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A NOTE ON THE THEOREM ON DIFFERENTIAL INEQUALITIES

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Abstract. It is proved that if a linear operator $l : C([a,b],R) \to L([a,b],R)$ is nonpositive and for the Cauchy problem $u''(t) = l(u)(t) + q(t)$, $u(a) = c$ the theorem on differential inequalities is valid, then l is a Volterra operator.

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The following notation will be used throughout the paper.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b];R)$ is the space of continuous functions $u : [a,b] \to R$ with the norm

$$\|u\|_C = \max\{|u(t)| : a \le t \le b\}.$$

$C([a,b];\mathcal{D})$, where $\mathcal{D} \subset R$, is the set of continuous functions $u : [a,b] \to \mathcal{D}$.

$\widetilde{C}([a,b];\mathcal{D})$ is the set of absolutely continuous functions $u : [a,b] \to \mathcal{D}$.

$L(]a,b[;R)$ is the space of integrable functions $p :]a,b[\to R$ with the norm

$$\|p\|_L = \int_a^b |p(s)| ds.$$

$L(]a,b[;\mathcal{D})$, where $\mathcal{D} \subset R$, is the set of integrable functions $p :]a,b[\to \mathcal{D}$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];R) \to L(]a,b[;R)$.

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \mathcal{L}_{ab}$ mapping $C([a,b];R_+)$ into $L(]a,b[;R_+)$.

An operator $\ell \in \mathcal{L}_{ab}$ (resp., $\ell : L(]a,b[;R) \to C([a,b];R)$) is said to be Volterra operator if for any $c \in]a,b[$ and $g \in C([a,b];R)$ (resp., $g \in L(]a,b[;R)$) such that $g(t) = 0$ for $a < t < c$, the equality $\ell(g)(t) = 0$ holds for $a < t < c$.

Consider the Cauchy problem

$$u'(t) = \ell(u)(t) + g(t), \tag{1}$$

$$u(a) = c, \tag{2}$$

where $\ell \in \mathcal{L}_{ab}$, $g \in L(]a,b[;R)$, and $c \in R$. By a solution of problem (1), (2) we understand a function $u \in \widetilde{C}([a,b];R)$ satisfying condition (2) and equality (1) almost everywhere in $]a,b[$.

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Definition 1. We say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\mathcal{S}_{ab}$ if the homogeneous problem

$$u'(t) = \ell(u)(t), \tag{1_0}$$

$$u(a) = 0 \tag{2_0}$$

has only the trivial solution and for arbitrary $c \geq 0$, $g \in L(]a,b[;R_+)$ the solution of problem (1), (2) is a nonnegative function.

Remark 1. Let problem (1_0), (2_0) have only the trivial solution. Then $\ell \in \mathcal{S}_{ab}$ if and only if the lemma on differential inequalities holds, i.e., for any $u, v \in \widetilde{C}([a,b];R)$ such that

$$u'(t) \leq \ell(u)(t) + g(t), \quad v'(t) \geq \ell(v)(t) + g(t) \quad \text{for } a < t < b,$$
$$u(a) \leq v(a),$$

the inequality $u(t) \leq v(t)$ is fulfilled for $a \leq t \leq b$.

In the paper [1], sufficient conditions are established guaranteeing the inclusion $\ell \in \mathcal{S}_{ab}$. In particular, in Theorem 1.1 iv) the following proposition is proved.

Proposition 1. *Let $-\ell \in \mathcal{P}_{ab}$, ℓ be Volterra operator, and let there exist $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ such that $\gamma'(t) \leq \ell(\gamma)(t)$ for $a < t < b$. Then $\ell \in \mathcal{S}_{ab}$.*

In [1], there is also an example showing that the condition on ℓ to be Volterra operator is essential and it cannot be weakened. Below we will prove (see Theorem 2) that in Proposition 1 the condition on ℓ to be Volterra operator is not only sufficient but even necessary.

Before we formulate the main results, let us introduce the following definition.

Definition 2. Let problem (1_0), (2_0) have only the trivial solution. Denote by Ω the operator that takes each function $g \in L(]a,b[;R)$ to the solution of problem (1), (2_0).

Remark 2. From Theorem 1 in [2] it follows that if problem (1_0), (2_0) has only the trivial solution, then for any $c \in R$ and $g \in L(]a,b[;R)$ problem (1), (2) has a unique solution. Therefore the operator Ω is well defined. It is also clear that Ω is a linear operator which maps the set $L(]a,b[;R)$ into the set $C([a,b];R)$.

Remark 3. From Theorem 1.4 in [3] it follows that if problem (1_0), (2_0) has only the trivial solution and there exists a function $h \in L(]a,b[;R_+)$ such that on the set $C([a,b];R)$ the inequality

$$|\ell(u)(t)| \leq h(t)\|u\|_C \quad \text{for } a < t < b \tag{3}$$

is fulfilled, then the operator Ω is continuous. Note that inequality (3) is fulfilled, e.g., if $-\ell \in \mathcal{P}_{ab}$.

Theorem 1. *Let $-\ell \in \mathcal{P}_{ab}$ and $\ell \in \mathcal{S}_{ab}$. Then Ω is Volterra operator.*

Proof. Let $g \in L(]a,b[;R)$, $c \in]a,b[$, and

$$g(t) = 0 \quad \text{for} \ \ a < t < c. \tag{4}$$

Show that

$$\Omega(g)(t) = 0 \quad \text{for} \ \ a \le t \le c. \tag{5}$$

Denote by u the solution of the problem (1), (2_0) and by v the solution of the equation

$$u'(t) = \ell(u)(t) + |g(t)| \tag{6}$$

satisfying condition (2_0).

In view of $\ell \in \mathcal{S}_{ab}$ and Remark 1 we have

$$v(t) \ge 0 \quad \text{for} \ \ a \le t \le b, \tag{7}$$

$$u(t) \le v(t) \quad \text{for} \ \ a \le t \le b. \tag{8}$$

Since $-\ell \in \mathcal{P}_{ab}$, (6) and (7) imply

$$v'(t) \le |g(t)| \quad \text{for} \ \ a < t < b.$$

Hence by (4) and (7) we obtain

$$v(t) = 0 \quad \text{for} \ \ a \le t \le c. \tag{9}$$

On the other hand, from (1), (6), and (8) on account of $-\ell \in \mathcal{P}_{ab}$, we get

$$(u(t) - v(t))' = \ell(u - v)(t) + g(t) - |g(t)| \ge g(t) - |g(t)| \quad \text{for} \ \ a < t < b.$$

Hence in view of (4) and (9) we have

$$u'(t) \ge v'(t) = 0 \quad \text{for} \ \ a < t < c.$$

This together with (8) and (9) results in

$$u(t) = 0 \quad \text{for} \ \ a \le t \le c.$$

Consequently (since $u(t) = \Omega(g)(t)$ for $a \le t \le b$), the equality (5) is fulfilled. □

Theorem 2. *Let $-\ell \in \mathcal{P}_{ab}$ and $\ell \in \mathcal{S}_{ab}$. Then ℓ is Volterra operator.*

Proof. Assume the contrary that there exist $v_0 \in C([a,b];R)$ and $c \in]a,b[$ such that

$$v_0(t) = 0 \quad \text{for} \ \ a \le t \le c, \qquad \text{mes}\{t \in]a,c[: \ell(v_0)(t) \ne 0\} > 0.$$

Without loss of generality we can assume that

$$\text{mes}\{t \in]a,c[: \ell(v_0)(t) < 0\} > 0. \tag{10}$$

First we show that

$$\Omega(\ell(|v_0|))(t) = 0 \quad \text{for} \ \ a \le t \le c. \tag{11}$$

Choose a sequence of functions $v_k \in \widetilde{C}([a,b];R)$, $k = 1,2,\dots$, such that

$$\lim_{k\to+\infty} \|v_k - |v_0|\|_C = 0, \tag{12}$$

$$v_k(t) = 0 \quad \text{for} \;\; a \le t \le c, \quad k = 1,2,\dots . \tag{13}$$

According to Remark 3 and (12), we have

$$\lim_{k\to+\infty} \|\Omega(\ell(v_k)) - \Omega(\ell(|v_0|))\|_C = 0. \tag{14}$$

It is clear that $v_k'(t) = \ell(v_k)(t) + g_k(t)$ for $a < t < b$, where $g_k(t) \overset{def}{=} v_k'(t) - \ell(v_k)(t)$ for $a < t < b$. Consequently,

$$v_k(t) = \Omega(g_k)(t) = \Omega(v_k')(t) - \Omega(\ell(v_k))(t) \quad \text{for} \;\; a \le t \le b, \quad k = 1,2,\dots .$$

Hence, taking into account the fact that Ω is Volterra operator (see Theorem 1) and condition (13), we obtain

$$\Omega(\ell(v_k))(t) = -v_k(t) = 0 \quad \text{for} \;\; a \le t \le c, \quad k = 1,2,\dots .$$

Now in view of (14) we get the equality (11) is fulfilled.

Denote by u the solution of problem (1), (2_0), where

$$g(t) = \begin{cases} -\ell(|v_0|)(t) & \text{for} \;\; a < t < c \\ 0 & \text{for} \;\; c < t < b \end{cases} . \tag{15}$$

It is evident that

$$g(t) \ge 0 \quad \text{for} \;\; a < t < b. \tag{16}$$

Furthermore, since $-\ell \in \mathcal{P}_{ab}$, the inequlity $\ell(|v_0|)(t) \le \ell(v_0)(t)$ holds for $a < t < b$, and consequently, due to (10),

$$\text{mes}\{t \in]a,c[: g(t) > 0\} > 0. \tag{17}$$

Since $u(t) = \Omega(g)(t)$ for $a \le t \le b$, on account of the fact that Ω is Volterra operator and condition (11), we get

$$u(t) = 0 \quad \text{for} \;\; a \le t \le c. \tag{18}$$

By $\ell \in \mathcal{S}_{ab}$ and Remark 1, from (1), (2_0), and (16) we have

$$u(t) \ge 0 \quad \text{for} \;\; a \le t \le b. \tag{19}$$

Now in view of $-\ell \in \mathcal{P}_{ab}$, (1) implies

$$u'(t) \le g(t) \quad \text{for} \;\; a < t < b.$$

Hence, on account of (15), we obtain

$$u'(t) \le 0 \quad \text{for} \;\; c < t < b.$$

The last inequality together with (18) and (19) results in $u(t) = 0$ for $a \le t \le b$. Thus (1) yields

$$g(t) = 0 \quad \text{for} \;\; a < t < c,$$

which contradicts (17). □

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