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On Certain Boundary Value Problems for Second-Order Linear Ordinary Differential Equations with Singularities

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For the differential equation

$$u'' = p_1(t)\, u + p_2(t)\, u' + p_0(t)$$

with locally integrable coefficients p_k: $]a, b[\to R$ $(k = 0, 1, 2)$ and for each of the following three types of boundary conditions

$$u(a+) = \alpha, \qquad \lim_{t \to b-} \frac{u'(t)}{\sigma(p_2)(t)} = \beta,$$

$$u(a+) = \alpha, \qquad u(b-) = \beta$$

and

$$u(a+) = \alpha, \qquad u(b-) = u(t_0) + \beta$$

where $-\infty < a < t_0 < b < +\infty$ and $\sigma(p_2)(t) = \exp(\int_{(a+b)/2}^{t} p_2(\tau)\, d\tau)$, the conditions of existence and uniqueness of a solution are established. These conditions extend the well-known results of Vallée Poussin [11] and cover the case when the functions p_k $(k = 0, 1, 2)$ are unintegrable on $[a, b]$ having singularities at the points a and b.

1. STATEMENT OF PROBLEMS AND NOTATION

Below we use the following notation.

R is the set of real numbers.

$L([a, b])$ is the set of functions p: $[a, b] \to R$ which are Lebesgue integrable on $[a, b]$.

$L_{loc}([a, b[)$ is the set of functions p: $[a, b[\to R$ which are Lebesgue integrable on $[a, b - \varepsilon]$ for any sufficiently small $\varepsilon > 0$.

$L_{loc}(]a, b[)$ is the set of functions p: $]a, b[\to R$ which are Lebesgue integrable on $[a + \varepsilon, b - \varepsilon]$ for any sufficiently small $\varepsilon > 0$.

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$\tilde{C}^1_{\text{loc}}(]a,b[)$ is the set of functions $u:]a,b[\to R$ having the absolutely continuous on $[a+\varepsilon, b-\varepsilon]$ first derivative for any sufficiently small $\varepsilon > 0$.

$\sigma: L_{\text{loc}}(]a,b[) \to L_{\text{loc}}(]a,b[)$ is the operator defined by the equality

$$\sigma(p)(t) = \exp\left[\int_{(a+b)/2}^t p(\tau)\, d\tau\right].$$

If $\sigma(p) \in L_{\text{loc}}([a,b[)$, then

$$\sigma_a(p)(t) = \frac{1}{\sigma(p)(t)} \int_a^t \sigma(p)(\tau)\, d\tau$$

and if $\sigma(p) \in L([a,b])$, then

$$\sigma_{ab}(p)(t) = \frac{1}{\sigma(p)(t)} \int_a^t \sigma(p)(\tau)\, d\tau \int_t^b \sigma(p)(\tau)\, d\tau.$$

$u(s+)$ and $u(s-)$ are the right-hand and the left-hand limits of the function u at the point s.

Under a solution of the differential equation

$$u'' = p_1(t)\, u + p_2(t)\, u' + p_0(t) \tag{1.1}$$

where $p_k \in L_{\text{loc}}(]a,b[)$ $(k = 0, 1, 2)$, we mean a function $u \in \tilde{C}^1_{\text{loc}}(]a,b[)$ which satisfies this equation almost everywhere in $]a,b[$.

For arbitrary $\alpha \in R$, $\beta \in R$, $t_0 \in]a,b[$ state the problems of finding a solution u of (1.1) satisfying the boundary condition of one of following three types:

$$u(a+) = \alpha, \qquad \lim_{t \to b-} \frac{u'(t)}{\sigma(p_2)(t)} = \beta, \tag{1.2}$$

$$u(a+) = \alpha, \qquad u(b-) = \beta \tag{1.3}$$

and

$$u(a+) = \alpha, \qquad u(b-) = u(t_0) + \beta. \tag{1.4}$$

Along with (1.1), (1.k) $(k = 2, 3, 4)$ consider the corresponding homogeneous problems

$$u'' = p_1(t)\, u + p_2(t)\, u', \tag{1.1$_0$}$$

$$u(a+) = 0, \qquad \lim_{t \to b-} \frac{u'(t)}{\sigma(p_2)(t)} = 0, \tag{1.2$_0$}$$

$$u(a+) = 0, \qquad u(b-) = 0, \tag{1.3$_0$}$$

$$u(a+) = 0, \qquad u(b-) = u(t_0). \tag{1.4$_0$}$$

Problems (1.1), (1.2) and (1.1), (1.3) are well-known boundary value problems and in the regular case when $p_k \in L([a, b])$ $(k = 0, 1, 2)$ they are studied with sufficient completeness (see [1, 4, 6, 11]). The conditions of unique solvability of the problem (1.1), (1.3) contained in [5, 7, 8 and 9] also cover the singular case when

$$p_k \notin L([a, b]), \qquad \sigma_{ab}(0)\, p_k \in L([a, b]) \qquad (k = 0, 1), \qquad p_2 \in L([a, b])$$

or

$$p_k \in L([a, b]) \qquad (k = 0, 1), \qquad p_2 \notin L([a, b]), \qquad \sigma(p_2) \in L([a, b]).$$

But in the case when all the functions p_0, p_1 and p_2 are unintegrable on $[a, b]$ the problem (1.1), (1.2) as well as the problem (1.1), (1.3) was not studied earlier.

In fact, the problem (1.1), (1.4) was not studied even in the regular case. It is worth mentioning that the similar problem, but for partial differential equations, which is known now as the Bitsadze–Samarskii problem, was first stated and solved in [3].

In this paper our aim is to investigate the question on unique solvability of the problems (1.1), (1.k) $(k = 2, 3, 4)$ when the case of $p_k \notin L([a, b])$ $(k = 0, 1, 2)$ is not excluded.

For each $k \in \{2, 3, 4\}$ we establish the unimprovable, in a certain sense, conditions under which the homogeneous problem (1.1_0), $(1.k_0)$ has only zero solution. This occurs to be necessary and sufficient for the unique solvability of the problem (1.1), (1.k) provided that either $k = 2$ and

$$\sigma(p_2) \in L([a, b]), \qquad \sigma_a(p_2)\, p_i \in L([a, b]) \qquad (i = 0, 1) \tag{1.5}$$

or $k \in \{3, 4\}$ and

$$\sigma(p_2) \in L([a, b]), \qquad \sigma_{ab}(p_2)\, p_i \in L([a, b]) \qquad (i = 0, 1).^{1} \tag{1.6}$$

2. Auxiliary Statements

In this section some properties of solutions of the homogeneous equation (1.1_0) are established. Here and in the sequel we assume that

$$p_i \in L_{\mathrm{loc}}(]a, b[) \qquad (i = 1, 2).$$

[1] The condition (1.5) (the condition (1.6)) holds if, e.g., the inequalities $|p_2(t)| \leqslant \lambda + \delta/(t-a)(b-t)$ and $|p_i(t)| \leqslant \lambda (t-a)^{-1-\delta}$ $(|p_i(t)| \leqslant \lambda [(t-a)(b-t)]^{-1-\delta})$ $(i = 0, 1)$, where $\lambda > 0$ and $0 \leqslant \delta < 1$, are fulfilled in $]a, b[$.

2.1. *The estimate of the Cauchy function of the equation* (1.1_0). C: $]a, b[\times]a, b[\to R$ is said to be the Cauchy function of Eq. (1.1_0) if for any $\tau \in]a, b[$ the function $u(t) = C(t, \tau)$ is a solution of (1.1_0) satisfying the initial conditions

$$u(\tau) = 0, \qquad u'(\tau) = 1.$$

If C is the Cauchy function of Eq. (1.1_0), then for any t and $\tau \in]a, b[$

$$C(t, \tau) = \frac{1}{\sigma(p_2)(\tau)} \int_\tau^t \sigma(p_2)(s)\, ds + \int_\tau^t \frac{1}{\sigma(p_2)(x)} \times \left(\int_x^t \sigma(p_2)(s)\, ds \right) p_1(x)\, C(x, \tau)\, dx. \tag{2.1}$$

LEMMA 2.1. *If*

$$\sigma(p_2) \in L([a, b]), \qquad \sigma_{ab}(p_2) p_1 \in L([a, b]), \tag{2.2}$$

then the Cauchy function of Eq. (1.1_0) *admits the estimate*

$$|C(t, \tau)| \leqslant \frac{r_0}{\sigma(p_2)(\tau)} \left| \int_\tau^t \sigma(p_2)(s)\, ds \right| \tag{2.3}$$

on the set $]a, b[\times]a, b[$, *where*

$$r_0 = \exp\left[\frac{2}{\int_a^b \sigma(p_2)(s)\, ds} \int_a^b \sigma_{ab}(p_2)(s)\, |p_1(s)|\, ds \right]. \tag{2.4}$$

Proof. Fix arbitrary $\tau \in]a, b[$. Setting

$$\rho(t) = \sigma(p_2)(\tau) \left| \int_\tau^t \sigma(p_2)(s)\, ds \right|^{-1} |C(t, \tau)| \qquad \text{for} \quad \tau \neq t, \quad \rho(\tau) = 1,$$

$$h(t, x) = \frac{1}{\sigma(p_2)(x)} \left| \int_x^t \sigma(p_2)(s)\, ds \int_\tau^x \sigma(p_2)(s)\, ds \right| \left| \int_\tau^t \sigma(p_2)(s)\, ds \right|^{-1} \qquad \text{for} \quad t \neq \tau$$

and $h(\tau, x) \equiv 0$, from (2.1) we obtain

$$\rho(t) \leqslant 1 + \left| \int_\tau^t h(t, x)\, |p_1(x)|\, \rho(x)\, dx \right| \qquad \text{for} \quad a < t < b.$$

On the other hand,

$$h(t, x) \leqslant \min\{\sigma_a(p_2)(x), \sigma_b(p_2)(x)\} \qquad \text{for} \quad (x - \tau)(t - x) \geqslant 0$$

where

$$\sigma_b(p_2)(x) = \frac{1}{\sigma(p_2)(x)} \int_x^b \sigma(p_2)(s)\, ds.$$

Thus

$$h(t, x) \leqslant r_1 \sigma_{ab}(p_2)(x) \qquad \text{for} \quad (x-\tau)(t-x) \geqslant 0$$

and

$$\rho(t) \leqslant 1 + r_1 \left| \int_\tau^t |p_1(x)|\, \sigma_{ab}(p_2)(x)\, \rho(x)\, dx \right| \qquad \text{for} \quad a < t < b$$

where

$$r_1 = 2 \left[\int_a^b \sigma(p_2)(s)\, ds \right]^{-1}.$$

According to the Bellman lemma [2, p. 46], the last inequality yields

$$\rho(t) \leqslant \exp \left(r_1 \left| \int_\tau^t \sigma_{ab}(p_2)(x)\, |p_1(x)|\, dx \right| \right) \leqslant r_0 \qquad \text{for} \quad a < t < b.$$

Hence the estimate (2.3) is valid. This completes the proof.

2.2. *The behavior of solutions of Eq.* (1.1_0) *in neighborhoods of the end-points of* $]a, b[$.

LEMMA 2.2. *If the conditions* (2.2) *hold, Eq.* (1.1_0) *has solutions* u_1 *and* u_2 *satisfying the initial conditions*

$$u_1(a+) = 0, \qquad \lim_{t \to a+} \frac{u_1'(t)}{\sigma(p_2)(t)} = 1, \tag{2.5}$$

and

$$u_2(b-) = 0, \qquad \lim_{t \to b-} \frac{u_2'(t)}{\sigma(p_2)(t)} = -1. \tag{2.6}$$

Furthermore, any solution u *of this equation linearly independent with* u_1 *(with* u_2*) has the finite nonzero limit* $u(a+)$ $(u(b-))$.

Proof. We shall carry out the proof of existence of u_1 only. The existence of u_2 may be proved similarly.

For any positive integer k set

$$t_k = a + (b-a)/k, \qquad v_k(t) = 0 \qquad \text{for} \quad a < t \leqslant t_k$$

and

$$v_k(t) = \sigma(p_2)(t_k)\, C(t, t_k) \qquad \text{for} \quad t_k < t < b$$

where C is the Cauchy function of Eq. (1.1_0).

According to Lemma 2.1

$$|v_k(t)| \leqslant r_0 \int_a^t \sigma(p_2)(\tau)\, d\tau \qquad \text{for} \quad a \leqslant t < b \tag{2.7}$$

where r_0 is the number given by (2.4). Thus, from the equality

$$v_k'(t) = \sigma(p_2)(t) \left[1 + \int_{t_k}^t p_1(\tau) \frac{v_k(\tau)}{\sigma(p_2)(\tau)}\, d\tau \right] \qquad \text{for} \quad t_k \leqslant t < b$$

we obtain

$$\left| \frac{v_k'(t)}{\sigma(p_2)(t)} - 1 \right| \leqslant r_0 \int_a^t |p_1(\tau)|\, \sigma_\alpha(p_2)(\tau)\, d\tau \qquad \text{for} \quad t_k \leqslant t < b. \tag{2.8}$$

From (2.7) and (2.8) it easily follows that the sequences $(v_k)_{k=1}^{+\infty}$ and $(v_k')_{k=1}^{+\infty}$ are uniformly bounded and equicontinuous on each compact interval, contained in $]a, b[$. So, without loss of generality, we may assume that they uniformly converge on each above-mentioned interval. Obviously,

$$u_1(t) = \lim_{k \to +\infty} v_k(t)$$

is a solution of Eq. (1.1_0). On the other hand, as follows from (2.7) and (2.8),

$$|u_1(t)| \leqslant r_0 \int_a^t \sigma(p_2)(\tau)\, d\tau,$$

$$\left| \frac{u_1'(t)}{\sigma(p_2)(t)} - 1 \right| \leqslant r_0 \int_a^t |p_1(\tau)|\, \sigma(p_2)(\tau)\, d\tau \qquad \text{for} \quad a < t < b.$$

Therefore, u_1 satisfies the initial conditions (2.5).

Let u be an arbitrary solution of Eq. (1.1_0) linearly independent with u_1. By (2.5) there exists $a_0 \in\,]a, b[$ such that

$$u_1(t) > 0 \qquad \text{for} \quad a < t \leqslant a_0.$$

Hence

$$u(t) = c_1 u_1(t) + c_2 u_1(t) \int_t^b \frac{\sigma(p_2)(\tau)}{u_1^2(\tau)}\, d\tau \qquad \text{for} \quad a < t \leqslant a_0,$$

$c_i = \text{const}$ $(i = 1, 2)$ and $c_2 \neq 0$. This implies

$$u(a+) = c_2 \lim_{t \to a+} \frac{\sigma(p_2)(t)}{u_1'(t)} = c_2.$$

We can similarly show that if u is linearly independent with u_2, then there exists the finite limit $u(b-) \neq 0$. This completes the proof.

LEMMA 2.3. *If the conditions* (2.2) *hold, then for any bounded continuously differentiable function* $v\colon |a, b| \to R$

$$\liminf_{t \to a+} \frac{|v'(t)|\, u_1(t)}{\sigma(p_2)(t)} = 0 \tag{2.9}$$

and

$$\liminf_{t \to b-} \frac{|v'(t)|\, u_2(t)}{\sigma(p_2)(t)} = 0 \tag{2.10}$$

where u_1 *and* u_2 *are the solutions of Eq.* (1.1_0) *satisfying the conditions* (2.5) *and* (2.6).

Proof. We shall prove only (2.9) since (2.10) may be proved similarly.

Admit the contrary, that (2.9) is not true. Then by (2.5) there exist numbers $\delta > 0$ and $a_0 \in [a, b]$ such that

$$u_1(t) > 0 \qquad \text{and} \qquad |v'(t)| > \delta \frac{u_1'(t)}{u_1(t)} \qquad \text{for} \quad a < t < a_0.$$

So

$$|v(t)| \geqslant \int_t^{a_0} |v'(\tau)|\, d\tau - |v(a_0)| \geqslant \delta \ln \frac{u_1(a_0)}{u_1(t)} - |v(a_0)| \qquad \text{for} \quad a < t < a_0$$

which is impossible since v is bounded. Thus (2.9) is valid. This completes the proof.

LEMMA 2.4. *If*

$$\sigma(p_2) \in L([a, b]), \qquad \sigma_a(p_2)\, p_1 \in L([a, b]), \tag{2.11}$$

then for any $\alpha, \beta \in R$ *the differential equation* (1.1_0) *has the unique solution* u *satisfying the initial conditions*

$$u(b-) = \alpha, \qquad \lim_{t \to b-} \frac{u'(t)}{\sigma(p_2)(t)} = \beta. \tag{2.12}$$

Proof. Since (2.11) guarantees the fulfillment of the condition (2.2), there exists the solution u_2 of Eq. (1.1_0) under the conditions (2.6). Let u_0 be a solution of Eq. (1.1_0) linearly independent with u_2. Then according to Lemma 2.2 there exists the finite limit $u_0(b-)$. Hence applying (2.11) and the equality

$$u_0'(t) = \sigma(p_2)(t)\left[\frac{u_0'(a_0)}{\sigma(p_2)(a_0)} + \int_{a_0}^{t} p_1(\tau)\frac{u_0(\tau)}{\sigma(p_2)(\tau)}\,d\tau\right]$$

where $a_0 \in \,]a, b[$, we conclude that there exists the finite limit

$$\gamma = \lim_{t \to b-} \frac{u_0'(t)}{\sigma(p_2)(t)}.$$

Obviously,

$$u(t) = \frac{\alpha}{u_0(b-)}\,u_0(t) + \left(\frac{\alpha}{u_0(b-)}\,\gamma - \beta\right) u_2(t)$$

is a solution of the problem (1.1_0), (2.12). On the other hand, Lemma 2.2 implies that this problem has at most one solution. This completes the proof.

2.3. *On the Green functions of the problems* (1.1_0), $(1, k_0)$ $(k = 2, 3, 4)$. g: $]a, b[\, \times \,]a, b[\, \to R$ is said to be the Green function of the problem (1.1_0), $(1, k_0)$ if for an arbitrarily fixed $\tau \in \,]a, b[$:

1. the function $u(t) = g(t, \tau)$ is continuous in $]a, b[$ and satisfies the boundary condition $(1.k_0)$;

2. the contractions of u to $]a, \tau[$ and to $]\tau, b[$ are solutions of Eq. (1.1_0);

3. $u'(\tau+) - u'(\tau-) = 1$.

In what follow by η we assume the function defined as

$$\eta(s, x) = \begin{cases} 1 & \text{for } s \leqslant x \\ 0 & \text{for } s > x. \end{cases}$$

LEMMA 2.5. *Let the conditions* (2.11) *hold, and let the problem* (1.1_0), (1.2_0) *have no nonzero solutions. Then there exists the unique Green function* g *of this problem and*

$$g(t, \tau) = -\frac{1}{u_2(a+)\,\sigma(p_2)(\tau)}\,[\eta(t, \tau)\,u_1(t)\,u_2(\tau) + \eta(\tau, t)\,u_2(t)\,u_1(\tau)] \tag{2.13}$$

where u_2 is the solution of the problem (1.1_0), (2.5) *and* u_2 *is the solution of Eq.* (1.1_0) *satisfying the initial condition*

$$u_2(b-)=1, \qquad \lim_{t\to b-} \frac{u_2'(t)}{\sigma(p_2)(t)}=0. \tag{2.14}$$

Proof. The solutions u_1 and u_2 appearing in the statement of the lemma exist by Lemmas 2.2 and 2.4. These solutions are linearly independent since the problem (1.1_0) and (1.2_0) has no nonzero solutions. Thus according to Lemma 2.2, $u(a+)\neq 0$.

It is clear that the function g given by (2.13) satisfies the first two items of the definition of the Green function. We shall show that it satisfies the third one as well, i.e., for any $\tau \in \,]a, b[$

$$w(\tau)\equiv \frac{u_1'(\tau)\,u_2(\tau)-u_2'(\tau)\,u_1(\tau)}{\sigma(p_2)(\tau)}=u_2(a+).$$

Really, by Lemmas 2.2 and 2.3, $w(a+)=u_2(a+)$. On the other hand, according to the Liouville formula $w(\tau)=\text{const}$. Thus g is a Green function of the problem (1.1_0), (1.2_0). The uniqueness of the Green function follows from the unique solvability of the problem. This completes the proof.

Applying Lemmas 2.2 and 2.3 it is easy to verify the validity of the following statements.

LEMMA 2.6. *Let the conditions* (2.2) *be fulfilled, and let the problem* (1.1_0), (1.3_0) *have no nonzero solutions. Then there exists the unique Green function of this problem and the representation* (2.13) *where* u_1 *and* u_2 *are the solutions of the problems* (1.1_0), (2.5) *and* (1.1_0), (2.6) *holds.*

LEMMA 2.7. *Let the conditions* (2.2) *be fulfilled, and let the problem* (1.1_0), (1.4_0) *have no nonzero solutions. Then there exists the unique Green function of this problem*

$$g(t,\tau)=\frac{u_1(t)}{u_1(b-)-u_1(t_0)}\,[\eta(\tau,t_0)\,C(t_0,\tau)-C(b-,\tau)] + \eta(\tau,t)\,C(t,\tau) \tag{2.15}$$

where u_1 *is the solution of the problem* (1.1_0), (2.5) *and* C *is the Cauchy function of Eq.* (1.1_0).

3. Green Formula

In this section we show that the well-known Green representation of the solution of the problem (1.1), (1.k), where $k \in \{2, 3, 4\}$, remains valid in the singular case when the functions p_i $(i = 0, 1, 2)$ have unintegrable singularities at the end-points of $]a, b[$.

Theorem 3.1. *Let either* $k = 2$ *and*

$$\sigma(p_2) \in L([a, b]), \qquad \sigma_a(p_2)\, p_i \in L([a, b]) \qquad (i = 0, 1)$$

or $k \in \{3, 4\}$ *and*

$$\sigma(p_2) \in L([a, b]), \qquad \sigma_{ab}(p_2)\, p_i \in L([a, b]) \qquad (i = 0, 1). \tag{3.1}$$

Then the problem (1.1), (1.k) *is uniquely solvable if and only if the corresponding homogeneous problem* (1.1_0), $(1.k_0)$ *has no nonzero solutions. If the last condition holds, then the solution* u *of the problem* (1.1), (1.k) *may be represented by the Green formula*

$$u(t) = u_0(t) + \int_a^b g(t, \tau)\, p_0(\tau)\, d\tau \tag{3.2}$$

where u_0 *is the solution of the problem* (1.1_0), (1.k) *and* g *is the Green function of the problem* (1.1_0), $(1.k_0)$.

Proof. We shall prove this theorem for the problem (1.1), (1.4). For the problems (1.1), (1.2) and (1.1), (1.3) it can be proved similarly.

It is obvious that if the problem (1.1_0), (1.4_0) has a nonzero solution, then the problem (1.1), (1.4) has either no solutions or infinitely many solutions. Assume that the conditions (3.1) hold and that the problem (1.1_0), (1.4_0) has no nonzero solutions. Then the problem (1.1), (1.4) has at most one solution. Thus it remains to verify that the function u given by (3.2) is a solution of the problem (1.1), (1.4).

Let u_1 be the solution of the problem (1.1), (2.5), and let u_2 be the solution of Eq. (1.1_0) linearly independent with u_1. By Lemma 2.2 without loss of generality we may assume that $u_2(a+) = 1$.

Let

$$\delta_i = u_i(b-) - u_i(t_0) \qquad (i = 1, 2).$$

As follows from the unique solvability of the homogeneous problem (1.1_0), (1.4_0), $\delta_1 \neq 0$ and the problem (1.1_0), (1.4) has the unique solution

$$u_0(t) = \left(\frac{\beta}{\delta_1} - \frac{\alpha\delta_2}{\delta_1}\right) u_1(t) + \alpha u_2(t).$$

On the other hand, according to Lemma 2.7 there exists the Green function of the problem (1.1_0), (1.4_0) and the equality (2.15) holds.

Taking into consideration that

$$C(t,\tau)=\frac{u_1(t)\,u_2(\tau)-u_2(t)\,u_1(\tau)}{\sigma(p_2)(\tau)},$$

from (2.15) we obtain

$$g(t,\tau)=\frac{u_1(\tau)}{\sigma(p_2)(\tau)}\left(\frac{\delta_2}{\delta_1}u_1(t)-u_2(t)\right)\quad \text{for}\quad a<\tau\leqslant t_0,\quad \tau\leqslant t<b, \tag{3.3}$$

$$g(t,\tau)=\frac{u_1(t)}{\sigma(p_2)(\tau)}\left(\frac{\delta_2}{\delta_1}u_1(\tau)-u_2(\tau)\right)\quad \text{for}\quad a<\tau\leqslant t_0,\quad a<t<\tau, \tag{3.4}$$

$$g(t,\tau)=-\frac{u_1(t)}{\delta_1}C(b-,\tau)+C(t,\tau)\quad \text{for}\quad t_0<\tau<b,\quad \tau\leqslant t<b \tag{3.5}$$

and

$$g(t,\tau)=-\frac{u_1(t)}{\delta_1}C(b-,\tau)\quad \text{for}\quad t_0<\tau<b,\quad a<t<\tau. \tag{3.6}$$

By Lemmas 2.1 and 2.2 there exists a number $r_0>1$ such that

$$|u_i(t)|<r_0\quad \text{for}\quad a<t<b\quad (i=1,2),$$

$$|u_1(t)|<r_0\gamma(t)\quad \text{for}\quad a<t<b$$

where

$$\gamma(t)=\int_a^t\sigma(p_2)(\tau)\,d\tau$$

and the estimate (2.3) holds. Hence, from the equalities (3.3)–(3.6) it follows that

$$|g(t,\tau)|\leqslant r\gamma(t)\frac{\sigma_{ab}(p_2)(\tau)}{\gamma(\tau)}\quad \text{for}\quad a<t<\tau<b \tag{3.7}$$

and

$$|g(t,\tau)|\leqslant r\sigma_{ab}(p_2)(\tau)\quad \text{for}\quad a<\tau<t<b \tag{3.8}$$

where

$$r=r_0^2\left(1+\frac{1+|\delta_2|}{|\delta_1|}\right)\left[\frac{1}{\gamma(t_0)}+\frac{1}{\gamma(b)-\gamma(t_0)}\right].$$

Applying the equalities (3.3)–(3.6) and the condition

$$\sigma_{ab}(p_2)\,p_0 \in L([a,b])$$

we conclude that the function

$$v(t) = \int_a^b g(t,\tau)\,p_0(\tau)\,d\tau$$

belongs to the class $\tilde{C}^1(]a,b[)$, is a solution of Eq. (1.1) and

$$v(b-) = v(t_0).$$

On the other hand, by (3.7) and (3.8)

$$|v(t)| \leqslant r\int_a^t \sigma_{ab}(p_2)(\tau)\,|\,p_0(\tau)|\,d\tau + r\gamma(t)\int_t^b \frac{\sigma_{ab}(p_2)(\tau)}{\gamma(\tau)}\,|\,p_0(\tau)|\,d\tau \qquad \text{for} \quad a < t < b.$$

Since $\gamma(t)$ monotonically tends to zero for $t \to a+$ and the condition (3.9) holds, we have

$$\gamma(t)\int_t^b \frac{\sigma_{ab}(p_2)(\tau)}{\gamma(\tau)}\,|\,p_0(\tau)|\,d\tau \to 0 \qquad \text{for} \quad t \to a+.$$

Thus $v(a+) = 0$. Therefore v is a solution of the problem (1.1), (1.4_0).

So the function u given by (3.2) as the sum of solutions of the problems (1.1_0), (1.4) and (1.1), (1.4_0) is the solution of the problem (1.1), (1.4). This completes the proof.

4. Uniqueness Theorems

Theorem 3.1 proved above reduces the question on unique solvability of each problem (1.1), $(1.k)$ $(k = 2, 3, 4)$ to the similar question for the corresponding homogeneous problem (1.1_0), $(1.k_0)$. In this section we give unimprovable, in a certain sense, conditions under which the problem (1.1), $(1.k)$ $(k = 2,3,4)$ has no nonzero solutions.

4.1. *The problem* (1.1_0), (1.2_0).

Theorem 4.1. *Let*

$$\sigma(p_2) \in L([a,b]), \qquad \sigma_a(p_2)\,p_1 \in L([a,b]) \tag{4.1}$$

and let there exist a bounded function $v \in \tilde{C}^1(]a, b[)$ *such that*

$$v(t) > 0, \qquad v'(t) > 0 \qquad \textit{for} \quad a < t < b, \tag{4.2}$$

$$D(v)(t) \equiv v''(t) - p_1(t)\, v(t) - p_2(t)\, v'(t) \leqslant 0 \qquad \textit{for} \quad a < t < b \tag{4.3}$$

and $D(v)(t) < 0$ *on a set of positive measure. Then the problem* (1.1_0), (1.2_0) *has no nontrivial solutions.*

Proof. Admit to the contrary that the problem (1.1_0), (1.2_0) has a nonzero solution u. According to Lemma 2.2 without loss of generality we assume that

$$\lim_{t \to a+} \frac{u'(t)}{\sigma(p_2)(t)} = 1.$$

This equality and (1.2_0) imply the existence of a point $b_0 \in |a, b|$ such that

$$u(t) > 0, \qquad u'(t) > 0 \qquad \text{for} \quad a < t < b_0 \tag{4.4}$$

and

$$\lim_{t \to b_0-} \frac{u'(t)}{\sigma(p_2)(t)} = 0. \tag{4.5}$$

Setting

$$\rho(t) = \frac{1}{\sigma(p_2)(t)} \left[v'(t)\, u(t) - u'(t)\, v(t)\right], \tag{4.6}$$

by (4.2) and (4.3) we obtain

$$\rho'(t) = \frac{1}{\sigma(p_2)(t)} D(v)(t)\, u(t) \leqslant 0 \qquad \text{for} \quad a < t < b. \tag{4.7}$$

Moreover

$$\text{if} \quad b_0 = b, \qquad \text{then} \quad \text{mes}\{t \in\,]a, b[: \rho'(t) < 0\} > 0. \tag{4.8}$$

As follows from (4.7), there exist finite or infinite limits $\rho(a+)$ and $\rho(b-)$. According to Lemma 2.3 and the conditions (4.2), (4.4) and (4.5) we have

$$\rho(a+) \leqslant 0, \qquad \rho(b_0 -) \geqslant 0 \qquad \text{and if } b_0 < b, \quad \text{then} \quad \rho(b_0) > 0.$$

But this is impossible because of the conditions (4.7) and (4.8), which shows that (1.1_0), (1.2_0) has no nonzero solutions. This completes the proof.

COROLLARY. *Let the conditions* (4.1) *hold, and let there exist functions*

$q_i \in L_{loc}(]a,b[)$ $(i=1,2)$ such that $\sigma(q_2) \in L([a,b])$, $\sigma_{ab}(q_2)\, q_1 \in L([a,b])$, the equation

$$v'' = q_1(t)\, v + q_2(t)\, v' \tag{4.9}$$

has a solution v satisfying the inequalities (4.2),

$$p_i(t) \geqslant q_i(t) \quad \textit{for} \quad a < t < b \quad (i=1,2) \tag{4.10}$$

and at least one of the inequalities (4.10) *is strict on a set of positive measure. Then the problem* (1.1_0), (1.2_0) *has no nonzero solutions.*

Proof. According to Lemma 2.2 the function v is bounded. On the other hand, by (4.2) and (4.10) the equality

$$D(v)(t) = [q_1(t) - p_1(t)]\, v(t) + [q_2(t) - p_2(t)]\, v'(t)$$

implies that $D(v)(t)$ satisfies the condition (4.3) and differs from zero on a set of positive measure. Thus all conditions of Theorem 4.1 hold. This completes the proof.

THEOREM 4.2. *Let the conditions* (4.1) *be fulfilled, and let there exist numbers $\lambda \in [0,1[$ and $l_i \in [0,+\infty[$ (i = 1, 2) such that*

$$\int_0^{+\infty} \frac{ds}{l_1 + l_2 s + s^2} \geqslant \frac{(b-a)^{1-\lambda}}{1-\lambda}, \tag{4.11}$$

$$(t-a)^{2\lambda}\, p_1(t) \geqslant -l_1, \qquad (t-a)^{\lambda} \left[p_2(t) + \frac{\lambda}{t-a} \right] \geqslant -l_2$$

$$\text{for} \quad a < t < b \tag{4.12}$$

and at least one of the inequalities (4.12) *is strict on a set of positive measure. Then the problem* (1.1_0), (1.2_0) *has no nonzero solutions.*

Proof. Let ρ be the function defined by the equality

$$\int_{\rho(t)}^{+\infty} \frac{ds}{l_1 + l_2 s + s^2} = \frac{(t-a)^{1-\lambda}}{1-\lambda}.$$

Then

$$\rho'(t) = -(t-a)^{-\lambda} [l_1 + l_2 \rho(t) + \rho^2(t)] \qquad \text{for} \quad a < t < b \tag{4.13}$$

and according to (4.11) we have

$$\rho(t) > 0 \qquad \text{for} \quad a < t < b. \tag{4.14}$$

Conditions (4.13) and (4.14) imply that

$$v(t) = \exp\left[-\int_t^b (\tau - a)^{-\lambda} \rho(\tau)\, d\tau\right]$$

is a solution of Eq. (4.9) where

$$q_1(t) = -l_1(t-a)^{-2\lambda}, \qquad q_2(t) = -l_2(t-a)^{-\lambda} - \frac{\lambda}{t-a}$$

and satisfies the inequalities (4.2). On the other hand, the inequalities (4.10) hold and at least one of them is strict on a set of positive measure. Thus according to Corollary of Theorem 4.1 the problem (1.1_0), (1.2_0) has no nonzero solutions. This completes the proof.

Remark 1. The condition (4.11) is a necessary, not only sufficient, condition for the problem (1.1_0), (1.2_0) to have no nonzero solutions for any $p_i \in L_{loc}(]a, b[)$ $(i = 1, 2)$ satisfying (4.1) and the inequalities

$$(t-a)^{2\lambda} p_1(t) > -l_1, \qquad (t-a)^{\lambda}\left[p_2(t) + \frac{\lambda}{t-a}\right] \geqslant -l_2$$
$$\text{for} \quad a < t < b. \tag{4.15}$$

Really, assume that (4.11) is violated. Then $l_1 > 0$. Choose numbers $l_1^* \in]0, l_1[$ and $l_2^* \in [0, l_2]$ such that

$$\int_0^{+\infty} \frac{ds}{l_1^* + l_2^* s + s^2} = \frac{(b-a)^{1-\lambda}}{1-\lambda}$$

and define the fuction ρ by the equality

$$\int_{\rho(t)}^{+\infty} \frac{ds}{l_1^* + l_2^* s + s^2} = \frac{(t-a)^{1-\lambda}}{1-\lambda} \qquad \text{for} \quad a < t < b.$$

Then

$$\rho(b+) = 0, \qquad \rho(t) > \frac{2}{1-\lambda}(t-a)^{\lambda-1} \qquad \text{for} \quad a < t < a + \varepsilon$$

where $\varepsilon > 0$ is a sufficiently small number. Hence the function

$$u(t) = \exp\left(-\int_t^b (\tau - a)^{-\lambda} \rho(\tau)\, d\tau\right)$$

is a nonzero solution of the problem (1.1_0), (1.2_0) where

$$p_1(t) = -l_1^*(t-a)^{-2\lambda}, \qquad p_2(t) = -\frac{\lambda}{t-a} - l_2^*(t-a)^{-\lambda},$$

although p_1 and p_2 satisfy the conditions (4.1) and (4.15).

Remark 2. According to Opial inequality [10], (4.11) holds if

$$l_1 h^2 + 2l_2 h \leqslant \pi^2$$

where $h = (2/(1-\lambda))(b-a)^{1-\lambda}$.

Remark 3. As is easy to verify in conditions of Theorem 4.1 or Theorem 4.2,

$$g(t, \tau) < 0, \qquad g(b-, \tau) < 0 \qquad \text{for} \quad a < t < b, \quad a < \tau < b,$$

$$C(t, \tau) > 0, \qquad \frac{\partial C(t, \tau)}{\partial t} > 0 \qquad \text{for} \quad a < \tau < t < b,$$

$$\lim_{t \to b-} \frac{1}{\sigma(p_2)(t)} \frac{\partial C(t, \tau)}{\partial t} > 0 \qquad \text{for} \quad a < \tau < b$$

where g is the Green function of the problem (1.1_0), (1.2_0) and C is the Cauchy function of Eq. (1.1_0).

According to the second inequality (4.12) we have $\sigma(p_2) \in L([a, b])$ and

$$\sigma_a(p_2)(t) \leqslant r(t-a) \qquad \text{for} \quad a < t < b$$

where $r = \text{const} > 0$. Thus Theorems 3.1 and 4.2 imply the following statement.

THEOREM 4.2′. *Let*

$$\int_a^b (t-a)\,|p_i(t)|\,dt < +\infty \qquad (i = 0, 1),$$

and let there exist numbers $\lambda \in [0, 1[$ *and* $l_i \in [0, +\infty[$ $(i = 1, 2)$ *such that the conditions* (4.11) *and* (4.12) *are fulfilled and at least one of the inequalities* (4.12) *is strict on a set of positive measure. Then the problem* (1.1), (1.2) *has one and only one solution.*

4.2. *The problem* (1.1_0), (1.3_0). The following theorem can be proved similarly to Theorem 2.2.

THEOREM 4.3. *Let*

$$\sigma(p_2) \in L([a,b]), \qquad \sigma_{ab}(p_2)\,p_1 \in L([a,b]), \tag{4.16}$$

and let there exist a bounded function $v \in \tilde{C}^1(]a,b[)$ *such that*

$$v(t) > 0 \qquad \text{for} \quad a < t < b,$$

the condition (4.3) *holds and* $D(v)(t) < 0$ *on a set of positive measure. Then the problem* (1.1_0), (1.3_0) *has no nonzero solutions.*

COROLLARY. *Let the condition* (4.16) *hold, and let there exist a number* $c \in]a,b[$ *and functions* $q_i \in L_{\text{loc}}(]a,b[)$ $(i = 1,2)$ *such that* $\sigma(q_2) \in L([a,b])$, $\sigma_{ab}(q_2)\,q_1 \in L([a,b])$, *Eq.* (4.9) *has a solution* v *satisfying the conditions*

$$v(t) > 0 \qquad \text{for} \quad a < t < b, \quad v'(t)(c-t) > 0$$
$$\text{for} \quad t \in]a,c[\,\cup\,]c,b[\tag{4.17}$$

and

$$p_1(t) \geqslant q_1(t), \qquad [p_2(t) - q_2(t)](c-t) \geqslant 0 \qquad \text{for} \quad a < t < b \tag{4.18}$$

and at least one of the inequalities (4.18) *is strict on a set of positive measure. Then the problem* (1.1_0), (1.3_0) *has no nonzero solutions.*

THEOREM 4.4. *Let the conditions* (4.16) *hold, and let there exist numbers* $\lambda_i \in [0,1[$, $l_{ij} \in [0,+\infty[$ $(i,j = 1,2)$ *and* $c \in]a,b[$ *such that*

$$\begin{aligned} \int_0^{+\infty} \frac{ds}{l_{11} + l_{12}s + s^2} &\geqslant \frac{(c-a)^{1-\lambda_1}}{1-\lambda_1}, \\ \int_0^{+\infty} \frac{ds}{l_{21} + l_{22}s + s^2} &\geqslant \frac{(b-c)^{1-\lambda_2}}{1-\lambda_2} \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} &(t-a)^{2\lambda_1} p_1(t) \geqslant -l_{11}, \qquad (t-a)^{\lambda_1}\left[p_2(t) + \frac{\lambda_1}{t-a}\right] \geqslant -l_{12} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad a < t < c, \\ &(b-t)^{2\lambda_1} p_1(t) \geqslant -l_{12}, \qquad (b-t)^{\lambda_1}\left[p_2(t) - \frac{\lambda_1}{b-t}\right] \leqslant l_{22} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad c < t < b \end{aligned} \tag{4.20}$$

and, in addition, at least one of the inequalities (4.20) *is strict on a set of positive measure. Then the problem* (1.1_0), (1.3_0) *has no nonzero solutions.*

Proof. In the case when $c=a$ or $c=b$, Theorem 4.4 can be proved similarly to Theorem 4.2. Let $c\in\,]a,b[$. Without loss of generality we assume that instead of (4.19) the equalities

$$\int_0^{+\infty}\frac{ds}{l_{11}+l_{12}s+s^2}=\frac{(c-a)^{1-\lambda_1}}{1-\lambda_1},\qquad \int_0^{+\infty}\frac{ds}{l_{21}+l_{22}s+s^2}=\frac{(b-c)^{1-\lambda_2}}{1-\lambda_2} \tag{4.21}$$

hold. Define functions ρ_1 and ρ_2 by the equalities

$$\int_{\rho_1(t)}^{+\infty}\frac{ds}{l_{11}+l_{12}s+s^2}=\frac{(t-a)^{1-\lambda_1}}{1-\lambda_1}\quad\text{for}\quad a<t\leqslant c,\qquad \int_{\rho_2(t)}^{+\infty}\frac{ds}{l_{21}+l_{22}s+s^2}=\frac{(b-t)^{1-\lambda_2}}{1-\lambda_2}\quad\text{for}\quad c\leqslant t<b. \tag{4.22}$$

Then according to (4.21)

$$\rho_1(t)>0\quad\text{for}\quad a<t<c,\qquad \rho_2(t)>0\quad\text{for}\quad c<t<b,\quad \rho_1(c)=\rho_2(c)=0. \tag{4.23}$$

Let

$$v(t)=\exp\left[-\int_t^c(\tau-a)^{-\lambda_1}\rho_1(\tau)\,d\tau\right]\quad\text{for}\quad a<t<c,\qquad v(t)=\exp\left[-\int_c^t(b-\tau)^{-\lambda_2}\rho_2(\tau)\,d\tau\right]\quad\text{for}\quad c<t<b. \tag{4.24}$$

As follows from (4.22) and (4.23), v is a solution of the equation (4.9) where

$$q_1(t)=-l_{11}(t-a)^{-2\lambda_1},\qquad q_2(t)=-l_{12}(t-a)^{-\lambda_1}-\lambda_2(t-a)^{-1}\quad\text{for}\quad a<t<c,$$
$$q_1(t)=-l_{21}(b-t)^{-2\lambda_2},\qquad q_2(t)=l_{22}(b-t)^{-\lambda_2}+\lambda_2(b-t)^{-1}\quad\text{for}\quad c<t<b \tag{4.25}$$

and satisfies the conditions (4.17). On the other hand, by (4.20) the inequalities (4.18) hold and at least one of these inequalities is strict on a set of positive measure. Thus according to Corollary of Theorem 4.3 the problem (1.1_0), (1.3_0) has no nonzero solution. This completes the proof.

Remark 1. The condition that at least one of the inequalities (4.20) should be strict on a set of positive measure is essential and cannot be omitted. Really, if $p_i(t) \equiv q_i(t)$, where q_1 and q_2 are the functions defined by (4.24), and if the equalities (4.21) are fulfilled then the function v given by the equality (4.24) is a nonzero solution of the problem (1.1_0), (1.3_0).

Remark 2. In conditions of Theorem 4.3 or Theorem 4.4 Eq. (1.1_0) is disconjugate on the segment $[a, b]$, i.e., each nontrivial solution of this equation has at most one zero on $[a, b]$,[2] and the Green function of the problem (1.1_0), (1.3_0) is negative in $]a, b[\times]a, b[$.

If the inequalities (4.20) hold, then $\sigma(p_2) \in L([a, b])$ and

$$\sigma_{ab}(p_2)(t) \leqslant r(t-a)(b-t) \qquad \text{for} \quad a < t < b$$

where $r = \text{const} > 0$. Hence Theorems 3.1 and 4.4 imply the following statement

THEOREM 4.4′. *Let*

$$\int_a^b (t-a)(b-t)\,|p_i(t)|\,dt < +\infty \qquad (i = 0, 1),$$

and let there exist numbers $\lambda_i \in [0, 1[$, $l_{ij} \in [0, +\infty[$ $(i, j = 1, 2)$ *and* $c \in]a, b[$ *such that the conditions* (4.19) *and* (4.20) *hold and at least one of the inequalities* (4.20) *is strict on a set of positive measure. Then the problem* (1.1), (1.3) *has one and only one solution.*

Theorems 4.3, 4.4 and 4.4′ extend the well-known Vallée Poussin theorems [11] to singular differential equations and generalize the results of [8]. Besides, one theorem of P. Jamet [7, Theorem 2.1] is a special case of Theorem 4.4′.

4.3. *The problem* (1.1_0), (1.4_0).

THEOREM 4.5. *Let the conditions* (4.16) *hold, and let there exist a bounded function* $v \in \tilde{C}^1(]a, b[)$ *having the left-hand limit* $v(b-)$ *and satisfying the inequalities*

$$v(t) > 0 \qquad \textit{for} \quad a < t < b, \quad v(b-) \geqslant v(t_0),$$

$$D(v)(t) \equiv v''(t) - p_1(t)\,v(t) - p_2(t)\,v'(t) \leqslant 0 \qquad \textit{for} \quad a < t < b$$

where $D(v)(t) < 0$ *on a set of positive measure. Then the problem* (1.1_0), (1.4_0) *has no nonzero solutions.*

[2] By the values of a solution u at the points a and b we mean $u(a+)$ and $u(b-)$, respectively.

Proof. Admit to the contrary that the problem (1.1_0), (1.4_0) has a nonzero solution u. According to Lemma 2.2 without loss of generality we may assume that

$$\lim_{t \to a+} \frac{u'(t)}{\sigma(p_2)(t)} = 1.$$

By Theorem 4.3

$$u(t) > 0 \qquad \text{for} \quad a < t < b.$$

Let ρ be the function defined by the equality (4.6). Then, since $D(v)$ is nonpositive, we have

$$\rho'(t) = \frac{1}{\sigma(p_2)(t)} u(t)\, D(v)(t) \leqslant 0$$
$$\text{for} \quad a < t < b, \quad \text{mes}\{t \in \,]a, b[\, : \rho'(t) < 0\} > 0. \tag{4.26}$$

Because of the monotonicity of ρ, Lemma 2.3 implies

$$\rho(a+) \leqslant 0. \tag{4.27}$$

According to (4.26) and (4.27) we obtain

$$\frac{u'(t)}{u(t)} \geqslant \frac{v'(t)}{v(t)} \qquad \text{for} \quad a < t < b,$$
$$\frac{u'(t)}{u(t)} > \frac{v'(t)}{v(t)} \qquad \text{for} \quad b - \varepsilon < t < b$$

where ε is a sufficiently small positive number. This yields

$$\frac{u(b-)}{u(t_0)} > \frac{v(b-)}{v(t_0)} \geqslant 1$$

which is impossible since u satisfies the boundary conditions (1.4_0). This completes the proof.

COROLLARY. *Let the conditions* (4.16) *hold, and let there exist a number* $c \in \,]t_0, b]$ *and functions* $q_i \in L_{\text{loc}}(]a, b[)$ $(i = 1, 2)$ *such that* $\sigma(q_2) \in L([a, b])$, $\sigma_{ab}(q_2)\, q_1 \in L([a, b])$, *Eq.* (4.9) *has a solution* v *satisfying the conditions*

$$v(a+) \geqslant 0, \quad v'(t)(c - t) > 0 \quad \textit{for} \quad t \in \,]a, c[\, \cup \,]c, b[, \quad v(b-) \geqslant v(t_0), \tag{4.28}$$

the inequalities (4.18) *are fulfilled and at least one of them is strict on a set of positive measure. Then the problem* (1.1_0), (1.4_0) *has no nonzero solutions.*

THEOREM 4.6. *Let the conditions* (4.16) *hold, and let there exist numbers* $\lambda_i \in [0, 1[$, $l_i \in [0, +\infty[$ $(i = 1, 2)$ *and* $c \in]t_0, b[$ *such that*

$$\frac{(c-a)^{1-\lambda_1}}{1-\lambda_1} - \frac{(b-c)^{1-\lambda_2}}{1-\lambda_2} \geqslant \frac{(t_0-a)^{1-\lambda_1}}{1-\lambda_1}, \tag{4.29}$$

$$\int_0^{+\infty} \frac{ds}{l_1 + l_2 s + s^2} \geqslant \frac{(c-a)^{1-\lambda_1}}{1-\lambda_1} \tag{4.30}$$

and

$$\begin{aligned} &(t-a)^{2\lambda_1} p_1(t) \geqslant -l_1, \qquad (t-a)^{\lambda_1}\left[p_2(t) + \frac{\lambda_1}{t-a}\right] \geqslant -l_2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \textit{for} \quad a < t < c, \\ &(b-t)^{2\lambda_2} p_1(t) \geqslant -l_1, \qquad (b-t)^{\lambda_2}\left[p_2(t) - \frac{\lambda_2}{b-t}\right] \leqslant l_2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \textit{for} \quad c < t < b \end{aligned} \tag{4.31}$$

where at least one of the inelqualities (4.31) *is strict on a set of positive measure. Then the problem* (1.1_0), (1.4_0) *has no nonzero solutions.*

Proof. Without loss of generality, we assume that the equality

$$\int_0^{+\infty} \frac{ds}{l_1 + l_2 s + s^2} = \frac{(c-a)^{1-\lambda_1}}{1-\lambda_1} \tag{4.32}$$

holds instead of (4.30).

Define the functions ρ_1 and ρ_2 by the equalities

$$\begin{aligned} &\int_0^{\rho_1(t)} \frac{ds}{l_1 + l_2 s + s^2} = \frac{(c-a)^{1-\lambda_1}}{1-\lambda_1} - \frac{(t-a)^{1-\lambda_1}}{1-\lambda_1} \quad \text{for} \quad a < t \leqslant c, \\ &\int_0^{\rho_2(t)} \frac{ds}{l_1 + l_2 s + s^2} = \frac{(b-c)^{1-\lambda_2}}{1-\lambda_2} - \frac{(b-t)^{1-\lambda_2}}{1-\lambda_2} \quad \text{for} \quad c \leqslant t < b. \end{aligned} \tag{4.33}$$

Then, according to (4.29) and (4.32), ρ_1 and ρ_2 satisfy the condition (4.23) and

$$\rho_1(t_0) \geqslant \rho_2(b).$$

Thus

$$\int_{t0}^{c} (\tau - a)^{-\lambda_1} \rho_1(\tau)\, d\tau = \int_0^{\rho_1(t_0)} \frac{s\, ds}{l_1 + l_2 s + s^2} \geqslant \int_0^{\rho_2(b)} \frac{s\, ds}{l_1 + l_2 s + s^2}$$
$$= \int_c^b (b - \tau)^{-\lambda_2} \rho_2(\tau)\, d\tau. \tag{4.34}$$

Conditions (4.23), (4.33) and (4.34) imply that the function v defined by the equalities (4.24) satisfies the conditions (4.28) and is a solution of Eq. (4.9) where

$$\begin{aligned} &q_1(t) = -l_1(t-a)^{-2\lambda_1}, \qquad q_2(t) = -l_2(t-a)^{-\lambda_1} - \lambda_1(t-a)^{-1} \\ &\qquad\qquad\qquad\qquad\qquad \text{for} \quad a < t < c, \\ &q_1(t) = -l_1(b-t)^{-2\lambda_2}, \qquad q_2(t) = l_2(b-t)^{-\lambda_2} + \lambda_2(b-t)^{-1} \\ &\qquad\qquad\qquad\qquad\qquad \text{for} \quad c < t < b. \end{aligned} \tag{4.35}$$

On the other hand, according to (4.31) the inequalities (4.18) hold and at least one of them is strict on a set of positive measure. Hence by Corollary of Theorem 4.5 the problem (1.1_0), (1.4_0) has no nonzero solutions. This completes the proof.

Remark 1. The condition that at least one of the inequalities (4.31) should be strict on a set of positive measure is essential and cannot be omitted. Really, if (4.32) holds,

$$\frac{(c-a)^{1-\lambda_1}}{1-\lambda_1} - \frac{(b-c)^{1-\lambda_2}}{1-\lambda_2} = \frac{(t_0-a)^{1-\lambda_2}}{1-\lambda_2}$$

and $p_i(t) \equiv q_i(t)$ $(i = 1, 2)$, where q_1 and q_2 are the functions defined by the equalities (4.35), then the function v defined by the equalities (4.24) is a nonzero solution of the problem (1.1_0), (1.4_0).

Remark 2. In conditions of Theorem 4.5 or Theorem 4.6 the Green function of the problem (1.1_0), (1.4_0) is negative on $]a, b[\times]a, b[$.

Theorems 3.1 and 4.6 imply the following statement.

THEOREM 4.6′. *Let*

$$\int_a^b (t-a)(b-t)\,|\,p_i(t)|\, dt < +\infty \qquad (i = 1, 2),$$

and let there exist numbers $\lambda_i \in [0, 1[$, $l_i \in [0, +\infty[$ $(i = 1, 2)$ *and* $c \in]t_0, b]$ *such that the conditions* (4.29), (4.30) *and* (4.31) *hold and at least one of the inequalities* (4.31) *is strict on a set of positive measure. Then the problem* (1.1), (1.4) *has one and only one solution.*

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