Perfect cliques and G_δ colorings of Polish spaces

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Abstract

A *coloring* of a set X is any subset C of $[X]^N$, where $N > 1$ is a natural number. We give some sufficient conditions for the existence of a perfect C-homogeneous set, in case where C is G_δ and X is a Polish space. In particular, we show that it is sufficient that there exist C-homogeneous sets of arbitrarily large countable Cantor-Bendixson rank. We apply our methods to show that an analytic subset of the plane contains a perfect 3-clique if it contains any uncountable k-clique, where k is a natural number or $\aleph_0$ (a set K is a k-*clique* in X if the convex hull of any of its k-element subsets is not contained in X).

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1 Introduction

For a set X and natural number N, $[X]^N$ denotes the collection of all N-element subsets of X. A (two-color) *coloring* of X is (represented by) a set $C \subseteq [X]^N$. We identify $[X]^N$ with a suitable subspace of the product X^N. We are interested in the following problem: find sufficient conditions for the existence of a perfect C-homogeneous set $P \subseteq X$, where X is a Polish space and $C \subseteq [X]^N$ is open (or more generally G_δ). A natural example for this problem is the following: let $X \subseteq \mathbb{R}^N$ be closed and $C = \{s \in [X]^k \colon \operatorname{conv} s \not\subseteq X\}$. Then C is open and a C-homogeneous set is called a k-clique in X. It is known (see [3]) that there exists a closed set $X \subseteq \mathbb{R}^2$ such that X is not a countable union of convex sets but every k-clique in X is countable for every $k < \omega$. On the other hand, it is proved in [3] that if a closed set $X \subseteq \mathbb{R}^2$ contains an uncountable k-clique for some k then it contains a perfect 3-clique.

We prove that if C is a G_δ coloring of a Polish space and there are no perfect C-homogeneous sets, then there is a countable ordinal γ such that the Cantor-Bendixson rank of every C-homogeneous set is $< \gamma$. In the context of cliques, this strengthens the result of Kojman [2] (see Theorem 3.1(a) below). From our result it follows that if C is a G_δ coloring of an analytic space then either there exists a perfect C-homogeneous set or all C-homogeneous sets are countable. This is not true for F_σ colorings: a result of Shelah [4] states that consistently there exist F_σ 2-colorings with uncountable but not perfect homogeneous sets. Concerning cliques, we investigate analytic subsets of the plane. We prove that if an analytic set $X \subseteq \mathbb{R}^2$ contains an uncountable $\aleph_0$-clique then X contains also a perfect 3-clique.

1.1 Notation

Any subset of $[X]^N$ is called a *coloring* (or *an N-coloring*) of X. We write $\neg C$ instead of $[X]^N \setminus C$. A set $S \subseteq X$ is *C-homogeneous* if $[A]^N \subseteq C$. We identify $[X]^N$ with the subspace of X^N consisting of all N-tuples $(x_0, \dots, x_{N-1})$ with $x_i \neq x_j$ for $i \neq j$. Thus we may consider topological properties of colorings. If $f \colon X \to Y$ is a function then we write $f[S]$ for the image of a set $S \subseteq X$ and $f(s)$ for the value at a point $s \in X$. By a *perfect set* we mean a compact, nonempty, topological space with no isolated points.

2 On colorings

First we recall a simple result on open 2-colorings of analytic spaces (see Todorčević-Farah's book [5, p. 81]). We present a proof for completeness.

Proposition 2.1. *Let X be an analytic space and let $C \subseteq [X]^2$ be open. Then either there exists a perfect C-homogeneous set or else X is a countable union of $\neg C$-homogeneous sets, i.e. $X = \bigcup_{n\in\omega} A_n$ where $[A_n]^2 \cap C = \emptyset$ for every $n \in \omega$.*

Proof. Let $f \colon \omega^\omega \to X$ be continuous and onto X. Define

$$C' = \{s \in [\omega^\omega]^2 \colon f[s] \in C\}.$$

Note that if $\{x, y\} \in C'$ then $f(x) \neq f(y)$. Now observe that if ω^ω is a union of countably many $\neg C'$-homogeneous sets, then the same holds for X. Also, if P is a compact, perfect, C'-homogeneous subset of ω^ω then $f \restriction P$ is one-to-one and hence $f[P]$ is a perfect C-homogeneous set. Thus we may assume that $X = \omega^\omega$ and that X cannot be covered by countably many $\neg C$-homogeneous sets.
Let V consist of all $x \in \omega^\omega$ such that some neighborhood of x is a countable union of $\neg C$-homogeneous sets. By assumption, it follows that $V \neq \omega^\omega$. Let $B = \omega^\omega \setminus V$. Now we are working in B: construct a tree $T = \{u_s \colon s \in 2^{<\omega}\}$ of open subsets of B such that T defines a Cantor set and $\{x, y\} \in C$ whenever $x \in u_s$, $y \in u_t$ and $s, t \in 2^k$ are distinct, $k < \omega$. Coming to split u_s, where $s \in 2^k$, we first find a pair $\{x, y\} \in [u_s]^2 \cap C$ (this is possible since u_s is not $\neg C$-homogeneous). Next, using the fact that C is open, enlarge x, y to open sets $u_{s^\frown 0}$, $u_{s^\frown 1}$, preserving C-homogeneity. The perfect set obtained from T is evidently C-homogeneous. $\square$

The above result is no longer valid when we replace the word "open" with "closed", see [5, p. 83]. Also, the above proposition cannot be strengthened for colorings of triples: there exists a clopen 3-coloring of 2^ω such that there are no uncountable homogeneous sets neither of this color nor of its complement, see Blass' example [1]. In this example, the Cantor-Bendixson rank of any homogeneous set is at most 1. Below we show that in this situation, there always exists a countable ordinal which bounds the Cantor-Bendixson ranks of all homogeneous sets. In fact this is true for G_δ colorings.

For a topological space Y and an ordinal α we denote by $Y^{(\alpha)}$ the α-derivative of Y; the *Cantor-Bendixson rank* of Y is the minimal ordinal γ such that $Y^{(\gamma+1)}$ is empty.

Theorem 2.2. *Let C be a G_δ N-coloring of a Polish space X. If for every countable ordinal γ there exists a C-homogeneous set of the Cantor-Bendixson rank $\geqslant \gamma$ then X contains a perfect C-homogeneous set.*

Proof. Fix a countable base $\mathcal{B}$ in X and fix a complete metric on X. Let $C = \bigcap_{n\in\omega} C_n$, where each C_n is open and $C_{n+1} \subseteq C_n$. We will construct a tree of open sets $T = \{u_s \colon s \in 2^{<\omega}\}$ with the following properties:

(i) $\operatorname{cl} u_{s^\frown i} \subseteq u_s$, $\operatorname{cl} u_s \cap \operatorname{cl} u_t = \emptyset$ if s, t are incompatible and $\operatorname{diam}(u_s) < 2^{-\operatorname{length}(s)}$;

(ii) if $k < \omega$ and $s_0, \ldots, s_{N-1} \in 2^k$ are pairwise distinct then

$$\{x_0, \ldots, x_{N-1}\} \in C_k$$

whenever $x_i \in u_{s_i}$, $i < N$;

(iii) if $k < \omega$ then for each $\gamma < \omega_1$ there exists a C-homogeneous set $P = P_{k,\gamma}$ such that $P^{(\gamma)} \cap u_s \neq \emptyset$ for each $s \in 2^k$.

We start with $u_\emptyset = X$. Suppose that u_s has been defined for all $s \in 2^{\leqslant k}$. Fix $\gamma < \omega_1$ and consider $P = P_{k,\gamma+1}$, as in (iii). Then for each $s \in 2^k$ the set $P^{(\gamma)} \cap u_s$ is infinite. Fix $S \subseteq P^{(\gamma)}$ such that $|S \cap u_s| = 2$ for each $s \in 2^k$. Next, enlarge each $x \in S \cap u_s$ to a small open set $v_x \in \mathcal{B}$, contained in u_s, such that $\{y_0, \ldots, y_{N-1}\} \in C_{k+1}$ whenever y_i are taken from pairwise distinct v_x's. This is possible, because C_{k+1} is open. Let $\varphi(\gamma) = \{v_x \colon x \in S\}$. This defines a mapping $\varphi \colon \omega_1 \to [\mathcal{B}]^{<\omega}$. As $\mathcal{B}$ is countable, there is unbounded $F \subseteq \omega_1$ such that $\varphi \restriction F$ is constant, say $\{v_{s^\frown i} \colon s \in 2^k, i < 2\}$, where $v_{s^\frown i} \subseteq u_s$. Set $u_{s^\frown i} = v_{s^\frown i}$. Observe that (i) holds if we let v_x's to be small enough. Also (ii) holds, by the definition of v_x's. Finally, (iii) holds, because $P^{(\gamma)}_{k,\gamma+1} \cap u_t \neq \emptyset$ for $t \in 2^{k+1}$ whenever $\gamma \in F$. By (ii) the perfect set obtained from this construction is C-homogeneous. □

Using the above theorem and arguments from the proof of Proposition 2.1 we obtain the following (see Shelah [4, Remark 1.14]):

Corollary 2.3. *Let $1 \leqslant N < \omega$ and let C be a G_δ N-coloring of an analytic space X. If there exists an uncountable C-homogeneous set then there exists also a perfect one.*

3 Applications to convexity

Let $X \subseteq E$, where E is a real vector space. A subset K of X is a *k-clique* (k can be a cardinal or just a natural number, we will use this notion for $k < \omega$ and $k = \aleph_0$) if $\operatorname{conv} S \nsubseteq X$ whenever $S \in [K]^k$. If E is finite-dimensional and $k > \dim E$ then we can define the notion of a *strong k-clique* replacing $\operatorname{conv} S$ by $\operatorname{int}\operatorname{conv} S$ in the definition. A finite set $S \subseteq X$ is (*strongly*) *defected* in X if $\operatorname{conv} S \nsubseteq X$ ($\operatorname{int}\operatorname{conv} S \nsubseteq X$). It is clear that the relation of strong defectedness is open and defectedness is open provided that X is closed.

Applying the results of the previous section we get the following:

Theorem 3.1. *(a) Let X be a closed set in a Polish linear space and let $N < \omega$. If X does not contain a perfect N-clique then all N-cliques in X are countable. Moreover, there exists an ordinal $\gamma < \omega_1$ which bounds the Cantor-Bendixson ranks of all N-cliques in X.*

(b) Let X be an analytic subset of $\mathbb{R}^m$. If $m < N < \omega$ and X contains an uncountable strong N-clique then X contains also a perfect one.

Theorem 3.1(a) was proved, under the stronger assumption that X is a countable union of convex sets, by Kojman in [2].

In [3] we proved, in particular, that in a closed planar set either all cliques are countable or there exists a perfect 3-clique. Here we prove the same for analytic sets, namely:

Theorem 3.2. *Let $X \subseteq \mathbb{R}^2$ be analytic. If X contains an uncountable $\aleph_0$-clique then X contains a perfect 3-clique.*

Proof. Fix a continuous function $f\colon \omega^\omega \to X$ onto X and fix an uncountable $\aleph_0$-clique $K \subseteq X$. We may assume that every line contains only countably many points of L: otherwise, for some line L, $X \cap L$ contains an uncountable $\aleph_0$-clique, so it contains a perfect 2-clique (Proposition 2.1), which is also a 3-clique in X. Fix uncountable $K' \subseteq \omega^\omega$ such that $f \restriction K'$ is a bijection onto K.

A finite collection $\{u_0, \dots, u_{k-1}\}$ of open subsets of ω^ω will be called *relevant* if each u_i contains uncountably many points of K', $\operatorname{cl} u_i \cap \operatorname{cl} u_j = \emptyset$ whenever $i < j < k$ and

$$\operatorname{int}\operatorname{conv}\{f(x_0), f(x_1), f(x_2)\} \nsubseteq X$$

whenever x_0, x_1, x_2 are taken from pairwise distinct u_i's. To find a perfect 3-clique in X, it suffices to construct a perfect tree of open sets in ω^ω with relevant levels. If P is a perfect set obtained from such a tree then $f \restriction P$ is one-to-one and $f[P]$ is a perfect strong 3-clique.

Suppose that we have a relevant collection $\{u_0, \dots, u_k\}$. We have to show that it is possible to split each u_i to obtain again a relevant collection. We will split u_k. Let $L = K' \cap u_k$ and pick $y_i \in u_i$ for $i < k$. Define $c_i\colon [L]^2 \to 2$ by letting $c_i(x_0, x_1) = 1$ iff $\operatorname{conv}\{f(x_0), f(x_1), f(y_i)\} \nsubseteq X$. Observe that there are no infinite c_i-homogeneous sets of color 0: if $S \subseteq L$ is infinite then, by Carathéodory's theorem, there is $s \in [S]^3$ such that $f[s]$ is defected in X (because $f[S]$ is defected) and hence for some $x_0, x_1 \in s$ we have $\operatorname{conv}\{f(x_0), f(x_1), f(y_i)\} \nsubseteq X$, because $\operatorname{conv} T \subseteq \bigcup_{x,y \in T} \operatorname{conv}\{x, y, p\}$ for $T \subseteq \mathbb{R}^2$, $p \in \mathbb{R}^2$. Using k times the theorem of Dushnik-Miller we obtain uncountable $L' \subseteq L$ which is c_i-homogeneous of color 1 for $i < k$. Shrinking L' we may assume that each nonempty open subset of L' is uncountable. Now choose disjoint open sets v_0, v_1 with $\mathrm{cl}\, v_j \subseteq u_k$ and $v_j \cap L' \neq \emptyset$ for $j < 2$. To finish the proof we need the following geometric property of the plane:

Claim 3.3. *Let $A, B \subseteq X \subseteq \mathbb{R}^2$ and $c \in \mathbb{R}^2$ be such that A, B are uncountable, each line contains countably many points of $A \cup B$ and $\mathrm{conv}\{a, b, c\} \nsubseteq X$ whenever $a \in A$, $b \in B$. Then there are $a_0 \in A$, $b_0 \in B$ such that $\mathrm{int}\,\mathrm{conv}\{a_0, b_0, c\} \nsubseteq X$.*

Proof. Suppose this is not true. Observe that, replacing A and B if necessary, we may assume that for some $b_0 \in B$, $[a, b_0] \cup [a, c] \nsubseteq X$ whenever $a \in A$. Indeed, if $[b, c] \subseteq X$ for some $b \in B$ then we take $b_0 = b$, otherwise we take any $a_0 \in A$ and we replace A and B. Now, without loss of generality, we may assume that $b_0 = (-1, 0)$, $c = (1, 0)$ and A is contained in $(-1, 1) \times (0, 1)$. Now, if some vertical line contains two elements of A then we are done: we take $a_0 \in A$ such that some $a_1 \in A$ is below a_0, then the relative interiors of segments $[b_0, a_1]$, $[c, a_1]$ are contained in the interior of $\mathrm{conv}\{a_0, b_0, c\}$.
Assume that each vertical line contains at most one element of A. As A is uncountable, there is $a_1 \in A$ such that arbitrarily close to a_1 there are uncountably many points both on the left and the right side of a_1. Suppose now that e.g. $\{b_0, a_1\}$ is defected in X. As $[b_0, a_1]$ contains only countably many points of A, we can find $a_2 \in A$ which is close enough to a_1, on the left side of a_1 and not in $[b_0, a_1]$. If a_2 is below $[b_0, a_1]$ then we can set $a_0 = a_1$, otherwise we can set $a_0 = a_2$. □

Let $i = 0$. Using Claim 3.3 for $A = f[v_0 \cap L']$, $B = f[v_1 \cap L']$ and $c = f(y_i)$ we get $x_j \in v_j$ such that $\mathrm{int}\,\mathrm{conv}\{f(x_0), f(x_1), f(y_i)\} \nsubseteq X$. By continuity, shrink v_0, v_1 and enlarge y_i to an open set $u'_i \subseteq u_i$ such that each triple selected from $f[v_0] \times f[v_1] \times f[u'_i]$ is (strongly) defected in X. Repeat the same argument for each $i < k$, obtaining a relevant collection $\{u'_0, \dots, u'_{k-1}, v'_0, v'_1\}$ which realizes the splitting of u_k. This completes the proof. □

Actually, we have proved that if an analytic planar set X contains any uncountable $\aleph_0$-clique then either X contains a perfect strong 3-clique or else, $X \cap L$ contains a perfect 2-clique for some line L.

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