A universal homogeneous Banach space of density continuum

Wiesław Kubiś

Czech Academy of Sciences, Prague
and
Jan Kochanowski University in Kielce

http://www.math.cas.cz/~kubis/

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Main result

Theorem

Assume $2^{\aleph_0} = \aleph_1$. There exists a Banach space $\mathbb{U}$ with the following properties.

- $\operatorname{dens}(\mathbb{U}) = \aleph_1$.
- *Given two linear isometric embeddings $i \colon X \to \mathbb{U}$ and $f \colon X \to Y$, where X, Y are separable Banach spaces, there exists a linear isometric embedding $g \colon Y \to \mathbb{U}$ such that $i = g \circ f$.*

Moreover, the above properties determine the space $\mathbb{U}$ uniquely up to a linear isometry. Further:

- *Every Banach space of density $\leqslant \aleph_1$ embeds isometrically into $\mathbb{U}$.*
- *Every linear isometry between separable subspaces of $\mathbb{U}$ extends to a linear isometry of $\mathbb{U}$.*

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Background: Fraïssé - Jónsson theory

Some definitions

- A category $\mathfrak{K}$ has the amalgamation property if for every arrows $f\colon x \to y$, $g\colon x \to z$ there are arrows $f'\colon y \to w$ and $g'\colon z \to w$ with $f' \circ f = g' \circ g$.

$$\begin{array}{ccc} z & \xrightarrow{g'} & w \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\ x & \xrightarrow[f]{} & y \end{array}$$

- A sequence in $\mathfrak{K}$ is a covariant functor from an ordinal into $\mathfrak{K}$.

$$x_0 \xrightarrow{x_0^1} x_1 \xrightarrow{x_1^2} x_2 \longrightarrow \cdots$$

- $\mathfrak{K}$ is κ-continuous if sequences of length $< \kappa$ have colimits in $\mathfrak{K}$.

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Let $\kappa = \operatorname{cf} \kappa \geqslant \aleph_0$ be a cardinal and let $\mathfrak{K}$ be a category with at most κ arrows. Assume

- *$\mathfrak{K}$ has amalgamations and $0 \in \mathfrak{K}$.*
- *$\mathfrak{K}$ is κ-continuous.*
- *$\mathfrak{K}$ has $\leqslant \kappa$ types of arrows.*

Then there exists a continuous sequence $\vec{u}$ in $\mathfrak{K}$ such that

- $\operatorname{length}(\vec{u}) = \kappa$,
- *for every $\xi < \kappa$ and for every arrow $f \colon u_\xi \to x$ there are $\eta \geqslant \xi$ and an arrow $g \colon x \to u_\eta$ satisfying $g \circ f = u_\xi^\eta$.*

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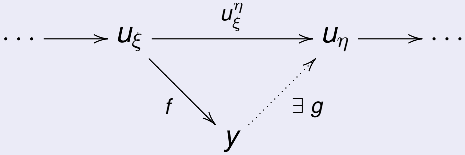

Let $\mathfrak{K} \subseteq \mathfrak{L}$ and let $\vec{x}$ be a sequence in $\mathfrak{K}$, $V \in \mathfrak{L}$. We say that V is the **exact colimit** of $\vec{x}$ if

- V is the colimit of $\vec{x}$, and
- for every $y \in \mathfrak{K}$ and for every arrow $f \in \mathfrak{L}(y, V)$ there are $\xi < \text{length}(\vec{x})$ and $f' \in \mathfrak{K}(y, x_\xi)$ such that $f = x_\xi^\infty \circ f'$.

A sequence satisfying the assertion of the previous theorem will be called **Fraïssé**.

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Assume further that $\mathfrak{K} \subseteq \mathfrak{L}$ and $\mathbb{U}$ is the exact colimit of $\vec{u}$. Then:

1. *For every sequence $\vec{x}$ in $\mathfrak{K}$ such that $\operatorname{length}(\vec{x}) \leqslant \kappa$ and $X = \lim(\vec{x})$ exists in $\mathfrak{L}$, there is an arrow $F \colon X \to \mathbb{U}$.*
2. *$\mathbb{U}$ is unique up to isomorphism.*
3. *For every $a, b \in \mathfrak{K}$ and for every $f \in \mathfrak{K}(a, b)$, $i \in \mathfrak{L}(a, \mathbb{U})$, $j \in \mathfrak{L}(b, \mathbb{U})$ there is an automorphism $H \colon \mathbb{U} \to \mathbb{U}$ such that the following diagram commutes.*

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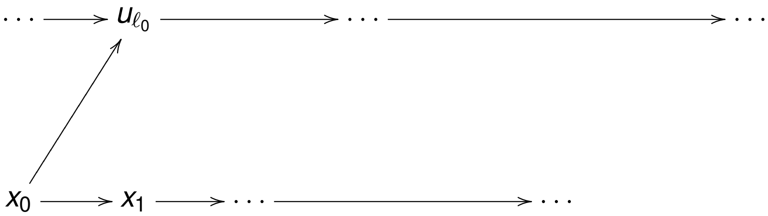
u_{ℓ_0}
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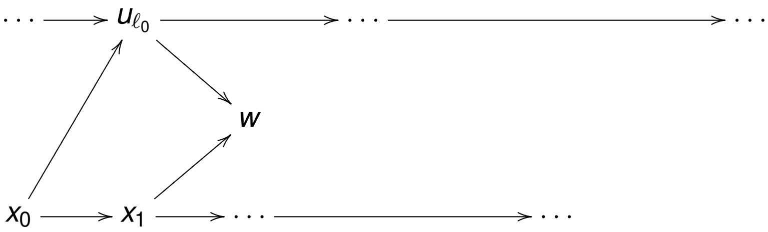

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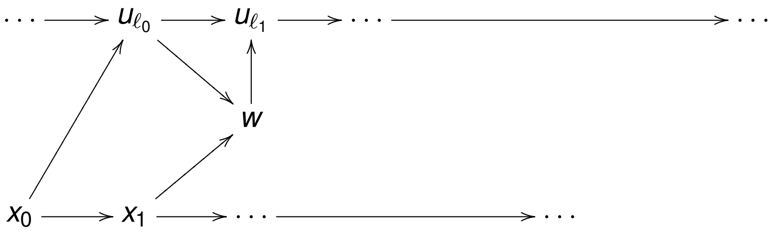

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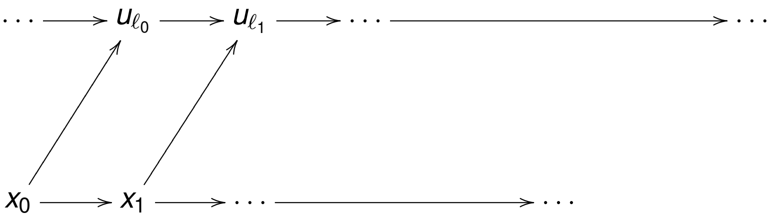
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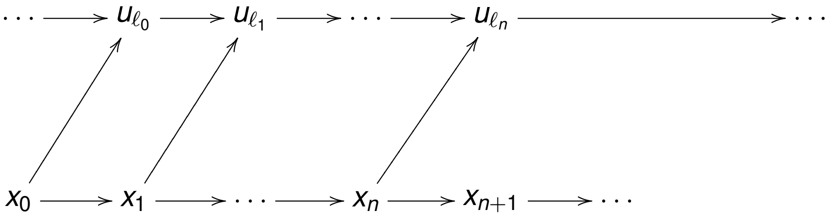
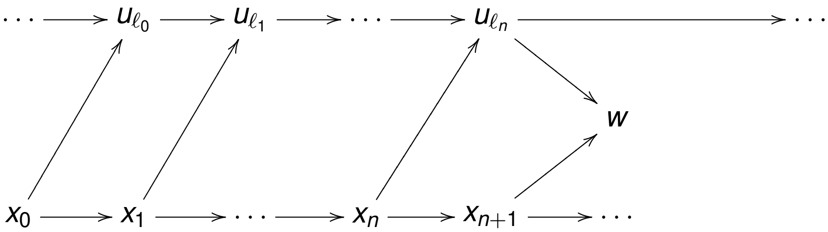

$\cdots \longrightarrow u_{\ell_0} \longrightarrow u_{\ell_1} \longrightarrow \cdots \longrightarrow u_{\ell_n} \longrightarrow \cdots$
$x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_n \longrightarrow x_{n+1} \longrightarrow \cdots$


$\cdots \to u_{\ell_0} \to u_{\ell_1} \to \cdots \to u_{\ell_n} \to \cdots$
w
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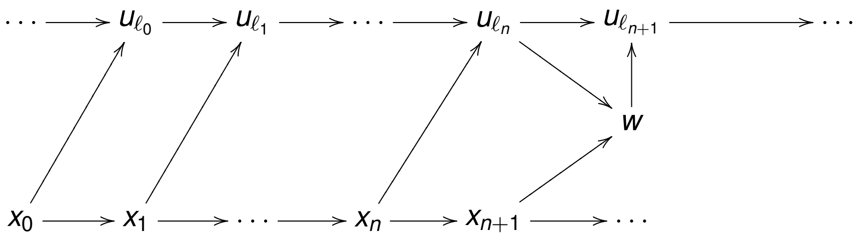
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$$
\begin{array}{ccccccccccccc}
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& \nearrow & & \nearrow & & & & \nearrow & & \nearrow & & & \\
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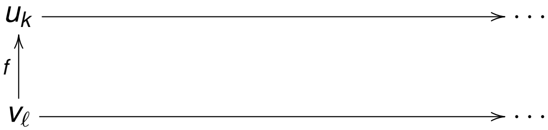
$$

The back-and-forth argument



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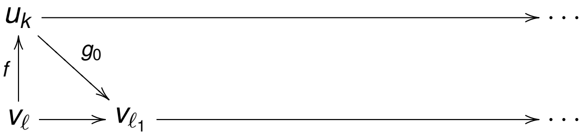

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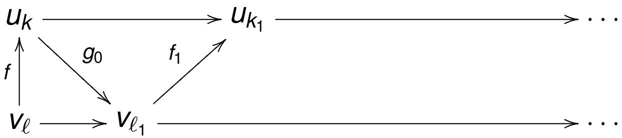

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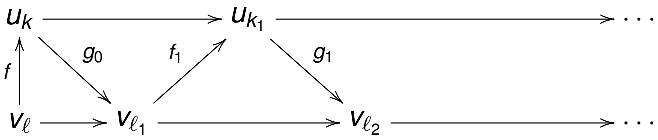

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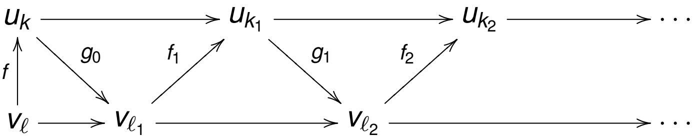

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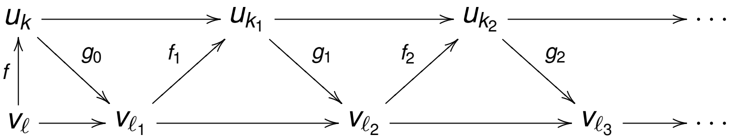

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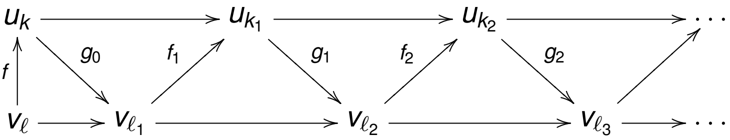

Some history

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- 1960: Jónsson (uncountable model theory)
- 1989: Droste & Göbel (category theory)

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Some natural categories

- Sets with one-to-one maps.
- Boolean algebras with injective homomorphisms.
- Nonempty compact spaces with quotient maps.
- Bounded distributive lattices with injective homomorphisms.
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Lemma

$\mathfrak{Lat}$ has the amalgamation property.

Theorem

Assume CH. There exists a unique bounded distributive lattice $\mathbb{L}$ such that $|\mathbb{L}| = \aleph_1$, every distributive lattice of cardinality $\leqslant \aleph_1$ is embeddable into $\mathbb{L}$ and every partial isomorphism between countable sublattices of $\mathbb{L}$ extends to an automorphism of $\mathbb{L}$.

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Let $\mathfrak{K}$ be the category of all nonempty compact metric spaces with quotient maps. We have a natural contravariant functor

$$\mathrm{Ult}\colon \mathfrak{Lat} \to \mathfrak{Comp}$$

Theorem

$\mathbb{L} = \mathcal{P}(\omega)/_{\mathrm{fin}}$ *and* $\mathrm{Ult}(\mathbb{L}) = \beta\omega \setminus \omega$.

Theorem

Let $\omega^* = \beta\omega \setminus \omega$.

1. *(Parovičenko) Every nonempty compact space of weight* $\leqslant \aleph_1$ *is a quotient of* ω^*.
2. *(Błaszczyk & Szymański) For every quotient maps* $q\colon \omega^* \to X$, $f\colon Y \to X$ *with* Y *compact metric, there exists a quotient map* $g\colon \omega^* \to Y$ *such that* $q = f \circ g$.

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Theorem

Assume $2^{\aleph_0} = \aleph_1$. There exists a Banach space $\mathbb{U}$ with the following properties.

- $\operatorname{dens}(\mathbb{U}) = \aleph_1$.
- *Given two linear isometric embeddings $i \colon X \to \mathbb{U}$ and $f \colon X \to Y$, where X, Y are separable Banach spaces, there exists a linear isometric embedding $g \colon Y \to \mathbb{U}$ such that $i = g \circ f$.*

Moreover, the above properties determine the space $\mathbb{U}$ uniquely up to a linear isometry. Further:

- *Every Banach space of density $\leqslant \aleph_1$ embeds isometrically into $\mathbb{U}$.*
- *Every linear isometry between separable subspaces of $\mathbb{U}$ extends to a linear isometry of $\mathbb{U}$.*

Problem

Find a "concrete" Banach space U of density continuum such that

$$ZFC \wedge CH \vdash U = \mathbb{U}.$$

Remark

$$\mathbb{U} \neq \ell_\infty / c_0.$$

Remark

The "continuous functions" functor is not so good as Ult. Namely: not all linear isometries of Banach spaces come from homeomorphisms of compact spaces.

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