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On a Nonlocal Boundary Value Problem for Second Order Linear Ordinary Differential Equations

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Existence and uniqueness criteria are established for the solution of the equation

$$u'' = p_1(t)u + p_2(t)u' + p_0(t),$$

satisfying the boundary conditions

$$u(a+) = c_1, \qquad u(b-) = \int_a^b u(x)\,d\mu(x) + c_2,$$

where the coefficients $p_k :]a, b[\to \mathbb{R}$ $(k = 0, 1, 2)$ are locally integrable and $\mu : [a, b] \to \mathbb{R}$ is a function of bounded variation. These criteria include the case when the functions $p_k :]a, b[\to \mathbb{R}$ $(k = 0, 1, 2)$ are not integrable on $[a, b]$, having singularities in a and b. © 1995 Academic Press, Inc.

1. STATEMENT OF MAIN RESULTS

The following notation is used throughout.

$\mathbb{R}$ is the set of all real numbers.

$\mathbb{L}([a, b])$ is the set of functions $p :]a, b[\to \mathbb{R}$ which are Lebesque integrable on $[a, b]$.

$\mathbb{L}_{loc}(]a, b[)$ is the set of functions $p :]a, b[\to \mathbb{R}$ which are Lebesque integrable on $[a + \varepsilon, b - \varepsilon]$ for any sufficiently small $\varepsilon > 0$.

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$\sigma : \mathbb{L}_{loc}(]a, b[) \to \mathbb{L}_{loc}(]a, b[)$ is the operator defined by

$$\sigma(p)(t) = \exp\left[\int_{(a+b)/2}^{t} p(s)\, ds\right].$$

If $\sigma(p) \in \mathbb{L}([a, b])$, $\alpha \in [a, b]$, and $\beta \in]a, b]$, then

$$\sigma_\alpha(p)(t) = \frac{1}{\sigma(p)(t)} \left|\int_\alpha^t \sigma p(s)\, ds\right|,$$

$$\sigma_{\alpha\beta}(p)(t) = \frac{1}{\sigma(p)(t)} \left|\int_\alpha^t \sigma p(s)\, ds\right| \left|\int_t^\beta \sigma p(s)\, ds\right|.$$

$u(s+)$ and $u(s-)$ are the right-hand and the left-hand limits of the function u in s.

$[p(t)]_+ = \frac{1}{2}(|p(t)| + p(t))$, $[p(t)]_- = \frac{1}{2}(|p(t)| - p(t))$.

By a solution of the equation

$$u'' = p_1(t)u + p_2(t)u' + p_0(t), \tag{1.1}$$

where $p_k :]a, b[\to \mathbb{R}$ $(k = 0, 1, 2)$, we mean a function $u :]a, b[\to \mathbb{R}$ which is absolutely continuous on any closed subinterval of $]a, b[$ along with its derivative and satisfies (1.1) almost everywhere in $]a, b[$.

Let $c_i \in \mathbb{R}$ $(i = 1, 2)$ and $\mu : [a, b] \to \mathbb{R}$ be a function of bounded variation. This paper is concerned with finding the a solution of Eq. (1.1) satisfying boundary conditions

$$u(a+) = c_1, \qquad u(b-) = \int_a^b u(s)\, d\mu(s) + c_2. \tag{1.2}$$

Parallel with (1.1), (1.2) we shall also consider the corresponding homogeneous problem

$$u'' = p_1(t)u + p_2(t)u', \tag{1.1$_0$}$$

$$u(a+) = 0, \qquad u(b-) = \int_a^b u(s)\, d\mu(s). \tag{1.2$_0$}$$

In the case when μ is a piecewise constant function, the problems of the type (1.1), (1.2) were studied by many authors (cf., for example, [1–3, 5, 6, 8] and the references indicated therein). In particular, in [1, 2, 8] the problem (1.1), (1.2) was studied in the regular case (i.e., when $p_k \in \mathbb{L}([a, b])$, $k = 0, 1, 2$) with boundary conditions

$$u(a+) = c_1, \qquad u(b-) = \sum_{i=1}^{n} \alpha_i u(t_i) + c_2,$$

where $a < t_1 < t_2 < \cdots < t_n < b$, $\alpha_i \in \mathbb{R}$ $(i = \overline{1, n})$, $c_i \in \mathbb{R}$ $(i = 1, 2)$.

The criteria of solvability of the problem (1.1), (1.2) given in [5, 6] include also the singular case (i.e., when, in general, $p_k \notin \mathbb{L}([a, b])$). But the boundary conditions there are of very specific type

$$u(a+) = c_1, \qquad u(b-) = \lambda u(t_0) + c_2,$$

where $a < t_0 < b$, $\lambda \in [0, \infty[$.

As for the general case (i.e., when μ, in general, is not piecewise constant), this problem has not been studied even in the regular case.

In this paper we are going to investigate the problem of the unique solvability of the problem (1.1), (1.2) including the possibility when $p_k \notin \mathbb{L}([a, b])$ $(k = 0, 1, 2)$.

Below we state the conditions under which the homogeneous problem (1.1_0), (1.2_0) has only the zero solution. It is also shown that the latter is necessary and sufficient for the unique solvability of the problem (1.1), (1.2) if only

$$\sigma(p_2) \in \mathbb{L}([a, b]), \qquad \sigma_{ab}(p_2)p_i \in \mathbb{L}([a, b]) \quad (i = 0, 1).$$

Before we go on to formulate the main results, we introduce the following definitions

DEFINITION 1.1. We say that a vector-function $(p_1, p_2) :]a, b[\to \mathbb{R}^2$ belongs to the class $\mathbb{V}(]a, b[)$ if

$$\sigma(p_2) \in \mathbb{L}([a, b]), \qquad \sigma_{ab}(p_2)p_1 \in \mathbb{L}([a, b]),$$

and Eq. (1.1_0) has no nonzero solution satisfying the conditions $u(a+) = 0$, $u(b-) = 0$.

DEFINITION 1.2. We say that a vector-function $(p_1, p_2) :]a, b[\to \mathbb{R}^2$ belongs to the class $\mathbb{U}(]a, b[)$ if $(p_1, p_2) \in \mathbb{V}(]a, b_1[)$ for any $b_1 \in]a, b]$.

Criteria for a vector-function $(p_1, p_2) :]a, b[\to \mathbb{R}^2$ to belong to the classes $\mathbb{V}(]a, b[)$ or $\mathbb{U}(]a, b[)$ can be found in [3, 4, 7].

DEFINITION 1.3. A function $g :]a, b[\times]a, b[\to \mathbb{R}$ is said to be the Green function of the problem (1.1_0), (1.2_0) if for any fixed $\tau \in]a, b[$:

(1) the function $u(t) = g(t, \tau)$ is continuous on $]a, b[$ and satisfies the boundary conditions (1.2_0);

(2) the restrictions of u on $]a, \tau[$ and $]\tau, b[$ are the solutions of (1.1_0);

(3) $u'(\tau+) - u'(\tau-) = 1$.

1.1. The Green Formula

THEOREM 1.1. *Let*

$$\sigma(p_2) \in \mathbb{L}([a, b]), \qquad \sigma_{ab}(p_2)p_i \in \mathbb{L}([a, b]) \quad (i = 0, 1). \tag{1.3}$$

Then the problem (1.1), (1.2) *is uniquely solvable if and only if the corresponding homogeneous problem* (1.1_0), (1.2_0) *has only the zero solution. If the latter is fulfilled, then the unique Green function g of the problem* (1.1_0), (1.2_0) *exists and the solution u of* (1.1), (1.2) *is represented by the Green formula*

$$u(t) = u_0(t) + \int_a^b g(t, \tau)p_0(\tau)\, d\tau \qquad \text{for } a < t < b,$$

where u_0 *is the solution of* (1.1_0), (1.2).

As it will be shown below (see Lemma 2.10), the Green function of the problem (1.1_0), (1.2_0) admits the estimate

$$|g(t, \tau)| \leq c\sigma_{ab}(p_2)(\tau) \qquad \text{for } a < t, \tau < b.$$

Therefore, the integral appearing in the right side of the Green formula does exist.

1.2. Existence and Uniqueness Theorems

For further convenience we introduce the following notation. Let μ be a function of bounded variation, $\sigma(p_2) \in \mathbb{L}([a, b])$, and

$$\delta = \int_a^b (1 + \mu(s) - \mu(b))\sigma(p_2)(s)\, ds \neq 0. \tag{1.4}$$

Then put

$$h_\mu(p_2)(t) = \delta^{-1}\left(\sigma_a(p_2)(t)\int_t^b \mu(s)\sigma(p_2)(s)\, ds - \sigma_b(p_2)(t)\int_a^t \mu(s)\sigma(p_2)(s)\, ds\right) \qquad \text{for } a < t < b.$$

THEOREM 1.2. *Let the conditions* (1.3), (1.4), *and*

$$\int_a^b \sigma(p_2)(s)\sigma_a(p_2)(s)[h_\mu(p_2)(s)p_1(s)]_- \, ds + \int_a^b \sigma_{ab}(p_2)(s)[p_1(s)]_- \, ds \leq \int_a^b \sigma(p_2)(s)\, ds \tag{1.5}$$

hold. Then the problem (1.1), (1.2) *has a unique solution.*

This theorem, in particular, implies that if (1.3) is fulfilled, $p_1(t) \geq 0$ for $a < t < b$, μ is increasing, and $\delta > 0$, then the problem (1.1), (1.2) has a unique solution.

Note that if the conditions of Theorem 1.2 are satisfied, then every nonzero solution of Eq. (1.1_0) has at most one zero in $[a, b]$,[1] i.e.,

$$(p_1, p_2) \in \mathbb{U}(]a, b[). \tag{1.6}$$

THEOREM 1.3. *Let the conditions* (1.3), (1.4), (1.6), *and*

$$0 \neq \int_a^b [h_\mu(p_2)(s)p_1(s)]_- \, ds \times \exp\left[\frac{1}{\int_a^b \sigma(p_2)(s)\, ds} \int_a^b \alpha_{ab}(p_2)(s)[p_1(s)]_- \, ds\right] < 1 \tag{1.7}$$

hold. Then the problem (1.1), (1.2) *has a unique solution.*

In the case when

$$p_1(t) \leq 0 \qquad \text{for } a < t < b \tag{1.8}$$

the condition (1.6) in Theorem 1.3 can somewhat be weakened. More precisely, the following theorem is valid

THEOREM 1.3′. *Let* $\gamma \in \{-1, 1\}$ *and the conditions* (1.3), (1.4), (1.8),

$$(p_1, p_2) \in \mathbb{V}(]a, b[), \tag{1.9}$$

[1] Cf., for example, [7, Theorem 1.1].

and

$$0 \neq \int_a^b [|p_1(s)|\gamma h_\mu(p_2)(s)]_- \, ds$$
$$\times \exp\left[\frac{1}{\int_a^b \sigma(p_2)(s)\,ds} \int_a^b \sigma_{ab}(p_2)(s)|p_1(s)|\,ds\right] < 1$$

hold. Then the problem (1.1), (1.2) *has a unique solution.*

Consider, as an example, the problem

$$u'' = p(t)u; \qquad u(a+) = 0, u(b-) = \lambda u(t_0) + c, \tag{1.10}$$

where $a < t_0 < b$, $c \neq 0$, $\lambda \in \mathbb{R}$, $k \in \{1, 2, 3, \dots\}$, and $-k^2\pi^2/(b-a)^2 < p(t) < -(k-1)^2\pi^2/(b-a)^2$ for $a < t < b$. According to Theorem 1.3′, the problem (1.10) is uniquely solvable if only either

$$0 < \lambda < \frac{2(b-a)^2}{t_0 - a}\left[2(b-a) + (b-t_0)k^2\pi^2 \exp\left(\frac{k^2\pi^2}{4(b-a)}\right)\right]^{-1}$$

or

$$\lambda > \frac{2(b-a)^2}{t_0 - a}\left[2(b-a) - (b-t_0)k^2\pi^2 \exp\left(\frac{k^2\pi^2}{4(b-a)}\right)\right]^{-1} > 0$$

or

$$\lambda < 0 \qquad \text{and} \qquad 2(b-a) < (b-t_0)k^2\pi^2 \exp\left(\frac{k^2\pi^2}{4(b-a)}\right)$$

or

$$0 > \lambda > \frac{2(b-a)^2}{t_0 - a}\left[2(b-a) - (b-t_0)k^2\pi^2 \exp\left(\frac{k^2\pi^2}{4(b-a)}\right)\right]^{-1}.$$

In addition the solution of the problem (1.10) has exactly k zeros in $[a, b[$.

THEOREM 1.3″. *Let the conditions* (1.3), (1.4), (1.9), *and*

$$\int_a^b |h_\mu(p_2)(s)p_1(s)|\,ds \times \exp\left[\frac{1}{\int_a^b \sigma(p_2)(s)\,ds}\int_a^b \sigma_{ab}(p_2)(s)|p_1(s)|\,ds\right] < 1$$

hold. Then the problem (1.1), (1.2) *has a unique solution.*

THEOREM 1.4. *Let μ be nondecreasing and the conditions* (1.3), (1.6), *and*

$$\int_a^b (\mu(b)-\mu(s))\sigma(p_2)(s)\,ds \times \exp\left[\frac{1}{\int_a^b \sigma(p_2)(s)\,ds}\int_a^b \sigma_{ab}(p_2)(s)[p_1(s)]_-\,ds\right] < \int_a^b \sigma(p_2)(s)\,ds \tag{1.11}$$

hold. Then the problem (1.1), (1.2) *has a unique solution.*

Consider, as an example, the problem

$$u'' = p(t)u; \qquad u(a+) = 0, \qquad u(b-) = \int_a^b u(s)\,d\mu(s) + c, \tag{1.12}$$

where μ is increasing and $|p(t)| < 2/(t-a)(b-t)$ for $a < t < b$. According to Theorem 1.4 the problem (1.12) is uniquely solvable if

$$\int_a^b (\mu(b)-\mu(s))\,ds \le (b-a)e^{-2}.$$

THEOREM 1.4′. *Let μ be nondecreasing and the conditions* (1.3), (1.9), *and*

$$\int_a^b (\mu(b)-\mu(s))\sigma(p_2)(s)\,ds \times \exp\left[\frac{1}{\int_a^b \sigma(p_2)(s)\,ds}\int_a^b \sigma_{ab}(p_2)(s)|p_1(s)|\,ds\right] < \int_a^b \sigma(p_2)(s)\,ds$$

hold. Then the problem (1.1), (1.2) *has a unique solution.*

THEOREM 1.5. *Let the conditions* (1.3), (1.6), (1.8) *hold and for some natural n one of the inequalities*

$$\int_a^b a_n(s)\,d\mu_2(s) + r\int_a^b \sigma(p_2)(s)\sigma_a(p_2)(s)\,d\mu_1(s) < 1, \tag{1.13}$$

$$\int_a^b a_n(s)\,d\mu_1(s) + r\int_a^b \sigma(p_2)(s)\sigma_a(p_2)(s)\,d\mu_2(s) > 1, \tag{1.14}$$

be fulfilled where $\mu_1(t) + \mu_2(t) = \mu(t)$ *for* $a \le t \le b$, μ_1 *and* μ_2 *are respectively nondecreasing and nonincreasing functions and*

$$r = \frac{1}{\int_a^b \sigma(p_2)(s)\,ds}\exp\left[\frac{\int_a^b \sigma_{ab}(p_2)(s)|p_1(s)|\,ds}{\int_a^b \sigma(p_2)(s)\,ds}\right],$$

$$a_1(t) = \frac{1}{\int_a^b \sigma(p_2)(s)\,ds}\int_a^t \sigma(p_2)(s)\,ds,$$

$$a_{k+1}(t) = a_1(t) + \frac{\sigma(p_2)(t)}{\int_a^b \sigma(p_2)(s)\,ds}\left[\sigma_b(p_2)(t)\int_a^t \sigma_a(p_2)(s)|p_1(s)|a_k(s)\,ds\right.$$

$$\left. + \sigma_a(p_2)(t)\int_t^b \sigma_b(p_2)(s)|p_1(s)|a_k(s)\,ds\right] \quad \textit{for } a < t < b. \tag{1.15}$$

Then the problem (1.1), (1.2) *has a unique solution.*

Consider, as an example, the problem

$$u'' = p(t)u; \qquad u(a+) = c_1, \qquad u(b-) = -\lambda u(t_0) + \int_a^b u(s)\,ds + c_2, \tag{1.16}$$

where $a < t_0 < b$, $\lambda > 0$, $c_1, c_2 \in \mathbb{R}$, and $0 \ge p(t) > -2/(t-a)(b-t)$ for $a < t < b$. According to Theorem 1.5, the problem (1.16) is uniquely solvable if only either

$$e^2(b-a)^2 - 2\lambda(t_0 - a) \le 2(b-a)$$

or

$$(b-a)^2 - 2\lambda(t_0 - a)e^2 \ge 2(b-a).$$

As an another example, consider the problem

$$u'' = p(t)u; \qquad u(a+) = c_1, \qquad u(b-) = \int_a^b u(s)\,ds + c_2,$$

where $p(t) \leq 0$ for $a < t < b$ and the function $t \mapsto (t-a)(b-t)p(t)$ is integrable. According to Theorem 1.5 this problem is uniquely solvable if only $b - a > 2$ and $(p, 0) \in \mathbb{U}(]a, b[)$, in particular, if

$$b - a > 2 \qquad \text{and} \qquad \int_a^b (s-a)(b-s)|p(s)|\,ds \leq b - a,$$

or $b - a < 2$, $(p, 0) \in \mathbb{U}(]a, b[)$, and

$$\int_a^b (s-a)(b-s)|p(s)|\,ds \leq (b-a)\ln\frac{2}{b-a}.$$

Remark 1.1. Let the conditions of either Theorem 1.2 or Theorem 1.3 be fulfilled. Then the solution u_1 of the problem

$$u'' = p_1(t)u + p_2(t)u'; \qquad u(a+) = 0, \qquad \lim_{t \to a+} \frac{u'(t)}{\sigma(p_2)(t)} = 1 \tag{1.17}$$

satisfies the inequality

$$\delta\left(u_1(b-) - \int_a^b u_1(s)\,d\mu(s)\right) > 0$$

and if the conditions of Theorem 1.4 are fulfilled, then

$$u_1(b-) > \int_a^b u_1(s)\,d\mu(s).$$

2. Auxiliary Statements

In this section some properties of solutions of Eqs. (1.1) and (1.1_0) are established. Here and in the sequel we assume that

$$p_i \in \mathbb{L}_{loc}(]a, b[) \qquad (i = 0, 1, 2).$$

LEMMA 2.1. *Let*

$$\sigma(p_2) \in \mathbb{L}([a,b]), \qquad \sigma_{ab}(p_2)p_1 \in \mathbb{L}([a,b]). \tag{2.1}$$

Then any solution u of Eq. (1.1_0) *having finite limits* $u(a+)$ *and* $u(b-)$ *satisfies*

$$\liminf_{t\to a+} \sigma_{ab}(p_2)(t)|u'(t)| = 0 \tag{2.2}$$

and

$$\liminf_{t\to b+} \sigma_{ab}(p_2)(t)|u'(t)| = 0 \tag{2.3}$$

Proof. We shall prove only the inequality (2.2); (2.3) can be proved analogously.

Suppose on the contrary, that (2.2) is not fulfilled. Then $\varepsilon > 0$ and $a_0 \in]a, b[$ can be found such that

$$|u'(t)| > \varepsilon\sigma(p_2)(t)\left[\int_a^t \sigma(p_2)(s)\,ds\right]^{-1} \qquad \text{for } a < t < a_0.$$

Thus

$$\begin{aligned} |u(t)| &\geq \int_t^{a_0} |u'(s)|\,ds - |u(a_0)| \\ &\geq \varepsilon\left[\ln\int_a^{a_0}\sigma(p_2)(s)\,ds - \ln\int_a^t \sigma(p_2)(s)\,ds\right] \qquad \text{for } a < t < a_0, \end{aligned}$$

which is impossible in view of boundedness of u. The contradiction thus obtained shows that (2.2) is true. And so the lemma is proved.

Remark 2.1. Taking into account Lemma 2.1 of [3], it is easy to be verify that

$$\lim_{t\to b+} \sigma_b(p_2)(t)|u_1'(t)| = 0,$$

where u_1 is a solution of the problem (1.17).

LEMMA 2.2. *Let the conditions* (1.4) *and* (2.1) *be fulfilled. Then any solution u of* (1.1_0), (1.2_0) *satisfies*

$$\int_a^b h_\mu(p_2)(s)p_1(s)u(s)\,ds = -u(b-). \tag{2.4}$$

Proof. It is easy to see that

$$|h_\mu(p_2)(t)| \le c\sigma_{ab}(p_2)(t) \qquad \text{for } a < t < b, \tag{2.5}$$

where $c > 0$ is a constant. Hence according to Lemma 2.1, the sequences $(a_k)_{k=1}^{+\infty}$ and $(b_k)_{k=1}^{+\infty}$ can be found such that they tend monotonically to a and b, respectively, and

$$\lim_{k\to+\infty} u'(a_k)h_\mu(p_2)(a_k) = \lim_{k\to+\infty} u'(b_k)h_\mu(p_2)(b_k) = 0. \tag{2.6}$$

Integration by parts and the differential equation now yield

$$\begin{aligned}
\int_{a_k}^{b_k} h_\mu(p_2)(s)p_1(s)u(s)\,ds &= -\int_{a_k}^{b_k} u'(s)h'_\mu(p_2)(s)\,ds \\
&\quad -\int_{a_k}^{b_k} p_2(s)u'(s)h_\mu(p_2)(s)\,ds + u'(b_k)h_\mu(p_2)(b_k) - u'(a_k)h_\mu(p_2)(a_k) \\
&= -\delta^{-1}(u(b_k) - u(a_k))\int_a^b \mu(s)\sigma(p_2)(s)\,ds \\
&\quad + \delta^{-1}\int_{a_k}^{b_k} \mu(s)u'(s)\,ds \int_a^b \sigma(p_2)(s)\,ds + u'(b_k)h_\mu(p_2)(b_k) \\
&\quad - u'(a_k)h_\mu(p_2)(a_k).
\end{aligned}$$

Then (2.5) and (2.6) imply

$$\begin{aligned}
&\int_a^b h_\mu(p_2)(s)p_1(s)u(s)\,ds \\
&\qquad = \frac{1}{\delta}\left(\int_a^b \mu(s)\,du(s)\int_a^b \sigma(p_2)(s)\,ds - u(b-)\int_a^b \mu(s)\sigma(p_2)(s)\,ds\right).
\end{aligned}$$

Since

$$\int_a^b \mu(s)\,du(s) = \mu(b)u(b-) - \int_a^b u(s)\,d\mu(s) = u(b-)(\mu(b) - 1),$$

the last inequality can be written as

$$\int_a^b h_\mu(p_2)(s)p_1(s)u(s)\,ds = -\delta^{-1}u(b-)\int_a^b (1+\mu(s)-\mu(b))\sigma(p_2)(s)\,ds = -u(b-).$$

Therefore, (2.4) is valid and the lemma is proved.

Before we go on to formulate the next lemma, we introduce the following notation. Let (2.1) hold and $q \in \mathbb{L}([a, b])$. Then put

$$\varphi_0(p_1, p_2) = \left[\int_a^b \sigma(p_2)(s)\,ds\right]^{-1} \exp\left[\frac{\int_a^b \sigma_{ab}(p_2)(s)p_1(s)\,ds}{\int_a^b \sigma(p_2)(s)\,ds}\right],$$

$$\varphi_1(p_1, p_2, q) = \varphi_0(p_1, p_2)\int_a^b q(s)\,ds.$$

LEMMA 2.3. *Let* (1.6) *be fulfilled. Then the solution u of the problem*

$$u'' = p_1(t)u + p_2(t)u'; \qquad u(a+) = 0, \qquad u(b-) = c \tag{2.7}$$

admits the estimate

$$|u(t)| \le |c|\varphi_0([p_1]_-, p_2)\int_a^t \sigma(p_2)(s)\,ds \qquad \textit{for } a \le t \le b. \tag{2.8}$$

Proof. Suppose, without loss of generality, that $c > 0$ and $u(t) > 0$ for $a < t \le b$. Then the Green formula (see [3]) implies

$$\begin{aligned} u(t) = \left(\int_a^b \sigma(p_2)(s)\,ds\right)^{-1} \sigma(p_2)(t)\Big(c\sigma_a(p_2)(t) \\ - \sigma_b(p_2)(t)\int_a^t \sigma_a(p_2)(s)p_1(s)u(s)\,ds \\ -\sigma_a(p_2)(t)\int_t^b \sigma_b(p_2)(s)p_1(s)u(s)\,ds\Big) \qquad \text{for } a \le t \le b. \end{aligned} \tag{2.9}$$

Hence we have

$$\begin{aligned} |u(t)|\left(\int_a^t \sigma(p_2)(s)\,ds\right)^{-1} \le \left(\int_a^b \sigma(p_2)(s)\,ds\right)^{-1}\Big[|c| \\ + \int_a^b \sigma_{ab}(p_2)(s)[p_1(s)]_-|u(s)|\left(\int_a^s \sigma(p_2)(\tau)\,d\tau\right)^{-1} ds\Big] \qquad \text{for } a < t < b. \end{aligned}$$

Applying the Gronwall-Bellman lemma we obtain the estimate (2.8). The lemma is proved.

The following four lemmas are proved analogously.

LEMMA 2.4. *Let* (1.6) *be fulfilled*, $q \in \mathbb{L}([a, b])$, *and* $q(t) \leq 0$ *for* $a < t < b$. *Then the solution* u *of the problem*

$$u'' = p_1(t)u + p_2(t)u' + q(t)\sigma(p_2)(t); \qquad u(a+) = 0, \qquad u(b-) = 0 \quad (2.10)$$

admits the estimate

$$|u(t)| \leq \varphi_1([p_1]_-, p_2, |q|)\sigma(p_2)(t)\sigma_{ab}(p_2)(t) \qquad \text{for } a \leq t \leq b.$$

LEMMA 2.5. *Let* (1.9) *be fulfilled. Then the solution* u *of* (2.7) *admits the estimate*

$$|u(t)| \leq \varphi_0(|p_1|, p_2) \int_a^t \sigma(p_2)(s)\, ds \qquad \text{for } a \leq t \leq b.$$

LEMMA 2.6. *Let* (1.9) *be fulfilled*, $q \in \mathbb{L}([a, b])$, *and* $p_1(t) \leq 0$ *for* $a < t < b$. *The solution* u *of* (2.10) *admits the estimate*

$$[u(t)]_+ \leq \varphi_1(|p_1|, p_2, [q]_-)\sigma(p_2)(t)\sigma_{ab}(p_2)(t) \qquad \text{for } a \leq t \leq b.$$

LEMMA 2.7. *Let* (1.9) *be fulfilled and* $q \in \mathbb{L}([a, b])$. *Then the solution* u *of* (2.10) *admits the estimate*

$$|u(t)| \leq \varphi_1(|p_1|, p_2, |q|)\sigma(p_2)(t)\sigma_{ab}(p_2)(t) \qquad \text{for } a \leq t \leq b.$$

LEMMA 2.8. *Let* (1.6) *and* (1.8) *be fulfilled. Then for any natural* n *the solution* u *of* (2.7) *admits the estimate*

$$|u(t)| \geq |c|a_n(t) \qquad \text{for } a \leq t \leq b,$$

where the functions a_k *are defined by the recurrent relations* (1.15).

Proof. We shall assume without loss of generality that $u(t) > 0$ for $a < t \leq b$. According to the Green formula the representation (2.9) is valid. Hence

$$|u(t)| \geq |c|a_1(t) \qquad \text{for } a \leq t \leq b.$$

Suppose now that for some $k \in \{1, 2, \ldots, n - 1\}$

$$|u(t)| \geq |c|a_k(t) \qquad \text{for } a \leq t \leq b.$$

Then in virtue of (2.9) we have

$$|u(t)| \geq |c|\sigma(p_2)(t)\left(\int_a^b \sigma(p_2)(s)\,ds\right)^{-1}\Big[\sigma_a(p_2)(t)$$

$$+\,\sigma_b(p_2)(t)\int_a^t \sigma_a(p_2)(s)|p_1(s)|a_k(s)\,ds$$

$$+\,\sigma_a(p_2)(t)\int_t^b \sigma_b(p_2)(s)|p_1(s)|a_k(s)\,ds\Big] = |c|a_{k+1}(t) \qquad \text{for } a \leq t \leq b.$$

Therefore, the lemma is proved by induction.

It can immediately be verified that the following lemma is valid.

LEMMA 2.9. *Let* (2.1) *be fulfilled and the problem* (1.1_0), (1.2_0) *have only the zero solution. Then there exists the unique Green function g of the problem and*

$$g(t,\tau) = \frac{u_1(t)}{u_1(b-) - \int_a^b u_1(s)\,d\mu(s)}\left[\int_\tau^b c(s,\tau)\,d\mu(s) - c(b-,\tau)\right]$$

$$+\,\eta(\tau,t)c(t,\tau) \qquad \textit{for } a < t, \tau < b,$$

where

$$\eta(t,x) = \begin{cases} 1 & \textit{for } t \leq x \\ 0 & \textit{for } t > x, \end{cases}$$

and u_1 *is the solution of* (1.17) *and c is the Cauchy function of Eq.* (1.1_0).

Taking into account Lemma 2.1 of [3], it is easy to verify that the following lemma is valid.

LEMMA 2.10. *Let* (2.1) *be fulfilled and the problem* (1.1_0), (1.2_0) *have only the zero solution. Then the Green function g of the problem admits the estimate*

$$|g(t,\tau)| \leq c\sigma_b(p_2)(\tau)\int_a^t \sigma(p_2)(s)\,ds \qquad \textit{for } a < t \leq \tau < b,$$

$$|g(t,\tau)| \leq c\sigma_{ab}(p_2)(\tau) \qquad \textit{for } a < \tau < t < b,$$

where $c > 0$ *is a constant.*

3. Proof of the Main Results

Proof of Theorem 1.1. It is clear that if the problem (1.1_0), (1.2_0) has a nonzero solution, then the problem (1.1), (1.2) has either no solution or infinitely many of them. Suppose now that the problem (1.1_0), (1.2_0) has only the zero solution. Then the problem (1.1), (1.2) has at most one solution. Therefore, it suffices to prove that the function v defined by

$$v(t) = \int_a^b g(t, \tau) p_0(\tau)\, d\tau \qquad \text{for } a < t < b$$

is the solution of the problem (1.1), (1.2_0).

Taking into account Lemmas 2.9 and 2.10, we find out that v is the solution of (1.1) and

$$v(b-) = \int_a^b v(s)\, d\mu(s).$$

On the other hand, according to Lemma 2.10,

$$|v(t)| \le c \int_a^t \sigma_{ab}(p_2)(s) |p_0(s)|\, ds$$
$$+ c \int_a^t \sigma(p_2)(s)\, ds \int_t^b \sigma_b(p_2)(s) |p_0(s)|\, ds \qquad \text{for } a < t < b.$$

It is easy to see that

$$\int_a^t \sigma(p_2)(s)\, ds \int_t^b \sigma_b(p_2)(s) |p_0(s)|\, ds \to 0 \qquad \text{for } t \to a+.$$

Hence $v(a+) = 0$. Therefore, v is the solution of (1.1), (1.2_0). And the theorem is proved.

Proof of Theorem 1.2. By (1.5) we have that

$$\int_a^b \sigma_{ab}(p_2)(s) [p_1(s)]_-\, ds \le \int_a^b \sigma(p_2)(s)\, ds.$$

Therefore (see [4, 7]) the solution u_1 of the problem (1.17) is positive in $]a, b]$.

Suppose now that Remark 1.1 is not true. Then for a positive c

$$u_1(b-) = c\int_a^b u_1(s)\,d\mu(s),$$

where $c \le 1$ if $\delta \ge 0$ and $c > 1$ if $\delta < 0$, δ being the number defined by (1.4).

According to the Green formula, the representation

$$u_1(t) = \int_a^b g_0(t,\tau)p_1(\tau)u_1(\tau)\,d\tau \qquad \text{for } a \le t \le b \tag{3.1}$$

is valid, where g_0 is the Green function of the problem

$$u'' = p_2(t)u; \qquad u(a+) = 0, \qquad u(b-) = c\int_a^b u(s)\,d\mu(s).$$

It immediately can be verified that

$$g_0(t,\tau) = h(\tau)\int_a^t \sigma(p_2)(s)\,ds - \frac{1}{\int_a^b \sigma(p_2)(s)\,ds}\sigma(p_2)(t)\sigma_a(p_2)(t)\sigma_b(p_2)(\tau) \qquad \text{for } a \le t \le \tau \le b,$$

$$g_0(t,\tau) = h(\tau)\int_a^t \sigma(p_2)(s)\,ds - \frac{1}{\int_a^b \sigma(p_2)(s)\,ds}\sigma(p_2)(t)\sigma_a(p_2)(\tau)\sigma_b(p_2)(t) \qquad \text{for } a \le \tau \le t \le b, \tag{3.2}$$

where

$$h(t) = \frac{c\delta}{c\delta + (1-c)\int_a^b \sigma(p_2)(s)\,ds}h_\mu(p_2)(t) \qquad \text{for } a < t < b. \tag{3.3}$$

and

$$0 < \frac{c\delta}{c\delta + (1-c)\int_a^b \sigma(p_2)(s)\,ds} < 1. \tag{3.4}$$

According to (1.5) and (3.2)–(3.4), from (3.1) we obtain the contradiction

$$\begin{aligned} v(t) &\le \int_a^b \sigma(p_2)(s)\sigma_a(p_2)(s)[h(s)p_1(s)]_- v(s)\,ds \\ &\quad + \left[\int_a^b \sigma(p_2)(s)\,ds\right]^{-1} \int_a^b \sigma_{ab}(p_2)(s)[p_1]_- v(s)\,ds \\ &< \lambda \left[\int_a^b \sigma(p_2)(s)\sigma_a(p_2)(s)[h_\mu(p_2)(s)]_-\,ds \right.\\ &\quad \left. + \left[\int_a^b \sigma(p_2)(s)\,ds\right]^{-1} \int_a^b \sigma_{ab}(p_2)(s)[p_1(s)]_-\,ds\right] \le \lambda, \end{aligned}$$

where $v(t) = u_1(t)(\int_a^t \sigma(p_2)(s)\,ds)^{-1}$ for $a < t < b$ and $\lambda = \sup\{v(t) : a < t < b\}$. The theorem is proved.

Proof of Theorem 1.3. In order to prove the theorem it suffices to be convinced that Remark 1.1 is valid. Assume, on the contrary, that Remark 1.1 is not true. Then by (1.6) for a positive c we have

$$u_1(t) < 0 \quad \text{for } a < t \le b, \qquad u_1(b-) = c\int_a^b u_1(s)\,d\mu(s),$$

where $c \le 1$ if $\delta > 0$ and $c > 1$ if $\delta < 0$, δ being the number defined by (1.4). Denote by v the solution of the problem

$$\begin{aligned} v'' &= p_1(t)v - p_2(t)v' - \sigma(p_2)(t)[h(t)p_1(t)]_- \\ v(a+) &= 0, \qquad v(b-) = 0, \end{aligned}$$

where h- is the function defined by (3.3).

Let

$$\rho(t) = \frac{u_1'(t)v(t) - u_1(t)v'(t)}{\sigma(p_2)(t)} \qquad \text{for } a < t < b.$$

It is easy to see that

$$\rho'(t) = [h(t)p_1(t)]_- u_1(t) \qquad \text{for } a < t < b. \tag{3.5}$$

Therefore, according to Lemma 2.3 of [3], the finite limit $\rho(a+) \ge 0$ exists. Integrating (3.5) and taking into account (3.3), (3.4), Remark 2.1, and Lemma 2.2, we find

$$\lim_{t\to b-} \frac{v'(t)}{\sigma(p_2)(t)} \le -1.$$

Hence, we see that for any n large enough there exists $\varepsilon_n \in]0, (b-a)/2[$ such that

$$v(t) > (1 - 1/n) \int_t^b \sigma(p_2)(s)\, ds \qquad \text{for } b - \varepsilon_n \le t \le b.$$

On the other hand according to (3.4) and Lemma 2.4 the estimate

$$|v(t) \le \varphi_1([p_1]_-, p_2, [h_\mu(p_2)p_1]_- \times \int_a^t \sigma(p_2)(s)\, ds \int_t^b \sigma(p_2)(s)\, ds \qquad \text{for } a \le t \le b$$

is valid. The last two inequalities imply that

$$\varphi_1([p_1]_-, p_2, [h_\mu(p_2)p_1]_-) \int_a^b \sigma(p_2)(s)\, ds \ge 1,$$

which contradict inequality (1.7). This contradiction proves the theorem.

Proof of Theorem 1.4. It suffices to be convinced that Remark 1.1 is valid. Suppose on the contrary that this remark is not true. Then the solution u_1 of the problem (1.7) satisfies the inequalities

$$u_1(t) > 0 \quad \text{for } a < t \le b, \qquad u_1(b-) \le \int_a^b u_1(s)\, d\mu(s).$$

According to Lemma 2.3, the estimate

$$u_1(t) \le u_1(b-)\varphi_0([p_1]_-, p_2) \int_a^t \sigma(p_2)(s)\, ds \qquad \text{for } a \le t \le b$$

holds. Integration by parts and the above two inequalities yield

$$u_1(b-) \le \int_a^b u_1(s)\, d\mu(s)$$
$$\le u_1(b-)\varphi_0([p_1]_-, p_2) \int_a^b (\mu(b) - \mu(s))\sigma(p_2)(s)\, ds.$$

This together with condition (1.11) and the fact that μ is nondecreasing yields the contradiction $1 < 1$. The theorem is proved.

Proof of Theorem 1.5. Suppose the contrary. Then the problem (1.1_0), (1.2_0) has a nonzero solution u. Without loss of generality we can assume that $u(t) > 0$ for $a < t \leq b$. We shall carry out the proof only in the case when (1.13) is fulfilled. The case when (1.14) is fulfilled can be treated analogously.

According to Lemmas 2.3 and 2.8, the estimates

$$u(b-)a_n(t) \leq u(t) \leq u(b-)\varphi_0(|p_1|, p_2) \int_a^t \sigma(p_2)(s)\, ds \qquad \text{for } a \leq t \leq b$$

hold. Hence we easily find that

$$\int_a^b u(s)\, d\mu_1(s) \leq u(b-)\varphi_0(|p_1|, p_2) \int_a^b \sigma(p_2)(s)\sigma_a(p_2)(s)\, d\mu_1(s)$$

and

$$\int_a^b u(s)\, d\mu_2(s) \leq u(b-) \int_a^b a_n(s)\, d\mu_2(s).$$

Adding these inequalities and taking into account (1.13), we obtain the contradiction $1 < 1$. The theorem is proved.

Theorems 1.3′ and 1.3″ can be proved similarly to Theorem 1.3. Instead of Lemma 2.4, Lemmas 2.6 and 2.7 are to be applied, respectively. Theorem 1.4′ can be proved similarly to Theorem 1.4. The only difference is that Lemma 2.5 is to be applied instead of Lemma 2.3.

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