Fraïssé-Jónsson limits

Category-theoretic approach

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Outline

Amalgamations

Let $\mathfrak{K}$ be a category.
We say that $\mathfrak{K}$ has the amalgamation property if

for every arrows $f\colon z \to x$, $g\colon z \to y$ there are arrows $f'\colon x \to w$ and $g'\colon y \to w$ such that $f' \circ f = g' \circ g$.

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$$\begin{array}{ccc} y & \xrightarrow{g'} & w \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\ z & \xrightarrow[f]{} & x \end{array}$$

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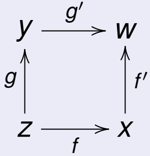

Fraïssé sequences

Definition

A sequence $\vec{u}\colon \kappa \to \mathfrak{K}$ is Fraïssé if it satisfies the following conditions.

1. $\vec{u}$ is cofinal in $\mathfrak{K}$, i.e. for every $x \in \mathfrak{K}$ there are $\alpha < \kappa$ and $f\colon x \to u_\alpha$ in $\mathfrak{K}$.
2. For every $\xi < \kappa$, for every $f\colon u_\xi \to y$, there exist $\eta \geqslant \xi$ and $g\colon y \to u_\eta$ such that $u_\xi^\eta = g \circ f$.

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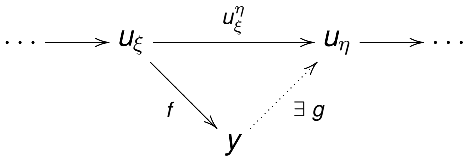

Notation:

$\mathfrak{S}_{\leqslant\kappa}(\mathfrak{K})$ = the category of sequences of length $\leqslant \kappa$ in $\mathfrak{K}$.

A category $\mathfrak{K}$ is κ-bounded if for every sequence $\vec{u} \in \mathfrak{S}_{<\kappa}(\mathfrak{K})$ there are $a \in \mathfrak{K}$ and an arrow of sequences $\vec{f}\colon \vec{u} \to a$.

A subcategory $\mathcal{F} \subseteq \mathfrak{K}$ is dominating in $\mathfrak{K}$ if

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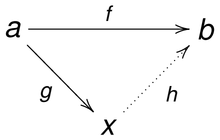

The existence

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Let $\kappa > 1$ be a regular cardinal and let $\mathfrak{K}$ be a κ-bounded category which has the amalgamation property and the joint embedding property. Assume further that $\mathfrak{K}$ has a dominating subcategory of cardinality $\leqslant \kappa$.
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Assume $\vec{u}$ is a Fraïssé sequence in a category with amalgamation $\mathfrak{K}$. Then for every countable sequence $\vec{x}$ in $\mathfrak{K}$ there exists an arrow $\vec{f} \colon \vec{x} \to \vec{u}$.

Corollary

Let $\vec{u}$ be a countable Fraïssé sequence in a category $\mathfrak{K}$. If $\mathfrak{K}$ has the amalgamation property then $\vec{u}$ is cofinal in $\mathfrak{S}_\omega(\mathfrak{K})$.

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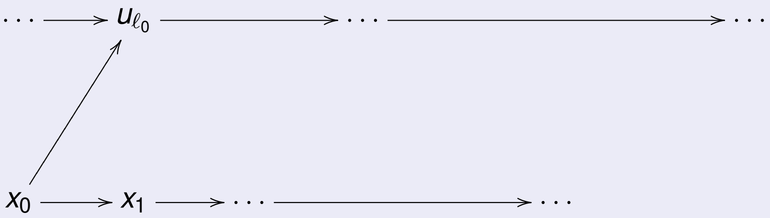
$$\cdots \longrightarrow u_{\ell_0} \longrightarrow \cdots \longrightarrow \cdots$$

$$x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow \cdots$$

Cofinality



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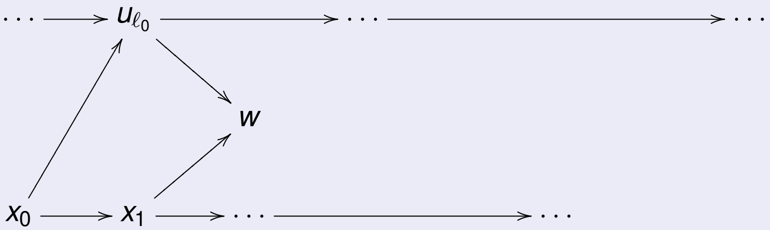

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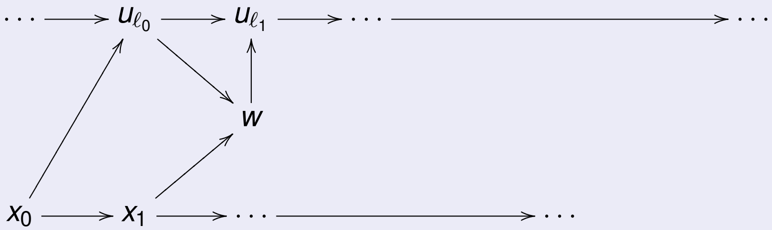
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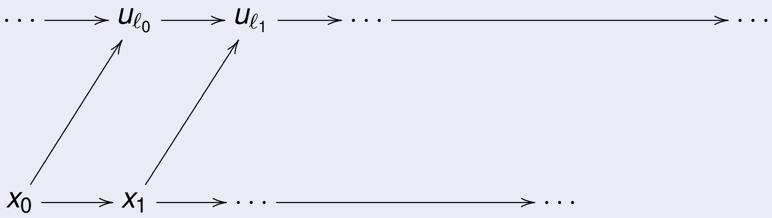

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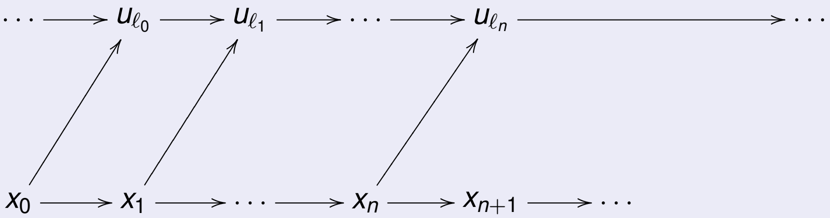

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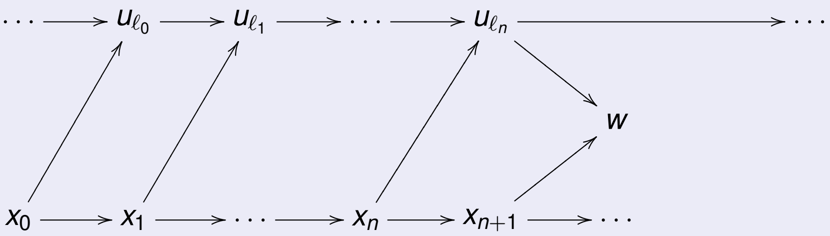

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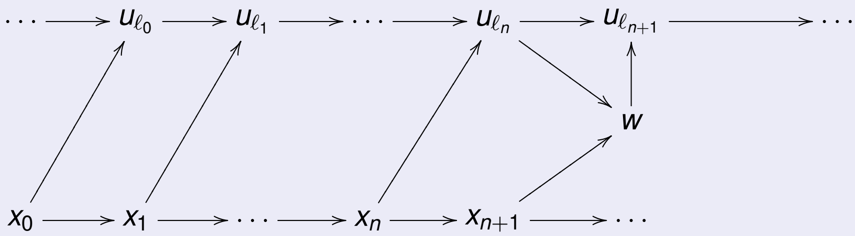

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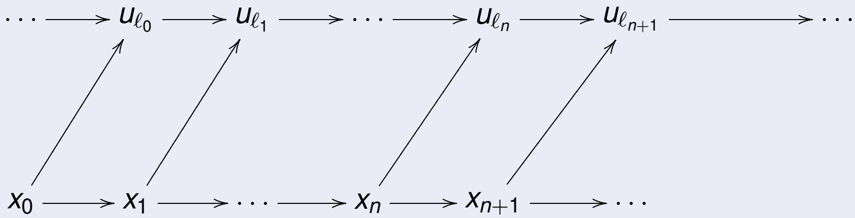
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Homogeneity & Uniqueness

Theorem

Assume that $\vec{u}$, $\vec{v}$ are ω-Fraïssé sequences in a fixed category $\mathfrak{K}$.

(a) *Let $f\colon u_k \to v_\ell$, where $k, \ell < \omega$. Then there exists an isomorphism $F\colon \vec{u} \to \vec{v}$ such that $F \circ u_k = v_\ell \circ f$. In particular $\vec{u} \approx \vec{v}$.*

(b) *Assume $\mathfrak{K}$ has the amalgamation property. Then for every $a, b \in \mathfrak{K}$ and for every arrows $f\colon a \to b$, $i\colon a \to \vec{u}$, $j\colon b \to \vec{v}$ there exists an isomorphism $F\colon \vec{u} \to \vec{v}$ such that $F \circ i = j \circ f$.*

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Back-and-forth argument

$$\begin{array}{ccc} u_k & \longrightarrow & \cdots \\ {\scriptstyle f}\uparrow & & \\ v_\ell & \longrightarrow & \cdots \end{array}$$

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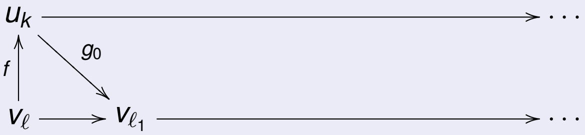

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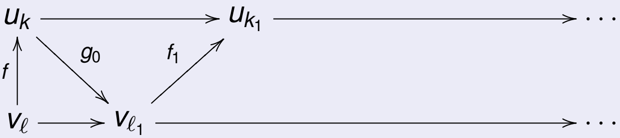

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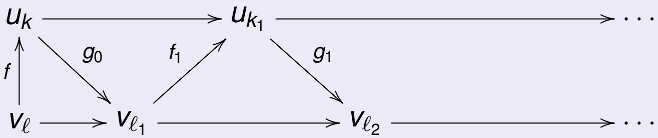

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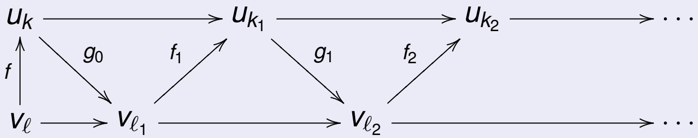

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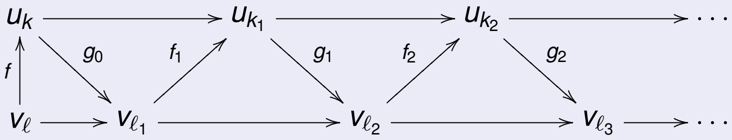

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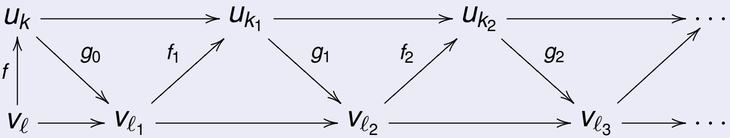

Some history

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Projection-embedding pairs

Fix a category $\mathfrak{K}$.
Define a new category $\mathfrak{K}^{\mathrm{PE}}$ as follows.

- The objects of $\mathfrak{K}^{\mathrm{PE}}$ are the objects of $\mathfrak{K}$.
- An arrow from $a \in \mathfrak{K}^{\mathrm{PE}}$ to $b \in \mathfrak{K}^{\mathrm{PE}}$ is a pair $\langle e, p \rangle$, where

$$a \xrightarrow{e} b \,, \qquad a \xleftarrow{p} b$$

  are arrows in $\mathfrak{K}$ satisfying $p \circ e = \mathrm{id}_a$.
- The composition is

$$\langle e, p \rangle \circ \langle e', p' \rangle = \langle e \circ e', p' \circ p \rangle.$$

There are two natural functors:

$$P(\langle e, p \rangle) = p, \qquad E(\langle e, p \rangle) = e.$$

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$\mathfrak{Set}$ = the category of nonempty finite sets.

Let $\vec{u}$ be a Fraïssé sequence in $\mathfrak{Set}^{\mathrm{PE}}$.
How to interpret its properties?
Let $\vec{x}$, $\vec{y}$ be sequences in $\mathfrak{Set}^{\mathrm{PE}}$ and fix $\vec{f}\colon \vec{x} \to \vec{y}$. Let

$$X = \varprojlim P[\vec{x}], \quad Y = \varprojlim P[\vec{y}]$$

and

$$D = \lim E[\vec{x}], \quad G = \lim E[\vec{y}].$$

Claim

X and Y are totally disconnected compact metric spaces, D is dense in X, G is dense in Y and $\vec{f}$ corresponds to a pair $\langle f, j\rangle$, where

- *$f\colon Y \to X$ is a quotient map,*
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$\mathfrak{Set}$ = the category of nonempty finite sets.

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The Fraïssé sequence in $\mathfrak{Set}^{\mathsf{PE}}$ *corresponds to a pair* $\langle 2^\omega, Q\rangle$*, where* Q *is a countable dense subset of the Cantor set* 2^ω*.*

Corollary

Let K *be a totally disconnected compact metric space and let* $D \subseteq K$ *be dense. Then there exists a retraction* $f \colon 2^\omega \to K$ *such that* $f \restriction Q$ *is a retraction onto* D*.*

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Let K *be a totally disconnected compact metric space wihout isolated points and let* $D \subseteq K$ *be countable and dense. Then*

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Homogeneity translates to the following:

Fact

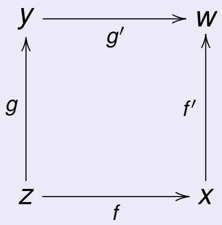
Let $\{U_0, \dots, U_{n-1}\}$ and $\{V_0, \dots, V_{n-1}\}$ be two partitions of 2^ω into clopen sets and let $a_i \in U_i$, $b_i \in V_i$ be fixed for each $i < n$. Then there exists a homeomorphism $h \colon 2^\omega \to 2^\omega$ satisfying

$$h(a_i) = b_i \quad \text{and} \quad h^{-1}[V_i] = U_i$$

for every $i < n$.

The arrows $\langle f', g' \rangle$ provide a pushout of $\langle f, g \rangle$ if moreover for every $\overline{f}, \overline{g}$ satisfying $\overline{f} \circ f = \overline{g} \circ g$ there exists a unique arrow h such that $h \circ f' = \overline{f}$ and $h \circ g' = \overline{g}$.

The pushout of $\langle f, g \rangle$



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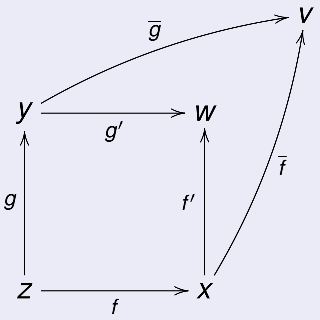

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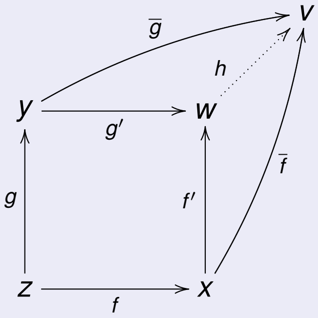

A general example

Fix a small category $\mathbb{B}$ and fix a covariant functor $F \colon \mathfrak{K} \to \mathfrak{L}$.
Define a new category $\mathfrak{fun}(\mathbb{B}, F)$ as follows.

- An object of $\mathfrak{fun}(\mathbb{B}, F)$ is a map $x \colon \mathrm{Ob}(\mathbb{B}) \to \mathrm{Ob}(\mathfrak{K})$ which after moving to $\mathfrak{L}$ via F "becomes" a covariant functor.
  That is, for each $b \in \mathrm{Ob}(\mathbb{B})$, $x(b) \in \mathrm{Ob}(\mathfrak{K})$ and for each arrow $f \colon a \to b$ in $\mathbb{B}$,
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F has the pushout property if for every $\mathfrak{K}$-arrows $f\colon z \to x$, $g\colon z \to y$, there exist $k\colon x \to w$, $\ell\colon y \to w$ such that

$$\begin{array}{ccc} y & \xrightarrow{\ \ell\ } & w \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle k} \\ z & \xrightarrow[\ f\ ]{} & x \end{array}$$

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Lemma

Assume F has the pushout property. Then $\mathfrak{fun}(\mathbb{B}, F)$ has the amalgamation property.

Example

Let $\mathbb{B} = \mathbb{Z}$ or $\mathbb{B} = \mathbb{N}$, treated as a monoidal category.
Let $\mathfrak{K}$ be the category of monomorphisms of a fixed category $\mathfrak{L}$.
Let F be the "inclusion" functor.
Under suitable assumptions, $\mathfrak{fun}\,(\mathbb{B}, F)$ has a Fraïssé sequence.

Corollary

There exists a nonexpansive homeomorphism $h\colon 2^\omega \to 2^\omega$ such that for every totally disconnected compact metric space K and for every nonexpansive homeomorphism $f\colon K \to K$ there is a quotient $q\colon 2^\omega \to 2^\omega$ for which the diagram

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Example

Let $\mathbb{B}$ have two objects $0 \neq 1$ and two arrows $e\colon 0 \to 1$, $p\colon 1 \to 0$, satisfying $p \circ e = \mathrm{id}_0$.
In particular, $\mathrm{End}(0) = \{\mathrm{id}_0\}$ and $\mathrm{End}(1) = \{\mathrm{id}_1, e \circ p\}$.
Let F be as before.

Assume

1. $\mathfrak{K}$ is a countable category.
2. $\mathfrak{K}$ has the initial object.
3. Monomorphisms admit pushouts in $\mathfrak{K}$.

Let $\mathfrak{K}^{\mathrm{mon}}$ be the category of all monomorphisms of $\mathfrak{K}$.

Proposition

$\mathfrak{K}^{\mathrm{mon}}$ has a unique (countable) Fraïssé sequence $\vec{u}$. Further,

- $\vec{u} \oplus \vec{u} \approx \vec{u}$.
- *The colimit of any countable sequence of monomorphisms*

$$\vec{u} \longrightarrow \vec{u} \longrightarrow \vec{u} \longrightarrow \cdots$$

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Theorem

There exists a PE-pair $\langle r, j\rangle \colon \vec{u} \to \vec{u}$ such that for every morphism $\langle p, e\rangle \colon \vec{x} \to \vec{y}$ in $\mathfrak{S}_\omega(\mathfrak{K})^{\mathrm{PE}}$ there are monomorphisms $k \colon \vec{x} \to \vec{u}$, $\ell \colon \vec{y} \to \vec{u}$ such that the diagrams

$$\begin{array}{ccc} \vec{u} & \xrightarrow{\;j\;} & \vec{u} \\ {\scriptstyle k}\uparrow & & \uparrow{\scriptstyle \ell} \\ \vec{x} & \xrightarrow[\;e\;]{} & \vec{y} \end{array} \qquad\qquad \begin{array}{ccc} \vec{u} & \xleftarrow{\;r\;} & \vec{u} \\ {\scriptstyle k}\uparrow & & \uparrow{\scriptstyle \ell} \\ \vec{x} & \xleftarrow[\;p\;]{} & \vec{y} \end{array}$$

commute.

The Gurariĭ space

Theorem (Gurariĭ, 1966)

There exists a separable Banach space $\mathbb{G}$ *with the following property:*

(∗) *Given finite-dimensional spaces* $Y \subseteq X$, $\varepsilon > 0$ *and an isometric embedding* $i\colon Y \to \mathbb{G}$ *there exists an embedding* $j\colon X \to \mathbb{G}$ *such that*

$$j \restriction Y = i \qquad \text{and} \qquad \max\{\|j\|, \|j^{-1}\|\} < 1 + \varepsilon.$$

Theorem (Lusky, 1976)

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Theorem

$\mathbb{G} \oplus_1 \mathbb{G} \approx \mathbb{G}$.

Explanation:

$X \oplus_1 Y$ is $X \times Y$ with the norm $\|\langle x, y\rangle\| = \|x\| + \|y\|$.

Corollary

The ℓ_1-sum of $\aleph_1$ many copies of the Gurariĭ space is a Gurariĭ space.

Remark

From the general theory of Fraïssé-Jónsson limits it follows that, under CH, there exists a unique Banach space U of density $\aleph_1$ such that for every separable spaces $E \subseteq F$, every isometric embedding $T \colon E \to U$ extends to an isometric embedding $\overline{T} \colon F \to U$.

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Projection-embedding pairs II

A sequence $\vec{x}$ in $\mathfrak{K}^{\mathrm{PE}}$ will be called semicontinuous if $E[\vec{x}]$ is continuous in $\mathfrak{K}$.

Theorem

Let $\mathfrak{K}$ be a category and let $\vec{u}$ and $\vec{v}$ be semicontinuous Fraïssé sequences in $\mathfrak{K}^{\mathrm{PE}}$ of the same regular length κ. Then for every arrow $f \colon u_0 \to \vec{v}$ in $\mathfrak{K}^{\mathrm{PE}}$ there exists an isomorphism of sequences $\vec{f} \colon \vec{u} \to \vec{v}$ such that $\vec{f} \circ u_0^{\infty} = f$.

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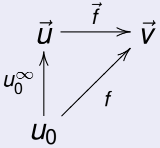

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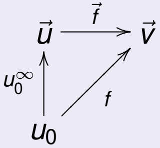

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Let $f\colon z \to x$, $g\colon z \to y$ be arrows in $\mathfrak{K}^{\mathrm{PE}}$. If $\langle E(f), E(g)\rangle$ has a pushout in $\mathfrak{K}$, then $\langle f, g\rangle$ has a proper amalgamation in $\mathfrak{K}^{\mathrm{PE}}$. That is, there exist arrows $h\colon x \to w$, $k\colon y \to w$ in $\mathfrak{K}^{\mathrm{PE}}$ such that the following diagrams commute in $\mathfrak{K}$.

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If $\mathfrak{K}$ has pullbacks or pushouts then $\mathfrak{K}^{\mathrm{PE}}$ has proper amalgamations.

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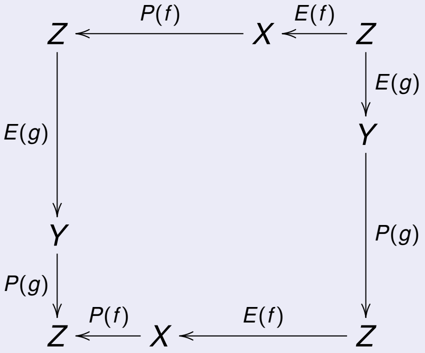

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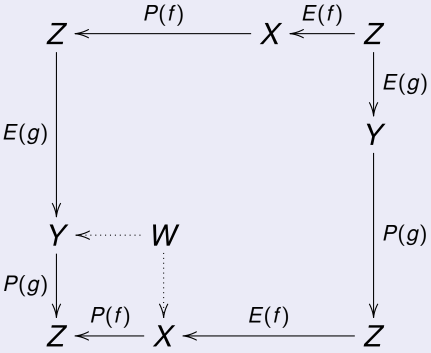

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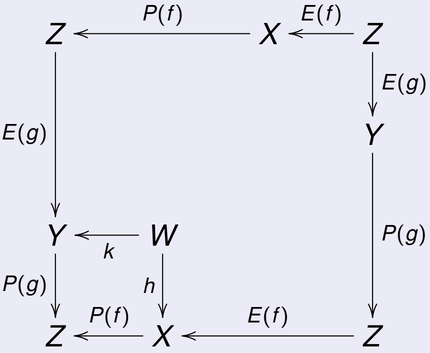

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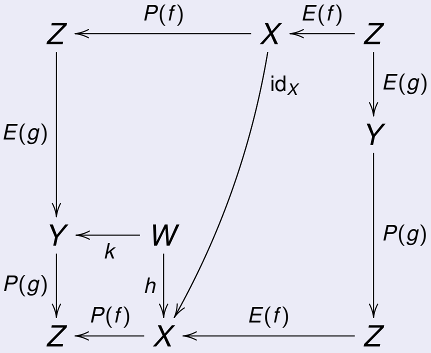

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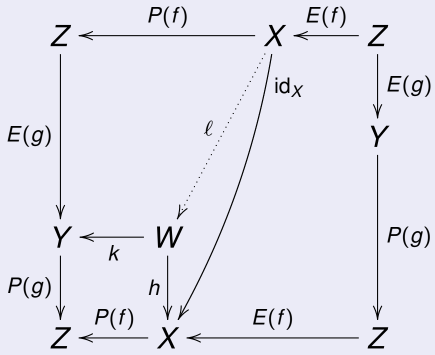

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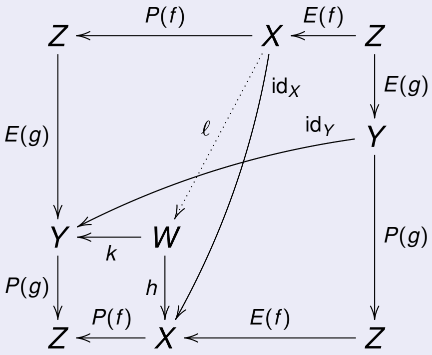

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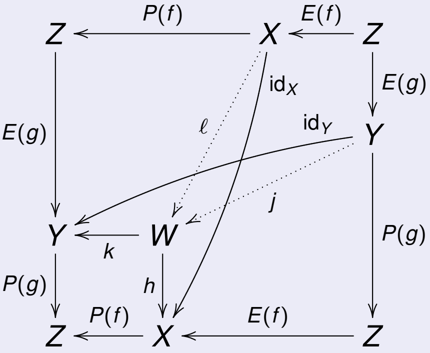

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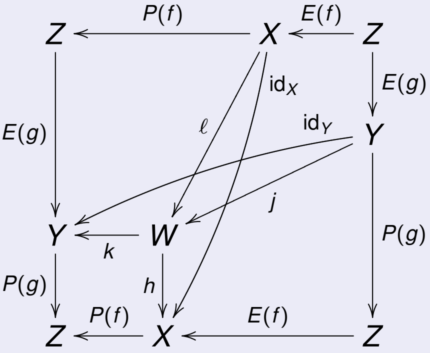

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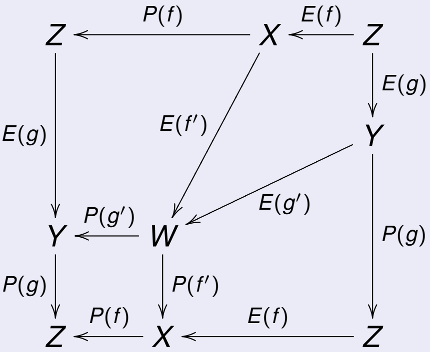

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Theorem

Let $\mathfrak{K}$ be a category such that $\mathfrak{K}^{\mathsf{PE}}$ has proper amalgamations. Assume $\vec{u}$ is a semi-continuous κ-Fraïssé sequence in $\mathfrak{K}^{\mathsf{PE}}$.

Then for every semi-continuous sequence $\vec{x} \in \mathfrak{S}_{\leqslant\kappa}(\mathfrak{K}^{\mathsf{PE}})$ there exists an arrow of sequences $\vec{f} \colon \vec{x} \to \vec{u}$ in $\mathfrak{K}^{\mathsf{PE}}$.

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Let $\mathfrak{B}_{\rm sep}$ be the category of all separable Banach spaces with linear transformations of norm $\leqslant 1$.

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Left-invertible arrows have pushouts in $\mathfrak{B}_{\rm sep}$.

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The category $\mathfrak{B}_{\rm sep}$ has $2^{\aleph_0}$ many isomorphic types of arrows.

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Assume $2^{\aleph_0} = \aleph_1$. Then there exists a semicontinuous ω_1-Fraïssé sequence in ${\mathfrak{B}_{\rm sep}}^{\rm PE}$.

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Let X be a Banach space of density $\aleph_1$. A projectional resolution of identity (PRI) is a sequence of norm one projections $\{P_\alpha\}_{\alpha<\omega_1}$ onto separable subspaces of X such that

1. $P_\xi P_\eta = P_\eta P_\xi = P_{\min\{\xi,\eta\}}$
2. $X = \bigcup_{\alpha<\omega_1} \operatorname{im} P_\alpha$
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Theorem

Assume $2^{\aleph_0} = \aleph_1$.
There exists a Banach space E *with a PRI* $\{P_\alpha\}_{\alpha<\omega_1}$ *and of density* $\aleph_1$*, which has the following properties:*

(a) *The family* $\{X \subseteq E : X \text{ is } 1\text{-complemented in } E\}$ *is, modulo linear isometries, the class of all Banach spaces of density* $\leqslant \aleph_1$ *with a PRI.*

(b) *Given separable subspaces* $X, Y \subseteq E$*, norm one projections* $P : E \to X$*,* $Q : E \to Y$*, both compatible with* $\{P_\alpha\}_{\alpha<\omega_1}$*, and given a linear isometry* $T : X \to Y$*, there exist a linear isometry* $H : E \to E$ *extending* T *and satisfying* $H \circ P = Q \circ H$*.*

Moreover, the above properties describe the space E *uniquely, up to a linear isometry.*

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THE END

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