Fraïssé sequences

Category-theoretic approach to universal homogeneous structures

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Motivations

- Fraïssé-Jónsson theory of universal homogeneous structures (1953)
  - Cantor's back-and-forth method
- Work of Droste & Göbel (1989)
- Reversed Fraïssé limits: Irwin & Solecki (2005)
- Compact spaces "generated" by retractions (Valdivia compacta)
- Banach spaces with a projectional resolution of the identity

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Categories

Let $\mathfrak{K}$ be a category.

We say that $\mathfrak{K}$ has the amalgamation property if for every arrows $f\colon Z \to X$ and $g\colon Z \to Y$ there are arrows $f'\colon X \to W$ and $g'\colon Y \to W$ such that $f'f = g'g$.

We say that $\mathfrak{K}$ has the joint embedding property if for every objects $X, Y \in \mathfrak{K}$ there exist $V \in \mathfrak{K}$ and arrows $f\colon X \to V$ and $g\colon Y \to V$.

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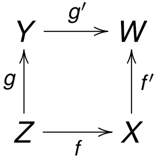

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$$\begin{array}{ccc} Y & \xrightarrow{g'} & W \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\ Z & \xrightarrow[f]{} & X \end{array}$$

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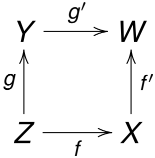

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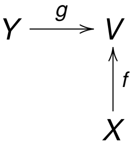

Sequences

By a sequence in a category $\mathfrak{K}$ we mean a (covariant) functor $\vec{x} \colon \lambda \to \mathfrak{K}$, where λ is an ordinal.

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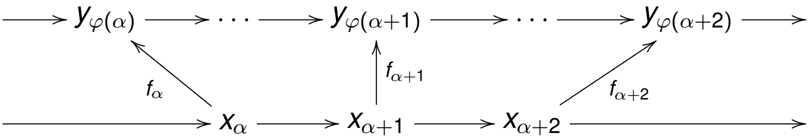

The κ-completion of a category

Let κ be a regular cardinal and let $\mathfrak{K}$ be a category. Denote by $\mathfrak{S}_\kappa(\mathfrak{K})$ the category of all sequences in $\mathfrak{K}$ of length $< \kappa$.

A category $\mathfrak{L}$ is κ-closed if sequences of length $< \kappa$ have colimits in $\mathfrak{L}$.

Theorem

1. *For every κ-closed category $\mathfrak{L}$, every covariant functor $F \colon \mathfrak{K} \to \mathfrak{L}$ has a unique extension $F' \colon \mathfrak{S}_\kappa(\mathfrak{K}) \to \mathfrak{L}$ to a κ-continuous functor.*

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Fraïssé sequences

A κ-Fraïssé sequence in $\mathfrak{K}$ is an inductive sequence $\vec{u}\colon \kappa \to \mathfrak{K}$ satisfying the following conditions:

(U) For every $x \in \mathfrak{K}$ there exists $\xi < \kappa$ such that $\mathfrak{K}(x, u_\xi) \neq \emptyset$.

(A) For every $\xi < \kappa$ and for every arrow $f \in \mathfrak{K}(u_\xi, y)$, where $y \in \mathfrak{K}$, there exist $\eta \geqslant \xi$ and $g \in \mathfrak{K}(y, u_\eta)$ such that $u_\xi^\eta = gf$.

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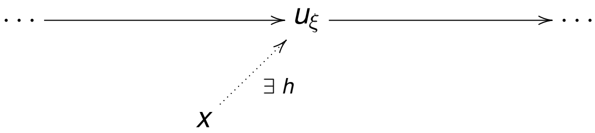

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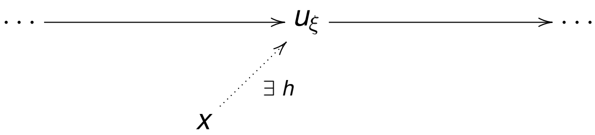

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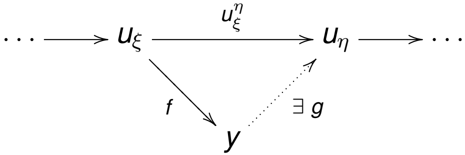

Dominating families of arrows

Let $\mathcal{F}$ be a set of arrows in $\mathfrak{K}$. Let $\mathrm{Dom}(\mathcal{F}) = \{\mathrm{dom}(f) \colon f \in \mathcal{F}\}$.

We say that $\mathcal{F}$ is dominating in $\mathfrak{K}$ if the following conditions are satisfied:

(D1) For every $x \in \mathfrak{K}$ there exists $a \in \mathrm{Dom}(\mathcal{F})$ such that $\mathfrak{K}(x, a) \neq \emptyset$.

(D2) For every arrow $g \colon a \to y$ in $\mathfrak{K}$ with $a \in \mathrm{Dom}(\mathcal{F})$ there exist arrows f, h in $\mathfrak{K}$ such that $f \in \mathcal{F}$ and $f = hg$.

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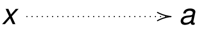

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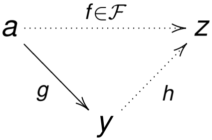

The existence

A category $\mathfrak{K}$ is κ-bounded if for every sequence $\vec{u} \in \mathfrak{S}_\kappa(\mathfrak{K})$ there are $a \in \mathfrak{K}$ and an arrow of sequences $F \colon \vec{u} \to a$.

Theorem

Let $\kappa > 1$ be a regular cardinal and let $\mathfrak{K}$ be a κ-bounded category which has the amalgamation property and the joint embedding property. Assume further that $\mathcal{F} \subseteq \mathrm{Arr}(\mathfrak{K})$ is dominating in $\mathfrak{K}$ and $|\mathcal{F}| \leqslant \kappa$.

Then there exists a Fraïssé sequence $\vec{u} \colon \kappa \to \mathfrak{K}$ such that $\{u_\alpha \colon \alpha < \kappa\} \subseteq \mathrm{Dom}(\mathcal{F})$.

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Countable Fraïssé sequences

Theorem (Countable Cofinality)

Assume $\vec{u}$ is a Fraïssé sequence in a category with amalgamation $\mathfrak{K}$. Then for every countable sequence $\vec{x}$ in $\mathfrak{K}$ there exists an arrow $\vec{f}\colon \vec{x} \to \vec{u}$.

Corollary

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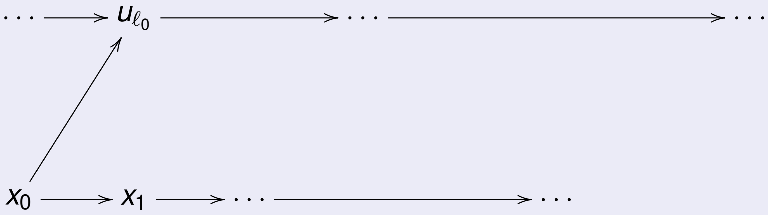

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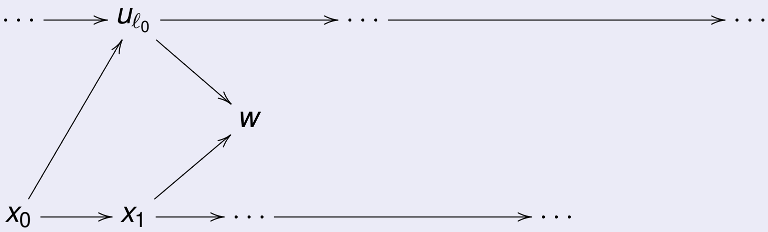
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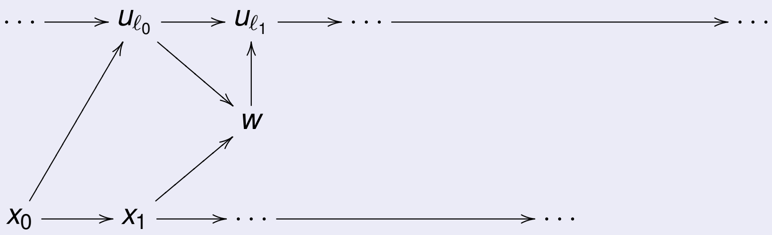

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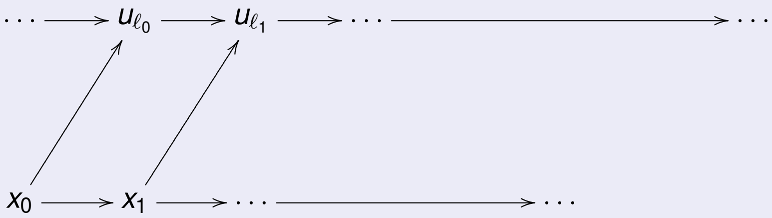
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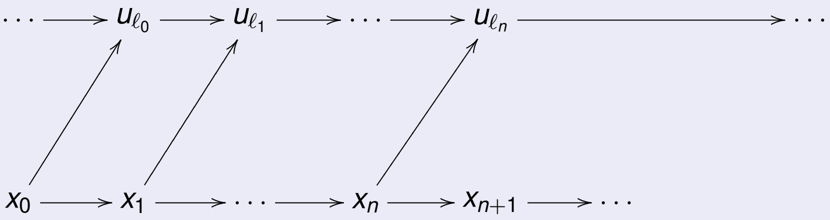
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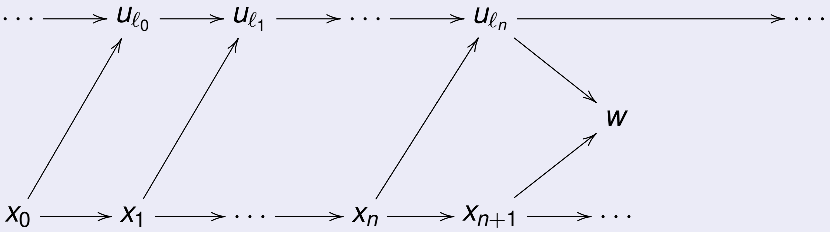

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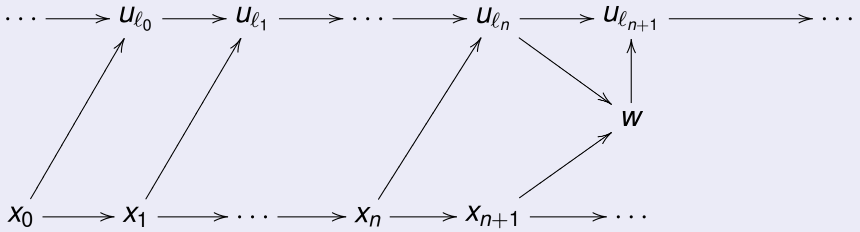

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Homogeneity & Uniqueness

Theorem

Assume that $\vec{u}$, $\vec{v}$ are countable Fraïssé sequences in a category $\mathfrak{K}$.

(a) *Let $f\colon u_k \to v_\ell$, where $k, \ell < \omega$. Then there exists an isomorphism $F\colon \vec{u} \to \vec{v}$ such that $F\vec{u}_k = \vec{v}_\ell f$. In particular $\vec{u} \approx \vec{v}$.*

(b) *Assume $\mathfrak{K}$ has the amalgamation property. Then for every $a, b \in \mathfrak{K}$ and for every arrows $f\colon a \to b$, $i\colon a \to \vec{u}$, $j\colon b \to \vec{v}$ there exists an isomorphism $F\colon \vec{u} \to \vec{v}$ such that $Fi = jf$.*

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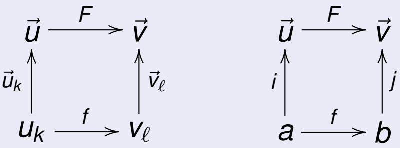

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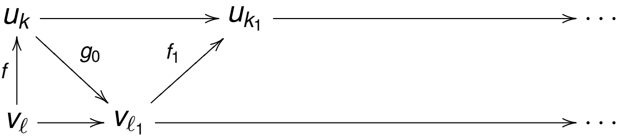

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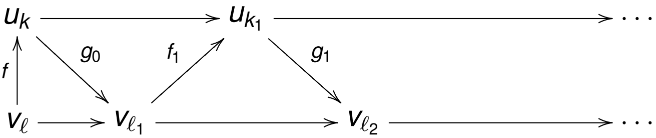

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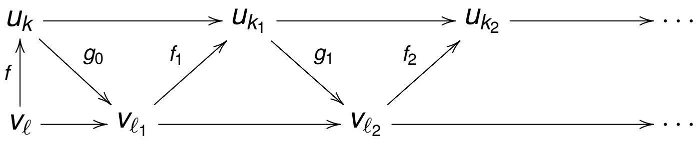

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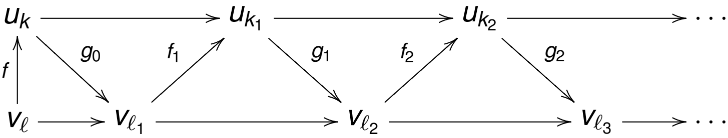

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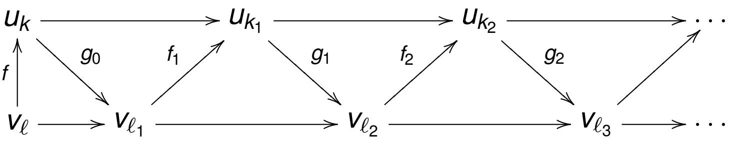

Uncountable Fraïssé sequences

Theorem

Let $\kappa > \aleph_0$ be regular and assume that $\mathfrak{K}$ is a full and cofinal subcategory of a κ-closed category $\mathfrak{L}$. If $\mathfrak{L}$ has the amalgamation property, then:

1. *There exists, up to isomorphism, at most one κ-Fraïssé sequence in $\mathfrak{K}$.*
2. *A κ-Fraïssé sequence in $\mathfrak{K}$ is also a Fraïssé sequence in $\mathfrak{L}$ and it is both $\mathfrak{L}$-homogeneous and $\mathfrak{S}_{\kappa^+}(\mathfrak{L})$-cofinal.*

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There exists an $\aleph_1$-bounded category with amalgamation which has many pairwise incomparable $\aleph_1$-Fraïssé sequences.

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Valdivia compacta

Definition: A space K of weight $\leqslant \aleph_1$ is Valdivia compact iff $K = \varprojlim \vec{s}$, where $\vec{s}$ is a continuous inverse sequence of metric compacta whose all bonding maps are retractions. We would like to prove that:

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There exists a Valdivia compact K of weight $\aleph_1$ such that:

- *Every nonempty Valdivia compact of weight $\leqslant \aleph_1$ is a retract of K.*
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Retractive pairs

Fix a category $\mathfrak{K}$. Denote by $\ddagger\mathfrak{K}$ the following category:

- The objects of $\ddagger\mathfrak{K}$ are the same as the objects of $\mathfrak{K}$.
- Given $a, b \in \ddagger\mathfrak{K}$, an arrow $f : a \to b$ in $\ddagger\mathfrak{K}$ is a pair $f = \langle r, e \rangle$, where $r : b \to a$ and $e : a \to b$ are arrows of $\mathfrak{K}$ such that $re = \mathrm{id}_a$. We shall write $r(f) = r$, $e(f) = e$.
- Given compatible arrows f, g in $\ddagger\mathfrak{K}$, their composition is

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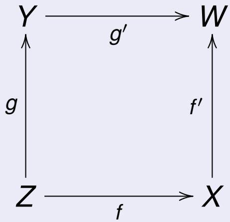
The pushout of $\langle f, g \rangle$



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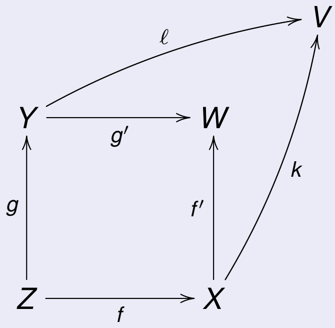

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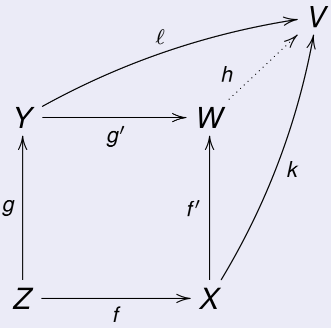

Given a category $\mathfrak{K}$, let $\Phi \colon \ddagger\mathfrak{K} \to \mathfrak{K}$ be the contravariant "forgetful" functor, i.e. $\Phi(f) = r(f)$ for every arrow f in $\ddagger\mathfrak{K}$.

We shall say that a sequence $\vec{x} \in \mathfrak{S}_\lambda(\ddagger\mathfrak{K})$ is semi-continuous if $\Phi[\vec{x}]$ is continuous.

Theorem

Let $\mathfrak{K}$ be a category and let $\vec{u}$ and $\vec{v}$ be semicontinuous Fraïssé sequences in $\ddagger\mathfrak{K}$ of the same regular length κ. Then for every arrow $f \colon u_0 \to \vec{v}$ there exists an isomorphism of sequences $\vec{f} \colon \vec{u} \to \vec{v}$ such that $\vec{f}\vec{u}_0 = f$.

In particular $\vec{u} \approx \vec{v}$.

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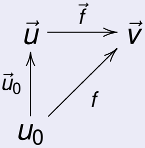

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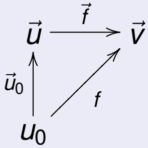

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Proper amalgamations

Let $f\colon Z \to X$ and $g\colon Z \to Y$ be arrows in $\ddagger\mathfrak{K}$.
We say that arrows $h\colon X \to W$, $k\colon Y \to W$ provide a proper amalgamation of f, g if $hf = kg$ and moreover $i(g)r(f) = r(k)i(h)$, $i(f)r(g) = r(h)i(k)$ hold.

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If $\mathfrak{K}$ has pullbacks or pushouts then $\ddagger\mathfrak{K}$ has proper amalgamations.

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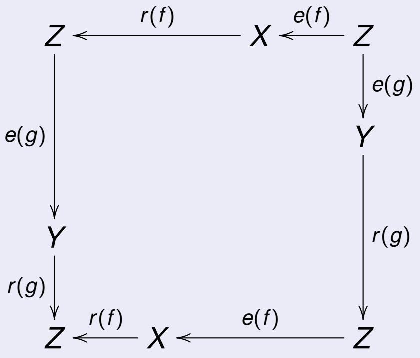

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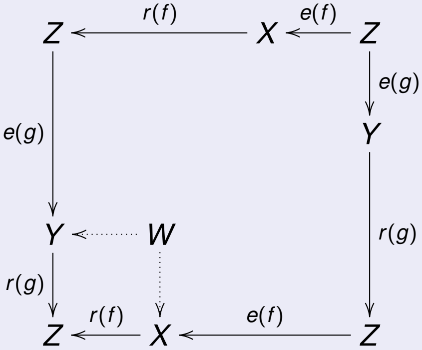

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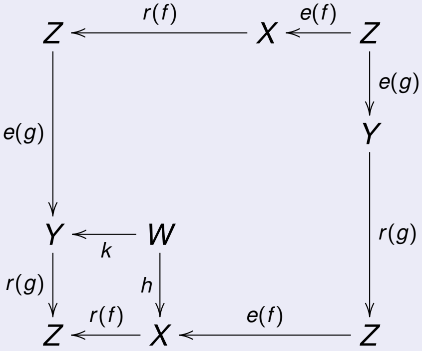

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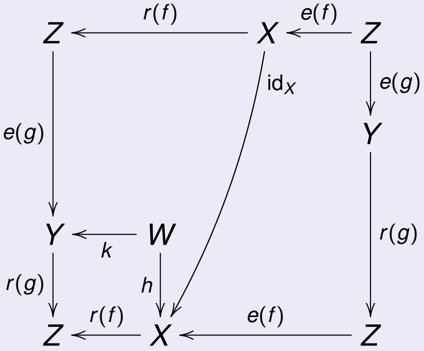

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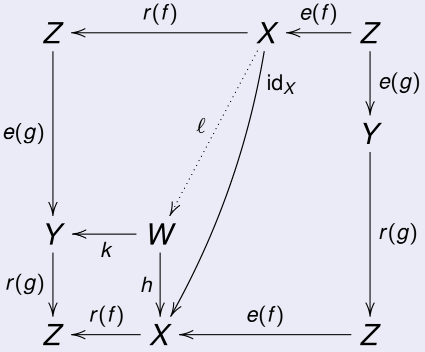

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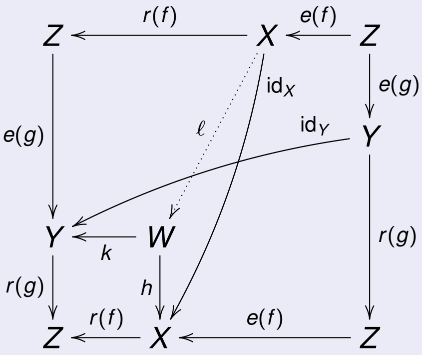

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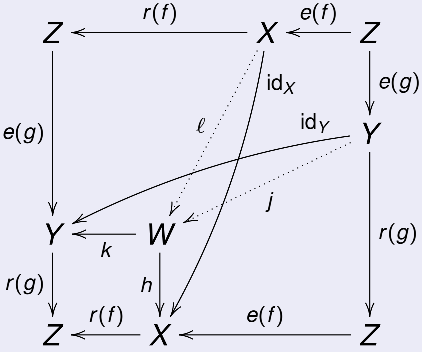

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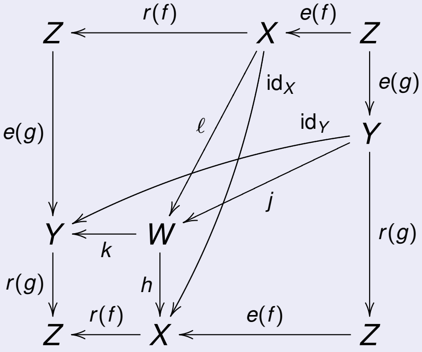

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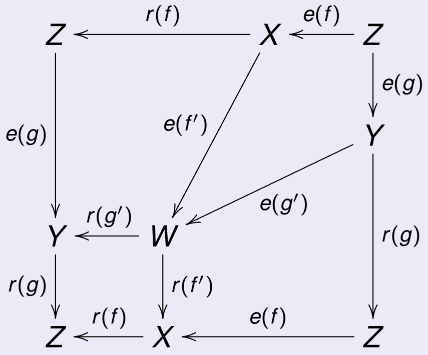

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Theorem

Let $\mathfrak{K}$ be a category such that $\ddagger\mathfrak{K}$ has proper amalgamations. Assume $\vec{u}$ is a semi-continuous κ-Fraïssé sequence in $\ddagger\mathfrak{K}$.
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Assume CH. Then there exists a Valdivia compact K of weight $\aleph_1$ such that every nonempty Valdivia compact of weight $\leqslant \aleph_1$ is a retract of K. Moreover, K is "retractively homogeneous".

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Banach spaces

Let $\mathfrak{B}_{\aleph_0}$ be the category of separable Banach spaces, with arrows being linear transformations of norm $\leqslant 1$.

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$\mathfrak{B}_{\aleph_0}$ is $\aleph_1$-closed and has pushouts.

Theorem

Under CH there exists a Banach space E of density $\aleph_1$ such that

- *E has a projectional resolution of the identity (PRI);*
- *every Banach space of density $\leqslant \aleph_1$ and with a PRI is linearly isometric to a one-complemented subspace of E.*
- *E is "projectively homogeneous".*

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- *E is "projectively homogeneous".*

Banach spaces

Let $\mathfrak{B}_{\aleph_0}$ be the category of separable Banach spaces, with arrows being linear transformations of norm $\leqslant 1$.

Claim

$\mathfrak{B}_{\aleph_0}$ is $\aleph_1$-closed and has pushouts.

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