On the Existence of Bifurcation Points for Periodic Problems to Distributional Differential Equations

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1 Introduction

The contribution is a continuation of the research started in [7] and [2]. We deal with periodic problems for nonlinear distributional (measure) differential equations with a parameter. In particular, we are interested in the existence of the bifurcation points for such problems.

The concept of distributional differential equations arose more or less together with the concept of systems with impulses. In general, they can describe some physical or biological problems, such as heartbeat, blood flow, pulse/frequency modulated systems, biological neural networks and/or models arising in control theory in which measures can be suitable controls, cf. e.g. [10]. Of course, differential equations with measures appear also in non-smooth mechanics. In these models, derivatives are understood in the sense of distributions and the solutions are generally discontinuous, but not too bad from another point of view, i.e. they are usually regulated or have bounded variation. For some early results, see e.g. [1] and references therein.

In this article we consider distributional differential systems of the form

$$Dx = f(\lambda, x, t) + g(x, t) \cdot Dh, \tag{1.1}$$

where D stands for the distributional derivatives and λ is a parameter. To this end, a handful tool are generalized ordinary differential equations (we write simply GODEs) introduced by Kurzweil in [3, 4] in the middle of 1950's. Since then, many authors have dealt with the potentialities of this theory (see e.g. [5, 9, 11] and references therein). In particular, measure differential equations of the form (1.1) as well as equations with impulses acting in fixed times are their special cases.

Throughout $G[0,T]$ is the Banach space of regulated functions (functions having all onesided limits) with values in $\mathbb{R}^n$ and equipped with the supremal norm and $BV[0,T] \subset G[0,T]$ is the space of functions with bounded variation on $[0,T]$. As usual, we denote $\Delta^+ x(s) = x(s+) - x(s)$ and $\Delta^- x(t) = x(t) - x(t-)$ for $x \in G[0,T]$. Our basic assumptions are the following:

Assumptions 1.1. $T \in (0,\infty)$, $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}$ are open sets, $f : \Lambda \times \Omega \times [0,T] \to \mathbb{R}^n$, $g : \Omega \times [0,T] \to \mathbb{R}^n$, $h : \mathbb{R} \to \mathbb{R}$ has a bounded variation on $[0,T]$ and is left-continuous on $[0,T]$, while $h(0-) = h(0)$ and $h(T+) = h(T)$.

2 Distributional differential equations

By distributions we understand linear continuous n-vector functionals on the topological vector space $\mathcal{D}^n$ of functions $\varphi:\mathbb{R}\to\mathbb{R}^n$ possessing for any $j\in N\cup\{0\}$ a derivative $\varphi^{(j)}$ of the order j which is continuous on $\mathbb{R}$ and such that $\varphi^{(j)}(t)=0$ if $t\notin(0,T)$. The space $\mathcal{D}^n$ is endowed by the topology in which the sequence $\varphi_k\in\mathcal{D}$ tends to $\varphi_0\in\mathcal{D}$ in $\mathcal{D}$ if and only if $\lim_k\|\varphi_k^{(j)}-\varphi_0^{(j)}\|_\infty=0$ for all non negative integers j.

The space of n-vector distributions on $[0,T]$ (dual of $\mathcal{D}^n$) is denoted by $\mathcal{D}^{n*}$. Instead of $\mathcal{D}^{1*}$ we write $\mathcal{D}^*$. Given a distribution $f\in\mathcal{D}^{n*}$ and a test function $\varphi\in\mathcal{D}^n$, $\langle f,\varphi\rangle$ is the value of the functional f on φ. Of course, reasonable real valued point functions are naturally included into distributions. The zero distribution $0\in\mathcal{D}^{n*}$ on $[0,T]$ can be identified with an arbitrary measurable function vanishing a.e. on $[0,T]$. Obviously, if $f\in G[0,T]$ is left-continuous on $(0,T]$, then $f=0\in\mathcal{D}^{*n}$ if and only if $f(t)\equiv 0$.

For $h\in\mathcal{D}^*$, the symbol Dh stands for its distributional derivative, i.e.

$$Dh:\varphi\in\mathcal{D}\to\langle Dh,\varphi\rangle=-\langle h,\varphi'\rangle\ \text{ for all }\ \varphi\in\mathcal{D}.$$

If $f\in AC[0,T]$, then $Df=f'$, of course.

The term $g(t,x)\cdot Dh$ on the right hand side of (1.1) is a symbol for the distributional product of the function $\widetilde{g}_x:t\in[0,T]\to g(x(t),t)$ and the derivative Dh of h. As in the Schwartz setting no general rule how to define a product of an arbitrary couple of distributions is available, some more explanation is desirable. In text-books one can find the trivial examples. However, the product occurring in (1.1) is not covered by these cases. Fortunately, it turned out that, for this aim, a good tool is provided by the Kurzweil–Stieltjes integral. The following definition has been introduced in [12] and was used in [9, Section 8.4], as well.

Definition 2.1. If $g:[0,T]\to\mathbb{R}^n$ and $h:[0,T]\to\mathbb{R}$ are such that the Kurzweil–Stieltjes integral $\int_0^T g\,dh$ exists, then the product $g\cdot Dh$ is the distributional derivative of the indefinite integral $H(t)=\int_0^t g\,dh$, i.e. $g\cdot Dh=DH$.

The multiplication operation given by Definition 2.1 has all the usual properties excepting that (cf. [12, Remark 4.1] and [9, Theorem 6.4.2]) the expected formula $D(f\cdot g)=Df\cdot g+f\cdot Dg$ does not hold, in general. More precisely, if f and g are regulated and at least one of them has a bounded variation, then

$$D(f\cdot g)=Df\cdot g+f\cdot Dg+Df\cdot\Delta^+\widetilde{g}-\Delta^-\widetilde{f}\cdot Dg,$$

where

$$\Delta^+\widetilde{g}(t)=\begin{cases}\Delta^+g(t) & \text{if } t<T,\\ 0 & \text{if } t=T\end{cases}\quad\text{and}\quad \Delta^-\widetilde{f}(t)=\begin{cases}0 & \text{if } t=0,\\ \Delta^-f(t) & \text{if } t>0.\end{cases}$$

Now, we can define solutions of (1.1) as follows:

Definition 2.2. A couple $(x,\lambda)\in G[0,T]\times\Lambda$ is a solution of (1.1) if x is left-continuous on $(0,T]$, $x(t)\in\Omega$ for all $t\in[0,T]$, the distributional product $\widetilde{g}_x\cdot Dh$ of the function $\widetilde{g}_x:t\in[0,T]\to g(x(t),t)\in\mathbb{R}^n$ with Dh has a sense and the equality (1.1) is satisfied in the distributional sense, i.e. $\langle Dx,\varphi\rangle=\langle\widetilde{f}_{\lambda,x},\varphi\rangle+\langle\widetilde{g}_x\cdot Dh,\varphi\rangle$ for all $\varphi\in\mathcal{D}^n$, where $\widetilde{f}_{\lambda,x}:t\in[0,T]\to f(\lambda,x(t),t)\in\mathbb{R}^n$.

Together with (1.1) let us consider two related equations

$$x(t) = x(0) + \int_0^t f(\lambda, x(s), s)\,ds + \int_0^t g(x(s), s)\,dh(s) \ \text{ for } \ t \in [0, T] \tag{2.1}$$

and (GODE)

$$x(t) = x(0) + \int_{t_0}^t DF(x(\tau), t) \ \Big(\text{i.e. } \frac{dx}{d\tau} = DF(x, t)\Big). \tag{2.2}$$

We have

Theorem 2.1. *Let Assumptions 1.1 and*

$$\begin{cases} f(\lambda, \cdot, \cdot) \ \textit{ is Carathéodory on } \ \Omega \times [0, T] \ \textit{ for any } \ \lambda \in \Lambda, \\ g(\cdot, t) \ \textit{ is continuous on } \Omega \textit{ for } t \in [0, T] \textit{ and there is } m_h \textit{ such that:} \\ \qquad \displaystyle\int_0^T m_h(s)\, d[\operatorname{var}_0^s h] < \infty \ \textit{ and } \ \|g(x, t)\| \le m_h(t) \ \textit{ for } \ (\lambda, x, t) \in \Lambda \times \Omega \times [0, T]. \end{cases}$$

hold and let

$$F(\lambda, x, t) = \int_0^t f(\lambda, x, s)\,ds + \int_0^t g(x, s)\,dh(s) \ \textit{ for } \ (\lambda, x, t) \in \Lambda \times \Omega \times [0, T].$$

Then the equations (1.1), (2.1) *and* (2.2) *are equivalent.*

3 Bifurcations

In the rest we assume that the assumptions of Theorem 2.1 are satisfied. Let us consider the equivalent periodic problems

$$Dx = f(\lambda, x, t) + g(x, t) \cdot Dh, \quad x(0) = x(T) \tag{3.1}$$

and

$$x(t) = x(T) + \int_0^t f(\lambda, x(s), s)\,ds + \int_0^t g(x(s), s)\,dh(s).$$

Put

$$\Phi(\lambda, x)(t) = x(T) + \int_0^t f(\lambda, x(s), s)\,ds + \int_0^t g(x(s), s)\,dh(s) \ \text{ for } \ \lambda \in \lambda, \ \ x \in B(x_0, \rho), \ \ t \in [0, T].$$

Then $\Phi(\lambda, \cdot)$ maps $B(x_0, \rho)$ into $G[0, T]$ for any $\lambda \in \Lambda$ and (3.1) is equivalent to finding couples (x, λ) such that $x = \Phi(\lambda, x)$.

Definition 3.1. Let x_0 be a solution of (3.1) for all $\lambda \in \Lambda$ and let $\rho > 0$ be such that $x(t) \in \Omega$ for all $t \in [0, T]$ whenever $\|x - x_0\| < \rho$. Then (λ_0, x_0) a bifurcation point of (3.1) if every its neighborhood in $\Lambda \times G[0, T]$ contains a solution (λ, x) such that $x \neq x_0$.

Next result is taken from [2].

Theorem 3.1. *In addition to the assumptions of Theorem 2.1, let x_0 and ρ be as in Definition 3.1 and*

$$\begin{cases} \text{there is a } \gamma:[0,T]\to\mathbb{R} \text{ nondecreasing and such that for any } \varepsilon>0 \text{ there is a } \delta>0 \\ \qquad \text{such that } \Big\|\displaystyle\int_s^t \big[f(\lambda_2,x,r)-f(\lambda_1,x,r)\big]\,dr\Big\| < \varepsilon|\gamma(t)-\gamma(s)| \\ \qquad\qquad \text{for } x\in\Omega,\ \ t,s\in[0,T] \text{ and } \lambda_1,\lambda_2\in\Lambda \text{ such that } |\lambda_1-\lambda_2|<\delta \end{cases}$$

and let $[\lambda_1^*,\lambda_2^*]\subset\Lambda$ be such that x_0 is an isolated fixed point of both $\Phi(\lambda_1^*,\,\cdot\,)$ and $\Phi(\lambda_2^*,\,\cdot\,)$ and

$$\deg_{LS}\big(Id-\Phi(\lambda_1^*,\,\cdot\,),B(x_0,\rho),0\big)\neq\deg_{LS}\big(Id-\Phi(\lambda_2^*,\,\cdot\,),B(x_0,\rho),0\big).$$

Then there is $\lambda_0\in[\lambda_1^*,\lambda_2^*]$ such that (x_0,λ_0) is a bifurcation point of (3.1).

The conditions necessary for the pair $(\lambda_0,x_9)\in\Lambda\times\Omega$ to be a bifurcation point of (3.1) are presented in our upcoming paper [8]. One of the equivalent formulations of the main result reads as follows:

Theorem 3.2. *Besides the assumptions of Theorem 3.1, let us assume also*

- *f has a total differential $f'_x(\lambda,x,t)$ for $(\lambda,x,t)\in\Lambda\times\Omega\times[0,T]$ fulfilling Carathéodory conditions withe respect to (x,t);*
- *g has a total differential $g'_x(x,t)$ for $(x,t)\in\Omega\times[0,T]$ which is bounded on $\Omega\times[0,T]$ and continuous with respect to $x\in\Omega$ for each $t\in[0,T]$ and there is $\Theta_h:[0,T]\to\mathbb{R}$ such that*

$$\int_0^T \Theta_h(s)\,d\,[\operatorname{var}_0^s h]<\infty \ \text{ and } \ \|g'_x(x,t)\|\le\Theta_h(t);$$

- *there is a nondecreasing function $\gamma:[0,T]\to\mathbb{R}$ such that for any $\varepsilon>0$ there is a $\delta>0$ such that*

$$\Big\|\int_s^t \big[f'_x(\lambda_1,x,r)-f'_x(\lambda_2,y,r)\big]\,dr+\int_s^t \big[g'_x(x,r)-g'_x(y,r)\big]\,dh(r)\Big\|<\varepsilon|\gamma(t)-\gamma(s)|,$$

  whenever $|\lambda_1-\lambda_2|+\|x-y\|<\delta$.

Then the couple $(\lambda_0,x_9)\in\Lambda\times\Omega$ is not a bifurcation point for (3.1) *whenever the homogeneous system*

$$z(r)=z(T)-\int_0^t f'_x(\lambda_0,x_0,\tau)z(\tau)\,d\tau-\int_0^t g'_x(x_0,\tau)z(\tau)\,dh(\tau),\ \ r\in[0,T]$$

have only trivial solutions.

Remark. It is worth noting that in the proofs of theorems 3.1 and 3.2, reformulating the given problem to GODEs proved useful.

Example 3.1. Consider the periodic impulse problem $x' = \lambda\, b(t)x + c(t)x^2$, $\Delta^+ x(\frac{1}{2}) = x^2(\frac{1}{2})$, $x(0) = x(1)$ with $b, c \in L^1[0,1]$ and $\int\limits_0^1 b\, ds \neq 0$. One can verify that, by Theorem 3.1, the couple $(0,0)$ is its bifurcation point, while by Theorem 3.2 the couple $(\lambda, 0)$ can not be a bifurcation point whenever $\lambda \neq 0$.

Example 3.2. One can verify that $u_0(t) = (2+\cos t)^3$ solves for all $\lambda \in \mathbb{R}$ the impulsive problem related to the Liebau valveless pumping phenomena

$$u'' = \lambda\big((2+\cos t)u' + 3(\sin t)u\big) + (6.6 - 5.7\cos t - 9\cos^2 t)u^{1/3} - 0.3u^{2/3},$$
$$\Delta^+ u'\Big(\frac{\pi}{2}\Big) = \Big(64 - u^2\Big(\frac{\pi}{2}\Big)\Big), \quad u(0) = u(2\pi), \ \ u'(0) = u'(2\pi).$$

By Theorem 3.2 and using the result by A. Lomtatidze (cf. [6, Theorem 11.1 and Remark 0.5]) and with some help of the software system Mathematica we can conclude that the couple $(x_0, 0)$ can not be its bifurcation point.

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