is path-wise connected.*

Let (X,d) be a separable, locally path-wise connected metric space. In [9], Sakai, Yaguchi and the second author proved that $(CL(X),\mathcal{T}_W)$ is an absolute neighborhood retract provided $X\setminus\bigcup\mathcal{B}$ has finitely many components, for every finite family $\mathcal{B}$ consisting of closed balls in (X,d). We give an example of an almost convex, locally path-wise connected, separable metric space, for which the Wijsman hyperspace is not locally connected.

Example 5.5. Let (X,d) be the separable hedgehog space, i.e.

$$X=\{\theta\}\cup\bigcup_{n\in\omega}(0,1]\times\{n\},$$

where $d(\theta,(t,n)) = t$, $d((t,n),(s,n)) = |t-s|$ and $d((t,n),(s,m)) = t+s$ for $n \neq m$. Then (X,d) is an almost convex metric space and it is an absolute retract. We claim that $(CL(X), \mathcal{T}_W)$ is not locally connected. Let $A_0 = \{(1,n)\colon n \in \omega\}$ and let $\mathcal{U} = \{A \in CL(X)\colon \operatorname{dist}(\theta, A) > 1/2\}$. Then $\mathcal{U}$ is a neighborhood of A_0. For each $n \in \omega$ define

$$\mathcal{V}_n^- = \{A \in CL(X)\colon \operatorname{dist}((1,n),A) < 1/2\}, \quad \mathcal{V}_n^+ = \{A \in CL(X)\colon \operatorname{dist}((1,n),A) > 1\}.$$

Clearly $\mathcal{V}_n^- \cap \mathcal{V}_n^+ = \emptyset$ and $\mathcal{V}_n^-, \mathcal{V}_n^+ \in \mathcal{T}_W$. Observe that $\mathcal{U} \subseteq \mathcal{V}_n^- \cup \mathcal{V}_n^+$. Also, $\mathcal{V}_n^-$ is a neighborhood of A_0. Now we claim that for every neighborhood $\mathcal{V}$ of A_0 with $\mathcal{V} \subseteq \mathcal{U}$, there is $n \in \omega$ such that $\mathcal{V}_n^+ \cap \mathcal{V} \neq \emptyset$, i.e. $\mathcal{V}_n^+$ disconnects $\mathcal{V}$. Indeed, observe that $A_0 = \lim_{n\to\infty} A_n$, where $A_n = \{(1,k)\colon k < n\}$. Thus, for every open $\mathcal{V} \subseteq \mathcal{U}$ with $A_0 \in \mathcal{V}$ there exists $n \in \omega$ such that $A_n \in \mathcal{V}$. On the other hand $\operatorname{dist}((1,n),A_n) = 2$ so $A_n \in \mathcal{V}_n^+$.

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