On perfect cliques with respect to infinitely many relations

(joint work with Martin Doležal)

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Motivation

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Every uncountable analytic set contains a perfect set.

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Definition

A set S is perfect if it is nonempty, dense-in-itself and completely metrizable.

Motivation (continued)

Theorem (Feng 1993)

Let X be an analytic space and let $C \subseteq [X]^2$ be open. Then either

- $X = \bigcup_{n\in\omega} X_n$, where $[X_n]^2 \cap C = \emptyset$ for every $n \in \omega$, or else
- there exists a perfect set P such that $[P]^2 \subseteq C$.

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Remark (Blass)

The theorem above fails when 2 is replaced by 3.

Definition

Let $\mathscr{R}$ be a family of relations on a set X. We say that $S \subseteq X$ is an $\mathscr{R}$-clique if for every $n \in \omega$, for every distinct $s_0, \dots, s_{n-1} \in S$ the relation

$$R(s_0, \dots, s_{n-1})$$

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Theorem (Mycielski 1964)

Let X be a dense-in-itself Polish space and let $\mathscr{R}$ be a countable family of co-meager relations on X. Then there exists a perfect $\mathscr{R}$-clique.

Theorem (Shelah 1999)

The following statement is consistent with ZFC:

- *For every analytic relation R on a Polish space X, either all R-cliques have cardinalities $\leqslant \aleph_1$ or else there exists a perfect R-clique.*

Theorem (Shelah 1999; Vejnar & K. 2012)

There exists a σ-compact symmetric binary relation E on the Cantor space 2^ω such that

1. *there exists an E-clique of cardinality $\aleph_1$,*
2. *there are no E-cliques of cardinality $> \aleph_1$, and*
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$$E = \{\langle x, y\rangle \in 2^\omega \times 2^\omega \colon (\exists\, n \in \omega) \quad \text{either } x = f_n(y) \text{ or } y = f_n(x)\},$$

where $\{f_n \colon 2^\omega \to 2^\omega\}_{n\in\omega}$ is a suitable family of continuous functions.

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Remark

The case $\mathscr{R} = \{R\}$, $\kappa = \aleph_0$ was proved in

- W. Kubiś, *Perfect cliques and G_δ colorings of Polish spaces*, Proc. Amer. Math. Soc. 131 (2003), 619–623.

Corollary

Let X be a complete metric space and let $\mathscr{R}$ be a countable family of G_δ relations on X. If there exists a nonempty dense-in-itself $\mathscr{R}$-clique then there exists also a perfect one.

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Corollary

Let X be an analytic space and let $\mathscr{R}$ be a countable family of G_δ relations on X. If there exists an uncountable $\mathscr{R}$-clique, then there exists also a perfect one.

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Theorem (Głąb & Strobin 2014)

Let $G = \prod_{n \in \omega} G_n$, where each G_n is a countable group. If G contains an uncountable free subgroup then it also contains a free subgroup of cardinality $2^{\aleph_0}$.

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Theorem

Let G be a Polish group. Then either all free subgroups of G are countable or else G contains a perfect set of free generators.

Definition

Let $w(x_0, \dots, x_{n-1})$ be a word, that is, an irreducible term in the language of groups. Given a group G and $g_0, \dots, g_{n-1} \in G$, define $R_w(g_0, \dots, g_{n-1})$ if and only if

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Claim

Let G be a topological group. Then for every word w, the relation R_w is open on G.

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Corollary

Let G be a completely metrizable topological group containing a nonempty dense-in-itself set of free generators. Then G contains a perfect set generating a free subgroup.

Thank you for your attention!

M. Doležal, W. Kubiś, *Perfect independent sets with respect to infinitely many relations*, preprint, http://arxiv.org/abs/1510.05127