COMPLETE METRIC ABSOLUTE NEIGHBORHOOD RETRACTS

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Abstract. We characterize complete metric absolute (neighborhood) retracts in terms of existence of certain maps of CW-polytopes. Using our result, we prove that a compact metric space with a convex and locally convex simplicial structure is an AR. This answers a question of Kulpa from [5]. As another application, we prove that the hyperspace of closed subsets of a separable Banach space endowed with the Wijsman topology is an absolute retract.

1. Introduction

A metrizable space X is an *absolute (neighborhood) retract* (briefly: AR (ANR)) if it is a retract of (an open subset of) a normed linear space containing X as a closed subset. There are several known characterizations of ANR's stated in terms of maps of CW-polytopes. Probably the most well-known is Dugundji-Lefschetz' theorem about realizations of polytopes. Another result in this spirit is due to Nhu [7].

We introduce a metric property (Property (B) below) which, roughly speaking, says that there is a sequence of maps of CW-polytopes with some 'compatibility' conditions, related to the metric. We prove (Section 2) that a complete metric space with this property is an ANR; a stronger version of Property (B) (called Property (B*)) implies that the space is an AR. It appears that Property (B) characterizes ANR's among complete metric spaces; we also give an example of a (non-complete) metric space with Property (B), which is not an ANR. Property (B) does not require the existence of extensions of any maps, which is required in Dugundji-Lefschetz' characterization. In Section 3 we show that the realization property of Dugundji-Lefschetz implies Property (B). This gives a proof of Dugundji-Lefschetz' theorem in the case of completely metrizable spaces.

The last section is devoted to applications. First we consider simplicial structures introduced by Kulpa [5] and we show that a compact metric space with a convex and locally convex simplicial structure is an AR. This solves Kulpa's problem from [5].

As a second application, we study hyperspaces of closed sets endowed with the Wijsman topology. This topology is important and useful in the analysis of set convergence and in optimization theory; for references see Beer's book [1]. We show that if a given metric space has the property that after removing finitely many closed balls, the remaining part is path-wise connected, then its Wijsman hyperspace has Property (B*). Consequently, the Wijsman hyperspace of a separable Banach space is an absolute retract.

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1.1. **Notation.** We denote by $[X]^{<\omega}$ and $[X]^n$ the collection of all finite and n-element subsets of X respectively; ω denotes the set of all nonnegative integers. Given any set S we shall denote by $\Sigma(S)$ the union of all geometric simplices with vertices in S, endowed with the CW-topology. More precisely, $\Sigma(S)$ is the set of all formal convex combinations of the form $\sum_{s\in S}\lambda_s s$, where $\lambda_s = 0$ for all but finitely many $s \in S$; a subset $U \subset \Sigma(S)$ is open if its intersection with any simplex σ of $\Sigma(S)$ is open in σ (with respect to the standard topology on σ). When S is finite, $\Sigma(S)$ is called the *geometric* or *abstract simplex* with the set of vertices S. A *subsimplex* (or a *face*) of a simplex $\Sigma(S)$ is, by definition, a simplex $\Sigma(T)$, where $T \subset S$. The *boundary* of a simplex $\sigma = \Sigma(S)$ is $\operatorname{bd}\sigma = \bigcup_{T\subset S,\, T\neq S}\Sigma(T)$.

An *abstract polytope* is a space of the form $P = \bigcup_{T\in\mathcal{A}}\Sigma(T)$, where $\mathcal{A}$ is any family of sets, endowed with the CW-topology, i.e. P is a subspace of $\Sigma(S)$, where $S = \bigcup\mathcal{A}$. S is the *set of vertices* of P and we write $S = \operatorname{vert} P$. Observe that P can also be written as $\bigcup_{T\in\mathcal{A}_0}\Sigma(T)$, where $\mathcal{A}_0 = \{T\colon (\exists\, T' \in \mathcal{A})\ T \in [T']^{<\omega}\}$. A *subpolytope* of an abstract polytope $P = \bigcup_{T\in\mathcal{A}}\Sigma(T)$ is a polytope $Q = \bigcup_{R\in\mathcal{B}}\Sigma(R)$ such that every simplex of Q (i.e. a face of $\Sigma(R)$ for some $R \in \mathcal{B}$) is also a simplex of P, i.e. for every $R \in \mathcal{B}$ there exists $T \in \mathcal{A}$ with $R \subset T$. A polytope P is *convex* if $P = \bigcup_{T\in[S]^{<\omega}}\Sigma(T)$, where $S = \operatorname{vert} P$. Clearly, this agrees with the definition of the convex hull of S, when we consider a polytope as a subset of a real linear space.

By a *polytope* in a topological space Y we mean a continuous map $\varphi\colon P \to Y$ of an abstract polytope P. We say that $S = \operatorname{vert} P$ is the *set of vertices* of φ and we write $S = \operatorname{vert}\varphi$. Sometimes φ is called a *singular polytope* in Y or a *realization of* P in Y. We define the notion of subpolytope, convex polytope and simplex like in the abstract case.

2. Main result

Let $\mathcal{U}$ be a collection of subsets of a topological space Y. We write $A \prec \mathcal{U}$ whenever A is a set contained in some element of $\mathcal{U}$. A polytope φ in Y is $\mathcal{U}$*-dense* if $U \cap \varphi[\operatorname{vert}\varphi] \neq \emptyset$ whenever $U \in \mathcal{U}\setminus\{\emptyset\}$. Let $\mathcal{U},\mathcal{V}$ be two open covers of Y. We say that a polytope φ is $(\mathcal{U},\mathcal{V})$*-compatible* if for every finite set $S \subset \operatorname{vert}\varphi$ we have $\Sigma(S) \subset \operatorname{dom}\varphi$ and $\varphi[\Sigma(S)] \prec \mathcal{V}$ whenever $\varphi[S] \prec \mathcal{U}$. By $\operatorname{mesh}\mathcal{U}$ we mean the supremum of diameters of members of $\mathcal{U}$.

We say that a metric space Y has *Property* (B) provided there exists a sequence of open covers $\{\mathcal{U}_n\}_{n\in\omega}$ satisfying the following conditions:

- (a) for each $n \in \omega$, $\mathcal{U}_{n+1}$ is a star-refinement of $\mathcal{U}_n$,
- (b) $\sum_{n\in\omega}\operatorname{mesh}\mathcal{U}_n < +\infty$,
- (c) for each $n \in \omega$ there is $m > n + 5$ and there exists a $\mathcal{U}_{m+1}$-dense polytope in Y, which is simultaneously $(\mathcal{U}_m,\mathcal{U}_{n+5})$- and $(\mathcal{U}_{n+1},\mathcal{U}_n)$-compatible.

If additionally, for some $n \in \omega$ there is a convex polytope satisfying (c) then we say that Y has *Property* (B*).

Theorem 1. *Every complete metric space with Property* (B) *is an absolute neighborhood retract. A complete metric space with Property* (B*) *is an absolute retract.*

Proof. We start with two lemmas. We assume here that Y is a complete metric space with Property (B), A is a closed subset of a metrizable space X and $f\colon A \to Y$ is a fixed continuous map.

Lemma 1. *Let $n > 0$ and suppose that $g\colon X \to Y$ is a continuous map such that $g|A$ is $\mathcal{U}_{n+3}$-close to f. Then there exists a continuous map $g'\colon X \to Y$ which is $\mathcal{U}_{n-1}$-close to g and $\mathcal{U}_{n+4}$-close to f on A.*

Proof. Let $m > n+5$ be as in condition (c) of Property (B). Set $\mathcal{U} = \mathcal{U}_{m+1}$. For $U \in \mathcal{U}$ define

$$U^* = f^{-1}[U] \cup (g^{-1}[\operatorname{star}(U,\mathcal{U}_{n+3})] \setminus A).$$

Observe that $\{U^*\}_{U\in\mathcal{U}}$ is an open cover of X. Let $\{h_U\}_{U\in\mathcal{U}}$ be a locally finite partition of unity such that $h_U^{-1}[(0,1]] \subset U^*$ for $U \in \mathcal{U}$. By condition (c) of Property (B), there exists a $\mathcal{U}$-dense polytope φ in Y which is simultaneously $(\mathcal{U}_m,\mathcal{U}_{n+5})$- and $(\mathcal{U}_{n+1},\mathcal{U}_n)$-compatible. For each $U \in \mathcal{U} \setminus \{\emptyset\}$ choose $y_U \in \operatorname{vert}\varphi$ such that $\varphi(y_U) \in U$.

Fix $t \in X$ and consider $\mathcal{U}_t = \{U \in \mathcal{U}\colon h_U(t) > 0\}$. Then $g(t) \in \operatorname{star}(U,\mathcal{U}_{n+3})$ for $U \in \mathcal{U}_t$. Let $S_t = \{y_U\colon U \in \mathcal{U}_t\}$. Then $\varphi[S_t] \subset \operatorname{star}(g(t),\mathcal{U}_{n+2}) \prec \mathcal{U}_{n+1}$ and hence $\Sigma(S_t) \subset \operatorname{dom}\varphi$. Define a map $g'\colon X \to Y$ by setting

$$g'(t) = \varphi\Big(\sum_{U\in\mathcal{U}} h_U(t)y_U\Big).$$

Clearly g' is continuous and $\mathcal{U}_{n-1}$-close to g since $\{g'(t)\}\cup\varphi[S_t] \prec \mathcal{U}_n$ and $\{g(t)\}\cup\varphi[S_t] \prec \mathcal{U}_{n+1}$.

Suppose now that $t \in A$. Then $f(t) \in \bigcap\mathcal{U}_t$ and consequently $\varphi[S_t]\cup\{f(t)\} \subset \operatorname{star}(f(t),\mathcal{U}) \prec \mathcal{U}_m$. Thus $\varphi[S_t] \cup \{g'(t)\} \in \varphi[\Sigma(S_t)] \prec \mathcal{U}_{n+5}$ which means that $g'|A$ is $\mathcal{U}_{n+4}$-close to f. □

Lemma 2. *There exists an open set $W \supset A$ and a continuous map $g\colon W \to Y$ which is $\mathcal{U}_4$-close to f on A. If Y has Property* (B*) *then we may assume that $W = X$.*

Proof. Applying condition (c) of Property (B) (for $n = 0$) we get $m > 5$ and a polytope φ which is $\mathcal{U}_{m+1}$-dense and $(\mathcal{U}_m,\mathcal{U}_5)$-compatible. Set $\mathcal{U} = \mathcal{U}_{m+1}$. By paracompactness, there is a locally finite open cover $\{H_U\}_{U\in\mathcal{U}}$ of X such that $A\cap \operatorname{cl} H_U \subset f^{-1}[U]$ for every $U \in \mathcal{U}$. Set

$$V_U = H_U \setminus \bigcup\{\operatorname{cl} H_G\colon G \in \mathcal{U} \;\&\; A \cap \operatorname{cl} H_G \cap H_U = \emptyset\}.$$

Observe that each V_U is open in X, $A \subset \bigcup_{U\in\mathcal{U}} V_U$ and $V_{U_1} \cap V_{U_2} \neq \emptyset$ implies $U_1 \cap U_2 \neq \emptyset$. The last property follows from the fact that if $V_{U_1} \cap V_{U_2} \neq \emptyset$ then there is $t \in A\cap \operatorname{cl} H_{U_1} \cap H_{U_2}$ and consequently $f(t) \in U_1 \cap U_2$. Let $W = \bigcup_{U\in\mathcal{U}} V_U$ and let $\{h_U\}_{U\in\mathcal{U}}$ be a locally finite partition of unity in W such that $h_U^{-1}[(0,1]] \subset V_U$ for every $U \in \mathcal{U}$.

Now, for each $U \in \mathcal{U} \setminus \{\emptyset\}$ choose $y_U \in \operatorname{vert}\varphi$ so that $\varphi(y_U) \in U$. Define

$$(*) \qquad g(t) = \varphi\Big(\sum_{U\in\mathcal{U}} h_U(t)y_U\Big), \qquad t \in W.$$

Observe that g is well-defined, since if $\mathcal{U}_t = \{U \in \mathcal{U}\colon h_U(t) > 0\}$ then $\{\varphi(y_U)\colon U \in \mathcal{U}_t\} \subset \operatorname{star}(U_0,\mathcal{U})$, where $U_0 \in \mathcal{U}_t$ is arbitrary (because $U_1 \cap U_2 \neq \emptyset$ whenever $U_1, U_2 \in \mathcal{U}_t$) and consequently $\Sigma(\{y_U : U \in \mathcal{U}_t\}) \subset \operatorname{dom}\varphi$. As in the proof of the previous Lemma, one can check that $g|A$ is $\mathcal{U}_4$-close to f.

Finally, if Y has Property (B*) then we may assume that φ is a convex polytope, so formula $(*)$ well defines a continuous map on the entire space X. Thus, in this case we can set $W = X$. $\square$

Theorem 1 follows immediately from Lemma 1 and Lemma 2. Indeed, using Lemma 2 we get a continuous map $g_0 : W \to Y$ which is $\mathcal{U}_4$-close to f, where $W \supset A$ is open. If Y has Property (B*) then $W = X$. Now we can use inductively Lemma 1 to obtain a sequence of continuous maps $g_n : W \to Y$ such that g_{n+1} is $\mathcal{U}_{n-1}$-close to g_n and $\mathcal{U}_{n+4}$-close to f on A. By condition (b) of Property (B), the sequence $\{g_n\}_{n\in\omega}$ converges uniformly to a continuous map $f' : W \to Y$ which is an extension of f (here we have used the completeness of Y). $\square$

We now show that every metric ANR/AR has Property (B)/(B*).

Proposition 1. *Let Y be a metric ANR. Then there exists a polytope φ in Y with $\operatorname{vert}\varphi = Y$ and there exists a sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open covers of Y such that for each $n \in \omega$, $\operatorname{mesh}\mathcal{U}_n \leqslant 2^{-n}$, $\mathcal{U}_{n+1}$ is a star-refinement of $\mathcal{U}_n$ and φ is $(\mathcal{U}_{n+1}, \mathcal{U}_n)$-compatible. If additionally, Y is an AR then φ is a convex polytope.*

Proof. By the theorem of Arens-Eells we can assume that Y is a closed subset of a normed linear space E. Let $r : W \to Y$ be a retraction, where $W \supset Y$ is open in E. Define

$$P = \bigcup\{\Sigma(S) : S \in [Y]^{<\omega} \ \&\ \operatorname{conv}_E S \subset W\} \subset \Sigma(Y).$$

Let $\psi : P \to E$ be the unique affine map with $\psi|Y = \mathrm{id}_Y$. Then $\varphi = r\psi$ is a polytope in Y with $\operatorname{vert}\varphi = Y$. Let $\mathcal{U}_0$ be any open cover of Y with mesh $\leqslant 1$. Suppose that covers $\mathcal{U}_0, \ldots, \mathcal{U}_n$ are already defined so that $\operatorname{mesh}\mathcal{U}_i < 2^{-(i+1)}$ and they satisfy conditions (a) and (b). By the continuity of r, there exists an open cover $\mathcal{V}$ of W, consisting of convex sets and such that $\{r[V] : V \in \mathcal{V}\}$ is a refinement of $\mathcal{U}_n$. Now let $\mathcal{U}_{n+1}$ be a star-refinement of $\mathcal{U}_n$ with mesh $\leqslant 2^{-(n+1)}$, which is also a refinement of $\mathcal{V}$. Then φ is $(\mathcal{U}_{n+1}, \mathcal{U}_n)$-compatible. Finally, if Y is an AR then $W = E$ and hence $P = \Sigma(Y)$. $\square$

Below we describe an example of a separable metric space with Property (B*), which is not an ANR. Thus, the completeness assumption in Theorem 1 is essential.

Example 1. Consider the Hilbert cube $Q = [0,1]^\omega$ endowed with the product metric. There exists a sequence $\{A_n\}_{n\in\omega}$ of pairwise disjoint dense convex subsets of Q. Indeed, if $\{B_n\}_{n\in\omega}$ is a decomposition of ω into infinite sets then we can set

$$A_n = \{x \in Q : \ \exists\, i \in B_n \ \big(x(i) > 0 \ \&\ (\forall\, j > i)\ x(j) = 0\big)\}.$$

Now, for each $n \in \omega$ choose finite $D_n \subset A_n$ which is $1/n$-dense in Q and define $Y = \bigcup_{n\in\omega} \operatorname{conv} D_n$. Clearly, Y is dense in Q, so Q is the completion of Y.

Let $\{\mathcal{U}_n\}_{n\in\omega}$ be a sequence of finite open covers of Y such that $\operatorname{mesh}\mathcal{U}_n \leqslant 2^{-n}$, $\mathcal{U}_{n+1}$ is a star-refinement of $\mathcal{U}_n$ and each element of $\mathcal{U}_n$ is of the form $U \cap Y$, where $U \subset Q$ is convex. Let $\varphi_k : \Sigma(D_k) \to \operatorname{conv} D_k \subset Y$ be the unique affine map which extends id_{D_k}.

Then φ_k is $(\mathcal{U}_n,\mathcal{U}_n)$-compatible for each $n \in \omega$; moreover φ_k is $\mathcal{U}_n$-dense for a sufficiently large k. It follows that Y has Property (B*). On the other hand, Y is not an ANR, since it is not locally path-wise connected at any point: as no continuum can be decomposed into countably many nonempty closed subsets, every path in Y is contained in $\operatorname{conv} D_n$ for some n, but these sets are pairwise disjoint and nowhere dense in Y.

3. A relation to Dugundji-Lefschetz' theorem

We recall the theorem of Lefschetz [6] and Dugundji [4] characterizing metric ANR's, stated in terms of realizations of polytopes. Let P be a CW-polytope and let Q be a subpolytope of P. A continuous map $\varphi\colon Q \to Y$ is a *partial realization of* P relative to a cover $\mathcal{U}$, provided Q contains all the vertices of P and for each simplex σ of P, $\varphi[Q \cap \sigma] \prec \mathcal{U}$. If $Q = P$ then φ is a *full realization* relative to $\mathcal{U}$. Dugundji-Lefschetz' theorem says that a metrizable space Y is an ANR if and only if every open cover $\mathcal{U}$ of Y has an open refinement $S(\mathcal{U})$ such that for every CW-polytope P, every partial realization of P relative to $S(\mathcal{U})$ can be extended to a full realization of P relative to $\mathcal{U}$.

We show that every metric space with the realization property stated above, has Property (B). This provides a proof of the "if" part of Dugundji-Lefschetz' theorem, in the case of completely metrizable spaces.

Fix a metric space Y with the above realization property. Let $\mathcal{U}_0$ be any open cover of Y with finite mesh and, inductively, let $\mathcal{U}_{n+1}$ be an open star-refinement of $S(\mathcal{U}_n)$ with mesh $\leqslant 2^{-n}$. Clearly, the sequence $\{\mathcal{U}_n\}_{n\in\omega}$ satisfies conditions (a), (b) of Property (B). We check (c). Fix $n \in \omega$ and set $m = n+6$. Choose any $\mathcal{U}_{m+1}$-dense set $S \subset Y$. Consider

$$P_1 = \bigcup\{\Sigma(T)\colon T \in [S]^{<\omega} \;\&\; T \prec \mathcal{U}_m\}.$$

Clearly, P_1 is a CW-polytope, S is a subpolytope of P_1 and the identity map $\mathrm{id}_S\colon S \to Y$ is a partial realization of P_1 relative to $\mathcal{U}_m$. As $\mathcal{U}_m$ is a refinement of $S(\mathcal{U}_{n+5})$, there exists a full realization $\varphi_1\colon P_1 \to Y$ of P_1 relative to $\mathcal{U}_{n+5}$ with $\varphi_1|S = \mathrm{id}_S$. Observe that φ_1 is $(\mathcal{U}_m,\mathcal{U}_{n+5})$-compatible. Define

$$P_2 = \bigcup\{\Sigma(T)\colon T \in [S]^{<\omega} \;\&\; \varphi_1[\Sigma(T) \cap P_1] \prec S(\mathcal{U}_n)\}.$$

Then P_1 is a subpolytope of P_2 and φ_1 is a partial realization of P_2 relative to $S(\mathcal{U}_n)$. Let $\varphi_2\colon P_2 \to Y$ be a full realization of P_2 relative to $\mathcal{U}_n$ which extends φ_1. Clearly, φ_2 is $(\mathcal{U}_m,\mathcal{U}_{n+5})$-compatible. Fix $T \in [S]^{<\omega}$ and $U \in \mathcal{U}_{n+1}$ with $T \subset U$. Set $Q = \Sigma(T) \cap P_1$. For each face σ of $\Sigma(T)$ with $\sigma \subset Q$ there is $W_\sigma \in \mathcal{U}_{n+5}$ with $\varphi_1[\sigma] \subset W_\sigma$. We have

$$\varphi_1[Q] \subset U \cup \bigcup\{W_\sigma\colon \sigma \text{ is a face of } \Sigma(T) \text{ with } \sigma \subset Q\} \subset \operatorname{star}(U,\mathcal{U}_{n+5}),$$

thus $\varphi_1[Q] \prec S(\mathcal{U}_n)$. Hence $\Sigma(T)$ is a simplex in P_2 and $\varphi_2[\Sigma(T)] \prec \mathcal{U}_n$. It follows that φ_2 is $(\mathcal{U}_{n+1},\mathcal{U}_n)$-compatible. This shows that condition (c) of Property (B) is satisfied.

4. Applications

4.1. **Simplicial structures.** Following Kulpa [5] we say that a collection $\mathcal{F}$ consisting of simplices in a space Y is a *simplicial structure* in Y provided $\sigma \in \mathcal{F}$ implies that $\operatorname{vert}\sigma \subset Y$, $\sigma|\operatorname{vert}\sigma = \mathrm{id}_{\operatorname{vert}\sigma}$ and every subsimplex of σ is in $\mathcal{F}$. The pair $(Y, \mathcal{F})$ is then called a *simplicial space*. We write $\operatorname{vert}\mathcal{F} = \{\operatorname{vert}\sigma \colon \sigma \in \mathcal{F}\}$. A simplicial space $(Y, \mathcal{F})$ is *locally convex* if for each $p \in Y$ and its neighborhood V there exists a smaller neighborhood U of p such that $[U]^{<\omega} \subset \operatorname{vert}\mathcal{F}$ and for every $\sigma \in \mathcal{F}$, $\operatorname{vert}\sigma \subset U$ implies $\operatorname{im}\sigma := \sigma[\Sigma(\operatorname{vert}\sigma)] \subset V$. A simplicial space $(Y, \mathcal{F})$ is *convex* if every finite subset of Y is in $\operatorname{vert}\mathcal{F}$. A theorem of Kulpa [5] says that every convex locally convex simplicial space has the fixed point property for continuous self-maps with compact images. We show that every compact metric space with such a property is an AR. This answers a question posed by Kulpa in [5].

Theorem 2. *Every compact metric space with a convex and locally convex simplicial structure is an AR.*

Proof. Fix an open cover $\mathcal{U}$ of a compact metric space Y with a convex, locally convex simplicial structure $\mathcal{F}$. Denote by $R(\mathcal{U})$ a fixed refinement $\mathcal{V}$ of $\mathcal{U}$ with the following property:

$$(\forall\, V \in \mathcal{V})(\exists\, U \in \mathcal{U})(\forall\, \sigma \in \mathcal{F})\, \operatorname{vert}\sigma \subset V \implies \operatorname{im}\sigma \subset U.$$

Now define a sequence of open covers $\mathcal{U}_n$ such that $\mathcal{U}_{n+1}$ is a finite star-refinement of $R(\mathcal{U}_n)$ with mesh $\leqslant 2^{-n}$. Clearly, the sequence $\{\mathcal{U}_n\}_{n\in\omega}$ satisfies conditions (a) and (b) of Property (B*). We check condition (c). Fix $n \in \omega$ and let $m = n + 6$. There exists a $\mathcal{U}_{m+1}$-dense simplex $\sigma \in \mathcal{F}$, since $\mathcal{F}$ is convex and Y is compact. Observe that σ is $(\mathcal{U}_{k+1}, \mathcal{U}_k)$-compatible for each $k \in \omega$. Indeed, if $S \subset \operatorname{vert}\sigma$ and $S \prec \mathcal{U}_{k+1}$ then $S \prec R(\mathcal{U}_k)$ so $\sigma[\Sigma(S)] \prec \mathcal{U}_k$. This shows that Y has Property (B*). By Theorem 1, Y is an AR. $\square$

4.2. **Hyperspaces.** For a topological space X we denote by $CL(X)$ the hyperspace of all nonempty closed subsets. We write $\mathcal{T}_V$ for the Vietoris topology on $CL(X)$. Let (X, d) be a metric space. The *Wijsman topology* is the least topology $\mathcal{T}_{W_d}$ on $CL(X)$ such that for each $p \in X$ the function $\operatorname{dist}(p, \,\cdot\,) \colon CL(X) \to \mathbb{R}$ is continuous. Equivalently, $\mathcal{T}_{W_d}$ is the topology generated by all sets of the form:

$$U^-(p, r) = \{A \in CL(X) \colon\ \operatorname{dist}(p, A) < r\},$$
$$U^+(p, r) = \{A \in CL(X) \colon\ \operatorname{dist}(p, A) > r\},$$

where $p \in X$ and $r > 0$. The Wijsman topology is weaker than the Vietoris one. Also, $(CL(X), \mathcal{T}_{W_d})$ is metrizable (Polish) iff (X, d) is separable (Polish) (Beer-Costantini's theorem, see [3]). For a survey on hyperspace topologies we refer to Beer's book [1].

Theorem 3. *Let (X, d) be a Polish space with the following property:*

($*$) *if $\mathcal{K}$ is a finite family of closed balls in X then $X \setminus \bigcup \mathcal{K}$ is path-wise connected.*

Then $(CL(X), \mathcal{T}_{W_d})$ is an absolute retract.

It is clear that to divide $\mathbb{R}^n$ we need at least $n+1$ compact convex sets. It follows that a finite union of bounded closed convex sets in an infinite dimensional normed space does not divide the space. Hence, applying Theorem 3, we get the following.

Corollary 1. *Let $(X, \|\cdot\|)$ be an infinite-dimensional separable Banach space. Then $(CL(X), \mathcal{T}_{W_{\|\cdot\|}})$ is an absolute retract.*

It has been proved by Sakai & Yang [8] that the Wijsman hyperspace of $\mathbb{R}^n$ is homeomorphic to the Hilbert cube minus a point (the authors of [8] consider hyperspaces with the *Fell topology* which, in the case of locally compact metric spaces, is equivalent to the Wijsman one). So the Wijsman hyperspace of every separable Banach space is an AR.

Proof of Theorem 3. Fix a Polish space (X, d) with property $(*)$. Denote by $\mathcal{B}$ the collection of all sets of the form $U^-(p_1, r_1) \cap \dots \cap U^-(p_k, r_k) \cap U^+(q_1, s_1) \cap \dots \cap U^+(q_l, s_l)$, where $p_1, \dots, p_k, q_1, \dots, q_l \in X$, $r_1, \dots, r_k, s_1, \dots, s_l > 0$. Clearly, $\mathcal{B}$ is an open base for $\mathcal{T}_{W_d}$. The following two lemmas refer to the Wijsman topology on $CL(X)$.

Lemma 3. *For each $W \in \mathcal{B}$ the set $[X]^{<\omega} \cap W$ is path-wise connected.*

Proof. Let $W = U^-(p_1, r_1) \cap \dots \cap U^-(p_k, r_k) \cap U^+(q_1, s_1) \cap \dots \cap U^+(q_l, s_l)$, where p_i, q_i, r_i, s_i are as above, and denote $G = X \setminus (\overline{\mathrm{B}}(q_1, s_1) \cup \dots \cup \overline{\mathrm{B}}(q_l, s_l))$, where $\overline{\mathrm{B}}(q, s)$ is the closed ball centered at $q \in X$ with radius $s > 0$. By $(*)$, G is path-wise connected. Fix $a, b \in [X]^{<\omega} \cap W$. For each $(x, y) \in a \times b$ choose a path $\gamma_{x,y} \colon [0,1] \to G$ with $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$. Define $\Gamma \colon [0,1] \to CL(X)$ by

$$\Gamma(t) = \begin{cases} \displaystyle\bigcup_{(x,y) \in a \times b} \{x, \gamma_{x,y}(2t)\}, & t \leqslant 1/2, \\ \displaystyle\bigcup_{(x,y) \in a \times b} \{\gamma_{x,y}(2-2t), y\}, & t \geqslant 1/2. \end{cases}$$

Clearly, $\Gamma(t) \in W$ for every $t \in [0,1]$ and $\Gamma(0) = a$ and $\Gamma(1) = b$. A routine verification shows that Γ is continuous (it is actually continuous with respect to the Vietoris topology). □

Lemma 4. *Let $\varphi \colon \mathrm{bd}\,\sigma \to CL(X)$ be a continuous map from the boundary of a geometric simplex σ. If the dimension of σ is at least 2 then there exists a continuous extension $\psi \colon \sigma \to CL(X)$ of φ such that for each $W \in \mathcal{B}$ we have $\psi[\sigma] \subset W$ whenever $\varphi[\mathrm{bd}\,\sigma] \subset W$.*

Proof. Take a Vietoris continuous map $r \colon \sigma \to CL(\mathrm{bd}\,\sigma)$ which extends the natural injection $i \colon \mathrm{bd}\,\sigma \to CL(\mathrm{bd}\,\sigma)$ (see [2, Lemma 3.3]). Define

$$\psi(s) = \mathrm{cl}_X \bigcup \varphi[r(s)], \qquad s \in \sigma.$$

An easy verification shows that ψ is continuous. Clearly, ψ is an extension of φ. If $\varphi[\mathrm{bd}\,\sigma] \subset U^-(p, r)$ then also $\psi[\sigma] \subset U^-(p, r)$. If $\varphi[\mathrm{bd}\,\sigma] \subset U^+(p, r)$ then for $s \in \sigma$ we have $\mathrm{dist}(p, \psi(s)) \geqslant r$ and, using the compactness of $r(s)$ and the continuity of $\mathrm{dist}(p, \varphi(\,\cdot\,))$, we get $\mathrm{dist}(p, \psi(s)) > r$. Thus also $\psi[\sigma] \subset U^+(p, r)$. □

Fix a complete metric ϱ in $(CL(X), \mathcal{T}_{W_d})$. We will show that $(CL(X), \varrho)$ has Property (B*). Let $\{\mathcal{U}_n\}_{n\in\omega}$ be a sequence of covers of $CL(X)$ such that for each $n \in \omega$, $\mathcal{U}_n \subset \mathcal{B}$, $\operatorname{mesh}\mathcal{U}_n \leqslant 2^{-n}$ and $\mathcal{U}_{n+1}$ is a star-refinement of $\mathcal{U}_n$. We show that condition (c) of Property (B*) is fulfilled.

Fix $n \in \omega$ and set $m = n + 6$. As $[X]^{<\omega}$ is dense in $(CL(X), \mathcal{T}_{W_d})$, we can find a set $S \subset [X]^{<\omega}$ which is $\mathcal{U}_{m+1}$-dense. Define

$$P_1 = \bigcup\{\Sigma(T)\colon T \in [S]^{<\omega} \ \&\ T \prec \mathcal{U}_m\}.$$

Denote by $P_1^{(1)}$ the 1-skeleton of P_1, i.e. the union of all at most 1-dimensional simplices of P_1. By Lemma 3, the identity map $\mathrm{id}\colon S \to S$ can be continuously extended to $\varphi^1\colon P_1^{(1)} \to CL(X)$, such that for each $T \in [S]^2$ we have $\varphi^1[\Sigma(T)] \subset W$ for some $W \in \mathcal{U}_m$, whenever $T \prec \mathcal{U}_m$. Now by Lemma 4, φ^1 can be extended to a continuous map $\varphi_1\colon P_1 \to CL(X)$, which is a $(\mathcal{U}_m, \mathcal{U}_{n+5})$-compatible polytope. Next define

$$P_2 = \bigcup\{\Sigma(T)\colon T \in [S]^{<\omega} \ \&\ T \prec \mathcal{U}_{n+1}\}.$$

Then P_1 is a subpolytope of P_2. Again by Lemma 3, φ_1 can be continuously extended to $\varphi^2\colon P_1 \cup P_2^{(1)} \to CL(X)$ with the property analogous to φ^1. Finally, as $\varphi^2[P_2^{(0)}] \subset [X]^{<\omega}$ and $[X]^{<\omega}$ is path-wise connected, so again using Lemma 4, φ^2 can be extended to a continuous map $\varphi_2\colon \Sigma(S) \to CL(X)$ which is $(\mathcal{U}_{n+1}, \mathcal{U}_n)$-compatible. So φ_2 is a $\mathcal{U}_{m+1}$-dense convex polytope in $(CL(X), \mathcal{T}_{W_d})$ which is both $(\mathcal{U}_m, \mathcal{U}_{n+5})$- and $(\mathcal{U}_{n+1}, \mathcal{U}_n)$-compatible. This shows that $(CL(X), \mathcal{T}_{W_d})$ has Property (B*). □

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