Duality for Stieltjes differential and integral equations

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Abstract

We propose new definitions of adjoint equations to first-order linear Stieltjes differential and integral equations. We investigate the existence and uniqueness of their solutions, provide explicit solution formulas, and obtain corresponding versions of Lagrange's identity. We show that our results are compatible with known results for dynamic equations on time scales, which are included as a special case.

Keywords: adjoint equation; Lagrange's identity; Stieltjes differential equation; generalized ordinary differential equation; Volterra–Stieltjes integral equation; dynamic equation

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1 Introduction

In the classical theory of ordinary differential equations, there is a close link between the pair of scalar first-order linear differential equations

$$x'(t) = p(t)x(t), \qquad y'(t) = -p(t)y(t),$$

where p is a continuous function. These equations are said to be adjoint or dual to each other. Their operator forms are

$$Lx(t) = 0, \qquad L^*y(t) = 0,$$

where L and L^* are linear operators given by $Lx(t) = x'(t) - p(t)x(t)$ and $L^*y(t) = y'(t) + p(t)y(t)$, respectively; the operator L^* is called the formal adjoint to the operator L (note that the concept of formal adjointness is much more general, and makes sense for any linear differential operator).

The solutions of both equations can be written in the explicit form

$$x(t) = x(t_0)\exp\left(\int_{t_0}^{t} p(s)\,\mathrm{d}s\right), \qquad y(t) = y(t_0)\exp\left(-\int_{t_0}^{t} p(s)\,\mathrm{d}s\right),$$

from which it is clear that the product $x \cdot y$ is a constant function. The same fact also follows from the Lagrange identity

$$(x \cdot y)' = yLx - xL^*y,$$

which holds for each pair of continuously differentiable functions x and y.

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Similar concepts and results have been established for scalar first-order linear dynamic equations on time scales, which provide a unified treatment of differential and difference equations; see [1, 2]. For Δ-dynamic equations, the pair of adjoint or dual equations has the form

$$x^{\Delta}(t) = p(t)x(t), \qquad y^{\Delta}(t) = -p(t)y(\sigma(t)), \tag{1.1}$$

where σ is the forward jump operator. Similarly, for ∇-dynamic equations, we have the pair of adjoint equations

$$x^{\nabla}(t) = p(t)x(t), \qquad y^{\nabla}(t) = -p(t)y(\rho(t)), \tag{1.2}$$

where ρ is the backward jump operator. The corresponding versions of Lagrange's identity for Δ- and ∇-dynamic equations can be found in [2, Theorem 2.68] and [1, Theorem 3.33].

The goal of the present paper is to investigate duality for more general classes of equations that involve Stieltjes derivatives and Stieltjes integrals, and whose solutions need not be continuous in the classical sense. The paper is organized as follows.

In Section 2, we begin by recalling the notion of the Stieltjes derivative of a function x with respect to a nondecreasing and left-continuous function $g\colon \mathbb{R} \to \mathbb{R}$, denoted by x'_g, as well as some of its basic properties. In Section 3, we focus on the linear first-order Stieltjes differential equation

$$x'_g(t) = p(t)x(t), \tag{1.3}$$

and introduce its adjoint equation,

$$y'_g(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)} y(t), \tag{1.4}$$

where $\Delta^+ g(t) = g(t+) - g(t)$ and $g(t+)$ denotes the right-hand side limit of g at t. We will show that solutions of both equations can be expressed in terms of the exponential function. The form of the adjoint equation is motivated by the following facts:

1. If x, y are solutions of Eq. (1.3) and Eq. (1.4), respectively, then the product $x \cdot y$ is a constant function.

2. The corresponding linear operators satisfy an analogue of Lagrange's identity.

3. It is known that Δ-dynamic equations represent a special case of Stieltjes differential equations with a suitable choice of g; in this case, Eq. (1.3) and Eq. (1.4) reduce to the dynamic equations given in (1.1). This fact will be established in Section 4.

We will see that Eq. (1.4) can be written in the equivalent form

$$y'_g(t) = -p(t)y(t+), \tag{1.5}$$

which is quite close to the second equation in Eq. (1.1). We will also consider the nonhomogeneous versions of Eq. (1.3) and Eq. (1.4), which generalize the nonhomogeneous Δ-dynamic equations from [2, Section 2.4].

In Section 5, we summarize some basic results on Stieltjes integral equations that are needed in the rest of the paper. In Section 6, we consider a function $P\colon [a, b] \to \mathbb{C}$ with bounded variation, and focus on the following pair of Stieltjes integral equations:

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s-)\,\mathrm{d}P(s), \tag{1.6}$$

$$y(t) = y(t_0) - \int_{t_0}^{t} y(s+)\,\mathrm{d}P(s). \tag{1.7}$$

A precise definition of solutions to these equations is given in Definition 6.1, where it is emphasized that the left/right limits in the integrands have to be replaced by the corresponding function value whenever s coincides with the left/right endpoint of the integration domain. We explain why it makes sense to refer to Eq. (1.6) and Eq. (1.7) as adjoint equations. The reasons are similar to the case of Stieltjes differential equations: The product of solutions is constant, and we have an analogue of Lagrange's identity.

We show that, in the special case when P is left-continuous or right-continuous, the solutions of Eq. (1.6) and Eq. (1.7) can be expressed in terms of the generalized exponential function introduced in [13]. In the general case, we no longer have explicit solution formulas, but we show that Eq. (1.6) and Eq. (1.7) can be rewritten as Volterra–Stieltjes integral equations with suitable kernels, and provide sufficient conditions for the existence and uniqueness of their solutions.

In Section 7, we show that the Stieltjes differential equations (1.3) and (1.4) represent special cases of Eq. (1.6) and Eq. (1.7) with a suitable left-continuous function P. Therefore, our theory for Stieltjes integral equations is compatible with the theory for Stieltjes differential equations developed in earlier sections.

Finally, in Section 8, we explain that the Δ-dynamic equations in (1.1) correspond to a special case of Eq. (1.6) and Eq. (1.7) with a suitable left-continuous function P, while the ∇-dynamic equations in (1.2) correspond to a special case of Eq. (1.6) and Eq. (1.7) with a suitable right-continuous function P. As in the case of Stieltjes differential equations, we do not restrict ourselves to homogeneous equations, and consider also the nonhomogeneous case.

Throughout the paper, we consider equations with complex-valued coefficients and solutions, but all results can be restricted to the real domain. The motivation for dealing with the complex-valued case stems from the fact that the complex g-exponential function, defined as the solution of Eq. (1.3), leads to the g-sine and g-cosine functions (see [4, Section 4]). Similarly, the complex generalized exponential function, defined as the solution of a certain Stieltjes integral equation, leads to the generalized trigonometric functions (see [13, Section 4] or [14, Section 8.5]). In other words, complex-valued solutions are useful even if we are primarily interested in solving equations in the real domain.

Certain parts of the present paper can be read independently. Readers interested only in Stieltjes differential equations can focus on Sections 2, 3, and 4. Although we recall all necessary concepts, some familiarity with Stieltjes derivatives and Stieltjes differential equations will be helpful.

Readers interested only in Stieltjes integral equations can jump directly to Section 5. Some basic familiarity with the Kurzweil–Stieltjes integral and generalized ordinary differential equations will be helpful, but we have summarized all necessary results in Appendix B and in the beginning of Section 5. For more information, see e.g. [14, 19].

A basic knowledge of the time scale calculus and dynamic equations will be helpful in Sections 4 and 8, but we try to make the paper self-contained by providing all the basic definitions in Appendix A. More information is available in [2].

Let us provide some additional references related to the topics of this paper. Stieltjes differential equations were studied in a number of recent papers, see e.g. [4, 5, 7, 8, 10, 11, 12], as well as some of the references therein. It is known that they generalize classical ordinary differential equations, impulsive differential equations, and Δ-dynamic equations on time scales.

Linear Stieltjes integral equations that we consider in the present paper are closely related to generalized ordinary differential equations studied e.g. in [14, 19, 21]. In fact, if P is left-continuous or right-continuous, then Eq. (1.6) and Eq. (1.7) reduce to linear generalized differential equations (see Theorem 6.7). In the general case, they are equivalent to certain Volterra–Stieltjes integral equations (see Theorem 6.8), which were studied e.g. in [21]. Note that generalized ordinary differential equations are, in fact, integral equations, and it is known that they generalize other types of equations, including classical ordinary differential equations, impulsive differential equations, Δ- and ∇-dynamic equations, as well as Stieltjes differential equations.

This is, to the best of our knowledge, the first paper dealing with duality for Stieltjes differential equations. On the other hand, adjoint equations to generalized ordinary differential equations were already studied in the context of boundary value problems, see [20], [21, Chapter 3], [23]. The definitions of adjoint equations provided there are different from our definitions, because they serve a different purpose. The form of adjoint equations that we consider here is motivated by the fact that the corresponding operators satisfy an analogue of Lagrange's identity. Moreover, we do not impose any requirements on the jumps or left/right-continuity of the right-hand sides, and our form of adjoint equations is compatible with the definitions used in the context of dynamic equations on time scales.

2 Preliminaries on Stieltjes derivatives

Let $g\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function. We shall denote by μ_g the Lebesgue–Stieltjes measure associated to g given by

$$\mu_g([c,d)) = g(d) - g(c), \quad c,d \in \mathbb{R},\ c < d,$$

see [3, 17, 18]. We will use the term "g-measurable" for a set or function to refer to μ_g-measurability in the corresponding sense, and we denote by $\mathcal{L}^1_g(X,\mathbb{C})$ the set of Lebesgue–Stieltjes μ_g-integrable functions on a g-measurable set X with values in $\mathbb{C}$, whose integral we denote by

$$\int_X f(s)\,\mathrm{d}\mu_g(s), \quad f \in \mathcal{L}^1_g(X,\mathbb{C}).$$

Similarly, we will talk about properties holding g-almost everywhere in a set X (shortened to g-a.e. in X), or holding for g-almost all (or simply, g–a.a.) $x \in X$, as a simplified way to express that they hold μ_g-almost everywhere in X or for μ_g-almost all $x \in X$, respectively.

Define

$$\begin{aligned} C_g &= \{t \in \mathbb{R} : g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\},\\ D_g &= \{t \in \mathbb{R} : \Delta^+ g(t) > 0\}. \end{aligned}$$

Observe that, as pointed out in [7], the set C_g has null g-measure and it is open in the usual topology, so it can be uniquely expressed as the countable union of open disjoint intervals, say

$$C_g = \bigcup_{n\in\mathbb{N}} (a_n, b_n),$$

for some $a_n, b_n \in [-\infty, +\infty]$, $n \in \mathbb{N}$. With this notation, following [9] we define $N_g = N_g^- \cup N_g^+$, with

$$\begin{aligned} N_g^- &= \{a_n \in \mathbb{R} : n \in \mathbb{N}\} \setminus D_g = \{a_n \in \mathbb{R} : \Delta^+ g(a_n) = 0\},\\ N_g^+ &= \{b_n \in \mathbb{R} : n \in \mathbb{N}\} \setminus D_g = \{b_n \in \mathbb{R} : \Delta^+ g(b_n) = 0\}. \end{aligned}$$

We are now in the position to introduce the definition of the Stieltjes derivative of a complex-valued function. This can be regarded as a generalization of the definition in [7, 10] or a particular case of the one in [4], although further assumptions are required in that case in order to consider the derivatives at the points of C_g, which we do not do as it is irrelevant in the work ahead given that $\mu_g(C_g) = 0$.

Definition 2.1. Let $f\colon \mathbb{R} \to \mathbb{C}$ and $t \in \mathbb{R} \setminus C_g$. We define the *Stieltjes derivative*, or *g-derivative*, of f at t as follows, provided the corresponding limit exists:

$$f'_g(t) = \begin{cases} \lim\limits_{s\to t} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_g \cup N_g,\\ \lim\limits_{s\to t^-} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \in N_g^-,\\ \lim\limits_{s\to t^+} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \in D_g \cup N_g^+, \end{cases}$$

In that case, we say that f is *g-differentiable at t*.

Remark 2.2. It is important to note that, as explained in [10, Remark 2.2], for $t \in N_g$ we have

$$f_g'(t) = \lim_{s \to t} \frac{f(s) - f(t)}{g(s) - g(t)},$$

as the domain of the quotient function gives the corresponding one-sided limit. Furthermore, since g is a regulated function, it follows that the g-derivative of f at a point $t \in D_g$ exists if and only if $f(t+)$ exists and, in that case,

$$f_g'(t) = \frac{\Delta^+ f(t)}{\Delta^+ g(t)}.$$

The following result contains some basic properties of Stieltjes derivatives such as linearity and the product rule. This result can be directly deduced from [10, Proposition 2.5], where the same properties are presented for real-valued functions.

Proposition 2.3. *Let $f_1, f_2 \colon [a, b] \to \mathbb{C}$ be two g-differentiable functions at $t \in \mathbb{R} \setminus C_g$. Then:*

- *The function $\lambda_1 f_1 + \lambda_2 f_2$ is g-differentiable at t for any $\lambda_1, \lambda_2 \in \mathbb{C}$ and*

$$\left(\lambda_1 f_1 + \lambda_2 f_2\right)_g' (t) = \lambda_1 \left(f_1\right)_g' (t) + \lambda_2 \left(f_2\right)_g' (t).$$

- *The product $f_1 f_2$ is g-differentiable at t and*

$$\left(f_1 f_2\right)_g' (t) = \left(f_1\right)_g' (t) f_2(t) + \left(f_2\right)_g' (t) f_1(t) + \left(f_1\right)_g' (t) \left(f_2\right)_g' (t) \Delta^+ g(t). \tag{2.1}$$

- *If $f_2(t)\left(f_2(t) + (f_2)_g'(t)\, \Delta^+ g(t)\right) \neq 0$, the quotient f_1/f_2 is g-differentiable at t and*

$$\left(\frac{f_1}{f_2}\right)_g' (t) = \frac{\left(f_1\right)_g' (t) f_2(t) - \left(f_2\right)_g' (t) f_1(t)}{f_2(t)\left(f_2(t) + (f_2)_g'(t)\, \Delta^+ g(t)\right)}.$$

Another fundamental concept in this context is that of continuity with respect to the map g, which we present in the following definition that can be found in [5].

Definition 2.4. A function $f \colon [a, b] \to \mathbb{C}$ is *g-continuous* at a point $t \in [a, b]$, or *continuous with respect to g* at t, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(s)| < \varepsilon, \quad \text{for all } s \in [a, b],\ |g(t) - g(s)| < \delta.$$

If f is g-continuous at every point $t \in A \subset [a, b]$, we say that f is g-continuous on A.

The following result describes some properties of g-continuous functions. It can be deduced directly from [5, Proposition 3.2], in which the same information is presented for real-valued functions.

Proposition 2.5. *If a function $f \colon [a, b] \to \mathbb{C}$ is g-continuous on $[a, b]$, then:*

- *f is continuous from the left at every $t \in (a, b]$;*
- *if g is continuous at $t \in [a, b)$, then so is f;*
- *if g is constant on some $[\alpha, \beta] \subset [a, b]$, then so is f.*

In particular, g-continuous functions on $[a, b]$ are continuous on $[a, b]$ when g is continuous on $[a, b)$.

As a consequence of Proposition 2.5 we can obtain the following result.

Proposition 2.6. *Let $u\colon [a,b] \to \mathbb{C}$ be a g-continuous function on $[a,b]$. Then, for each $t \in [a,b)$ such that $u'_g(t)$ exists, we have*

$$\Delta^+ u(t) = u'_g(t)\Delta^+ g(t). \tag{2.2}$$

Proof. Let $t \in [a,b)$ be such that $u'_g(t)$ exists. If $t \in [a,b) \setminus D_g$, then the equality becomes trivial as $\Delta^+ g(t) = 0$ and Proposition 2.5 ensures that u is continuous at t. Otherwise, $t \in [a,b) \cap D_g$ and, in that case, the equality follows from Remark 2.2. $\square$

Similarly to Proposition 2.5, the following result can be obtained from [5, Corollary 3.5] by separating complex-valued functions into real and imaginary parts.

Proposition 2.7. *If $f\colon [a,b] \to \mathbb{C}$ is a g-continuous function on $[a,b]$, then it is Borel-measurable and, in particular, g-measurable.*

Lastly, we present the concept of g-absolute continuity introduced in [7], as well as some of its properties. For simplicity, we introduce such concept as part of the following result, [7, Proposition 5.4]. Observe that the original result is stated for real-valued functions, but it can easily be adapted to complex-valued functions.

Theorem 2.8. *Let $F\colon [a,b] \to \mathbb{C}$. The following conditions are equivalent:*

1. *The function F is g–*absolutely continuous *on $[a,b]$ according to the following definition: for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every open pairwise disjoint family of subintervals $\{(a_n,b_n)\}_{n=1}^m$ satisfying*

$$\sum_{n=1}^m (g(b_n) - g(a_n)) < \delta,$$

we have

$$\sum_{n=1}^m |F(b_n) - F(a_n)| < \varepsilon.$$

2. *The function F satisfies the following conditions:*
   (i) *there exists $F'_g(t)$ for g-a.a. $t \in [a,b)$;*
   (ii) *$F'_g \in \mathcal{L}^1_g([a,b),\mathbb{C})$;*
   (iii) *for each $t \in [a,b]$,*

$$F(t) = F(a) + \int_{[a,t)} F'_g(s)\,\mathrm{d}\mu_g(s). \tag{2.3}$$

Remark 2.9. Observe that the equality in Eq. (2.3) is, indeed, true for $t = a$ as we are considering the integral over the empty set, which makes the integral null.

We denote by $\mathcal{AC}_g([a,b],\mathbb{C})$ the set of g-absolutely continuous functions in $[a,b]$ with values on $\mathbb{C}$. It is important to note that, as in the usual case, every g-absolutely continuous function is also g-continuous, see [5, Proposition 5.5].

Finally, we present a result that, in a way, is the converse of Theorem 2.8. This result follows from [7, Proposition 2.4] by separating the real and imaginary parts of the complex-valued function.

Theorem 2.10. *Let $f \in \mathcal{L}^1_g([a,b),\mathbb{C})$. Then, the function $F\colon [a,b] \to \mathbb{C}$ defined as*

$$F(t) = \int_{[a,t)} f(s)\,\mathrm{d}g(s),$$

is an element of $\mathcal{AC}_g([a,b],\mathbb{C})$ and $F'_g(t) = f(t)$ for g-a.a. $t \in [a,b)$.

3 Duality for equations with Stieltjes derivatives

As in the previous section, we assume that $g\colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing and left-continuous function and $a, b \in \mathbb{R}$, $a < b$. We now focus on the relations between the pair of homogeneous linear equations with Stieltjes derivatives

$$x'_g(t) = p(t)x(t), \tag{3.1}$$

$$y'_g(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)} y(t), \tag{3.2}$$

where we assume that $p \in \mathcal{L}^1_g([a,b], \mathbb{C})$. Naturally, for Eq. (3.2) to make sense we need to assume that

$$1 + p(t)\Delta^+ g(t) \neq 0 \text{ for all } t \in [a,b] \cap D_g. \tag{3.3}$$

This condition is not new, as it is required in [4, 5] to obtain the solution of Eq. (3.1). Furthermore, observe that under this condition, Eq. (3.2) is equivalent to

$$y'_g(t)(1 + p(t)\Delta^+ g(t)) = -p(t)y(t). \tag{3.4}$$

In order to properly study Eq. (3.1) and Eq. (3.2), let us introduce the concept of solution for each of the two problems.

Definition 3.1. A solution of Eq. (3.1) in the interval $[a,b]$ is a function $x \in \mathcal{AC}_g([a,b], \mathbb{C})$ such that

$$x'_g(t) = p(t)x(t) \quad \text{ for } g\text{-a.a. } t \in [a,b).$$

Similarly, a solution of Eq. (3.2) in the interval $[a,b]$ is a function $y \in \mathcal{AC}_g([a,b], \mathbb{C})$ such that

$$y'_g(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)} y(t) \quad \text{ for } g\text{-a.a. } t \in [a,b).$$

Remark 3.2. Observe that we are not including the endpoint b in the equalities that the solutions must satisfy as either $b \notin D_g$, and thus $\mu_g(\{b\}) = 0$, or $b \in D_g$ and the corresponding derivatives cannot be defined as the functions are not defined beyond b. For $t = a$, the derivative is not defined when $a \in N_g^-$ and, in that case, $\mu_g(\{a\}) = 0$. Nevertheless, we cannot exclude the point a from these equalities as we might have that $a \in D_g$, in which case $\mu_g(\{a\}) > 0$.

Remark 3.3. Since every solution of Eq. (3.2) is g-continuous, we can use Proposition 2.6 to observe that Eq. (3.2) is equivalent to the equation

$$y'_g(t) = -p(t)y(t+). \tag{3.5}$$

Nevertheless, we will continue to use Eq. (3.2) as it is more convenient to obtain Lagrange's identity.

Existence and uniqueness of solutions to Eq. (3.1) and Eq. (3.2) will be proved in Proposition 3.8. However, let us first explain why it is reasonable to call Eq. (3.2) to be adjoint to Eq. (3.1), and vice versa. We need the following auxiliary result.

Lemma 3.4. *Let $p \in \mathcal{L}^1_g([a,b], \mathbb{C})$. Then, the following statements hold:*

(i) *The sum $\sum_{t \in [a,b] \cap D_g} |p(t)|\Delta^+ g(t)$ is finite.*

(ii) *The map $t \mapsto p(t)\Delta^+ g(t)$ is bounded on $[a,b]$.*

(iii) *If p satisfies condition* (3.3) *then, for any $q \in \mathcal{L}^1_g([a,b], \mathbb{C})$,*

$$Q(t) = \frac{q(t)}{1 + p(t)\Delta^+ g(t)} \in \mathcal{L}^1_g([a,b], \mathbb{C}).$$

Proof. Let $p \in \mathcal{L}^1_g([a,b],\mathbb{C})$. Observe that (i) follows directly from the g-integrability of p as

$$\sum_{t\in[a,b]\cap D_g} |p(t)|\Delta^+ g(t) = \sum_{t\in[a,b]\cap D_g} \int_{\{t\}} |p(s)|\,\mathrm{d}\mu_g(s) = \int_{[a,b]\cap D_g} |p(s)|\,\mathrm{d}\mu_g(s) \le \int_{[a,b]} |p(s)|\,\mathrm{d}\mu_g(s) < +\infty.$$

Note that (ii) follows from (i).

Now, for (iii), observe that $Q = q$ on $[a,b]\setminus D_g$ so, given that D_g is at most countable, it follows that Q is g-measurable. In order to show that Q is integrable, we decompose

$$\begin{aligned}\int_{[a,b]} |Q(t)|\,\mathrm{d}\mu_g(s) &= \int_{[a,b]\setminus D_g} |Q(t)|\,\mathrm{d}\mu_g(s) + \int_{[a,b]\cap D_g} |Q(t)|\,\mathrm{d}\mu_g(s)\\ &= \int_{[a,b]\setminus D_g} |q(t)|\,\mathrm{d}\mu_g(s) + \sum_{t\in[a,b]\cap D_g} |Q(t)|\Delta^+ g(t),\end{aligned}$$

so, since $q \in \mathcal{L}^1_g([a,b],\mathbb{C})$, it suffices to show that the last sum is finite.

Define $A = \{t \in [a,b]\cap D_g : |p(t)|\Delta^+ g(t) > 1/2\}$ and $B = ([a,b]\cap D_g)\setminus A$. Observe that A has finite cardinality as

$$\sum_{t\in A} \frac{1}{2} < \sum_{t\in A} |p(t)|\Delta^+ g(t) \le \sum_{t\in[a,b]\cap D_g} |p(t)|\Delta^+ g(t),$$

which we know to be finite by (i). On the other hand, for $t \in B$, we have

$$|1 + p(t)\Delta^+ g(t)| \ge 1 - |p(t)|\Delta^+ g(t) \ge 1/2,$$

so $|Q(t)| \le 2|q(t)|$ for $t \in B$. Thus,

$$\sum_{t\in[a,b]\cap D_g} |Q(t)|\Delta^+ g(t) = \sum_{t\in A} |Q(t)|\Delta^+ g(t) + \sum_{t\in B} |Q(t)|\Delta^+ g(t) \le \sum_{t\in A} |Q(t)|\Delta^+ g(t) + 2\sum_{t\in B} |q(t)|\Delta^+ g(t).$$

Now, the finiteness of the sum follows from (i) and the fact that A has finite cardinality. □

Let us consider now the linear operators associated with Eq. (3.1) and Eq. (3.4), which we denote by L and L^*, respectively. In that case, we have that $L, L^*\colon \mathcal{AC}_g([a,b],\mathbb{C}) \to \mathcal{L}^1_g([a,b),\mathbb{C})$ are given by

$$Lu(t) = u'_g(t) - p(t)u(t) \quad \text{for } g\text{-a.a. } t \in [a,b), \tag{3.6}$$

$$L^* v(t) = v'_g(t)(1 + p(t)\Delta^+ g(t)) + p(t)v(t) \quad \text{for } g\text{-a.a. } t \in [a,b). \tag{3.7}$$

Observe that L and L^* do, indeed, map $\mathcal{AC}_g([a,b],\mathbb{C})$ to $\mathcal{L}^1_g([a,b],\mathbb{C})$. For L, this is clear from Proposition 2.7 and Theorem 2.8. A similar reasoning works for L^* noting that the map $1 + p(t)\Delta^+ g(t^*)$ is bounded, see Lemma 3.4.

We are now in the position to establish a version of Lagrange's identity for differential equations with Stieltjes derivatives which, to some extent, justifies calling Eq. (3.1) and Eq. (3.2) adjoint equations.

Theorem 3.5. *For all functions $x, y \in \mathcal{AC}_g([a,b],\mathbb{C})$, we have*

$$(x\cdot y)'_g(t) = \big(y(t) + y'_g(t)\Delta^+ g(t)\big)\, Lx(t) + x(t)L^* y(t) \quad \textit{for } g\textit{-a.a. } t \in [a,b). \tag{3.8}$$

In particular, if $x, y \in \mathcal{AC}_g([a,b],\mathbb{C})$ are functions satisfying $Lx = 0$ and $L^* y = 0$ g-a.e. in $[a,b)$, then

$$x(t)y(t) = x(a)y(a) \quad \textit{for all } t \in [a,b].$$

Proof. Since $x, y \in \mathcal{AC}_g([a,b],\mathbb{C})$, there exists $E \subset [a,b)$ such that $\mu_g(E) = 0$ and $x'_g(t), y'_g(t)$ exist for all $t \in [a,b) \setminus E$. Using Proposition 2.3 and Eq. (2.1), we obtain

$$\begin{aligned}\left(y(t) + y'_g(t)\Delta^+ g(t)\right) Lx(t) + x(t)L^*y(t) &= \left(y(t) + y'_g(t)\Delta^+ g(t)\right)\left(x'_g(t) - p(t)x(t)\right)\\ &\quad + x(t)\left(y'_g(t)(1 + p(t)\Delta^+ g(t)) + p(t)y(t)\right)\\ &= y(t)x'_g(t) - p(t)y(t)x(t) + y'_g(t)\Delta^+ g(t)\left(x'_g(t) - p(t)x(t)\right)\\ &\quad + x(t)y'_g(t) + p(t)x(t)y'_g(t)\Delta^+ g(t) + p(t)x(t)y(t)\\ &= x'_g(t)y(t) + y'_g(t)x(t) + x'_g(t)y'_g(t)\Delta^+ g(t)\\ &= (x \cdot y)'_g(t)\end{aligned}$$

for all $t \in [a,b) \setminus E$. If $Lx = 0$ and $L^*y = 0$ g-a.e. in $[a,b)$, it follows from the previous calculation that $(x \cdot y)'_g(t) = 0$ for g-a.a. $t \in [a,b)$, and therefore, since $x \cdot y \in \mathcal{AC}_g([a,b],\mathbb{C})$, the function $x \cdot y$ is constant on $[a,b]$. □

A closely related result is the following one.

Corollary 3.6. *Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $x, y \in \mathcal{AC}_g([a,b],\mathbb{C})$ be functions such that $x(t)y(t) = \alpha$ for all $t \in [a,b]$. If $Lx = 0$ g-a.e. in $[a,b)$, then $L^*y = 0$ g-a.e. in $[a,b)$. Conversely, if $L^*y = 0$ g-a.e. in $[a,b)$ and condition* (3.3) *is satisfied, then $Lx = 0$ g-a.e. in $[a,b)$.*

Proof. First, note that the hypotheses guarantee that $x(t), y(t) \neq 0$, $t \in [a,b]$. Furthermore, Eq. (3.8) holds, so we can find $E \subset [a,b)$ with $\mu_g(E) = 0$ such that for all $t \in [a,b) \setminus E$,

$$(x \cdot y)'_g(t) = \left(y(t) + y'_g(t)\Delta^+ g(t)\right) Lx(t) + x(t)L^*y(t).$$

First, assume that $Lx = 0$ g-a.e. in $[a,b)$. In that case, there is $E_x \subset [a,b)$ such that $\mu_g(E_x) = 0$ and $Lx(t) = 0$ for all $t \in [a,b) \setminus E_x$. Consider $N = E \cup E_x$. Then $\mu_g(N) = 0$ and for all $t \in [a,b) \setminus N$,

$$(x \cdot y)'_g(t) = \left(y(t) + y'_g(t)\Delta^+ g(t)\right) Lx(t) + x(t)L^*y(t) = x(t)L^*y(t).$$

Hence, since $x(t)y(t) = \alpha$, $t \in [a,b]$, we have that $(x \cdot y)'_g(t) = 0$ for all $t \in [a,b) \setminus N$ so, given that $x(t) \neq 0$ for $t \in [a,b) \setminus N$, we necessarily have that $L^*y(t) = 0$ for all $t \in [a,b) \setminus N$.

Conversely, suppose that $L^*y = 0$ g-a.e. in $[a,b)$ and condition (3.3) is satisfied. In that case, there is $E_y \subset [a,b)$ such that $\mu_g(E_y) = 0$ and $Ly(t) = 0$ for all $t \in [a,b) \setminus E_x$. In particular, we have that

$$y'_g(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)} y(t), \quad t \in [a,b) \setminus E_y,$$

which guarantees that

$$y(t) + y'_g(t)\Delta^+ g(t) = y(t) - \frac{p(t)}{1 + p(t)\Delta^+ g(t)} y(t)\Delta^+ g(t) = \frac{y(t)}{1 + p(t)\Delta^+ g(t)}, \quad t \in [a,b) \setminus E_y.$$

Now, define $M = E \cup E_y$. Then $\mu_g(M) = 0$ and for all $t \in [a,b) \setminus M$,

$$(x \cdot y)'_g(t) = \left(y(t) + y'_g(t)\Delta^+ g(t)\right) Lx(t) + x(t)L^*y(t) = \left(y(t) + y'_g(t)\Delta^+ g(t)\right) Lx(t) = \frac{y(t)}{1 + p(t)\Delta^+ g(t)} Lx(t).$$

Again, since $x(t)y(t) = \alpha$, $t \in [a,b]$, we have that $(x \cdot y)'_g(t) = 0$ for all $t \in [a,b) \setminus M$ so, since $y(t) \neq 0$ for $t \in [a,b) \setminus M$ and condition (3.3) holds, we must have that $Lx(t) = 0$, $t \in [a,b) \setminus M$. □

Interestingly, the solutions of homogeneous linear equations with Stieltjes derivatives have already been completely determined in [5, 10] on the real line and, in [4], in the complex plane under the assumption that $a \notin D_g \cup N_g^-$ and $b \notin D_g \cup C_g \cup N_g^+$. Here, we shall show that the solution for the complex case is the same even when we do not impose such conditions. To that end, we present the following result that can easily be derived from [11, Proposition 4.28].

Proposition 3.7. *Let $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, $x_0 \in \mathbb{C}$, $f\colon [t_0, t_1] \times \mathbb{C} \to \mathbb{C}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function such that $t_0 \in D_g$. Define $\widetilde{g}\colon \mathbb{R} \to \mathbb{R}$ as*

$$\widetilde{g}(t) = \begin{cases} g(t_0+) & \text{if } t \leq t_0, \\ g(t) & \text{if } t > t_0. \end{cases} \tag{3.9}$$

If $y\colon [t_0, t_1] \to \mathbb{C}$ is such that $y(t_0) = x_0 + f(t_0, x_0)\Delta^+ g(t_0)$ and

$$y'_{\widetilde{g}}(t) = f(t, y(t)) \quad \textit{for } \widetilde{g}\textit{-a.a. } t \in [t_0, t_1),$$

then the map $x\colon [t_0, t_1] \to \mathbb{C}$ defined as

$$x(t) = \begin{cases} x_0 & \text{if } t = t_0, \\ y(t) & \text{if } t \in (t_0, t_1], \end{cases}$$

is such that

$$x'_g(t) = f(t, x(t)) \quad \textit{for } g\textit{-a.a. } t \in [t_0, t_1).$$

In order to give the explicit expression of the solutions of Eq. (3.1) and Eq. (3.2) we need to introduce some functions.

Given $\beta \in \mathcal{L}^1_g([a,b], \mathbb{C})$, we define $\widehat{\beta}\colon [a,b] \to \mathbb{C}$ as

$$\widehat{\beta}(t) = \begin{cases} \beta(t), & t \in [a,b] \setminus D_g, \\ \dfrac{\ln(1 + \beta(t)\Delta^+ g(t))}{\Delta^+ g(t)}, & t \in [a,b] \cap D_g, \end{cases} \tag{3.10}$$

where ln denotes the principal branch of the complex logarithm; and $\exp_g(\beta, \cdot)\colon [a,b] \to \mathbb{C}$ as

$$\exp_g(\beta, t) = \exp\left(\int_{[a,t)} \widehat{\beta}(s)\, \mathrm{d}\mu_g(s) \right), \quad t \in [a,b]. \tag{3.11}$$

We are now in the position to present the solutions of the adjoint homogeneous linear equations with Stieltjes derivatives.

Proposition 3.8. *Let $p \in \mathcal{L}^1_g([a,b], \mathbb{C})$ be such that condition (3.3) holds and $x_a, y_a \in \mathbb{C}$. Then:*

1. *The unique solution of Eq. (3.1) satisfying $x(a) = x_a$ is given by*

$$x(t) = x_a \exp_g(p, t), \quad t \in [a,b]. \tag{3.12}$$

2. *The unique solution of Eq. (3.2) satisfying $y(a) = y_a$ is given by*

$$y(t) = y_a \exp_g(p, t)^{-1}, \quad t \in [a,b]. \tag{3.13}$$

Proof. First, observe that the hypotheses and Lemma 3.4 ensure that

$$\frac{-p(t)}{1 + p(t)\Delta^+ g(t)} \in \mathcal{L}^1_g([a,b], \mathbb{C}).$$

Furthermore, for $t \in [a,b) \cap D_g$, we have that

$$1 - \frac{p(t)}{1 + p(t)\Delta^+ g(t)} \Delta^+ g(t) = \frac{1}{1 + p(t)\Delta^+ g(t)} \neq 0.$$

Hence, it is enough to prove the first statement as Eq. (3.13) follows from the equality $\exp_g\left(\frac{-p}{1+p\Delta^+ g}, \cdot\right) = \exp_g(p, \cdot)^{-1}$, see [4, Proposition 4.6].

Now, Theorem 4.2 and Remark 4.3 in [4] show that the first statement is true when $a \notin D_g \cup N_g^-$ and $b \notin D_g \cup C_g \cup N_g^+$, so it remains to prove the remaining cases. Nevertheless, since our definition of solution excludes the point b, we only need to prove the result for $a \in D_g \cup N_g^-$. Observe, however, that if $a \in N_g^-$, then $\mu_g(\{a\}) = 0$ so the result still holds as solutions must only satisfy the equation g-a.e. in $[a, b)$. This means that we only need to study the case $a \in D_g$.

Let us assume that g is such that $a \in D_g$ and let $\widetilde{g}$ be as in Eq. (3.9) with $t_0 = a$. Theorem 4.2 and Remark 4.3 in [4] guarantee that the map

$$\widetilde{x}(t) = x_a(1 + p(t)\Delta^+ g(t)) \exp_{\widetilde{g}}(p, t), \quad t \in [a, b],$$

is the unique solution of $\widetilde{x}'_{\widetilde{g}}(t) = p(t)\widetilde{x}(t)$ satisfying $\widetilde{x}(a) = x_a(1 + p(t)\Delta^+ g(t))$. Thus, Proposition 3.7 guarantees that the solution of Eq. (3.1) is the map $x\colon [a, b] \to \mathbb{C}$ given by

$$x(t) = \begin{cases} x_a, & t = a, \\ x_a(1 + p(t)\Delta^+ g(t)) \exp_{\widetilde{g}}(p, t), & t \in (a, b]. \end{cases}$$

Let us show that $x(t) = x_a \exp_g(p, t)$ for all $t \in [a, b]$. The equality is obvious for $t = a$, so it is enough to show that $\exp_g(p, t) = (1 + p(t)\Delta^+ g(t)) \exp_{\widetilde{g}}(p, t)$ for all $t \in (a, b]$ or, equivalently,

$$\frac{\exp_g(p, t)}{1 + p(t)\Delta^+ g(t)} = \exp_{\widetilde{g}}(p, t), \quad t \in (a, b]. \tag{3.14}$$

Let us denote by $\widehat{p}$ the corresponding modification of p in (3.10) for g; and by $\widetilde{p}$, the one associated to $\widetilde{g}$. Observe that, for $t \in (a, b]$,

$$\begin{aligned}\exp_g(p, t) &= \exp\left(\int_{[a,t)} \widehat{p}(s)\, \mathrm{d}\mu_g(s)\right) = \exp\left(\int_{\{a\}} \widehat{p}(s)\, \mathrm{d}\mu_g(s) + \int_{(a,t)} \widehat{p}(s)\, \mathrm{d}\mu_g(s)\right) \\ &= \exp\left(\int_{\{a\}} \widehat{p}(s)\, \mathrm{d}\mu_g(s)\right) \exp\left(\int_{(a,t)} \widehat{p}(s)\, \mathrm{d}\mu_g(s)\right) = (1 + p(t)\Delta^+ g(t)) \exp\left(\int_{(a,t)} \widehat{p}(s)\, \mathrm{d}\mu_g(s)\right).\end{aligned}$$

Hence, noting that $g = \widetilde{g}$ and $\widehat{p} = \widetilde{p}$ on $(a, b]$, we have that for $t \in (a, b]$,

$$\frac{\exp_g(p, t)}{1 + p(t)\Delta^+ g(t)} = \exp\left(\int_{(a,t)} \widehat{p}(s)\, \mathrm{d}\mu_g(s)\right) = \exp\left(\int_{(a,t)} \widetilde{p}(s)\, \mathrm{d}\mu_{\widetilde{g}}(s)\right),$$

from which Eq. (3.14) follows since $\mu_{\widetilde{g}}(\{a\}) = 0$. □

Remark 3.9. If $x_a \cdot y_a \neq 0$, an alternative way of obtaining Eq. (3.13) is to combine Eq. (3.12) with Corollary 3.6.

Finally, one might be interested in the nonhomogeneous versions of Eq. (3.1) and Eq. (3.2). A possible formulation could be

$$x'_g(t) = p(t)x(t) + f(t), \tag{3.15}$$

$$y'_g(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)} y(t) + \frac{f(t)}{1 + p(t)\Delta^+ g(t)}, \tag{3.16}$$

with $p, f \in \mathcal{L}^1_g([a, b], \mathbb{C})$ such that condition (3.3) holds. The justification of (3.16) comes from the definition of the linear operator L^*. Furthermore, as we shall show in the next section, this formulation allows us to establish connections with other problems.

For completeness, we include the following result that provides the solutions of Eq. (3.15) and Eq. (3.16). A similar result for the solution of Eq. (3.15) is stated in [4, Proposition 4.12] under the assumption that the initial point is a continuity point of the map g. Here, we shall prove the result without such restriction following the ideas in [10, Proposition 3.5].

Proposition 3.10. *Let $p, f \in \mathcal{L}^1_g([a,b],\mathbb{C})$ be such that condition (3.3) holds and $x_a, y_a \in \mathbb{C}$. Then:*

1. *The unique g-absolutely continuous solution of Eq. (3.15) satisfying $x(a) = x_a$ is the function $x\colon [a,b] \to \mathbb{C}$ given by*
$$x(t) = x_a \exp_g(p,t) + \exp_g(p,t) \int_{[a,t)} \frac{f(s)}{1+p(s)\Delta^+ g(s)} \exp_g(p,s)^{-1} \,\mathrm{d}\mu_g(s). \tag{3.17}$$
2. *The unique g-absolutely continuous solution of Eq. (3.16) satisfying $y(a) = y_a$ is the function $y\colon [a,b] \to \mathbb{C}$ given by*
$$y(t) = y_a \exp_g(p,t)^{-1} + \exp_g(p,t)^{-1} \int_{[a,t)} f(s) \exp_g(p,s) d\mu_g(s).$$

Proof. In order to prove the first statement, let us first show that

$$h(t) = \frac{f(t)}{1+p(t)\Delta^+ g(t)} \exp_g(p,t)^{-1} \in \mathcal{L}^1_g([a,b],\mathbb{C}).$$

Indeed, it is clear that $t \mapsto \exp_g(p,t)$ is g-measurable and, for each $t \in [a,b]$,

$$|\exp_g(p,t)| \geq \exp\left(\int_{[a,t)} -|\widehat{p}(s)| \,\mathrm{d}\mu_g(s) \right) \geq \exp\left(\int_{[a,b]} -|\widehat{p}(s)| \,\mathrm{d}\mu_g(s) \right) := m.$$

Hence, since $m > 0$, it follows that h is g-measurable and, using Lemma 3.4, g-integrable on $[a,b]$ as

$$\int_{[a,b]} |h(t)| \,\mathrm{d}\mu_g(t) = \int_{[a,b]} \left| \frac{f(t)}{1+p(t)\Delta^+ g(t)} \right| \left| \frac{1}{\exp_g(p,t)} \right| \mathrm{d}\mu_g(t) \leq \frac{1}{m} \int_{[a,b]} \left| \frac{f(t)}{1+p(t)\Delta^+ g(t)} \right|.$$

This means that the map $H(t) = \int_{[a,t)} h(s) \,\mathrm{d}\mu_g(s)$, $t \in [a,b]$, is g-absolutely continuous, see Theorem 2.10, and, as a consequence, so is the map x in Eq. (3.17).

Since $x \in \mathcal{AC}_g([a,b],\mathbb{C})$, we know that there exists $N \subset [a,b]$ such that $\mu_g(N) = 0$ and $x'_g(t)$ exists for all $t \in [a,b) \setminus N$. Then, Eq. (2.1), Theorem 2.10 and Proposition 3.8 guarantee that, for each $t \in [a,b] \setminus N$,

$$\begin{aligned} x'_g(t) &= x_a p(t) \exp_g(p,t) + p(t) \exp_g(p,t) \int_{[a,t)} \frac{f(s)}{1+p(s)\Delta^+ g(s)} \exp_g(p,s)^{-1} \,\mathrm{d}\mu_g(s) \\ &\quad + \exp_g(p,t) \frac{f(t)}{1+p(t)\Delta^+ g(t)} \exp_g(p,t)^{-1} + p(t) \exp_g(p,t)) \frac{f(t)}{1+p(t)\Delta^+ g(t)} \exp_g(p,t)^{-1} \Delta^+ g(t) \\ &= p(t)x(t) + f(t), \end{aligned}$$

which finishes the proof of the first statement.

Now, for the second statement, note that Lemma 3.4 guarantees that the maps

$$P(t) := \frac{-p(t)}{1+p(t)\Delta^+ g(t)}, \quad F(t) := \frac{f(t)}{1+p(t)\Delta^+ g(t)},$$

are g-integrable on $[a,b]$. In particular, this means that Eq. (3.16) might be understood as a particular case of Eq. (3.15). Hence, noting that

$$\frac{F(t)}{1+P(t)\Delta^+ g(t)} = f(t),$$

and keeping in mind that $\exp_g\left(\frac{-p}{1+p\Delta^+ g}, \cdot\right) = \exp_g(p,\cdot)^{-1}$, see [4, Proposition 4.6], the result follows. $\square$

4 Relations between Stieltjes differential equations and dynamic equations

Readers who are not familiar with the time scale calculus should consult Appendix A before reading the present section.

Given a time scale $\mathbb{T} \subset \mathbb{R}$, the aim of this section is to study possible relations between the pair of adjoint equations (3.1)–(3.2) and the corresponding adjoint Δ-dynamic equations

$$x^{\Delta}(t) = p(t)x(t) + f(t), \qquad t \in [a,b)_{\mathbb{T}}, \tag{4.1}$$

$$y^{\Delta}(t) = -p(t)y(\sigma(t)) + f(t), \quad t \in [a,b)_{\mathbb{T}}, \tag{4.2}$$

where $a, b \in \mathbb{T}$ and $p, f\colon [a,b] \to \mathbb{C}$.

As pointed out in [11, Theorems 3.49 and 3.51], Eq. (3.15) and Eq. (4.1) are, in some sense, equivalent when we consider the nondecreasing and left-continuous map $g\colon \mathbb{R} \to \mathbb{R}$ defined as

$$g(t) = \begin{cases} a, & t \le a, \\ \inf\{s \in \mathbb{T} : s \ge t\}, & a < t \le b, \\ b, & t > b. \end{cases} \tag{4.3}$$

The mentioned equivalence should be understood as follows: given a solution of one of the problems, we can construct a solution of the other one. We shall show that Eq. (3.16) and Eq. (4.2) also share this property, and they do so under the same circumstances as Eq. (3.15) and Eq. (4.1).

Proposition 4.1. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be as in (4.3) and $y\colon \mathbb{T} \to \mathbb{C}$ be such that*

$$y^{\Delta}(t) = -p(t)y(\sigma(t)) + f(t), \quad t \in [a,b)_{\mathbb{T}}.$$

Then, if $1 + p(t)(\sigma(t) - t) \neq 0$, $t \in [a,b)_{\mathbb{T}}$, the map $\widetilde{y}(t) = y(g(t))$, $t \in [a,b)$, is such that

$$\widetilde{y}\,'_g(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)}\widetilde{y}(t) + \frac{f(t)}{1 + p(t)\Delta^+ g(t)} \quad \text{for } g\text{-a.a. } t \in [a,b).$$

Proof. First, observe $g(\mathbb{R}) = [a,b]_{\mathbb{T}}$ and $g(t) = t$ for all $t \in [a,b]_{\mathbb{T}}$, which ensure that $\widetilde{y}$ is well-defined. Furthermore, we also have $\Delta^+ g(t) = \sigma(t) - t$ for all $t \in [a,b)_{\mathbb{T}}$ and $C_g \cap (a,b) = (a,b) \setminus \mathbb{T}$, see [11, Section 3.3.3]. Hence, it is enough to show that the equality holds for all $t \in [a,b)_{\mathbb{T}}$.

Let $t \in [a,b)_{\mathbb{T}}$. By repeating the arguments in [11, Theorem 3.49] we obtain that $\widetilde{y}\,'_g(t)$ exists and $\widetilde{y}\,'_g(t) = y^{\Delta}(t)$. On the other hand, [2, Theorem 1.16 (iv)] guarantees that

$$y(\sigma(t)) = y(t) + y^{\Delta}(t)(\sigma(t) - t) = y(t) + (f(t) - p(t)y(\sigma(t)))(\sigma(t) - t) = y(t) + (f(t) - p(t)y(\sigma(t))\Delta^+ g(t),$$

from which we obtain that

$$y(\sigma(t)) = \frac{y(t) + f(t)\Delta^+ g(t)}{1 + p(t)\Delta^+ g(t)}.$$

Hence, it follows that

$$y^{\Delta}(t) = -p(t)\frac{y(t) + f(t)\Delta^+ g(t)}{1 + p(t)\Delta^+ g(t)} + f(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)}y(t) + \frac{f(t)}{1 + p(t)\Delta^+ g(t)},$$

and so, since $\widetilde{y}\,'_g(t) = y^{\Delta}(t)$ and $g(t) = t$,

$$\widetilde{y}\,'_g(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)}\widetilde{y}(t) + \frac{f(t)}{1 + p(t)\Delta^+ g(t)}. \qquad \square$$

Remark 4.2. Observe that the hypotheses required for Proposition 4.1 are not the same as in [11, Theorem 3.49]; the requirement for the solution of the dynamic equation to be continuous from the left at every right-scattered point is missing. This is because such condition is always redundant as every Δ-differentiable function is continuous, see [2, Theorem 1.16 (i)].

The following result completes the equivalence between the two problems by showing that every solution of the nonhomogenous adjoint Stieltjes differential equation provides a solution of the nonhomogenous dynamic equation.

Proposition 4.3. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be as in (4.3), $p\colon [a,b] \to \mathbb{C}$ be such that $1+p(t)\Delta^+g(t) \neq 0$, $t \in [a,b)_{\mathbb{T}}$, and $y\colon [a,b] \to \mathbb{C}$ be a g-continuous function such that*

$$y'_g(t) = -\frac{p(t)}{1+p(t)\Delta^+g(t)}y(t) + \frac{f(t)}{1+p(t)\Delta^+g(t)} \quad \text{for all } t \in [a,b)_{\mathbb{T}}.$$

Then, the map $\widetilde{y} = y|_{\mathbb{T}}$ is such that

$$\widetilde{y}^{\Delta}(t) = -p(t)\widetilde{y}(\sigma(t)) + f(t), \quad t \in [a,b)_{\mathbb{T}}.$$

Proof. Let $t \in [a,b)_{\mathbb{T}}$. Following the reasonings in [11, Theorem 3.51], we can see that $\widetilde{y}^{\Delta}(t)$ exists and $\widetilde{y}^{\Delta}(t) = y'_g(t)$. Using [2, Theorem 1.16 (iv)] once again we have that

$$\widetilde{y}(\sigma(t)) = \widetilde{y}(t) + \widetilde{y}^{\Delta}(t)(\sigma(t)-t) = y(t) + y'_g(t)\Delta^+g(t).$$

Thus,

$$\begin{aligned} -p(t)\widetilde{y}(\sigma(t)) + f(t) &= -p(t)y(t) + p(t)\left(\frac{p(t)}{1+p(t)\Delta^+g(t)}y(t) - \frac{f(t)}{1+p(t)\Delta^+g(t)}\right)\Delta^+g(t) + f(t) \\ &= -\frac{p(t)}{1+p(t)\Delta^+g(t)}y(t) + \frac{f(t)}{1+p(t)\Delta^+g(t)} = y'_g(t) = \widetilde{y}^{\Delta}(t). \end{aligned}$$

□

5 Preliminaries on Stieltjes integral equations

We now leave the topic of Stieltjes differential equations, and focus on Stieltjes integral equations. In the rest of the paper, the symbol $\int_a^b f\,\mathrm{d}g$ always denotes the Kurzweil–Stieltjes integral of a function $f\colon [a,b] \to \mathbb{C}$ with respect to a function $g\colon [a,b] \to \mathbb{C}$. Note that we no longer assume that g is left-continuous or nondecreasing. Also, unlike Section 2, the function g need not be defined on the whole $\mathbb{R}$, and we make no assumptions on the endpoints of the interval $[a,b]$. Readers who are not familiar with the Kurzweil–Stieltjes integral and its basic properties should consult Appendix B before reading the rest of the paper.

The next theorem is a basic result dealing with the existence and uniqueness of solutions to scalar linear Stieltjes integral equations, also known as generalized ordinary differential equations (see [13, Theorem 2.7] or [14, Theorem 8.5.1]).

Theorem 5.1. *Consider a function $P\colon [a,b] \to \mathbb{C}$, which has bounded variation on $[a,b]$. Let $t_0 \in [a,b]$ and assume that $1+\Delta^+P(t) \neq 0$ for every $t \in [a,t_0)$, and $1-\Delta^-P(t) \neq 0$ for every $t \in (t_0,b]$. Then, for every $z_0 \in \mathbb{C}$, there exists a unique function $z\colon [a,b] \to \mathbb{C}$ such that*

$$z(t) = z_0 + \int_{t_0}^t z(s)\,\mathrm{d}P(s), \quad t \in [a,b]. \tag{5.1}$$

The function z has bounded variation on $[a,b]$. If P and z_0 are real, then z is real as well.

Thanks to the previous theorem, it makes sense to introduce the generalized exponential function as follows, see [13, Definition 3.1] or [14, Definition 8.5.2].

Definition 5.2. Consider a function $P\colon [a,b] \to \mathbb{C}$ of bounded variation on $[a,b]$. Let $t_0 \in [a,b]$ and assume that $1+\Delta^+P(t) \neq 0$ for every $t \in [a,t_0)$, and $1-\Delta^-P(t) \neq 0$ for every $t \in (t_0,b]$. Then we define the generalized exponential function $t \mapsto e_{\mathrm{d}P}(t,t_0)$, $t \in [a,b]$, as the unique solution $z\colon [a,b] \to \mathbb{C}$ of the generalized linear differential equation

$$z(t) = 1 + \int_{t_0}^{t} z(s)\,\mathrm{d}P(s).$$

The basic properties of the generalized exponential function can be found in [13] and [14, Section 8.5].

The next theorem shows that the reciprocal values of the generalized exponential function correspond to the values of another generalized exponential function (see [13, Theorem 3.4] or [14, Theorem 8.5.8]).

Theorem 5.3. *Assume that $P\colon [a,b] \to \mathbb{C}$ has bounded variation, $1+\Delta^+P(t) \neq 0$ for every $t \in [a,b)$, and $1-\Delta^-P(t) \neq 0$ for every $t \in (a,b]$. Then*

$$(e_{\mathrm{d}P}(t,t_0))^{-1} = e_{\mathrm{d}(\ominus P)}(t,t_0), \quad t \in [a,b],$$

where

$$(\ominus P)(t) = -P(t) + \sum_{s\in[t_0,t)} \frac{(\Delta^+P(s))^2}{1+\Delta^+P(s)} - \sum_{s\in(t_0,t]} \frac{(\Delta^-P(s))^2}{1-\Delta^-P(s)}.$$

We can also consider the nonhomogeneous linear integral equation

$$z(t) = z_0 + \int_{t_0}^{t} z(s)\,\mathrm{d}P(s) + F(t) - F(t_0), \quad t \in [a,b]. \tag{5.2}$$

Its solution is given by the variation of constants formula, see [14, Theorem 7.8.4]. The next theorem shows that if F is left- or right-continuous, then the formulas are particularly simple. (The result for a left-continuous F and $t_0 = a$ can be found in [14, Corollary 7.8.6].)

Theorem 5.4. *Suppose that functions $P\colon [a,b] \to \mathbb{C}$ and $F\colon [a,b] \to \mathbb{C}$ have bounded variation on $[a,b]$. Let $t_0 \in [a,b]$ and assume that $1+\Delta^+P(t) \neq 0$ for every $t \in [a,t_0)$, and $1-\Delta^-P(t) \neq 0$ for every $t \in (t_0,b]$.*

- *If F is left-continuous, then the unique solution of Eq.* (5.2) *is given by*

$$z(t) = e_{\mathrm{d}P}(t,t_0)z_0 + \int_{t_0}^{t} e_{\mathrm{d}P}(t,s+)\,\mathrm{d}F(s).$$

- *If F is right-continuous, then the unique solution of Eq.* (5.2) *is given by*

$$z(t) = e_{\mathrm{d}P}(t,t_0)z_0 + \int_{t_0}^{t} e_{\mathrm{d}P}(t,s-)\,\mathrm{d}F(s).$$

Proof. According to the general variation of constants formula ([14, Theorem 7.8.4]), the unique solution of Eq. (5.2) is given by

$$z(t) = e_{\mathrm{d}P}(t,t_0)z_0 + F(t) - F(t_0) - e_{\mathrm{d}P}(t,t_0)\int_{t_0}^{t} (F(s)-F(t_0))\,\mathrm{d}e_{\mathrm{d}P}(s,t_0)^{-1}.$$

If F is left-continuous, the integration by parts formula of Theorem B.6 yields

$$z(t) = e_{\mathrm{d}P}(t,t_0)z_0 + F(t) - F(t_0) - e_{\mathrm{d}P}(t,t_0)\left((F(t)-F(t_0))e_{\mathrm{d}P}(t,t_0)^{-1} - \int_{t_0}^{t} e_{\mathrm{d}P}(s+,t_0)^{-1}\,\mathrm{d}F(s)\right)$$

$$= e_{\mathrm{d}P}(t,t_0)z_0 + e_{\mathrm{d}P}(t,t_0)\int_{t_0}^{t} e_{\mathrm{d}P}(t_0,s+)\,\mathrm{d}F(s) = e_{\mathrm{d}P}(t,t_0)z_0 + \int_{t_0}^{t} e_{\mathrm{d}P}(t,s+)\,\mathrm{d}F(s).$$

Note that according to Theorem B.6, the value $e_{\mathrm{d}P}(t,s+)$ should be replaced by $e_{\mathrm{d}P}(t,s)$ if $s=\max(t_0,t)$; however, since F is left-continuous, the value of the integrand at the right endpoint does not matter, and therefore the convention might be dropped.

Similarly, if F is right-continuous, integration by parts gives

$$z(t) = e_{\mathrm{d}P}(t,t_0)z_0 + F(t) - F(t_0) - e_{\mathrm{d}P}(t,t_0)\left((F(t)-F(t_0))e_{\mathrm{d}P}(t,t_0)^{-1} - \int_{t_0}^{t} e_{\mathrm{d}P}(s-,t_0)^{-1}\,\mathrm{d}F(s)\right)$$
$$= e_{\mathrm{d}P}(t,t_0)z_0 + e_{\mathrm{d}P}(t,t_0)\int_{t_0}^{t} e_{\mathrm{d}P}(t_0,s-)\,\mathrm{d}F(s) = e_{\mathrm{d}P}(t,t_0)z_0 + \int_{t_0}^{t} e_{\mathrm{d}P}(t,s-)\,\mathrm{d}F(s),$$

where the value of the integrand at the left endpoint does not matter because F is right-continuous. $\square$

In connection with the previous result, it is natural to ask: Given the function

$$z(t) = e_{\mathrm{d}P}(t,t_0)z_0 + \int_{t_0}^{t} e_{\mathrm{d}P}(t,s)\,\mathrm{d}F(s), \quad t\in[a,b], \tag{5.3}$$

is there a Stieltjes integral equation whose solution is z? The next theorem provides the answer for left-continuous or right-continuous functions F, and shows what happens if P is replaced by $\ominus P$.

Theorem 5.5. *Suppose that functions $P\colon [a,b]\to\mathbb{C}$ and $F\colon[a,b]\to\mathbb{C}$ have bounded variation on $[a,b]$, $t_0\in[a,b]$, $z_0,w_0\in\mathbb{C}$, $1+\Delta^+P(t)\neq 0$ for every $t\in[a,b)$, and $1-\Delta^-P(t)\neq 0$ for every $t\in(a,b]$. Let*

$$z(t) = e_{\mathrm{d}P}(t,t_0)z_0 + \int_{t_0}^{t} e_{\mathrm{d}P}(t,s)\,\mathrm{d}F(s), \qquad t\in[a,b],$$
$$w(t) = e_{\mathrm{d}\ominus P}(t,t_0)w_0 + \int_{t_0}^{t} e_{\mathrm{d}\ominus P}(t,s)\,\mathrm{d}F(s), \quad t\in[a,b].$$

- *If F is left-continuous, then*

$$z(t) = z_0 + \int_{t_0}^{t} z(s)\,\mathrm{d}P(s) + F_1(t) - F_1(t_0), \qquad t\in[a,b], \tag{5.4}$$
$$w(t) = w_0 + \int_{t_0}^{t} w(s)\,\mathrm{d}(\ominus P)(s) + F_2(t) - F_2(t_0), \quad t\in[a,b], \tag{5.5}$$

  where $F_1(t)=\int_{t_0}^{t}(1+\Delta^+P(s))\,\mathrm{d}F(s)$ and $F_2(t)=\int_{t_0}^{t}\frac{1}{1+\Delta^+P(s)}\,\mathrm{d}F(s)$ for all $t\in[a,b]$.

- *If F is right-continuous, then*

$$z(t) = z_0 + \int_{t_0}^{t} z(s)\,\mathrm{d}P(s) + G_1(t) - G_1(t_0), \qquad t\in[a,b], \tag{5.6}$$
$$w(t) = w_0 + \int_{t_0}^{t} w(s)\,\mathrm{d}(\ominus P)(s) + G_2(t) - G_2(t_0), \quad t\in[a,b], \tag{5.7}$$

  where $G_1(t)=\int_{t_0}^{t}(1-\Delta^-P(s))\,\mathrm{d}F(s)$ and $G_2(t)=\int_{t_0}^{t}\frac{1}{1-\Delta^-P(s)}\,\mathrm{d}F(s)$ for all $t\in[a,b]$.

Proof. The generalized exponential function satisfies

$$e_{\mathrm{d}P}(t,s) = (1+\Delta^+P(s))e_{\mathrm{d}P}(t,s+), \qquad e_{\mathrm{d}P}(t,s) = (1-\Delta^-P(s))e_{\mathrm{d}P}(t,s-)$$

(see [13, Theorem 3.2] or [14, Theorem 8.5.3]). Taking the reciprocal values, we get

$$e_{\mathrm{d}\ominus P}(t,s)=\frac{e_{\mathrm{d}\ominus P}(t,s+)}{1+\Delta^+P(s)},\qquad e_{\mathrm{d}\ominus P}(t,s)=\frac{e_{\mathrm{d}\ominus P}(t,s-)}{1-\Delta^-P(s)}.$$

The previous four identities and the definitions of z, w and F_1,F_2,G_1,G_2 imply

$$\begin{aligned}
z(t)&=e_{\mathrm{d}P}(t,t_0)z_0+\int_{t_0}^t(1+\Delta^+P(s))e_{\mathrm{d}P}(t,s+)\,\mathrm{d}F(s)=e_{\mathrm{d}P}(t,t_0)z_0+\int_{t_0}^t e_{\mathrm{d}P}(t,s+)\,\mathrm{d}F_1(s),\\
w(t)&=e_{\mathrm{d}\ominus P}(t,t_0)w_0+\int_{t_0}^t\frac{e_{\mathrm{d}\ominus P}(t,s+)}{1+\Delta^+P(s)}\,\mathrm{d}F(s)=e_{\mathrm{d}\ominus P}(t,t_0)w_0+\int_{t_0}^t e_{\mathrm{d}\ominus P}(t,s+)\,\mathrm{d}F_2(s),\\
z(t)&=e_{\mathrm{d}P}(t,t_0)z_0+\int_{t_0}^t(1-\Delta^-P(s))e_{\mathrm{d}P}(t,s-)\,\mathrm{d}F(s)=e_{\mathrm{d}P}(t,t_0)z_0+\int_{t_0}^t e_{\mathrm{d}P}(t,s-)\,\mathrm{d}G_1(s),\\
w(t)&=e_{\mathrm{d}\ominus P}(t,t_0)w_0+\int_{t_0}^t\frac{e_{\mathrm{d}\ominus P}(t,s-)}{1-\Delta^-P(s)}\,\mathrm{d}F(s)=e_{\mathrm{d}\ominus P}(t,t_0)w_0+\int_{t_0}^t e_{\mathrm{d}\ominus P}(t,s-)\,\mathrm{d}G_2(s)
\end{aligned}$$

for all $t\in[a,b]$. If F is left-continuous, then F_1,F_2 are left-continuous as well, and the first part of Theorem 5.4 implies that z,w satisfy Eq. (5.4) and Eq. (5.5), respectively.

Similarly, if F is right-continuous, then G_1,G_2 are right-continuous as well, and the second part of Theorem 5.4 implies that z,w satisfy Eq. (5.6) and Eq. (5.7). □

The next result is concerned with the solvability of Volterra–Stieltjes integral equations; its real-valued version can be found in [21, Chapter II, Corollary 3.12], and can be easily generalized into the complex-valued setting that we present here.

Theorem 5.6. *Suppose that $t_0\in[a,b]$, $f\colon[a,b]\to\mathbb{C}$ has bounded variation, and $K\colon[a,b]\times[a,b]\to\mathbb{C}$ satisfies the following conditions:*

- $1+K(t,t+)-K(t,t)\neq 0$ *for each* $t\in[a,t_0)$.
- $1-K(t,t)+K(t,t-)\neq 0$ *for each* $t\in(t_0,b]$.
- $\operatorname{var}(K(t_0,\cdot),[a,b])$ *is finite.*
- *The Vitali variation of K over $[a,b]\times[a,b]$ is finite.*

Then the integral equation

$$x(t)=f(t)+\int_{t_0}^t x(s)\,\mathrm{d}K(t,s),\quad t\in[a,b],$$

has a unique solution $x\colon[a,b]\to\mathbb{C}$.

6 Duality for Stieltjes integral equations

Consider an interval $[a,b]\subset\mathbb{R}$ and a function $P\colon[a,b]\to\mathbb{C}$ with bounded variation. We focus on the following pair of integral equations:

$$x(t)=x(t_0)+\int_{t_0}^t x(s-)\,\mathrm{d}P(s),\quad t\in[a,b], \tag{6.1}$$

$$y(t)=y(t_0)-\int_{t_0}^t y(s+)\,\mathrm{d}P(s),\quad t\in[a,b]. \tag{6.2}$$

These equations have to be interpreted carefully: In the first integral, the value $x(s-)$ should be understood as $x(s)$ when s coincides with $\min(t_0,t)$. Similarly, in the second integral, the value $y(s+)$ should be understood as $y(s)$ when s coincides with $\max(t_0,t)$. This convention leads to the following definition of solutions.

Definition 6.1. A regulated function $x\colon [a,b]\to\mathbb{C}$ is called a solution of Eq. (6.1) if it satisfies

$$x(t)=\begin{cases} x(t_0)+\int_{t_0}^{t}\big(x(s-)\chi_{(t_0,t]}(s)+x(t_0)\chi_{\{t_0\}}(s)\big)\,\mathrm{d}P(s), & t\ge t_0,\\ x(t_0)-\int_{t}^{t_0}\big(x(s-)\chi_{(t,t_0]}(s)+x(t)\chi_{\{t\}}(s)\big)\,\mathrm{d}P(s), & t\le t_0.\end{cases}$$

A regulated function $y\colon [a,b]\to\mathbb{C}$ is called a solution of Eq. (6.2) if it satisfies

$$y(t)=\begin{cases} y(t_0)-\int_{t_0}^{t}\big(y(s+)\chi_{[t_0,t)}(s)+y(t)\chi_{\{t\}}(s)\big)\,\mathrm{d}P(s), & t\ge t_0,\\ y(t_0)+\int_{t}^{t_0}\big(y(s+)\chi_{[t,t_0)}(s)+y(t_0)\chi_{\{t_0\}}(s)\big)\,\mathrm{d}P(s), & t\le t_0.\end{cases}$$

The reader might wonder why we need the previous convention regarding the values at the endpoints, and the corresponding definition of a solution. What would happen if Eq. (6.1) and Eq. (6.2) were interpreted literally without this convention? Although the equations would still make sense, their solutions would lack properties that we expect from solutions of dual equations. We will return to this question in Remark 6.3.

As in Section 3, we will postpone the question of the existence and uniqueness of solutions to Eq. (6.1) and (6.2) for later (see Theorem 6.8), and begin by deriving an analogue of Lagrange's identity, which will explain why it makes sense to call Eq. (6.1) and (6.2) dual (or adjoint) to each other.

The operator forms of equations (6.1) and (6.2) are $Lx=0$ and $L^*y=0$, where L and L^* are linear operators on $\mathcal{G}([a,b],\mathbb{C})$, the space of regulated functions, given by the formulas

$$Lx(t)=\begin{cases} x(t)-x(t_0)-\int_{t_0}^{t}\big(x(s-)\chi_{(t_0,t]}(s)+x(t_0)\chi_{\{t_0\}}(s)\big)\,\mathrm{d}P(s), & t\ge t_0,\\ x(t)-x(t_0)+\int_{t}^{t_0}\big(x(s-)\chi_{(t,t_0]}(s)+x(t)\chi_{\{t\}}(s)\big)\,\mathrm{d}P(s), & t\le t_0,\end{cases}$$

$$L^*y(t)=\begin{cases} y(t)-y(t_0)+\int_{t_0}^{t}\big(y(s+)\chi_{[t_0,t)}(s)+y(t)\chi_{\{t\}}(s)\big)\,\mathrm{d}P(s), & t\ge t_0,\\ y(t)-y(t_0)-\int_{t}^{t_0}\big(y(s+)\chi_{[t,t_0)}(s)+y(t_0)\chi_{\{t_0\}}(s)\big)\,\mathrm{d}P(s), & t\le t_0.\end{cases}$$

The next result generalizes the Lagrange identity, and implies that the product of a solution of Eq. (6.1) and a solution of Eq. (6.2) is a constant function.

Theorem 6.2. *If $x,y\colon [a,b]\to\mathbb{C}$ are functions of bounded variation and $t_0\in[a,b]$, then for each $T\in[a,b]$ we have*

$$x(T)y(T)-x(t_0)y(t_0)=\int_{t_0}^{T} y(t+)\,\mathrm{d}Lx(t)+\int_{t_0}^{T} x(t-)\,\mathrm{d}L^*y(t), \tag{6.3}$$

with the convention that $y(t+)=y(t)$ if $t=\max(t_0,T)$, and $x(t-)=x(t)$ if $t=\min(t_0,T)$.

In particular, if $Lx=0$ and $L^*y=0$ on $[a,b]$, then $x\cdot y$ is a constant function on $[a,b]$.

Proof. The validity of Eq. (6.3) is clear for $T=t_0$, when both sides vanish. Consider the case $T>t_0$. The definitions of L and L^* imply

$$x(t)=Lx(t)+x(t_0)+\int_{t_0}^{t}\big(x(s-)\chi_{(t_0,t]}(s)+x(t_0)\chi_{\{t_0\}}(s)\big)\,\mathrm{d}P(s)=Lx(t)+x(t_0)+g_1(t),\quad t\ge t_0,$$

$$y(t)=L^*y(t)+y(t_0)-\int_{t_0}^{t}\big(y(s+)\chi_{[t_0,t)}(s)+y(t)\chi_{\{t\}}(s)\big)\,\mathrm{d}P(s)=L^*y(t)+y(t_0)-g_2(t),\quad t\ge t_0,$$

where

$$g_1(t)=\int_{t_0}^{t}\big(x(s-)\chi_{(t_0,t]}(s)+x(t_0)\chi_{\{t_0\}}(s)\big)\,\mathrm{d}P(s),\quad t\ge t_0,$$

$$g_2(t) = \int_{t_0}^{t} \big(y(s+)\chi_{[t_0,t)}(s) + y(t)\chi_{\{t\}}(s)\big)\,\mathrm{d}P(s), \qquad t \geq t_0.$$

Denoting

$$\begin{aligned} f_1(t) &= y(t+)\chi_{[t_0,T)}(t) + y(T)\chi_{\{T\}}(t), \quad t \geq t_0, \\ f_2(t) &= x(t-)\chi_{(t_0,T]}(t) + x(t_0)\chi_{\{t_0\}}(t), \quad t \geq t_0, \end{aligned}$$

integration by parts (Theorem B.6) yields

$$\begin{aligned} x(T)y(T) - x(t_0)y(t_0) &= \int_{t_0}^{T} f_1(t)\,\mathrm{d}x(t) + \int_{t_0}^{T} f_2(t)\,\mathrm{d}y(t) \\ &= \int_{t_0}^{T} f_1(t)\,\mathrm{d}Lx(t) + \int_{t_0}^{T} f_2(t)\,\mathrm{d}L^*y(t) + \int_{t_0}^{T} f_1(t)\,\mathrm{d}g_1(t) - \int_{t_0}^{T} f_2(t)\,\mathrm{d}g_2(t), \end{aligned}$$

where the constant terms $x(t_0)$ and $y(t_0)$ were omitted, since they have no effect in the integrators.

To finish the proof, we need to show that the last two terms cancel each other out. Let us focus on the first one, split the integral over $[t_0, T]$ into two integrals over $[t_0, t_0 + \delta]$ and $[t_0 + \delta, T]$, where $t_0 < t_0 + \delta < T$, and finally pass to the limit as $\delta \to 0+$:

$$\int_{t_0}^{T} f_1(t)\,\mathrm{d}g_1(t) = \lim_{\delta\to 0+} \int_{t_0}^{t_0+\delta} f_1(t)\,\mathrm{d}g_1(t) + \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} f_1(t)\,\mathrm{d}g_1(t).$$

In the second integral, we have $t > t_0$, and the integrator g_1 can be simplified using Lemma B.2:

$$\begin{aligned} g_1(t) &= \int_{t_0}^{t} \big(x(s-)\chi_{(t_0,t]}(s) + x(t_0)\chi_{\{t_0\}}(s)\big)\,\mathrm{d}P(s) = \int_{t_0}^{t} x(s-)\chi_{(t_0,t]}(s)\,\mathrm{d}P(s) + x(t_0)\Delta^+P(t_0) \\ &= \int_{t_0}^{t} x(s-)\,\mathrm{d}P(s) - x(t_0-)\Delta^+P(t_0) + x(t_0)\Delta^+P(t_0) = \int_{t_0}^{t} x(s-)\,\mathrm{d}P(s) + \Delta^-x(t_0)\Delta^+P(t_0) \end{aligned} \tag{6.4}$$

(note that this identity holds only for $t > t_0$, because $g_1(t_0) = 0$). Observing that the last term does not depend on t and using Theorem B.7, we get

$$\begin{aligned} \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} f_1(t)\,\mathrm{d}g_1(t) &= \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} f_1(t)x(t-)\,\mathrm{d}P(t) \\ &= \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} \big(y(t+)\chi_{[t_0,T)}(t) + y(T)\chi_{\{T\}}(t)\big)\,x(t-)\,\mathrm{d}P(t) \\ &= \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} y(t+)\chi_{[t_0,T)}(t)x(t-)\,\mathrm{d}P(t) + y(T)x(T-)\Delta^-P(T). \end{aligned}$$

On the other hand, using Theorem B.8 and Eq. (6.4), we get

$$\begin{aligned} \lim_{\delta\to 0+} \int_{t_0}^{t_0+\delta} f_1(t)\,\mathrm{d}g_1(t) &= f_1(t_0)g_1(t_0+) = f_1(t_0) \lim_{\varepsilon\to 0+} \left(\int_{t_0}^{t_0+\varepsilon} x(s-)\,\mathrm{d}P(s) + \Delta^-x(t_0)\Delta^+P(t_0) \right) \\ &= f_1(t_0)(x(t_0-)\Delta^+P(t_0) + \Delta^-x(t_0)\Delta^+P(t_0)) = f_1(t_0)x(t_0)\Delta^+P(t_0). \end{aligned}$$

In a similar way, we express

$$\int_{t_0}^{T} f_2(t)\,\mathrm{d}g_2(t) = \lim_{\delta\to 0+} \int_{t_0}^{t_0+\delta} f_2(t)\,\mathrm{d}g_2(t) + \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} f_2(t)\,\mathrm{d}g_2(t).$$

For $t > t_0$, the integrator g_2 can be simplified using Lemma B.2:

$$g_2(t) = \int_{t_0}^{t} \big(y(s+)\chi_{[t_0,t)}(s) + y(t)\chi_{\{t\}}(s)\big)\,\mathrm{d}P(s) = \int_{t_0}^{t} y(s+)\chi_{[t_0,t)}(s)\,\mathrm{d}P(s) + y(t)\Delta^- P(t)$$
$$= \int_{t_0}^{t} y(s+)\,\mathrm{d}P(s) - y(t+)\Delta^- P(t) + y(t)\Delta^- P(t) = \int_{t_0}^{t} y(s+)\,\mathrm{d}P(s) - \Delta^+ y(t)\Delta^- P(t). \tag{6.5}$$

Thus, using Theorem B.7, we get

$$\lim_{\delta\to 0+} \int_{t_0+\delta}^{T} f_2(t)\,\mathrm{d}g_2(t) = \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} f_2(t)y(t+)\,\mathrm{d}P(t) - \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} f_2(t)\,\mathrm{d}\Delta^+ y(t)\Delta^- P(t).$$

Since $\Delta^+ y(t)\Delta^- P(t) = 0$ for all t with at most countably many exceptions, we can use Lemma B.4 to evaluate the last integral, getting

$$\int_{t_0+\delta}^{T} f_2(t)\,\mathrm{d}\Delta^+ y(t)\Delta^- P(t) = f_2(T)\Delta^+ y(T)\Delta^- P(T) - f_2(t_0+\delta)\Delta^+ y(t_0+\delta)\Delta^- P(t_0+\delta).$$

Therefore, since the latter term goes to zero for $\delta \to 0+$, we obtain

$$\lim_{\delta\to 0+} \int_{t_0+\delta}^{T} f_2(t)\,\mathrm{d}g_2(t) = \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} f_2(t)y(t+)\,\mathrm{d}P(t) - f_2(T)\Delta^+ y(T)\Delta^- P(T).$$

On the other hand, using Theorem B.8 and Eq. (6.5), we get

$$\begin{aligned}\lim_{\delta\to 0+} \int_{t_0}^{t_0+\delta} f_2(t)\,\mathrm{d}g_2(t) &= f_2(t_0)g_2(t_0+)\\ &= f_2(t_0)\lim_{\varepsilon\to 0+}\left(\int_{t_0}^{t_0+\varepsilon} y(s+)\,\mathrm{d}P(s) - \Delta^+ y(t_0+\varepsilon)\Delta^- P(t_0+\varepsilon)\right)\\ &= f_2(t_0)y(t_0+)\Delta^+ P(t_0).\end{aligned}$$

Collecting the previous results and using the definitions of f_1 and f_2, we obtain

$$\begin{aligned}\int_{t_0}^{T} f_1(t)\,\mathrm{d}g_1(t) - \int_{t_0}^{T} f_2(t)\,\mathrm{d}g_2(t) &= \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} y(t+)\chi_{[t_0,T)}(t)x(t-)\,\mathrm{d}P(t) + y(T)x(T-)\Delta^- P(T)\\ &\quad + f_1(t_0)x(t_0)\Delta^+ P(t_0) - \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} f_2(t)y(t+)\,\mathrm{d}P(t)\\ &\quad + f_2(T)\Delta^+ y(T)\Delta^- P(T) - f_2(t_0)y(t_0+)\Delta^+ P(t_0)\\ &= \lim_{\delta\to 0+} \int_{t_0+\delta}^{T} \big(y(t+)x(t-)\chi_{[t_0,T)}(t) - y(t+)x(t-)\chi_{(t_0,T]}(t)\big)\,\mathrm{d}P(t)\\ &\quad + y(T)x(T-)\Delta^- P(T) + y(t_0+)x(t_0)\Delta^+ P(t_0)\\ &\quad + x(T-)\Delta^+ y(T)\Delta^- P(T) - x(t_0)y(t_0+)\Delta^+ P(t_0)\\ &= -\lim_{\delta\to 0+} \int_{t_0+\delta}^{T} y(t+)x(t-)\chi_{\{T\}}(t)\,\mathrm{d}P(t) + y(T+)x(T-)\Delta^- P(T) = 0,\end{aligned}$$

which completes the proof for $T > t_0$. The proof for $T < t_0$ is similar and is left to the reader. □

Remark 6.3. Let us return to the question posed after Definition 6.1 – what would happen if the definition was abandoned and Eq. (6.1) and Eq. (6.2) were interpreted literally without the convention regarding the values at the endpoints? Let us show that the Lagrange identity (6.3) would be no longer valid. It suffices to show that the product of two solutions x, y of Eq. (6.1) and Eq. (6.2), respectively, would not be necessarily constant.

Focusing again on the case $T > t_0$ and, as in the proof of the previous theorem, denoting

$$\begin{aligned} f_1(t) &= y(t+)\chi_{[t_0,T)}(t) + y(T)\chi_{\{T\}}(t), \quad t \ge t_0, \\ f_2(t) &= x(t-)\chi_{(t_0,T]}(t) + x(t_0)\chi_{\{t_0\}}(t), \quad t \ge t_0, \end{aligned}$$

integration by parts (Theorem B.6) yields

$$x(T)y(T) - x(t_0)y(t_0) = \int_{t_0}^{T} f_1(t)\,\mathrm{d}x(t) + \int_{t_0}^{T} f_2(t)\,\mathrm{d}y(t).$$

The operators L and L^* are now given by the formulas

$$Lx(t) = x(t) - x(t_0) - \int_{t_0}^{t} x(s-)\,\mathrm{d}P(s), \qquad L^*y(t) = y(t) - y(t_0) + \int_{t_0}^{t} y(s+)\,\mathrm{d}P(s).$$

Therefore, since $x(t) = Lx(t) + x(t_0) + \int_{t_0}^{t} x(s-)\mathrm{d}P(s)$ and $y(t) = L^*y(t) + y(t_0) - \int_{t_0}^{t} y(s+)\,\mathrm{d}P(s)$, we get

$$\begin{aligned} x(T)y(T) - x(t_0)y(t_0) &= \int_{t_0}^{T} f_1(t)\,\mathrm{d}Lx(t) + \int_{t_0}^{T} f_2(t)\,\mathrm{d}L^*y(t) \\ &\quad + \int_{t_0}^{T} f_1(t)\,\mathrm{d}\left(\int_{t_0}^{t} x(s-)\mathrm{d}P(s)\right) - \int_{t_0}^{T} f_2(t)\,\mathrm{d}\left(\int_{t_0}^{t} y(s+)\,\mathrm{d}P(s)\right). \end{aligned}$$

Let us examine whether the last two terms cancel each other. Simplification using Theorem B.7 leads to

$$\begin{aligned} &\int_{t_0}^{T} f_1(t)\,\mathrm{d}\left(\int_{t_0}^{t} x(s-)\mathrm{d}P(s)\right) - \int_{t_0}^{T} f_2(t)\,\mathrm{d}\left(\int_{t_0}^{t} y(s+)\,\mathrm{d}P(s)\right) \\ &= \int_{t_0}^{T} f_1(t)x(t-)\,\mathrm{d}P(t) - \int_{t_0}^{T} f_2(t)y(t+)\,\mathrm{d}P(t) \\ &= \int_{t_0}^{T} \big(y(t+)x(t-)\chi_{[t_0,T)}(t) + y(T)x(t-)\chi_{\{T\}}(t)\big)\,\mathrm{d}P(t) \\ &\quad - \int_{t_0}^{T} \big(x(t-)y(t+)\chi_{(t_0,T]}(t) + x(t_0)y(t+)\chi_{\{t_0\}}(t)\big)\,\mathrm{d}P(t) \\ &= \int_{t_0}^{T} \big(y(t+)x(t-)\chi_{\{t_0\}}(t) + y(T)x(t-)\chi_{\{T\}}(t) - x(t-)y(t+)\chi_{\{T\}}(t) - x(t_0)y(t+)\chi_{\{t_0\}}(t)\big)\,\mathrm{d}P(t) \\ &= y(t_0+)x(t_0-)\Delta^+P(t_0) + y(T)x(T-)\Delta^-P(T) - x(T-)y(T+)\Delta^-P(T) - x(t_0)y(t_0+)\Delta^+P(t_0) \\ &= -\Delta^+P(t_0)y(t_0+)\Delta^-x(t_0) - \Delta^-P(T)x(T-)\Delta^+y(T), \end{aligned}$$

which is not necessarily zero. Thus, we see that even if $Lx = 0$ and $L^*y = 0$, the product $x \cdot y$ need not be a constant function.

Remark 6.4. A function $y\colon [a,b] \to \mathbb{C}$ is a solution of the equation

$$y(t) = y(t_0) - \int_{t_0}^{t} y(s+)\,\mathrm{d}P(s), \quad t \in [a,b],$$

if and only if the function $x\colon [-b,-a] \to \mathbb{C}$ given by $x(t) = y(-t)$ is a solution of the equation

$$x(t) = x(-t_0) + \int_{-t_0}^{t} x(s-)\,\mathrm{d}\widetilde{P}(s), \quad t \in [-b,-a],$$

where $\widetilde{P}(s) = -P(-s)$ for $s \in [-b, -a]$, and both solutions are understood in the sense of Definition 6.1. Indeed, the latter equation is equivalent to

$$x(-t) = x(-t_0) + \int_{-t_0}^{-t} x(s-)\,\mathrm{d}\widetilde{P}(s), \quad t \in [a, b].$$

According to the definition of a solution, the last integral should be understood as $\int_{-t_0}^{-t} f(s)\,\mathrm{d}\widetilde{P}(s)$, where

$$f(s) = \begin{cases} x(s) & \text{if } s = \min(-t_0, -t), \\ x(s-) & \text{otherwise.} \end{cases}$$

Note that

$$f(-s) = \begin{cases} x(-s) = y(s) & \text{if } -s = \min(-t_0, -t), \text{ i.e., if } s = \max(t_0, t), \\ x(-s-) = y(s+) & \text{otherwise.} \end{cases}$$

Thus, letting $\phi(s) = -s$ and using the substitution theorem $\int_{\phi(c)}^{\phi(d)} f(s)\,\mathrm{d}g(s) = \int_c^d f(\phi(t))\,\mathrm{d}g(\phi(t))$ (see [14, Theorem 6.6.5 and Exercise 6.6.6]), we obtain

$$\int_{-t_0}^{-t} x(s-)\,\mathrm{d}\widetilde{P}(s) = \int_{\phi(t_0)}^{\phi(t)} f(s)\,\mathrm{d}\widetilde{P}(s) = \int_{t_0}^{t} f(\phi(s))\,\mathrm{d}\widetilde{P}(\phi(s)) = \int_{t_0}^{t} f(-s)\,\mathrm{d}\widetilde{P}(-s) = -\int_{t_0}^{t} y(s+)\,\mathrm{d}P(s).$$

Hence, we see that a solution of Eq. (6.1) can be transformed to a solution of Eq. (6.2) with P replaced by $\widetilde{P}$, and vice versa. It will be useful to observe that the jumps of the function $\widetilde{P}$ are

$$\begin{aligned} \Delta^+\widetilde{P}(t) &= \widetilde{P}(t+) - \widetilde{P}(t) = -P(-t-) + P(-t) = \Delta^- P(-t), \quad t \in [-b, -a), \\ \Delta^-\widetilde{P}(t) &= \widetilde{P}(t) - \widetilde{P}(t-) = -P(-t) + P(-t+) = \Delta^+ P(-t), \quad t \in (-b, -a]. \end{aligned}$$

Given a solution x of Eq. (6.1), we can express $x(t-)$ in terms of $x(t)$ provided certain conditions are met. A similar property can be obtained for a solution of Eq. (6.2) as presented in the next lemma.

Lemma 6.5. *Suppose that a function $P\colon [a, b] \to \mathbb{C}$ has bounded variation and $t_0 \in [a, b]$.*

1. *If $x\colon [a, b] \to \mathbb{C}$ is a solution of Eq.* (6.1) *on $[a, b]$, then*

$$x(t-) = \begin{cases} x(t)\dfrac{1 + \Delta^+ P(t)}{1 + \Delta P(t)} & \textit{if } t \in (a, t_0) \textit{ and } 1 + \Delta P(t) \neq 0, \\ x(t)\dfrac{1}{1 + \Delta^- P(t)} & \textit{if } t \in [t_0, b] \textit{ and } 1 + \Delta^- P(t) \neq 0. \end{cases}$$

2. *If $y\colon [a, b] \to \mathbb{C}$ is a solution of Eq.* (6.2) *on $[a, b]$, then*

$$y(t+) = \begin{cases} y(t)\dfrac{1}{1 + \Delta^+ P(t)} & \textit{if } t \in [a, t_0] \textit{ and } 1 + \Delta^+ P(t) \neq 0, \\ y(t)\dfrac{1 + \Delta^- P(t)}{1 + \Delta P(t)} & \textit{if } t \in (t_0, b) \textit{ and } 1 + \Delta P(t) \neq 0. \end{cases}$$

Proof. Let us prove the first statement. We begin with the case $t \in (a, t_0)$. Using the definition of solution, we have

$$x(t-) = \lim_{\delta \to 0+} x(t - \delta) = \lim_{\delta \to 0+} \left(x(t_0) - \int_{t-\delta}^{t_0} x(s-)\chi_{(t-\delta, t_0]}(s)\,\mathrm{d}P(s) - \int_{t-\delta}^{t_0} x(t - \delta)\chi_{\{t-\delta\}}(s)\,\mathrm{d}P(s) \right).$$

According to Lemma B.2, the value of the second integral is $x(t-\delta)\Delta^+P(t-\delta)$. In the first integral, we can extend the integration domain from $[t-\delta, t_0]$ to $[a, t_0]$:

$$x(t-) = \lim_{\delta\to 0+}\left(x(t_0) - \int_a^{t_0} x(s-)\chi_{(t-\delta,t_0]}(s)\,\mathrm{d}P(s) - x(t-\delta)\Delta^+P(t-\delta)\right)$$

Using the bounded convergence theorem (Theorem B.9) and the fact that $\lim_{\delta\to 0+}\Delta^+P(t-\delta) = \lim_{\delta\to 0+}(P(t-\delta+) - P(t-\delta)) = P(t-) - P(t-) = 0$ (see [14, Corollary 4.1.9]), we get

$$\begin{aligned}
x(t-) &= x(t_0) - \int_a^{t_0} x(s-)\chi_{[t,t_0]}(s)\,\mathrm{d}P(s)\\
&= x(t) - \int_a^{t_0} x(s-)\chi_{\{t\}}(s)\,\mathrm{d}P(s) + \int_t^{t_0} x(s)\chi_{\{t\}}(s)\,\mathrm{d}P(s) = x(t) - x(t-)\Delta P(t) + x(t)\Delta^+P(t),
\end{aligned}$$

i.e., $x(t-)(1+\Delta P(t)) = x(t)(1+\Delta^+P(t))$, which proves the first statement for $t \in (a, t_0)$.

Suppose next that $t = t_0$. Performing a similar calculation as before with $t-\delta$ replaced by $t_0-\delta$, we obtain

$$\begin{aligned}
x(t_0-) &= \lim_{\delta\to 0+}\left(x(t_0) - \int_a^{t_0} x(s-)\chi_{(t_0-\delta,t_0]}(s)\,\mathrm{d}P(s) - x(t_0-\delta)\Delta^+P(t_0-\delta)\right)\\
&= x(t_0) - \int_a^{t_0} x(s-)\chi_{\{t_0\}}(s)\,\mathrm{d}P(s) = x(t_0) - x(t_0-)\Delta^-P(t_0),
\end{aligned}$$

and therefore $x(t_0-)(1+\Delta^-P(t_0)) = x(t_0)$, which settles the case $t = t_0$.

Finally, if $t \in (t_0, b]$, we have

$$\begin{aligned}
x(t-) &= \lim_{\delta\to 0+}\left(x(t_0) + \int_{t_0}^{t-\delta} \left(x(s-)\chi_{(t_0,t-\delta]}(s) + x(t_0)\chi_{\{t_0\}}(s)\right)\mathrm{d}P(s)\right)\\
&= \lim_{\delta\to 0+}\left(x(t_0) + \int_{t_0}^{b} \left(x(s-)\chi_{(t_0,t-\delta)}(s) + x(t_0)\chi_{\{t_0\}}(s)\right)\mathrm{d}P(s) + x(t-\delta-)\Delta^-P(t-\delta)\right)\\
&= x(t_0) + \int_{t_0}^{b} \left(x(s-)\chi_{(t_0,t)}(s) + x(t_0)\chi_{\{t_0\}}(s)\right)\mathrm{d}P(s)\\
&= x(t) - \int_{t_0}^{t} x(s-)\chi_{\{t\}}(s)\,\mathrm{d}P(s) = x(t) - x(t-)\Delta^-P(t).
\end{aligned}$$

Thus, $x(t-)(1+\Delta^-P(t)) = x(t)$, which completes the proof of the first statement.

In the proof of the second statement, we use the fact that, according to Remark 6.4, the function x given by $x(t) = y(-t)$ is a solution of the equation

$$x(t) = x(-t_0) + \int_{-t_0}^{t} x(s-)\,\mathrm{d}\widetilde{P}(s), \quad t \in [-b, -a],$$

where $\widetilde{P}(s) = -P(-s)$ for $s \in [-b, -a]$. We can use the first part of the lemma to calculate $y(t+) = x(-t-)$ for $t \in [a, b)$. For $t \in (t_0, b)$, we have $-t \in (-b, -t_0)$, and therefore

$$y(t+) = x(-t-) = x(-t)\frac{1+\Delta^+\widetilde{P}(-t)}{1+\Delta\widetilde{P}(-t)} = y(t)\frac{1+\Delta^-P(t)}{1+\Delta P(t)}.$$

Similarly, for $t \in [a, t_0]$, we have $-t \in [-t_0, -b]$, and therefore

$$y(t+) = x(-t-) = x(-t)\frac{1}{1+\Delta^-\widetilde{P}(-t)} = y(t)\frac{1}{1+\Delta^+P(t)}.$$

□

The next theorem shows that Eq. (6.1) and Eq. (6.2) can be transformed into linear integral equations that no longer involve the terms $x(s-)$ and $y(s+)$.

Theorem 6.6. *Assume that a function $P\colon [a,b] \to \mathbb{C}$ has bounded variation and $t_0 \in [a,b]$.*

1. *If $1 + \Delta^- P(t) \neq 0$ for each $t \in [t_0, b]$ and $1 + \Delta P(t) \neq 0$ for each $t \in (a, t_0)$, then a function $x\colon [a,b] \to \mathbb{C}$ is a solution of Eq. (6.1) if and only if it satisfies the following conditions:*
   - *For every $t \in (t_0, b]$,*
   $$x(t) = x(t_0) + \int_{t_0}^{t} x(s) \left(\frac{1}{1+\Delta^- P(s)} \chi_{(t_0,t]}(s) + \chi_{\{t_0\}}(s) \right) \mathrm{d}P(s).$$
   - *For every $t \in [a, t_0)$,*
   $$x(t) = x(t_0) - \int_{t}^{t_0} x(s) \left(\frac{1+\Delta^+ P(s)}{1+\Delta P(s)} \chi_{(t,t_0)}(s) + \chi_{\{t\}}(s) + \frac{1}{1+\Delta^- P(t_0)} \chi_{\{t_0\}}(s) \right) \mathrm{d}P(s).$$
2. *If $1 + \Delta^+ P(t) \neq 0$ for each $t \in [a, t_0]$ and $1 + \Delta P(t) \neq 0$ for each $t \in (t_0, b)$, then a function $y\colon [a,b] \to \mathbb{C}$ is a solution of Eq. (6.2) if and only if it satisfies the following conditions:*
   - *For every $t \in (t_0, b]$,*
   $$y(t) = y(t_0) - \int_{t_0}^{t} y(s) \left(\frac{1+\Delta^- P(s)}{1+\Delta P(s)} \chi_{(t_0,t)}(s) + \chi_{\{t\}}(s) + \frac{1}{1+\Delta^+ P(t_0)} \chi_{\{t_0\}}(s) \right) \mathrm{d}P(s).$$
   - *For every $t \in [a, t_0)$,*
   $$y(t) = y(t_0) + \int_{t}^{t_0} y(s) \left(\frac{1}{1+\Delta^+ P(s)} \chi_{[t,t_0)}(s) + \chi_{\{t_0\}}(s) \right) \mathrm{d}P(s).$$

Proof. Let us prove the first statement. From Lemma 6.5 and from the definition of a solution to Eq. (6.1), it is clear that a solution of Eq. (6.1) necessarily satisfies both conditions of the first statement.

Suppose, conversely, that $x\colon [a,b] \to \mathbb{C}$ satisfies both conditions of the first statement. It suffices to show that
$$x(t-) = \begin{cases} x(t) \dfrac{1+\Delta^+ P(t)}{1+\Delta P(t)}, & t \in (a, t_0), \\ x(t) \dfrac{1}{1+\Delta^- P(t)}, & t \in [t_0, b]. \end{cases}$$
The proof is similar to the one of Lemma 6.5. For $t \in (a, t_0)$, we have
$$\begin{aligned}
x(t-) &= \lim_{\delta \to 0+} \left(x(t_0) - \int_{t-\delta}^{t_0} x(s) \left(\frac{1+\Delta^+ P(s)}{1+\Delta P(s)} \chi_{(t-\delta,t_0)}(s) + \chi_{\{t-\delta\}}(s) + \frac{1}{1+\Delta^- P(t_0)} \chi_{\{t_0\}}(s) \right) \mathrm{d}P(s) \right) \\
&= x(t_0) - \lim_{\delta \to 0+} x(t-\delta) \Delta^+ P(t-\delta) \\
&\quad - \lim_{\delta \to 0+} \int_{a}^{t_0} x(s) \left(\frac{1+\Delta^+ P(s)}{1+\Delta P(s)} \chi_{(t-\delta,t_0)}(s) + \frac{1}{1+\Delta^- P(t_0)} \chi_{\{t_0\}}(s) \right) \mathrm{d}P(s) \\
&= x(t_0) - \int_{a}^{t_0} x(s) \left(\frac{1+\Delta^+ P(s)}{1+\Delta P(s)} \chi_{[t,t_0)}(s) + \frac{1}{1+\Delta^- P(t_0)} \chi_{\{t_0\}}(s) \right) \mathrm{d}P(s) \\
&= x(t_0) - x(t) \frac{1+\Delta^+ P(t)}{1+\Delta P(t)} \Delta P(t) - \int_{t}^{t_0} x(s) \left(\frac{1+\Delta^+ P(s)}{1+\Delta P(s)} \chi_{(t,t_0)}(s) + \frac{1}{1+\Delta^- P(t_0)} \chi_{\{t_0\}}(s) \right) \mathrm{d}P(s)
\end{aligned}$$

$$
\begin{aligned}
&= x(t) - x(t)\frac{1+\Delta^+P(t)}{1+\Delta P(t)}\Delta P(t) + \int_t^{t_0} x(s)\chi_{\{t\}}(s)\,\mathrm{d}P(s)\\
&= x(t) - x(t)\frac{1+\Delta^+P(t)}{1+\Delta P(t)}\Delta P(t) + x(t)\Delta^+P(t)\\
&= x(t)(1+\Delta^+P(t)) - x(t)\frac{1+\Delta^+P(t)}{1+\Delta P(t)}\Delta P(t) = x(t)\frac{1+\Delta^+P(t)}{1+\Delta P(t)}.
\end{aligned}
$$

For $t = t_0$, we perform a similar calculation as before with t replaced by t_0, but noting that $\chi_{[t_0,t_0)}(s) = 0$ for each s. Therefore,

$$
\begin{aligned}
x(t_0-) &= x(t_0) - \int_a^{t_0} x(s)\frac{1}{1+\Delta^-P(t_0)}\chi_{\{t_0\}}(s)\,\mathrm{d}P(s)\\
&= x(t_0) - x(t_0)\frac{1}{1+\Delta^-P(t_0)}\Delta^-P(t_0) = x(t_0)\frac{1}{1+\Delta^-P(t_0)}.
\end{aligned}
$$

Finally, for $t \in (t_0, b]$, we calculate

$$
\begin{aligned}
x(t-) &= \lim_{\delta\to 0+}\left(x(t_0) + \int_{t_0}^{t-\delta} x(s)\left(\chi_{(t_0,t-\delta]}(s)\frac{1}{1+\Delta^-P(s)} + \chi_{\{t_0\}}(s)\right)\mathrm{d}P(s)\right)\\
&= \lim_{\delta\to 0+}\left(x(t_0) + \int_{t_0}^{b} x(s)\left(\chi_{(t_0,t-\delta)}(s)\frac{1}{1+\Delta^-P(s)} + \chi_{\{t_0\}}(s)\right)\mathrm{d}P(s) + x(t-\delta)\frac{\Delta^-P(t-\delta)}{1+\Delta^-P(t-\delta)}\right)\\
&= x(t_0) + \int_{t_0}^{b} x(s)\left(\chi_{(t_0,t)}(s)\frac{1}{1+\Delta^-P(s)} + \chi_{\{t_0\}}(s)\right)\mathrm{d}P(s)\\
&= x(t) - \int_{t_0}^{t} x(s)\chi_{\{t\}}(s)\frac{1}{1+\Delta^-P(s)}\,\mathrm{d}P(s) = x(t) - x(t)\frac{\Delta^-P(t)}{1+\Delta^-P(t)} = x(t)\frac{1}{1+\Delta^-P(t)},
\end{aligned}
$$

which completes the proof of the first statement. The proof of the second statement is similar and is left to the reader. □

The next result describes what happens if P is left-continuous, right-continuous, or continuous.

Theorem 6.7. *Assume that a function $P\colon [a,b] \to \mathbb{C}$ has bounded variation and $t_0 \in [a,b]$.*

1. *If P is left-continuous and $1 + \Delta^+P(t) \neq 0$ for each $t \in [a, t_0)$, then $x\colon [a,b] \to \mathbb{C}$ is a solution of Eq.* (6.1) *if and only if*

   $$x(t) = x(t_0) + \int_{t_0}^{t} x(s)\,\mathrm{d}P(s), \quad t \in [a,b].$$

   Moreover, all solutions of Eq. (6.1) *have the form $x(t) = x(t_0)e_{\mathrm{d}P}(t,t_0)$.*

   If we in addition assume that $1 + \Delta^+P(t) \neq 0$ for each $t \in [t_0, b)$, then $y\colon [a,b] \to \mathbb{C}$ is a solution of Eq. (6.2) *if and only if*

   $$y(t) = y(t_0) - \int_{t_0}^{t} \frac{y(s)}{1+\Delta^+P(s)}\,\mathrm{d}P(s), \quad t \in [a,b],$$

   which holds if and only if

   $$y(t) = y(t_0) + \int_{t_0}^{t} y(s)\,\mathrm{d}(\ominus P)(s), \quad t \in [a,b].$$

   Moreover, all solutions of Eq. (6.2) *have the form $y(t) = y(t_0)e_{\mathrm{d}(\ominus P)}(t,t_0) = y(t_0)e_{\mathrm{d}P}(t,t_0)^{-1}$.*

2. *If P is right-continuous and $1+\Delta^- P(t)\neq 0$ for each $t\in(t_0,b]$, then $y\colon[a,b]\to\mathbb{C}$ is a solution of Eq.* (6.2) *if and only if*

$$y(t)=y(t_0)-\int_{t_0}^t y(s)\,\mathrm{d}P(s),\quad t\in[a,b].$$

Moreover, all solutions of Eq. (6.2) *have the form $y(t)=y(t_0)e_{\mathrm{d}(-P)}(t,t_0)$.*

If we in addition assume that $1+\Delta^- P(t)\neq 0$ for each $t\in(a,t_0]$, then $x\colon[a,b]\to\mathbb{C}$ is a solution of Eq. (6.1) *if and only if*

$$x(t)=x(t_0)+\int_{t_0}^t \frac{x(s)}{1+\Delta^- P(s)}\,\mathrm{d}P(s),\quad t\in[a,b],$$

which holds if and only if

$$x(t)=x(t_0)+\int_{t_0}^t x(s)\,\mathrm{d}(\ominus(-P))(s),\quad t\in[a,b].$$

Moreover, all solutions of Eq. (6.1) *have the form $x(t)=x(t_0)e_{\mathrm{d}(\ominus(-P))}(t,t_0)=x(t_0)e_{\mathrm{d}(-P)}(t,t_0)^{-1}$.*

3. *If P is continuous, then $x\colon[a,b]\to\mathbb{C}$ is a solution of Eq.* (6.1) *if and only if*

$$x(t)=x(t_0)+\int_{t_0}^t x(s)\,\mathrm{d}P(s),\quad t\in[a,b],$$

and $y\colon[a,b]\to\mathbb{C}$ is a solution of Eq. (6.2) *if and only if*

$$y(t)=y(t_0)-\int_{t_0}^t y(s)\,\mathrm{d}P(s),\quad t\in[a,b].$$

Moreover, all solutions have the form

$$x(t)=x(t_0)e_{\mathrm{d}P}(t,t_0)=x(t_0)e^{P(t)-P(t_0)},$$
$$y(t)=y(t_0)e_{\mathrm{d}(-P)}(t,t_0)=y(t_0)e^{P(t_0)-P(t)}.$$

Proof. To prove the first statement, assume that P is left-continuous. Noticing that $\Delta^- P(s)=0$ for every $s\in(a,b]$ and $\Delta P(s)=\Delta^+ P(s)$ for every $s\in[a,b)$, and using the first part of Theorem 6.6, we see that Eq. (6.1) is indeed equivalent to the equation

$$x(t)=x(t_0)+\int_{t_0}^t x(s)\,\mathrm{d}P(s),\quad t\in[a,b].$$

Clearly, all solutions of this equation have the form $x(t)=x(t_0)e_{\mathrm{d}P}(t,t_0)$.

Eq. (6.2) is equivalent to the two equations in the second part of Theorem 6.6, which in the left-continuous case read as follows:

- For every $t\in(t_0,b]$,

$$y(t)=y(t_0)-\int_{t_0}^t y(s)\left(\frac{1}{1+\Delta^+ P(s)}\chi_{(t_0,t)}(s)+\chi_{\{t\}}(s)+\frac{1}{1+\Delta^+ P(t_0)}\chi_{\{t_0\}}(s)\right)\mathrm{d}P(s).$$

- For every $t\in[a,t_0)$,

$$y(t)=y(t_0)+\int_t^{t_0} y(s)\left(\frac{1}{1+\Delta^+ P(s)}\chi_{[t,t_0)}(s)+\chi_{\{t_0\}}(s)\right)\mathrm{d}P(s).$$

However, note that in the first case, we have

$$\int_{t_0}^{t} y(s)\chi_{\{t\}}(s)\,\mathrm{d}P(s) = 0 = \int_{t_0}^{t} \frac{y(s)}{1+\Delta^+ P(s)}\chi_{\{t\}}(s)\,\mathrm{d}P(s),$$

because $\Delta^- P(t) = 0$. Similarly, in the second case, we

$$\int_{t}^{t_0} y(s)\chi_{\{t_0\}}(s)\,\mathrm{d}P(s) = 0 = \int_{t}^{t_0} \frac{y(s)}{1+\Delta^+ P(s)}\chi_{\{t_0\}}(s)\,\mathrm{d}P(s),$$

because $\Delta^- P(t_0) = 0$. Thus, we see that Eq. (6.2) is indeed equivalent to the equation

$$y(t) = y(t_0) - \int_{t_0}^{t} \frac{y(s)}{1+\Delta^+ P(s)}\,\mathrm{d}P(s), \quad t \in [a,b]. \tag{6.6}$$

Note that by Theorem B.7, we have

$$\begin{aligned}
-\int_{t_0}^{t} \frac{y(s)}{1+\Delta^+ P(s)}\,\mathrm{d}P(s) &= \int_{t_0}^{t} y(s)\,\mathrm{d}\left(\int_{t_0}^{s} \frac{-1}{1+\Delta^+ P(\tau)}\,\mathrm{d}P(\tau)\right) \\
&= \int_{t_0}^{t} y(s)\,\mathrm{d}\left(-P(t_0) - \int_{t_0}^{s} \frac{1}{1+\Delta^+ P(\tau)}\,\mathrm{d}P(\tau)\right).
\end{aligned}$$

However, since $(1+\Delta^+ P(\tau))^{-1} = 1$ for all τ with at most countably many exceptions, Lemma B.3 yields

$$\begin{aligned}
\int_{t_0}^{s} \frac{1}{1+\Delta^+ P(\tau)}\,\mathrm{d}P(\tau) &= P(s) - P(t_0) + \left(\frac{1}{1+\Delta^+ P(t_0)} - 1\right)\Delta^+ P(t_0) \\
&\quad + \sum_{\tau\in(t_0,s)} \left(\frac{1}{1+\Delta^+ P(\tau)} - 1\right)\Delta P(\tau) + \left(\frac{1}{1+\Delta^+ P(s)} - 1\right)\Delta^- P(s) \\
&= P(s) - P(t_0) - \sum_{\tau\in[t_0,s)} \frac{\Delta^+ P(\tau)^2}{1+\Delta^+ P(\tau)} = -(\ominus P)(s) - P(t_0),
\end{aligned}$$

and therefore Eq. (6.6) is indeed equivalent to the equation

$$y(t) = y(t_0) + \int_{t_0}^{t} y(s)\,\mathrm{d}(\ominus P)(s), \quad t \in [a,b].$$

Clearly, all solutions of this equation have the form $y(t) = y(t_0)e_{\mathrm{d}(\ominus P)}(t,t_0) = y(t_0)e_{\mathrm{d}P}(t,t_0)^{-1}$.

The proof of the second statement is similar, and is left to the reader. The third statement follows from the first and second one, and from the fact that if P is continuous, then $e_{\mathrm{d}P}(t,t_0) = e^{P(t)-P(t_0)}$ (see [13, Theorem 3.2] or [14, Theorem 8.5.3]). $\square$

We now return back to the general case when P need not be left-continuous or right-continuous, and show that Eq. (6.1) and Eq. (6.2) are equivalent to certain Volterra–Stieltjes integral equations. At the same time, we provide sufficient conditions for the existence and uniqueness of their solutions.

Theorem 6.8. *Assume that a function $P\colon [a,b] \to \mathbb{C}$ has bounded variation and $t_0 \in [a,b]$.*

1. *If $1 + \Delta^- P(t) \neq 0$ for each $t \in [t_0, b]$ and $1 + \Delta P(t) \neq 0$ for each $t \in (a, t_0)$, then a function $x\colon [a,b] \to \mathbb{C}$ is a solution of Eq. (6.1) if and only if it satisfies*

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s)\,\mathrm{d}K(t,s), \quad t \in [a,b], \tag{6.7}$$

where $K\colon [a,b]\times[a,b]\to\mathbb{C}$ *is given by*

$$K(t,s)=\begin{cases}\displaystyle\int_{t_0}^{\min(s,t)}\left(\frac{1}{1+\Delta^-P(\tau)}\chi_{(t_0,b]}(\tau)+\chi_{\{t_0\}}(\tau)\right)\mathrm{d}P(\tau) & t,s\in[t_0,b],\\ \displaystyle\int_{t_0}^{\max(s,t)}\left(\frac{1+\Delta^+P(\tau)}{1+\Delta P(\tau)}\chi_{(t,t_0)}(\tau)+\chi_{\{t\}}(\tau)+\frac{1}{1+\Delta^-P(t_0)}\chi_{\{t_0\}}(\tau)\right)\mathrm{d}P(\tau) & t,s\in[a,t_0],\\ 0 & \textit{otherwise.}\end{cases}$$

If we in addition assume that $1+\Delta^+P(t)\neq 0$ *for each* $t\in[a,t_0)$*, then for each* $x_0\in\mathbb{C}$*, Eq.* (6.7) *has a unique solution* $x\colon[a,b]\to\mathbb{C}$ *satisfying* $x(t_0)=x_0$*.*

2. *If* $1+\Delta^+P(t)\neq 0$ *for each* $t\in[a,t_0]$ *and* $1+\Delta P(t)\neq 0$ *for each* $t\in(t_0,b)$*, then a function* $y\colon[a,b]\to\mathbb{C}$ *is a solution of Eq.* (6.2) *if and only if it satisfies*

$$y(t)=y(t_0)+\int_{t_0}^{t}y(s)\,\mathrm{d}L(t,s),\quad t\in[a,b],\tag{6.8}$$

where $L\colon[a,b]\times[a,b]\to\mathbb{C}$ *is given by*

$$L(t,s)=\begin{cases}\displaystyle -\int_{t_0}^{\min(s,t)}\left(\frac{1+\Delta^-P(\tau)}{1+\Delta P(\tau)}\chi_{(t_0,t)}(\tau)+\chi_{\{t\}}(\tau)+\frac{1}{1+\Delta^+P(t_0)}\chi_{\{t_0\}}(\tau)\right)\mathrm{d}P(\tau) & t,s\in[t_0,b],\\ \displaystyle -\int_{t_0}^{\max(s,t)}\left(\frac{1}{1+\Delta^+P(\tau)}\chi_{[a,t_0)}(\tau)+\chi_{\{t_0\}}(\tau)\right)\mathrm{d}P(\tau) & t,s\in[a,t_0],\\ 0 & \textit{otherwise.}\end{cases}$$

If we in addition assume that $1+\Delta^-P(t)\neq 0$ *for each* $t\in(t_0,b]$*, then for each* $y_0\in\mathbb{C}$*, Eq.* (6.7) *has a unique solution* $y\colon[a,b]\to\mathbb{C}$ *satisfying* $y(t_0)=y_0$*.*

Proof. Let the assumptions of the first statement be satisfied. According to Theorem 6.6, a function $x\colon[t_0,b]\to\mathbb{C}$ is a solution of Eq. (6.1) if and only if for each $t\in(t_0,b]$, we have

$$x(t)=x(t_0)+\int_{t_0}^{t}x(s)\left(\frac{1}{1+\Delta^-P(s)}\chi_{(t_0,t]}(s)+\chi_{\{t_0\}}(s)\right)\mathrm{d}P(s).$$

Inside the integral we have $s\le t$, and therefore $\chi_{(t_0,t]}$ can be replaced by $\chi_{(t_0,b]}$. Using Theorem B.7, we get that, for $t>t_0$,

$$\begin{aligned}\int_{t_0}^{t}x(s)\left(\frac{\chi_{(t_0,b]}(s)}{1+\Delta^-P(s)}+\chi_{\{t_0\}}(s)\right)\mathrm{d}P(s)&=\int_{t_0}^{t}x(s)\,\mathrm{d}\left(\int_{t_0}^{s}\left(\frac{\chi_{(t_0,b]}(\tau)}{1+\Delta^-P(\tau)}+\chi_{\{t_0\}}(\tau)\right)\mathrm{d}P(\tau)\right)\\&=\int_{t_0}^{t}x(s)\,\mathrm{d}K(t,s).\end{aligned}$$

Similarly, according to Theorem 6.6, a function $x\colon[a,t_0]\to\mathbb{C}$ is a solution of Eq. (6.1) if and only if for each $t\in[a,t_0)$, we have

$$x(t)=x(t_0)-\int_{t}^{t_0}x(s)\left(\frac{1+\Delta^+P(s)}{1+\Delta P(s)}\chi_{(t,t_0)}(s)+\chi_{\{t\}}(s)+\frac{1}{1+\Delta^-P(t_0)}\chi_{\{t_0\}}(s)\right)\mathrm{d}P(s).$$

For $t<t_0$, using Theorem B.7, we get

$$-\int_{t}^{t_0}x(s)\left(\frac{1+\Delta^+P(s)}{1+\Delta P(s)}\chi_{(t,t_0)}(s)+\chi_{\{t\}}(s)+\frac{1}{1+\Delta^-P(t_0)}\chi_{\{t_0\}}(s)\right)\mathrm{d}P(s)=-\int_{t}^{t_0}x(s)\,\mathrm{d}K(t,s).$$

Thus, we have shown that Eq. (6.1) is indeed equivalent to Eq. (6.7).

Our next goal is to use Theorem 5.6 to investigate the existence and uniqueness of solutions on $[a, b]$. In this part of the proof, we assume that $1 + \Delta^+ P(t) \neq 0$ for each $t \in [a, t_0)$; this also implies that $1 + \Delta P(a) = 1 + \Delta^+ P(a) \neq 0$. Since

$$1 = \frac{1 + \Delta^+ P(\tau)}{1 + \Delta P(\tau)} + \frac{\Delta^- P(\tau)}{1 + \Delta P(\tau)},$$

we can write $K(t, s)$ in the alternative form

$$K(t,s) = \int_{t_0}^{\max(s,t)} \left(\frac{1 + \Delta^+ P(\tau)}{1 + \Delta P(\tau)} \chi_{[a,t_0)}(\tau) + \frac{\Delta^- P(\tau)}{1 + \Delta P(\tau)} \chi_{\{t\}}(\tau) + \frac{1}{1 + \Delta^- P(t_0)} \chi_{\{t_0\}}(\tau) \right) \mathrm{d}P(\tau)$$

for all $t, s \in [a, t_0]$. Note that we have also replaced $\chi_{[t,t_0)}(\tau)$ by $\chi_{[a,t_0)}(\tau)$ as $\tau \in [t, t_0]$. Moreover, we use the decomposition

$$K(t, s) = K_1(t, s) + K_2(t, s),$$

where

$$\begin{aligned}
K_1(t,s) &= \int_{t_0}^{\max(s,t)} \left(\frac{1 + \Delta^+ P(\tau)}{1 + \Delta P(\tau)} \chi_{[a,t_0)}(\tau) + \frac{1}{1 + \Delta^- P(t_0)} \chi_{\{t_0\}}(\tau) \right) \mathrm{d}P(\tau), \\
K_2(t,s) &= \int_{t_0}^{\max(s,t)} \frac{\Delta^- P(\tau)}{1 + \Delta P(\tau)} \chi_{\{t\}}(\tau) \,\mathrm{d}P(\tau).
\end{aligned}$$

For each $t \in [a, t_0)$, it follows from Theorem B.8 that

$$K_1(t, t+) - K_1(t, t) = \frac{1 + \Delta^+ P(t)}{1 + \Delta P(t)} \Delta^+ P(t).$$

Moreover, a simple calculation shows that

$$\begin{aligned}
K_2(t, t+) - K_2(t, t) &= \lim_{\delta \to 0+} \left(- \int_{t+\delta}^{t_0} \frac{\Delta^- P(\tau)}{1 + \Delta P(\tau)} \chi_{\{t\}}(\tau) \,\mathrm{d}P(\tau) + \int_t^{t_0} \frac{\Delta^- P(\tau)}{1 + \Delta P(\tau)} \chi_{\{t\}}(\tau) \,\mathrm{d}P(\tau) \right) \\
&= \frac{\Delta^- P(t)}{1 + \Delta P(t)} \Delta^+ P(t),
\end{aligned}$$

as the integral over $[t + \delta, t_0]$ equals zero, since $t < t + \delta$. Therefore,

$$1 + K(t, t+) - K(t, t) = 1 + \frac{1 + \Delta^+ P(t)}{1 + \Delta P(t)} \Delta^+ P(t) + \frac{\Delta^- P(t)}{1 + \Delta P(t)} \Delta^+ P(t) = 1 + \Delta^+ P(t) \neq 0.$$

On the other hand, for each $t \in (t_0, b]$, it follows from the definition of K and from Theorem B.8 that

$$K(t, t) - K(t, t-) = \frac{\Delta^- P(t)}{1 + \Delta^- P(t)},$$

and therefore

$$1 - K(t, t) + K(t, t-) = 1 - \frac{\Delta^- P(t)}{1 + \Delta^- P(t)} = \frac{1}{1 + \Delta^- P(t)} \neq 0.$$

Clearly, $\operatorname{var}(K(t_0, \cdot), [a, b])$ is finite, because $K(t_0, s) = 0$ for all $s \in [a, b]$.

Next, we verify that the Vitali variation of K over $[t_0, b] \times [t_0, b]$ is finite. Let $t_0 = \tau_0 < \tau_1 < \cdots < \tau_k = b$ and $t_0 = \sigma_0 < \sigma_1 < \cdots < \sigma_l = b$ be two partitions of $[t_0, b]$. We need to show that the expression

$$\sum_{i=1}^{k} \sum_{j=1}^{l} |K(\tau_i, \sigma_j) - K(\tau_{i-1}, \sigma_j) - K(\tau_i, \sigma_{j-1}) + K(\tau_{i-1}, \sigma_{j-1})|$$

has a finite upper bound that does not depend on the choice of the two partitions.

Note that the value $K(t,s)$ is given by an integral in which the integrand depends neither on t nor on s, and we are integrating over the interval $[t_0, \min(s,t)] = [t_0, s] \cap [t_0, t]$. In particular, $K(\tau_i, \sigma_j)$ is an integral over $[t_0, \tau_i] \cap [t_0, \sigma_j]$, $K(\tau_{i-1}, \sigma_j)$ is an integral over $[t_0, \tau_{i-1}] \cap [t_0, \sigma_j]$, and therefore $K(\tau_i, \sigma_j) - K(\tau_{i-1}, \sigma_j)$ is an integral over $[\tau_{i-1}, \tau_i] \cap [t_0, \sigma_j]$. Similarly, $K(\tau_i, \sigma_{j-1}) - K(\tau_{i-1}, \sigma_{j-1})$ is an integral over $[\tau_{i-1}, \tau_i] \cap [t_0, \sigma_{j-1}]$. Therefore,

$$K(\tau_i, \sigma_j) - K(\tau_{i-1}, \sigma_j) - K(\tau_i, \sigma_{j-1}) + K(\tau_{i-1}, \sigma_{j-1})$$

is an integral over $[\tau_{i-1}, \tau_i] \cap [\sigma_{j-1}, \sigma_j]$. Thus, if we denote

$$M_1 = \sup_{\tau \in [t_0, b]} \left| \frac{1}{1 + \Delta^- P(\tau)} \chi_{(t_0, b]}(\tau) + \chi_{\{t_0\}}(\tau) \right|,$$

Theorem B.8 yields the estimate

$$|K(\tau_i, \sigma_j) - K(\tau_{i-1}, \sigma_j) - K(\tau_i, \sigma_{j-1}) + K(\tau_{i-1}, \sigma_{j-1})| \le M_1 \operatorname{var}(P, [\tau_{i-1}, \tau_i] \cap [\sigma_{j-1}, \sigma_j]),$$

and therefore

$$\begin{aligned}\sum_{i=1}^k \sum_{j=1}^l |K(\tau_i, \sigma_j) - K(\tau_{i-1}, \sigma_j) - K(\tau_i, \sigma_{j-1}) + K(\tau_{i-1}, \sigma_{j-1})| &\le M_1 \sum_{i=1}^k \sum_{j=1}^l \operatorname{var}(P, [\tau_{i-1}, \tau_i] \cap [\sigma_{j-1}, \sigma_j]) \\ &= M_1 \operatorname{var}(P, [t_0, b]),\end{aligned}$$

which shows that the Vitali variation of K over $[t_0, b] \times [t_0, b]$ is finite. Thus, by Theorem 5.6 applied to the interval $[t_0, b]$, we see that for each $x_0 \in \mathbb{C}$, Eq. (6.7) has a unique solution $x\colon [t_0, b] \to \mathbb{C}$ satisfying $x(t_0) = x_0$.

It remains to consider the Vitali variation of K over $[a, t_0] \times [a, t_0]$. Using the decomposition

$$K(t,s) = -K_3(t,s) - K_4(t,s),$$

where

$$\begin{aligned} K_3(t,s) &= \int_{\max(s,t)}^{t_0} \left(\frac{1 + \Delta^+ P(\tau)}{1 + \Delta P(\tau)} \chi_{[a, t_0)}(\tau) + \frac{1}{1 + \Delta^- P(t_0)} \chi_{\{t_0\}}(\tau) \right) \mathrm{d}P(\tau), \\ K_4(t,s) &= \int_{\max(s,t)}^{t_0} \frac{\Delta^- P(\tau)}{1 + \Delta P(\tau)} \chi_{\{t\}}(\tau) \, \mathrm{d}P(\tau), \end{aligned}$$

it suffices to show that the Vitali variations of K_3 and K_4 over $[a, t_0] \times [a, t_0]$ are finite.

Note that the value $K_3(t,s)$ is given by an integral in which the integrand depends neither on t nor on s, and we are integrating over the interval $[\max(s,t), t_0] = [s, t_0] \cap [t, t_0]$. Thus, arguing as in the previous part of the proof, if we denote

$$M_2 = \sup_{\tau \in [a, t_0]} \left| \frac{1 + \Delta^+ P(\tau)}{1 + \Delta P(\tau)} \chi_{[a, t_0)}(\tau) + \frac{1}{1 + \Delta^- P(t_0)} \chi_{\{t_0\}}(\tau) \right|,$$

and choose partitions $a = \tau_0 < \tau_1 < \cdots < \tau_k = t_0$ and $a = \sigma_0 < \sigma_1 < \cdots < \sigma_l = t_0$, we obtain the estimate

$$|K_3(\tau_i, \sigma_j) - K_3(\tau_{i-1}, \sigma_j) - K_3(\tau_i, \sigma_{j-1}) + K_3(\tau_{i-1}, \sigma_{j-1})| \le M_2 \operatorname{var}(P, [\tau_{i-1}, \tau_i] \cap [\sigma_{j-1}, \sigma_j]).$$

Consequently,

$$\sum_{i=1}^k \sum_{j=1}^l |K_3(\tau_i, \sigma_j) - K_3(\tau_{i-1}, \sigma_j) - K_3(\tau_i, \sigma_{j-1}) + K_3(\tau_{i-1}, \sigma_{j-1})| \le M_2 \sum_{i=1}^k \sum_{j=1}^l \operatorname{var}(P, [\tau_{i-1}, \tau_i] \cap [\sigma_{j-1}, \sigma_j])$$

$$= M_2 \operatorname{var}(P,[a,t_0]),$$

which shows that the Vitali variation of K_3 over $[a,t_0]\times[a,t_0]$ is finite.

Concerning K_4, we observe that

$$K_4(t,s) = \begin{cases} \dfrac{\Delta^- P(t)\Delta^+ P(t)}{1+\Delta P(t)} & \text{if } t \geq s, \\ 0 & \text{if } t < s. \end{cases}$$

Therefore,

$$K_4(\tau_i,\sigma_j) - K_4(\tau_i,\sigma_{j-1}) = \begin{cases} -\dfrac{\Delta^- P(\tau_i)\Delta^+ P(\tau_i)}{1+\Delta P(\tau_i)} & \text{if } \sigma_{j-1} \leq \tau_i < \sigma_j, \\ 0 & \text{if } \tau_i \geq \sigma_j \text{ or } \tau_i < \sigma_{j-1}. \end{cases}$$

Note that for a fixed $i \in \{1,\dots,k\}$, the condition $\sigma_{j-1} \leq \tau_i < \sigma_j$ holds for exactly one $j \in \{1,\dots,l\}$. Consequently,

$$\sum_{j=1}^{l} |K_4(\tau_i,\sigma_j) - K_4(\tau_i,\sigma_{j-1})| = \left|\frac{\Delta^- P(\tau_i)\Delta^+ P(\tau_i)}{1+\Delta P(\tau_i)}\right|.$$

Similarly,

$$\sum_{j=1}^{l} |K_4(\tau_{i-1},\sigma_j) - K_4(\tau_{i-1},\sigma_{j-1})| = \left|\frac{\Delta^- P(\tau_{i-1})\Delta^+ P(\tau_{i-1})}{1+\Delta P(\tau_{i-1})}\right|.$$

It follows that

$$\sum_{i=1}^{k}\sum_{j=1}^{l} |K_4(\tau_i,\sigma_j) - K_4(\tau_{i-1},\sigma_j) - K_4(\tau_i,\sigma_{j-1}) + K_4(\tau_{i-1},\sigma_{j-1})| \leq 2\sum_{i=0}^{k} \left|\frac{\Delta^- P(\tau_i)\Delta^+ P(\tau_i)}{1+\Delta P(\tau_i)}\right|.$$

Therefore, denoting

$$M_3 = \sup_{\tau\in[a,t_0]} \left|\frac{\Delta^- P(\tau)}{1+\Delta P(\tau)}\right|,$$

we conclude that

$$\begin{aligned} \sum_{i=1}^{k}\sum_{j=1}^{l} |K_4(\tau_i,\sigma_j) - K_4(\tau_{i-1},\sigma_j) - K_4(\tau_i,\sigma_{j-1}) + K_4(\tau_{i-1},\sigma_{j-1})| &\leq 2M_3 \sum_{i=0}^{k} |\Delta^+ P(\tau_i)| \\ &\leq 2M_3 \operatorname{var}(P,[a,t_0]), \end{aligned}$$

which shows that the Vitali variation of K_4 over $[a,t_0]\times[a,t_0]$ is finite.

Thus, by Theorem 5.6 applied to the interval $[a,t_0]$, we see that for each $x_0 \in \mathbb{C}$, Eq. (6.7) has a unique solution $x\colon [a,t_0] \to \mathbb{C}$ satisfying $x(t_0) = x_0$.

We leave the proof of the second part to the reader; one can perform similar calculations as in the first part, or use the symmetry described in Remark 6.4. □

Remark 6.9. The proof of the previous theorem shows that the kernels K and L satisfy the assumptions of Theorem 5.6. Thus, if $f\colon [a,b] \to \mathbb{C}$ has bounded variation, the assumptions of Theorem 6.8 guarantee the existence and uniqueness of solutions also for the nonhomogeneous Volterra–Stieltjes integral equations

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s)\,\mathrm{d}K(t,s) + f(t) - f(t_0), \quad t \in [a,b],$$

$$y(t) = y(t_0) + \int_{t_0}^{t} y(s)\,\mathrm{d}L(t,s) + f(t) - f(t_0), \quad t \in [a,b],$$

or, equivalently, the nonhomogeneous versions of Eq. (6.1) and Eq. (6.2), namely

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s-)\,\mathrm{d}P(s) + f(t) - f(t_0), \quad t \in [a,b],$$
$$y(t) = y(t_0) - \int_{t_0}^{t} y(s+)\,\mathrm{d}P(s) + f(t) - f(t_0), \quad t \in [a,b],$$

where we are using the same convention regarding the values at the endpoints as in Eq. (6.1) and Eq. (6.2).

7 Relations between Stieltjes integral equations and Stieltjes differential equations

Let us verify that the adjoint Stieltjes differential equations introduced in Section 3 represent a special case of the adjoint Stieltjes integral equations studied in Section 6.

As in Section 2, let $g\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function, and consider an interval $[a,b] \subset \mathbb{R}$.

Proposition 7.1. *Given a function $p \in \mathcal{L}^1_g([a,b],\mathbb{C})$, choose an arbitrary $t_0 \in [a,b]$ and let*

$$P(t) = \int_{t_0}^{t} p(s)\,\mathrm{d}g(s), \quad t \in [a,b].$$

Then the following statements hold:

1. *A function $x\colon [a,b] \to \mathbb{C}$ is a solution of the Stieltjes differential equation*

$$x'_g(t) = p(t)x(t) \tag{7.1}$$

   if and only if

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s)\,\mathrm{d}P(s), \quad t \in [a,b], \tag{7.2}$$

   which holds if and only if x is a solution of Eq. (6.1)*.*

2. *Suppose that $1 + p(t)\Delta^+ g(t) \neq 0$ for all $t \in [a,b) \cap D_g$. Then a function $y\colon [a,b] \to \mathbb{C}$ is a solution of the Stieltjes differential equation*

$$y'_g(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)}\,y(t), \quad t \in [a,b], \tag{7.3}$$

   if and only if

$$y(t) = y(t_0) - \int_{t_0}^{t} \frac{y(s)}{1 + \Delta^+ P(s)}\,\mathrm{d}P(s), \quad t \in [a,b], \tag{7.4}$$

   which holds if and only if y is a solution of Eq. (6.2)*.*

Proof. Note that g is left-continuous, and therefore P has the same property (see Theorem B.8).

Let us prove the first statement.

First, observe that, according to Definition 3.1 and Theorems 2.8 and 2.10, a function $x\colon [a,b] \to \mathbb{C}$ is a solution of Eq. (7.1) if and only if

$$x(t) = x(a) + \int_{[a,t)} p(s)x(s)\,\mathrm{d}\mu_g(s), \quad t \in [a,b],$$

which is equivalent to

$$x(t) - x(t_0) = \int_{[a,t)} p(s)x(s)\,\mathrm{d}\mu_g(s) - \int_{[a,t_0)} p(s)x(s)\,\mathrm{d}\mu_g(s), \quad t \in [a,b].$$

Hence, it follows that $x\colon [a,b] \to \mathbb{C}$ is a solution of Eq. (7.1) if and only if

$$x(t) = \begin{cases} x(t_0) - \displaystyle\int_{[t,t_0)} p(s)x(s)\,\mathrm{d}\mu_g(s) & \text{if } t \le t_0, \\ x(t_0) + \displaystyle\int_{[t_0,t)} p(s)x(s)\,\mathrm{d}\mu_g(s) & \text{if } t > t_0. \end{cases} \tag{7.5}$$

We claim that Eq. (7.5) holds if and only

$$x(t) = x(t_0) + \int_{t_0}^{t} p(s)x(s)\,\mathrm{d}g(s), \quad t \in [a,b]. \tag{7.6}$$

Indeed, Eq. (7.5) implies Eq. (7.6), because Lebesgue–Stieltjes integrability implies Kurzweil–Stieltjes integrability. On the other hand, if Eq. (7.6) holds, then x is regulated (see Theorem B.8), therefore bounded and μ_g-measurable. Consequently, the function $p \cdot x$ is μ_g-integrable (being the product of an integrable and a bounded measurable function), and Eq. (7.5) holds.

Now, by Theorem B.7, Eq. (7.6) is equivalent to Eq. (7.2). The fact that Eq. (7.2) holds if and only if x is a solution of Eq. (6.1) follows from the fact that P is left-continuous, and from the first part of Theorem 6.7.

The proof of the second statement is similar. One merely needs to note that $\frac{-p}{1+p\Delta^+ g}$ is μ_g-integrable (see Lemma 3.4), and that $p(s)\Delta^+ g(s) = \Delta^+ P(s)$ for each $s \in [a,b)$. ☐

8 Relations between Stieltjes integral equations and dynamic equations

Finally, let us describe the relation between dynamic equations on time scales and the Stieltjes integral equations studied in Section 6. In comparison with Section 4, we will discuss dynamic equations with Δ-derivatives as well as ∇-derivatives.

Let $\mathbb{T}$ be a time scale. Throughout this section, we work with a fixed time scale interval $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$, where $a, b \in \mathbb{T}$, $a < b$. We introduce the functions

$$\begin{aligned} g(t) &= \inf\{s \in \mathbb{T} : s \ge t\}, \quad t \in [a,b], \\ h(t) &= \sup\{s \in \mathbb{T} : s \le t\}, \quad t \in [a,b]. \end{aligned}$$

Note that g is the restriction of the function introduced in Eq. (4.3) to the interval $[a,b]$. We have $g(t) = h(t) = t$ for every $t \in [a,b]_{\mathbb{T}}$, the function g is left-continuous, and the function h is right-continuous.

The homogeneous linear Δ and ∇-dynamic equations have the form

$$\begin{aligned} x^{\Delta}(t) &= p(t)x(t), \\ y^{\nabla}(t) &= q(t)y(t). \end{aligned}$$

For our purposes, it is more convenient to consider their integral forms

$$x(t) = x(t_0) + \int_{t_0}^{t} p(s)x(s)\,\Delta s, \quad t \in [a,b]_{\mathbb{T}}, \tag{8.1}$$

$$y(t) = y(t_0) + \int_{t_0}^{t} q(s)y(s)\,\nabla s, \quad t \in [a,b]_{\mathbb{T}}, \tag{8.2}$$

where the integrals on the right-hand sides are the Kurzweil Δ- and ∇-integrals (see Appendix A). For simplicity, we restrict ourselves to real-valued dynamic equations, but note that all considerations remain valid for complex-valued dynamic equations.

The next theorem, which follows from the results of [14, Section 8.7] (see Theorem 8.7.1 and the discussion after Lemma 8.7.2 there), shows that the dynamic equations (8.1) and (8.2) are equivalent to certain generalized ODEs.

Theorem 8.1. *The following statements hold:*

1. *Suppose that $p\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ is a Kurzweil Δ-integrable function, and let $\widetilde{p}\colon [a,b] \to \mathbb{R}$ be an arbitrary function such that $\widetilde{p}(t) = p(t)$ for all $t \in [a,b]_{\mathbb{T}}$. If $x\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ is a solution of Eq.* (8.1)*, then the function $x^*\colon [a,b] \to \mathbb{R}$ given by $x^*(t) = x(g(t))$ satisfies*

$$x^*(t) = x^*(t_0) + \int_{t_0}^{t} \widetilde{p}(s)x^*(s)\,\mathrm{d}g(s), \quad t \in [a,b], \tag{8.3}$$

   as well as

$$x^*(t) = x^*(t_0) + \int_{t_0}^{t} x^*(s)\,\mathrm{d}P(s), \quad t \in [a,b], \tag{8.4}$$

   where $P\colon [a,b] \to \mathbb{R}$ is given by

$$P(t) = \int_{t_0}^{t} \widetilde{p}(s)\,\mathrm{d}g(s), \quad t \in [a,b].$$

   Conversely, each function $x^*\colon [a,b] \to \mathbb{R}$ satisfying Eq. (8.3) *or Eq.* (8.4) *has the form $x^*(t) = x(g(t))$, where $x\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ is a solution of Eq.* (8.1)*.*

2. *Suppose that $q\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ is a Kurzweil ∇-integrable function, and let $\widetilde{q}\colon [a,b] \to \mathbb{R}$ be an arbitrary function such that $\widetilde{q}(t) = q(t)$ for all $t \in [a,b]_{\mathbb{T}}$. If $y\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ is a solution of Eq.* (8.2)*, then the function $y_*\colon [a,b] \to \mathbb{R}$ given by $y_*(t) = y(h(t))$ satisfies*

$$y_*(t) = y_*(t_0) + \int_{t_0}^{t} \widetilde{q}(s)y_*(s)\,\mathrm{d}h(s), \quad t \in [a,b], \tag{8.5}$$

   as well as

$$y_*(t) = y_*(t_0) + \int_{t_0}^{t} y_*(s)\,\mathrm{d}Q(s), \quad t \in [a,b], \tag{8.6}$$

   where $Q\colon [a,b] \to \mathbb{R}$ is given by

$$Q(t) = \int_{t_0}^{t} \widetilde{q}(s)\,\mathrm{d}h(s), \quad t \in [a,b].$$

   Conversely, each function $y_*\colon [a,b] \to \mathbb{R}$ satisfying Eq. (8.5) *or Eq.* (8.6) *has the form $y_*(t) = y(h(t))$, where $y\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ is a solution of Eq.* (8.2)*.*

The next lemma describes some properties of the functions P, Q introduced in the previous theorem.

Lemma 8.2. *Suppose that $p, q\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ and $P, Q\colon [a,b] \to \mathbb{R}$ are as in* Theorem 8.1.

1. *If there exists a Kurzweil Δ-integrable function $m\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ such that*

$$\Big|\int_u^v p(s)\,\Delta s\Big| \le \int_u^v m(s)\,\Delta s \quad \textit{whenever } u,v \in [a,b]_{\mathbb{T}},\ u \le v,$$

then P has bounded variation, it is left-continuous, and

$$\Delta^+ P(t) = \begin{cases} p(t)\,\mu(t) & \textit{if } \ t \in [a,b)_{\mathbb{T}}, \\ 0 & \textit{if } \ t \in [a,b)\backslash\mathbb{T}. \end{cases}$$

2. *If there exists a Kurzweil ∇-integrable function $l\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ such that*

$$\Big|\int_u^v q(s)\,\nabla s\Big| \le \int_u^v l(s)\,\nabla s \quad \textit{whenever } u,v \in [a,b]_{\mathbb{T}},\ u \le v,$$

then Q has bounded variation, it is right-continuous, and

$$\Delta^- Q(t) = \begin{cases} p(t)\,\nu(t) & \textit{if } \ t \in [a,b)_{\mathbb{T}}, \\ 0 & \textit{if } \ t \in [a,b)\backslash\mathbb{T}. \end{cases}$$

Using the results developed in the previous section, let us find the adjoint equations to Eq. (8.1) and Eq. (8.2).

First, let us focus on Δ-dynamic equations. Assume that $p\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Kurzweil Δ-integrable and satisfies the integrable majorant condition of Lemma 8.2. We know that Eq. (8.1) is equivalent (in the sense of Theorem 8.1) to the generalized ODE

$$x^*(t) = x^*(t_0) + \int_{t_0}^t x^*(s)\,\mathrm{d}P(s), \quad t \in [a,b].$$

According to Lemma 8.2, the function P is left-continuous and has bounded variation. Interpreting the last equation as a special case of Eq. (6.1), the adjoint equation has the form

$$y^*(t) = y^*(t_0) - \int_{t_0}^t y^*(s+)\,\mathrm{d}P(s), \quad t \in [a,b],$$

in which $y^*(s+)$ need not be replaced by $y^*(s)$ for $s = \max(t,t_0)$, because P is left-continuous and therefore the value of the integrand at $\max(t,t_0)$ does not matter. By Theorem B.7, the last equation is equivalent to

$$y^*(t) = y^*(t_0) - \int_{t_0}^t \widetilde{p}(s) y^*(s+)\,\mathrm{d}g(s), \quad t \in [a,b].$$

Note that g is constant on each interval $(\alpha,\beta] \subset [a,b]$ such that $(\alpha,\beta) \cap \mathbb{T} = \emptyset$. Thus, y^* has the same property, and $y^*(s+) = y^*(\sigma(s))$ for each $s \in [a,b]_{\mathbb{T}}$. Consequently, the last equation is equivalent to the dynamic equation

$$y(t) = y(t_0) - \int_{t_0}^t p(s) y(\sigma(s))\,\Delta s, \quad t \in [a,b]. \tag{8.7}$$

The assumption of the first part of Theorem 6.7, namely the condition $1 + \Delta^+ P(t) \neq 0$ for all $t \in [a,b)$, holds if and only if $1 + p(t)\mu(t) \neq 0$ for all $t \in [a,b)_{\mathbb{T}}$; this condition is well known as the μ-regressivity. If this condition holds, then according to Theorem 6.7, the adjoint equation can be also written in the form

$$y^*(t) = y^*(t_0) - \int_{t_0}^t \frac{y^*(s)}{1 + \Delta^+ P(s)}\,\mathrm{d}P(s), \quad t \in [a,b],$$

or equivalently

$$y^*(t) = y^*(t_0) - \int_{t_0}^{t} \frac{\widetilde{p}(s)y^*(s)}{1+\Delta^+P(s)}\,\mathrm{d}g(s), \quad t \in [a,b].$$

This equation is equivalent (in the sense of Theorem 8.1) to the dynamic equation

$$y(t) = y(t_0) - \int_{t_0}^{t} \frac{p(s)y(s)}{1+p(s)\mu(s)}\,\Delta s, \quad t \in [a,b]_{\mathbb{T}}. \tag{8.8}$$

The solutions of Eq. (8.1) and its adjoint equation (8.7) or (8.8) can be expressed in terms of the generalized exponential function $e_{\mathrm{d}P}$, which coincides with the time scale Δ-exponential function e_p; we get

$$\begin{aligned} x(t) &= x(t_0)e_{\mathrm{d}P}(t,t_0) = x(t_0)e_p(t,t_0),\\ y(t) &= y(t_0)e_{\mathrm{d}P}(t,t_0)^{-1} = y(t_0)e_p(t,t_0)^{-1}. \end{aligned}$$

Note that Eq. (8.7) and Eq. (8.8) are integral forms of the dynamic equations

$$y^{\Delta}(t) = -p(t)y(\sigma(t)), \qquad y^{\Delta}(t) = -\frac{p(t)y(t)}{1+p(t)\mu(t)}.$$

Thus, our results agree with the known results for linear Δ-dynamic equations that can be found e.g. in [2, Section 2.4]. Note that the authors of [2] also deal with the nonhomogeneous equation

$$x^{\Delta}(t) = p(t)x(t) + f(t), \tag{8.9}$$

whose integral form is

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s) + f(s))\,\Delta s, \quad t \in [a,b]_{\mathbb{T}}. \tag{8.10}$$

The corresponding generalized ODE is

$$x^*(t) = x^*(t_0) + \int_{t_0}^{t} x^*(s)\,\mathrm{d}P(s) + F(t) - F(t_0),$$

where

$$F(t) = \int_{t_0}^{t} \widetilde{f}(s)\,\mathrm{d}g(s), \quad t \in [a,b],$$

and $\widetilde{f}\colon [a,b] \to \mathbb{R}$ is an arbitrary function such that $\widetilde{f}(t) = f(t)$ for all $t \in [a,b]_{\mathbb{T}}$. Since F is left-continuous, the variation of constants formula of Theorem 5.4 yields

$$x^*(t) = x^*(t_0)e_{\mathrm{d}P}(t,t_0) + \int_{t_0}^{t} e_{\mathrm{d}P}(t,s+)\,\mathrm{d}F(s) = x^*(t_0)e_{\mathrm{d}P}(t,t_0) + \int_{t_0}^{t} e_{\mathrm{d}P}(t,s+)\widetilde{f}(s)\,\mathrm{d}g(s).$$

Switching back to the time scale setting, we see that the solution of Eq. (8.10) is

$$x(t) = x(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(s))f(s)\,\Delta s,$$

which agrees with [2, Theorem 2.77], but note that our approach is more general and does not require p and f to be rd-continuous.

According to [2], the adjoint equation to Eq. (8.9) is defined to be

$$y^{\Delta}(t) = -p(t)y(\sigma(t)) + f(t).$$

Using the formula $y(\sigma(t)) = y(t) + y^{\Delta}(t)\mu(t)$, the last equation can be rewritten in the equivalent form

$$y^{\Delta}(t) = -\frac{p(t)}{1+p(t)\mu(t)}y(t) + \frac{f(t)}{1+p(t)\mu(t)}.$$

Denote $q(t) = -\frac{p(t)}{1+p(t)\mu(t)}$, $t \in [a,b]_{\mathbb{T}}$. In the time scale calculus, this function is traditionally denoted by $q(t) = \ominus p(t)$. However, we avoid this notation because we use the symbol $\ominus$ in a different meaning (cf. Theorem 5.3). The integral form of the previous equation is

$$y(t) = y(t_0) + \int_{t_0}^{t} q(s)y(s)\,\Delta s + \int_{t_0}^{t} \frac{f(s)}{1+p(s)\mu(s)}\,\Delta s, \quad t \in [a,b]_{\mathbb{T}}. \tag{8.11}$$

The corresponding generalized ODE is

$$y^*(t) = y^*(t_0) + \int_{t_0}^{t} y^*(s)\,\mathrm{d}Q(s) + F_2(t) - F_2(t_0), \quad t \in [a,b], \tag{8.12}$$

where (note that $1 + \Delta^+ P(s) = 1 + p(s)\mu(s)$ for each $t \in [a,b)_{\mathbb{T}}$)

$$F_2(t) = \int_{t_0}^{t} \frac{\widetilde{f}(s)}{1+\Delta^+ P(s)}\,\mathrm{d}g(s) = \int_{t_0}^{t} \frac{1}{1+\Delta^+ P(s)}\,\mathrm{d}F(s), \qquad t \in [a,b],$$
$$Q(t) = \int_{t_0}^{t} \frac{-\widetilde{p}(s)}{1+\Delta^+ P(s)}\,\mathrm{d}g(s) = -\int_{t_0}^{t} \frac{1}{1+\Delta^+ P(s)}\,\mathrm{d}P(s), \quad t \in [a,b].$$

We observe that $Q = \ominus P$; this fact is easily verified by calculating the last integral using Lemma B.3, recalling that P is left-continuous and $P(t_0) = 0$ (alternatively, see the proofs of [13, Theorem 5.5] or [14, Theorem 8.7.8]).

Since F_2 is left-continuous, Eq. (8.12) is a special case of Eq. (5.5) from Theorem 5.5. According to this theorem, the unique solution is

$$y^*(t) = y^*(t_0)e_{\mathrm{d}Q}(t,t_0) + \int_{t_0}^{t} e_{\mathrm{d}Q}(t,s)\,\mathrm{d}F(s) = y^*(t_0)e_{\mathrm{d}Q}(t,t_0) + \int_{t_0}^{t} e_{\mathrm{d}Q}(t,s)\widetilde{f}(s)\,\mathrm{d}g(s).$$

Returning to the time scale setting, we obtain the solution of nonhomogeneous adjoint equation (8.11):

$$y(t) = y(t_0)e_q(t,t_0) + \int_{t_0}^{t} e_q(t,s)f(s)\,\Delta s, \quad t \in [a,b]_{\mathbb{T}}.$$

This formula agrees with the result of [2, Theorem 2.74].

A similar analysis can be performed for ∇-dynamic equations. Assume that $q\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Kurzweil ∇-integrable and satisfies the integrable majorant condition of Lemma 8.2. Eq. (8.2) is equivalent (in the sense of Theorem 8.1) to the generalized ODE

$$y_*(t) = y_*(t_0) + \int_{t_0}^{t} y_*(s)\,\mathrm{d}Q(s), \quad t \in [a,b].$$

According to Lemma 8.2, the function Q is right-continuous and has bounded variation. Interpreting the last equation as a special case of Eq. (6.2) with P replaced by $-Q$, the adjoint equation has the form

$$x_*(t) = x_*(t_0) - \int_{t_0}^{t} x_*(s-)\,\mathrm{d}Q(s), \quad t \in [a,b],$$

in which $x_*(s-)$ need not be replaced by $x_*(s)$ for $s = \min(t, t_0)$, because Q is right-continuous and therefore the value of the integrand at $\min(t, t_0)$ does not matter. By Theorem B.7, the last equation is equivalent to

$$x_*(t) = x_*(t_0) - \int_{t_0}^{t} \widetilde{q}(s)x_*(s-)\,\mathrm{d}h(s), \quad t \in [a,b].$$

Note that h is constant on each interval $[\alpha, \beta) \subset [a,b]$ such that $(\alpha, \beta) \cap \mathbb{T} = \emptyset$. Thus, x_* has the same property, and $x_*(s-) = x_*(\rho(s))$ for each $s \in [a,b]_{\mathbb{T}}$. Consequently, the last equation is equivalent to the dynamic equation

$$x(t) = x(t_0) - \int_{t_0}^{t} q(s)x(\rho(s))\,\nabla s, \quad t \in [a,b]. \tag{8.13}$$

The assumption of the second part of Theorem 6.7 with P replaced by $-Q$, namely the condition $1 - \Delta^- Q(t) \neq 0$ for all $t \in (a,b]$, holds if and only if $1 - q(t)\nu(t) \neq 0$ for all $t \in (a,b]_{\mathbb{T}}$; this condition is known as the ν-regressivity. If this condition holds, then according to Theorem 6.7, the adjoint equation can be also written in the form

$$x_*(t) = x_*(t_0) - \int_{t_0}^{t} \frac{x_*(s)}{1 - \Delta^- Q(s)}\,\mathrm{d}Q(s), \quad t \in [a,b],$$

or equivalently

$$x_*(t) = x_*(t_0) - \int_{t_0}^{t} \frac{\widetilde{q}(s)x_*(s)}{1 - \Delta^- Q(s)}\,\mathrm{d}h(s), \quad t \in [a,b].$$

This equation is equivalent (in the sense of Theorem 8.1) to the dynamic equation

$$x(t) = x(t_0) - \int_{t_0}^{t} \frac{q(s)x(s)}{1 - q(s)\nu(s)}\,\nabla s, \quad t \in [a,b]_{\mathbb{T}}. \tag{8.14}$$

The solutions of Eq. (8.2) and its adjoint equation can be expressed in terms of the generalized exponential function $e_{\mathrm{d}Q}$, which coincides with the time scale ∇-exponential function $\widehat{e}_q$; we get

$$\begin{aligned} y(t) &= y(t_0)e_{\mathrm{d}Q}(t,t_0) = y(t_0)\widehat{e}_q(t,t_0), \\ x(t) &= x(t_0)e_{\mathrm{d}Q}(t,t_0)^{-1} = x(t_0)\widehat{e}_q(t,t_0)^{-1}. \end{aligned}$$

Note that Eq. (8.13) and Eq. (8.14) are integral forms of the dynamic equations

$$x^{\nabla}(t) = -q(t)x(\rho(t)), \qquad x^{\nabla}(t) = -\frac{q(t)x(t)}{1 - q(t)\rho(t)}.$$

Once again, our results agree with the known results for linear ∇-dynamic equations that can be found e.g. in [1].

One can also consider the nonhomogeneous ∇-dynamic equation

$$y^{\nabla}(t) = q(t)y(t) + f(t). \tag{8.15}$$

According to [1], the adjoint equation is defined to be

$$x^{\nabla}(t) = -q(t)x(\rho(t)) + f(t).$$

Using the formula $x(\rho(t)) = x(t) - x^{\nabla}(t)\nu(t)$, the last equation can be rewritten in the equivalent form

$$x^{\nabla}(t) = -\frac{q(t)}{1 - q(t)\nu(t)}x(t) + \frac{f(t)}{1 - q(t)\nu(t)}. \tag{8.16}$$

Rewriting equations (8.15) and (8.16) as nonhomogeneous generalized linear ODEs and using Theorems 5.4 and 5.5 (the right-continuous case), one can recover the solutions formulas presented in [1, Theorem 3.42] and [1, Theorem 3.39]; the calculations are similar to those we have performed for Δ-dynamic equations, and we leave them to the reader.

We point out that the integral forms of the dynamic equations we have dealt with do not require rd-continuity or ld-continuity of the functions p and q, respectively. Indeed, it suffices if p is μ-regressive and satisfies the assumption of the first part of Lemma 8.2 (or, equivalently, it is Lebesgue Δ-integrable), and q is ν-regressive and satisfies the assumption of the second part of Lemma 8.2 (or, equivalently, it is Lebesgue ∇-integrable). Under these hypotheses, the Δ- and ∇-exponential functions e_p and $\widehat{e}_q$ can be introduced as the unique solutions of appropriate integral equations; see [13, Theorem 5.4] or [14, Theorem 8.7.7].

Appendix A Basic notions of the time scale calculus

The time scale calculus, which originated in the work of S. Hilger [6], is a popular tool that provides a unification of the continuous and discrete calculus, as well as differential and difference equations. It is concerned with functions $f\colon \mathbb{T} \to \mathbb{R}$, where $\mathbb{T}$ is a time scale – an arbitrary nonempty closed set $\mathbb{T} \subset \mathbb{R}$. The choice $\mathbb{T} = \mathbb{R}$ leads to the classical continuous calculus, while $\mathbb{T} = \mathbb{Z}$ corresponds to the discrete calculus. Another frequently studied time scale is $\mathbb{T} = q^{\mathbb{Z}} = \{q^n : n \in \mathbb{Z}\}$, where $q > 1$; this leads to the quantum calculus. The basic operations of the time scale calculus are the Δ-derivative, ∇-derivative, Δ-integral, and ∇-integral. We refer to these concepts in Sections 4 and 8. To make the paper self-contained, we recall their definitions in this appendix. More information can be found in [2].

We begin with the forward/backward jump operators and the graininess functions. If $t \in \mathbb{T}$ and $t < \sup \mathbb{T}$, we define the forward jump σ and the forward graininess μ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \mu(t) = \sigma(t) - t. \tag{A.1}$$

If $t = \sup \mathbb{T} < \infty$, we define $\sigma(t) = t$, $\mu(t) = 0$.

Similarly, if $t \in \mathbb{T}$ and $t > \inf \mathbb{T}$, we define the backward jump ρ and the backward graininess ν by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \qquad \nu(t) = t - \rho(t). \tag{A.2}$$

If $t = \inf \mathbb{T} > -\infty$, we define $\rho(t) = t$, $\nu(t) = 0$.

For an arbitrary interval $I \subset \mathbb{R}$, we denote $I_{\mathbb{T}} = I \cap \mathbb{T}$.

Given a function $f\colon \mathbb{T} \to \mathbb{R}$, we introduce the Δ-derivative and the ∇-derivative of f at a point $t \in \mathbb{T}$ as follows.

Definition A.1. Consider a function $f\colon \mathbb{T} \to \mathbb{R}$ and a point $t \in \mathbb{T}$.

(i) Suppose that $t < \sup \mathbb{T}$, or $t = \sup \mathbb{T}$ and $\rho(t) = t$. We say that the Δ-derivative $f^{\Delta}(t)$ exists and equals $D \in \mathbb{R}$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(\sigma(t)) - f(s) - D(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|$$

for all $s \in (t - \delta, t + \delta)_{\mathbb{T}}$.

(ii) Suppose that $t > \inf \mathbb{T}$, or $t = \inf \mathbb{T}$ and $\sigma(t) = t$. We say that the ∇-derivative $f^{\nabla}(t)$ exists and equals $D \in \mathbb{R}$, if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(\rho(t)) - f(s) - D(\rho(t) - s)| < \varepsilon|\rho(t) - s|$$

for all $s \in (t - \delta, t + \delta)_{\mathbb{T}}$.

The following remarks should help to clarify the meaning of Δ- and ∇-derivatives:

- If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f^\nabla(t) = f'(t)$, i.e., both derivatives coincide with the classical derivative.
- If $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = f(t+1) - f(t)$ and $f^\nabla(t) = f(t) - f(t-1)$, i.e., the Δ- and ∇-derivative reduce to the forward difference and backward difference, respectively.
- More generally, if $t \in \mathbb{T}$ satisfies $\sigma(t) > t$, then

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

  Similarly, if $t \in \mathbb{T}$ satisfies $\rho(t) < t$, then

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{t - \rho(t)} = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

We now turn our attention to the Δ- and ∇-integrals of a function $f\colon [a,b]_\mathbb{T} \to \mathbb{R}$. For our purposes, it is most convenient to deal with the Kurzweil integrals, which were introduced in [16] (see also [14, Section 8.6]), and are based on the following concept of δ-fine partitions.

Definition A.2. Consider a pair of functions $\delta_L, \delta_R\colon [a,b]_\mathbb{T} \to (0,\infty)$. Then $\delta = (\delta_L, \delta_R)$ is called a Δ-gauge on $[a,b]_\mathbb{T}$ if $\delta_R(t) \geq \mu(t)$ for all $t \in [a,b)_\mathbb{T}$, and a ∇-gauge on $[a,b]_\mathbb{T}$ if $\delta_L(t) \geq \nu(t)$ for all $t \in (a,b]_\mathbb{T}$. Suppose that $\delta = (\delta_L, \delta_R)$ is either a Δ-gauge or a ∇-gauge on $[a,b]_\mathbb{T}$. A partition $a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b$, where $\alpha_j \in \mathbb{T}$ for all $j \in \{0,\ldots,m\}$, together with a collection of tags $\xi_j \in [\alpha_{j-1}, \alpha_j]_\mathbb{T}$, $j \in \{1,\ldots,m\}$, is called δ-fine if

$$[\alpha_{j-1}, \alpha_j] \subset [\xi_j - \delta_L(\xi_j), \xi_j + \delta_R(\xi_j)] \quad \text{for all } j \in \{1,\ldots,m\}.$$

Definition A.3. Consider a function $f\colon [a,b]_\mathbb{T} \to \mathbb{R}$.

(i) We say that the Kurzweil Δ-integral $\int_a^b f(t)\,\Delta t$ exists and equals $I \in \mathbb{R}$, if for every $\varepsilon > 0$, there is a Δ-gauge δ on $[a,b]_\mathbb{T}$ such that

$$\left| \sum_{j=1}^m f(\xi_j)(\alpha_j - \alpha_{j-1}) - I \right| < \varepsilon$$

for all δ-fine partitions $a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b$, where $\alpha_j \in \mathbb{T}$ for all $j \in \{0,\ldots,m\}$, with tags $\xi_j \in [\alpha_{j-1}, \alpha_j]_\mathbb{T}$, $j \in \{1,\ldots,m\}$.

(ii) We say that the Kurzweil ∇-integral $\int_a^b f(t)\,\nabla t$ exists and equals $I \in \mathbb{R}$, if for every $\varepsilon > 0$, there is a ∇-gauge δ on $[a,b]_\mathbb{T}$ such that

$$\left| \sum_{j=1}^m f(\xi_j)(\alpha_j - \alpha_{j-1}) - I \right| < \varepsilon$$

for all δ-fine partitions $a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b$, where $\alpha_j \in \mathbb{T}$ for all $j \in \{0,\ldots,m\}$, with tags $\xi_j \in [\alpha_{j-1}, \alpha_j]_\mathbb{T}$, $j \in \{1,\ldots,m\}$.

We have dealt only with real-valued functions; however, the derivative and integral of a complex-valued function are easily obtained by decomposition into a real and imaginary part.

Appendix B Kurzweil–Stieltjes integrals

This appendix is intended for readers who are not familiar with the Kurzweil–Stieltjes integral. We recall the definition of the integral and some of its basic properties that were needed throughout the paper. For more information, see e.g. [14, 19].

Definition B.1. Consider a pair of functions $f, g\colon [a,b] \to \mathbb{R}$. We say that the Kurzweil–Stieltjes integral $\int_a^b f \,\mathrm{d}g$ exists and equals $I \in \mathbb{R}$, if for every $\varepsilon > 0$, there is a function $\delta : [a,b] \to (0,\infty)$ such that

$$\left| \sum_{j=1}^{m} f(\xi_j)(g(\alpha_j) - g(\alpha_{j-1})) - I \right| < \varepsilon$$

for all partitions $a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b$ with tags $\xi_j \in [\alpha_{j-1}, \alpha_j]$, $j \in \{1, \ldots, m\}$, such that $[\alpha_{j-1}, \alpha_j] \subset [\xi_j - \delta(\xi_j), \xi_j - \delta(\xi_j)]$.

A simple and useful sufficient condition for the existence of the Kurzweil–Stieltjes integral is as follows: If the functions f and g are regulated and one of them has bounded variation, then $\int_a^b f \,\mathrm{d}g$ exists (cf. [14, Theorem 6.13.1]). This sufficient condition is always satisfied in Sections 6–8.

Given a regulated function $g\colon [a,b] \to \mathbb{C}$, we use the following notation:

$$\Delta^+ g(t) = \begin{cases} g(t+) - g(t) & \text{if } t \in [a,b), \\ 0 & \text{if } t = b, \end{cases} \qquad \Delta^- g(t) = \begin{cases} g(t) - g(t-) & \text{if } t \in (a,b], \\ 0 & \text{if } t = a. \end{cases} \tag{B.1}$$

Also, we let $\Delta g(t) = \Delta^+ g(t) + \Delta^- g(t)$ for $t \in [a,b]$.

If $I \subset \mathbb{R}$ is an interval, $h\colon I \to \mathbb{R}$ is a function which is zero except on a countable set $\{t_1, t_2, \ldots\} \subset I$, and the sum $S = \sum_i h(t_i)$ is absolutely convergent, we write $S = \sum_{x \in I} h(x)$.

We sometimes encounter functions of the following type:

$$F_1(t) = \sum_{s \in [t_0, t)} f(s), \qquad F_2(t) = \sum_{s \in (t_0, t]} f(s), \qquad F_3(t) = \sum_{s \in (t_0, t)} f(s),$$

where $t_0 \in [a,b]$ is fixed and $t \in [a,b]$. For $t < t_0$, the above sums should be interpreted as follows:

$$F_1(t) = -\sum_{s \in [t, t_0)} f(s), \qquad F_2(t) = -\sum_{s \in (t, t_0]} f(s), \qquad F_3(t) = -\sum_{s \in (t, t_0)} f(s),$$

The next three lemmas show how to evaluate certain types of Kurzweil–Stieltjes integrals. The first result can be found in [22, Proposition 2.12].

Lemma B.2. *Let $f\colon [a,b] \to \mathbb{R}$ be a function which is zero except on a countable set $\{t_1, t_2, \ldots\} \subset [a,b]$ and $\sum_i f(t_i)$ be absolutely convergent. Then, for every regulated function $g\colon [a,b] \to \mathbb{R}$, we have*

$$\int_a^b f \,\mathrm{d}g = \sum_{x \in [a,b]} f(x) \Delta g(x),$$

with the convention that $\Delta g(a) = \Delta^+ g(a)$ and $\Delta g(b) = \Delta^- g(b)$.

Every function $h\colon [a,b] \to \mathbb{R}$ with bounded variation has at most countably many discontinuities, and $\sum_{x \in [a,b]} (|\Delta^+ h(x)| + |\Delta^- h(x)|)$ is finite (see [14, Corollary 2.3.8]). Hence, both $f = \Delta^+ h$ and $f = \Delta^- h$ satisfy the assumptions of Lemma B.2.

The second result is taken from [14, Lemma 6.3.16].

Lemma B.3. *Let $f\colon [a,b] \to \mathbb{R}$ be a regulated function such that $f(x) = c$ for all $x \in [a,b] \setminus D$, where $D \subset [a,b]$ is at most countable. If $g\colon [a,b] \to \mathbb{R}$ has bounded variation, then*

$$\int_a^b f \,\mathrm{d}g = c(g(b) - g(a)) + (f(a) - c)\Delta^+ g(a) + \sum_{x \in D \cap (a,b)} (f(x) - c)\Delta g(x) + (f(b) - c)\Delta^- g(b).$$

The third result follows from [14, Lemmas 6.3.15 and 6.3.16].

Lemma B.4. *Let $f, g\colon [a,b] \to \mathbb{R}$ be regulated functions and suppose at least one of them has bounded variation. If $g(x) = 0$ for all $x \in [a,b] \setminus D$, where $D \subset [a,b]$ is at most countable, then*

$$\int_a^b f \,\mathrm{d}g = f(b)g(b) - f(a)g(a).$$

The following version of the integration by parts formula for the Kurzweil–Stieltjes integral can be found in [13] (see Theorem 2.3 and Remark 2.4 there); it is obtained by a simple manipulation from the more common version of the integration by parts formula presented in [14, Theorem 6.4.2].

Theorem B.5. *If $f, g\colon [a,b] \to \mathbb{R}$ are regulated and at least one of them has bounded variation, then*

$$\int_a^b f \,\mathrm{d}g + \int_a^b g \,\mathrm{d}f = f(b)g(b) - f(a)g(a) + \sum_{x \in [a,b]} (\Delta^- f(x)\Delta g(x) - \Delta f(x)\Delta^+ g(x))$$

with the convention that $\Delta^- f(a) = 0$ and $\Delta^+ g(b) = 0$.

Combining Theorem B.5 with Lemma B.2, we obtain yet another version of the integration by parts formula.

Theorem B.6. *If $f, g\colon [a,b] \to \mathbb{R}$ have bounded variation, then*

$$\int_a^b f(x-) \,\mathrm{d}g(x) + \int_a^b g(x+) \,\mathrm{d}f(x) = f(b)g(b) - f(a)g(a)$$

with the convention that $f(a-) = f(a)$ and $g(b+) = g(b)$, i.e.,

$$\int_a^b \big(f(x-)\chi_{(a,b]}(x) + f(a)\chi_{\{a\}}(x)\big) \,\mathrm{d}g(x) + \int_a^b \big(g(x+)\chi_{[a,b)}(x) + g(b)\chi_{\{b\}}(x)\big) \,\mathrm{d}f(x) = f(b)g(b) - f(a)g(a).$$

Proof. By Lemma B.2, we have

$$\sum_{x \in [a,b]} \Delta^- f(x)\Delta g(x) = \int_a^b \Delta^- f(x) \,\mathrm{d}g(x), \qquad \sum_{x \in [a,b]} \Delta f(x)\Delta^+ g(x) = \int_a^b \Delta^+ g(x) \,\mathrm{d}f(x).$$

The proof is finished by substituting these results into the identity from Theorem B.5 and recalling that $\Delta^- f(x) = f(x) - f(x-)$ for $x \in (a,b]$, and $\Delta^+ g(x) = g(x+) - g(x)$ for $x \in [a,b)$. □

The substitution theorem for the Kurzweil–Stieltjes integral reads as follows (see [14, Theorem 6.6.1]).

Theorem B.7. *Assume that $h\colon [a,b] \to \mathbb{R}$ is a bounded function and $f, g\colon [a,b] \to \mathbb{R}$ are such that $\int_a^b f \,\mathrm{d}g$ exists. Then*

$$\int_a^b h(x) \,\mathrm{d}\left(\int_a^x f(z) \,\mathrm{d}g(z)\right) = \int_a^b h(x) f(x) \,\mathrm{d}g(x),$$

whenever either side of the equation exists.

The next result describes the properties of indefinite Kurzweil–Stieltjes integrals (see [14, Corollary 6.5.5]).

Theorem B.8. *Consider a pair of functions $f, g\colon [a,b] \to \mathbb{R}$ such that g is regulated and $\int_a^b f\,\mathrm{d}g$ exists. Then, for every $t_0 \in [a,b]$, the function*

$$h(t) = \int_{t_0}^t f\,\mathrm{d}g, \quad t \in [a,b]$$

is regulated and satisfies

$$\begin{aligned} h(t+) &= h(t) + f(t)\Delta^+ g(t), \quad t \in [a,b), \\ h(t-) &= h(t) - f(t)\Delta^- g(t), \quad t \in (a,b]. \end{aligned}$$

Moreover, if f is regulated and g has bounded variation, then h has bounded variation, and

$$\left|\int_a^b f\,\mathrm{d}g\right| \le \sup_{t\in[a,b]} |f(t)| \cdot \operatorname{var}(g,[a,b]).$$

A simple but useful convergence theorem for the Kurzweil–Stieltjes integral is the following bounded convergence theorem (see [14, Theorem 6.8.13] or [15, Theorem 6.3]).

Theorem B.9. *Suppose that a function $g\colon [a,b] \to \mathbb{R}$ has bounded variation and let $f_n\colon [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions satisfying the following conditions:*

(i) *The integral $\int_a^b f_n\,\mathrm{d}g$ exists for each $n \in \mathbb{N}$.*

(ii) $\lim_{n\to\infty} f_n(t) = f(t)$ *for all $t \in [a,b]$.*

(iii) *There exists a constant $M \ge 0$ such that $|f_n(t)| \le M$ for all $n \in \mathbb{N}$ and $t \in [a,b]$.*

Then the integral $\int_a^b f\,\mathrm{d}g$ exists and

$$\lim_{n\to\infty} \left(\sup_{t\in[a,b]} \left| \int_a^t f_n\,\mathrm{d}g - \int_a^t f\,\mathrm{d}g \right| \right) = 0.$$

Given a pair of functions $f, g\colon [a,b] \to \mathbb{C}$ with real parts f_1, g_1 and imaginary parts f_2, g_2, we define

$$\int_a^b f\,\mathrm{d}g = \int_a^b (f_1 + i f_2)\,\mathrm{d}(g_1 + i g_2) = \int_a^b f_1\,\mathrm{d}g_1 - \int_a^b f_2\,\mathrm{d}g_2 + i\left(\int_a^b f_1\,\mathrm{d}g_2 + \int_a^b f_2\,\mathrm{d}g_1\right)$$

whenever the integrals on the right-hand side exist. All results mentioned in this appendix are still valid for complex-valued functions.

Appendix C List of symbols

The present appendix provides a list of symbols used throughout the paper, including links to places where they are introduced or mentioned.

χ_A	characteristic (indicator) function of a set $A \subset \mathbb{R}$
$f(t+)$, $f(t-)$	one-sided limits of a function f at a point t

$\Delta^+ f(t)$, $\Delta^- f(t)$, $\Delta f(t)$	jumps of a function f at a point t (Appendix B, Eq. (B.1))
$\mathrm{var}(f, [a, b])$	variation of a function f on interval $[a, b]$
C_g, D_g, N_g^+, N_g^-, N_g	classification of points with respect to a function g (Section 2)
$\sum_{x\in I} f(x)$	sum of all nonzero values of f on interval I (Appendix B)
f'_g	Stieltjes derivative of f with respect to g (Definition 2.1)
μ_g	Lebesgue–Stieltjes measure corresponding to a function g (Section 2)
$\int_X f \,\mathrm{d}\mu_g$	Lebesgue–Stieltjes integral of f with respect to μ_g (Section 2)
$\int_a^b f \,\mathrm{d}g$	Kurzweil–Stieltjes integral of f with respect to g (Definition B.1)
$\mathcal{L}^1_g(X, \mathbb{C})$	space of Lebesgue–Stieltjes integrable functions on X (Section 2)
$\mathcal{AC}_g([a, b], \mathbb{C})$	space of g-absolutely continuous functions on $[a, b]$ (Section 2)
$\mathcal{G}([a, b], \mathbb{C})$	space of regulated functions on $[a, b]$ (Section 6)
$\exp_g$	g-exponential function (Section 3, Eq. (3.11))
$e_{\mathrm{d}P}$	generalized exponential function (Definition 5.2)
$\ominus P$	a function such that $e_{\mathrm{d}P} \cdot e_{\mathrm{d}\ominus P} = 1$ (Theorem 5.3)
$\mathbb{T}$	time scale, closed nonempty subset of $\mathbb{R}$ (Appendix A)
$I_{\mathbb{T}}$	time scale interval $I \cap \mathbb{T}$ (Appendix A)
σ, μ	forward jump and graininess (Appendix A, Eq. (A.1))
ρ, ν	backward jump and graininess (Appendix A, Eq. (A.2))
f^Δ, f^∇	Δ- and ∇-derivatives of a function f (Definition A.1)
$\int_a^b f(t)\,\Delta t$, $\int_a^b f(t)\,\nabla t$	Kurzweil Δ- and ∇-integrals of a function f (Definition A.3)

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