# FRÉCHET TYPE THEOREM AND ITS APPLICATIONS TO MULTIFUNCTIONS

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ABSTRACT. We state a Fréchet type theorem for measurable maps with values in an almost arcwise connected metrizable space. As an application, we obtain some results on continuous approximation of measurable multifunctions.

## 1. Introduction

The well known Fréchet theorem says that a Lebesgue measurable extended real-valued map is the almost everywhere pointwise limit of a sequence of finite continuous maps. Recently different generalizations of this classical result were obtained by Aldaz [1], Kawabe [3], Nowak [5] and Wiśniewski [7, 8]. All these authors assumed that the value space is a locally convex linear topological space. It is not the case when we deal with multifunctions, since hyperspaces have no linear structure.

The main result of this paper is a Fréchet type theorem for measurable maps with values in a separable almost arcwise connected metrizable space. Since we need no linear structure of the value space, we can apply the result to multifunctions. In this way we obtain some theorems on the approximation of measurable multifunctions by multifunctions continuous with respect to the Wijsman topology or the Hausdorff metric. The study of Fréchet type theorems for multifunctions was initiated by the first author in [4].

## 2. Preliminaries

In this section we introduce the notation and terminology used throughout the paper.

According to [2] we shall denote by $CL(Y)$ ($K(Y)$, $CLB(Y)$) the space of all nonempty closed (nonempty compact, nonempty closed bounded) subsets of a topological (metric) space $Y$. Furthermore, $CLC(Y)$ will denote the space of all nonempty closed convex subsets of a normed space $Y$. Let $(Y, \varrho)$ be a metric space. We shall denote by $\mathrm{B}(y_0, \varepsilon)$ the open ball with center $y_0$ and radius $\varepsilon$. *The Wijsman topology* on $CL(Y)$ is the least topology on $CL(Y)$ such that all the functions $\operatorname{dist}(y, \cdot\,)\colon CL(Y) \to \mathbb{R}$ are continuous. Equivalently, this is the topology generated by all sets of the form

$$U(y,\varepsilon) = \{A \in CL(Y) :\ \operatorname{dist}(y,A) < \varepsilon\},$$
$$V(y,\varepsilon) = \{A \in CL(Y) :\ \operatorname{dist}(y,A) > \varepsilon\},$$

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where $y \in Y$ and $\varepsilon > 0$, see [2]. Recall that *the Hausdorff distance* of two closed bounded sets $A, B \subset Y$ is

$$d_H(A,B) = \max\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\},$$

and $d_H$ is a metric on $CLB(Y)$. Since $|\operatorname{dist}(y,A) - \operatorname{dist}(y,B)| \leqslant d_H(A,B)$, the topology induced by the Hausdorff distance is stronger than the Wijsman topology on $CLB(Y)$.

By *a multifunction* we mean any map $\varphi\colon T \to \mathcal{P}(Y) \setminus \{\emptyset\}$, where $T, Y$ are arbitrary sets. If $(T, \mathfrak{M})$ is a measurable space and $Y$ is a topological space then a multifunction $\varphi\colon T \to CL(Y)$ is *measurable* provided for each open $V \subset Y$ the set $\{t \in T: \ \varphi(t) \cap V \neq \emptyset\}$ is measurable.

Let $(T, \mathfrak{M}, \mu)$ be a measure space and let $X$ be a topological space. A map $f\colon T \to X$ is *almost separably-valued* if there exists a measure zero set $M$ such that $f \mid (T \setminus M)$ is *separably-valued*, i.e. $f(T \setminus M)$ is separable. We say that $f$ is *simple* provided $f$ is measurable and $f(T)$ is finite. A map $f$ is *Bochner measurable* if there exists a sequence of simple maps almost everywhere pointwise converging to $f$. These notions will be applied for multifunctions as for single-valued maps. Suppose $T$ is a topological space and $\mathfrak{M}$ contains the Borel $\sigma$-field on $T$. Then a measure $\mu$ on $\mathfrak{M}$ is *regular* if for any $E \in \mathfrak{M}$ and $\varepsilon > 0$ there exists a closed set $F \subset E$ such that $\mu(E \setminus F) < \varepsilon$.

A topological space $X$ will be called *almost arcwise connected* provided there exists a dense set $D \subset X$ such that each two points $a, b \in D$ can be joined by an arc in $X$, i.e. there exists a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = a$ and $\gamma(1) = b$.

We shall use the following result which is well-known for real-valued maps (cf. e.g. [6, Thm. VII.4.5]).

**The Diagonal Lemma .** *Let $(T, \mathfrak{M}, \mu)$ be a space with a $\sigma$-finite measure and let $Z$ be a separable metrizable space. Furthermore, let $f_n^k, f^k, f\colon T \to Z$ be measurable maps such that $\lim_{n\to\infty} f^n(t) = f(t)$ a.e. and $\lim_{n\to\infty} f_n^k(t) = f^k(t)$ a.e. for each $k \in \mathbb{N}$. Then there exists an increasing function $\tau\colon \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} f^n_{\tau(n)}(t) = f(t)$ almost everywhere.*

*Proof.* There exists a finite measure defined on $\mathfrak{M}$ with the same sets of measure zero. Thus we may assume that $\mu$ is finite. Also, we may assume that $\lim_{n\to\infty} f_n^k = f^k$ and $\lim_{n\to\infty} f^n = f$ everywhere on $T$. Set

$$A_m(n) = \{t \in T: \ \varrho(f_i^n(t), f^n(t)) \leqslant \varrho(f^n(t), f(t)) + \frac{1}{n}, \text{ for each } i \geqslant m\},$$

where $\varrho$ is a metric compatible with the topology of $Z$. Note that $A_m(n)$ is measurable, since $Z$ is separable. Moreover $A_m(n) \subset A_{m+1}(n)$ and $\bigcup_{m\in\mathbb{N}} A_m(n) = T$. Fix $j \in \mathbb{N}$. As $\mu(T) < \infty$, for each $n \in \mathbb{N}$ there exists $\sigma_j(n) \in \mathbb{N}$ such that $\mu(T \setminus A_{\sigma_j(n)}(n)) \leqslant \frac{1}{j} 2^{-n}$. Let $A_j = \bigcap_{n\in\mathbb{N}} A_{\sigma_j(n)}(n)$. Then $\mu(T \setminus A_j) \leqslant \frac{1}{j}$. Now let $A = \bigcup_{j\in\mathbb{N}} A_j$ and let $\tau(n) = \max\{\sigma_j(n) : \ j \leqslant n\}$. Clearly $\mu(T \setminus A) = 0$. Fix $t \in A$. There exists $j \in \mathbb{N}$ with $t \in A_{\sigma_j(n)}(n)$ for every $n \in \mathbb{N}$; thus for $i \geqslant \sigma_j(n)$ we have $\varrho(f_i^n(t), f^n(t)) \leqslant \varrho(f^n(t), f(t)) + \frac{1}{n}$. Hence for $n \geqslant j$ we get

$$\varrho(f^n_{\tau(n)}(t), f(t)) \leqslant \varrho(f^n_{\tau(n)}(t), f^n(t)) + \varrho(f^n(t), f(t)) \leqslant 2\varrho(f^n(t), f(t)) + \frac{1}{n}.$$

It follows that $\lim_{n\to\infty} f^n_{\tau(n)}(t) = f(t)$ for $t \in A$. □

## 3. Main result

In this section we prove a general Fréchet type theorem. We start with an auxiliary lemma.

**Lemma 1.** *Let $(T, \mathfrak{M}, \mu)$ be a normal space with a regular measure and let $Y$ be an almost arcwise connected metrizable space. Then for each simple map $f \colon T \to Y$ there exists a sequence of continuous separably-valued maps $f_n \colon T \to Y$ almost everywhere pointwise convergent to $f$.*

*Proof.* Let $\{E^i : i = 1, \ldots, k\}$ be a measurable partition of $T$ such that $f \mid E^i = y^i \in Y$. By the regularity of $\mu$ there exist closed sets $F_n^i \subset E^i$ such that $\mu(E^i \setminus F_n^i) < \frac{1}{n}$ and $F_n^i \subset F_{n+1}^i$. Fix $r^1 = 0 < r^2 < \cdots < r^k = 1$ and define $h_n \colon \bigcup_{i=1}^k F_n^i \to [0,1]$ by setting $h_n \mid F_n^i = r^i$. By the the Tietze-Urysohn theorem, $h_n$ can be extended to a continuous map $H_n \colon T \to [0,1]$. There exists a dense set $D \subset Y$ such that each two points of $D$ can be joined by an arc in $Y$. Now we can find sequences $(y_n^i)_{n \in \mathbb{N}} \subset D$ and continuous maps $G_n \colon [0,1] \to Y$ such that $\lim_{n\to\infty} y_n^i = y^i$ and $G_n(r^i) = y_n^i$ for $i = 1, \ldots, k; n \in \mathbb{N}$. Set $f_n = G_n \circ H_n$. Clearly $f_n$ is continuous and separably-valued. Observe that the set $M = \bigcup_{i=1}^k (E^i \setminus \bigcup_{n\in\mathbb{N}} F_n^i)$ has measure zero. For $t \in T \setminus M$ we have $t \in F_n^i$ for almost all $n \in \mathbb{N}$ and consequently $f_n(t) = y_n^i @ >> n \to \infty > y^i = f(t)$. □

**Theorem 2.** *Let $T$ be a normal space with a regular $\sigma$-finite measure and let $X$ be an almost arcwise connected metrizable space. Assume that $f \colon T \to X$ is a measurable almost separably-valued map. Then there exists a sequence of continuous maps $f_n \colon T \to X$ almost everywhere pointwise convergent to $f$.*

*Proof.* Without loss of generality we may assume that $f(T)$ is separable. There exists a sequence of simple measurable maps $g^n \colon T \to X$ a.e. pointwise convergent to $f$. By Lemma 1 for each $n \in \mathbb{N}$ there exists a sequence of continuous maps $g_k^n \colon T \to X$ such that $\lim_{k\to\infty} g_k^n(t) = g^n(t)$ almost everywhere and $g_k^n(T)$ is separable. Applying the diagonal lemma for $Z = f(T) \cup \bigcup_{n\in\mathbb{N}} g_n(T) \cup \bigcup_{n,k\in\mathbb{N}} g_k^n(T)$ we obtain a map $\tau \colon \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} g_{\tau(n)}^n(t) = f(t)$ a.e. Set $f_n = g_{\tau(n)}^n$. □

Theorem 2 generalizes results obtained in [3, 5, 7, 8]. In particular, it implies the classical Fréchet theorem.

## 4. Applications to multifunctions

In this section we apply Theorem 2 for some hyperspaces. We start with the space $CL(Y)$ endowed with the Wijsman topology.

**Proposition 3.** *If $Y$ is an arcwise connected metric space then $CL(Y)$ with the Wijsman topology is almost arcwise connected.*

*Proof.* Observe that the collection of finite subsets of $Y$ forms a dense set in $CL(Y)$. It remains to show that for fixed two finite sets $A, B \subset Y$ there exists an arc in $CL(Y)$ joining $A, B$. For each $a \in A, b \in B$ choose a continuous map $h_{a,b} \colon [0,1] \to Y$ with $h_{a,b}(0) = a$ and $h_{a,b}(1) = b$. Define $\gamma(\lambda) = \{h_{a,b}(\lambda) : \ a \in A, b \in B\}, \lambda \in [0,1]$. It is easy to show that $\gamma$ is continuous with respect to the Wijsman topology. □

**Theorem 4.** *Let $T$ be a normal space with a regular $\sigma$-finite measure and let $Y$ be an arcwise connected separable metric space. Then for each measurable multifunction $\varphi \colon T \to CL(Y)$ there exists a sequence of multifunctions $\varphi_n \colon T \to CL(Y)$, which are continuous with respect to the Wijsman topology, almost everywhere convergent to $\varphi$.*

*Proof.* By [2, Theorem 2.1.5], $CL(Y)$ with the Wijsman topology is metrizable and separable, by Proposition 3 it is almost arcwise connected. Moreover, $\varphi$ is measurable with respect to the Wijsman topology, see [2, Hess' Theorem 6.5.14]. Thus we may apply Theorem 2. □

Next we give a Fréchet type theorem for closed convex-valued multifunctions.

**Proposition 5.** *If $Y$ is a normed space then $CLC(Y)$ with the Wijsman topology is almost arcwise connected.*

*Proof.* We first show that each two closed convex and bounded sets $A, B \subset Y$ can be joined by an arc in $CLC(Y)$. Define $\gamma\colon [0,1] \to CLC(Y)$ by setting $\gamma(\lambda) = \operatorname{cl}((1-\lambda)A + \lambda B)$. It is enough to verify that the superposition $\operatorname{dist}(y, \cdot) \circ \gamma$ is continuous for every $y \in Y$. This follows from the inequality $|\operatorname{dist}(y, \gamma(\lambda)) - \operatorname{dist}(y, \gamma(\delta))| \leqslant M|\lambda - \delta|$, where $M = \sup_{a \in A, b \in B} \|a - b\|$. Indeed, for each $n \in \mathbb{N}$ there exists $c_n = (1-\lambda)a_n + \lambda b_n$, where $a_n \in A, b_n \in B$, such that $\|y - c_n\| \leqslant \operatorname{dist}(y, \gamma(\lambda)) + \frac{1}{n}$. We have

$$\begin{aligned} \operatorname{dist}(y, \gamma(\delta)) &\leqslant \|y - (1-\delta)a_n - \delta b_n\| \leqslant \|y - c_n\| + \|a_n(\delta - \lambda) + b_n(\lambda - \delta)\| \\ &\leqslant \operatorname{dist}(y, \gamma(\lambda)) + \frac{1}{n} + |\lambda - \delta| \|a_n - b_n\| \leqslant \operatorname{dist}(y, \gamma(\lambda)) + \frac{1}{n} + M|\lambda - \delta|. \end{aligned}$$

Hence $\operatorname{dist}(y, \gamma(\delta)) - \operatorname{dist}(y, \gamma(\lambda)) \leqslant M|\lambda - \delta|$. By the same argument,

$$\operatorname{dist}(y, \gamma(\lambda)) - \operatorname{dist}(y, \gamma(\delta)) \leqslant M|\lambda - \delta|.$$

Now it remains to show that the collection $\{\operatorname{conv} S \ : \ S \subset Y \text{ is nonempty finite }\}$ is dense in $CLC(Y)$. Fix a closed convex set $F \in \bigcap_{i \leqslant k} U(y_i, \alpha_i) \cap \bigcap_{i \leqslant l} V(z_i, \beta_i)$, where $\alpha_i, \beta_i > 0$. For every $i \leqslant k$ there exists $s_i \in F \cap \mathrm{B}(y_i, \alpha_i)$. Set $A = \operatorname{conv}\{s_1, \ldots, s_k\}$. Then $A \in \bigcap_{i \leqslant k} U(y_i, \alpha_i) \cap \bigcap_{i \leqslant l} V(z_i, \beta_i)$. □

By Theorem 2 and Proposition 5 we obtain

**Theorem 6.** *Let $T$ be a normal space with a regular $\sigma$-finite measure and let $Y$ be a separable normed space. Then for each measurable multifunction $\varphi\colon T \to CLC(Y)$ there exists a sequence of multifunctions $\varphi_n\colon T \to CLC(Y)$, which are continuous with respect to the Wijsman topology, almost everywhere convergent to $\varphi$.*

We now consider the space of closed bounded sets with the Hausdorff distance.

**Proposition 7.** *If $Y$ is a normed space then $CLB(Y)$ with the Hausdorff distance is arcwise connected.*

*Proof.* Fix $A, B \in CLB(Y)$. Define $\gamma(\lambda) = \operatorname{cl}((1-\lambda)A + \lambda B)$, $\lambda \in [0,1]$. We show that $d_H(\gamma(\lambda), \gamma(\delta)) \leqslant M|\lambda - \delta|$, where $M = \sup_{a \in A, b \in B} \|a - b\|$. Fix $r > M|\lambda - \delta|$. For $a \in A, b \in B$ we have

$$\operatorname{dist}((1-\lambda)a + \lambda b, \gamma(\delta)) \leqslant \|(1-\lambda)a + \lambda b - (1-\delta)a - \delta b\| \leqslant M|\lambda - \delta|.$$

Hence $\gamma(\lambda) \subset \mathrm{B}(\gamma(\delta), r)$. By the same argument, $\gamma(\delta) \subset \mathrm{B}(\gamma(\lambda), r)$. Thus $d_H(\gamma(\lambda), \gamma(\delta)) \leqslant r$. □

**Theorem 8.** *Let $T$ be a normal space with a regular complete $\sigma$-finite measure and let $Y$ be a normed space. If $\varphi\colon T \to CLB(Y)$ is a Bochner measurable multifunction then there exists a sequence of continuous, with respect to the Hausdorff distance, multifunctions $\varphi_n\colon T \to CLB(Y)$ almost everywhere convergent to $\varphi$.*

*Proof.* It is enough to observe that $\varphi$ is almost separably-valued and, by the completeness of measure, $\varphi$ is measurable with respect to the topology induced by the Hausdorff distance. $\square$

As an another application of Theorem 2 we obtain a strengthening of a result from [4] on continuous approximation of compact-valued multifunctions.

**Theorem 9.** *Let $T$ be a normal space with a regular $\sigma$-finite measure and let $Y$ be an arcwise connected separable metrizable space. Then for each measurable multifunction $\varphi\colon T \to K(Y)$ there exists a sequence of multifunctions $\varphi_n\colon T \to K(Y)$ which are continuous with respect to the Hausdorff distance, almost everywhere pointwise converging to $\varphi$.*

*Proof.* The same arguments as in the proof of Proposition 3 show that $K(Y)$ with the topology generated by the Hausdorff distance is almost arcwise connected. On the other hand, a measurable multifunction is measurable with respect to this topology. Thus we can apply Theorem 2. $\square$

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