T. Chantladze; Alexander Lomtatidze; Douglas Ugulava

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CONJUGACY AND DISCONJUGACY CRITERIA FOR SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

T. CHANTLADZE, A. LOMTATIDZE, AND D. UGULAVA

Abstract. Conjugacy and disconjugacy criteria are established for the equation

$$u'' + p(t)u = 0\,,$$

where $p :]-\infty, +\infty[\to]-\infty, +\infty[$ is a locally summable function.

Introduction

Consider the equation

$$u'' + p(t)u = 0\,, \tag{1}$$

where $p : R \to R$ ($R =]-\infty, +\infty[$) is Lebesgue integrable on every compact subinterval of R. By a solution of the equation (1) we understand the function $u : R \to R$ which is absolutely continuous together with its first derivative on every segment contained in R, and almost everywhere satisfies the equation (1).

The equation (1) is said to be *conjugate* if it has a nontrivial solution with at least two zeros, and *disconjugate* otherwise. If there exists a nontrivial solution of (1) with a sequence of zeros tending to $+\infty$ (to $-\infty$), then the equation (1) is said to be *oscillatory* in the neighbourhood of $+\infty$ (in the neighbourhood of $-\infty$).

The history of the problem on conjugacy of (1) begins from the paper of S. W. Hawking and R. Penrose [8], where has been proved that conditions $p(t) \geq 0$ for $t \in R$ and $p \not\equiv 0$, guarantee conjugacy of (1). In [13] F. J. Tipler pointed out an interesting relevance of the study of conjugacy of (1) to general relativity and improved Hawking–Penrose's criterion, showing that (1) is conjugate whenever

$$\liminf_{t\to-\infty}\ \liminf_{\tau\to+\infty} \int_t^\tau p(s)\,ds > 0$$

is fulfilled. Later these results were generalized in various directions (see, e.g., [1, 2, 5, 6, 9 - 12]). However, the investigation of conjugacy of the equation (1) is not

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still completed. Below we establish new conditions for conjugacy and disconjugacy of the equation (1) which generalize and make the results obtained in [1, 5, 6, 8 - 10] more complete.

Put

$$c(t)=\frac{1}{|t|}\int_0^t\int_0^s p(\xi)\,d\xi\,ds \qquad \text{for } \ t\in R\setminus\{0\}\,.$$

According to Theorem 7.3 in [7, p. 367], if either

$$\lim_{t\to+\infty} c(t)=+\infty \quad \text{or} \quad -\infty<\liminf_{t\to+\infty} c(t)<\limsup_{t\to+\infty} c(t)\,,$$

resp. if either

$$\lim_{t\to-\infty} c(t)=+\infty \quad \text{or} \quad -\infty<\liminf_{t\to-\infty} c(t)<\limsup_{t\to-\infty} c(t)\,,$$

then the equation (1) is oscillatory in the neighbourhood of $+\infty$, resp. in the neighbourhood of $-\infty$. Therefore, below we suppose that there exist finite limits

$$c(-\infty)\overset{def}{=}\lim_{t\to-\infty} c(t)\,, \qquad c(+\infty)\overset{def}{=}\lim_{t\to+\infty} c(t)\,.$$

Denote

$$Q_1(t)=\int_0^t p(s)\,ds-c(+\infty)\,, \qquad Q_2(t)=\int_0^t p(s)\,ds+c(-\infty) \qquad \text{for } \ t\in R\,,$$

$$F_i(t)=\exp\left(2\int_0^t Q_i(s)\,ds\right) \qquad \text{for } \ t\in R\,, \quad i=1,2\,.$$

To formulate the main results, we need two propositions. The first one is the special case of Theorem 1.2 proved in [14], and the second one we prove in §3.

Proposition 1. *Let*

$$\int^{+\infty}\frac{Q_1^2(s)}{F_1(s)}\,ds=+\infty\,, \quad \textit{resp.} \quad \int_{-\infty}\frac{Q_2^2(s)}{F_2(s)}\,ds=+\infty\,.$$

Then the equation (1) *is oscillatory in the neighbourhood of* $+\infty$*, resp. in the neighbourhood of* $-\infty$*.*

Proposition 2. *If*

$$\int_{-\infty} F_1(s)\,ds<+\infty\,, \quad \textit{resp.} \quad \int^{+\infty} F_2(s)\,ds<+\infty \tag{2}$$

and

$$\int_{-\infty} \frac{\left[Q_1(s)\int_{-\infty}^{s} F_1(\xi)\,d\xi\right]^2}{F_1(s)}\,ds = +\infty\,, \quad \textit{resp.} \tag{3}$$
$$\int^{+\infty} \frac{\left[Q_2(s)\int_{s}^{+\infty} F_2(\xi)\,d\xi\right]^2}{F_2(s)}\,ds = +\infty\,,$$

then the equation (1) *is oscillatory in the neighbourhood of* $-\infty$*, resp. in the neighbourhood of* $+\infty$*.*

1. Main Results

Theorem 1. *Let* $p \not\equiv 0$ *and*

$$c(-\infty) + c(+\infty) \geq 0\,. \tag{4}$$

Then the equation (1) *is conjugate.*

Remark 1. It can be easily seen that Theorem 1 generalizes above-mentioned Tipler's result as well as Corollary 1 in [6] and Corollary 3.2 and Proposition 3 in [9].

Remark 2. From Theorem 1 and above-mentioned Theorem 7.3 in [7] it follows that at least one of the equations $u'' + p(t)u = 0$ and $u'' - p(t)u = 0$, where $p \not\equiv 0$, is conjugate whenever either

$$\liminf_{t\to-\infty} c(t) > -\infty \quad \text{or} \quad \limsup_{t\to-\infty} c(t) < +\infty$$

or

$$\liminf_{t\to+\infty} c(t) > -\infty \quad \text{or} \quad \limsup_{t\to+\infty} c(t) < +\infty\,.$$

According to Theorem 1, it is natural to assume in the following that

$$c(-\infty) + c(+\infty) < 0\,. \tag{5}$$

Note that the inequality (5) guarantees the conditions (2). Consequently, in view of Propositions 1 and 2 it is also natural to assume that

$$\int^{+\infty} \frac{Q_1^2(s)}{F_1(s)}\,ds < +\infty, \qquad \int_{-\infty} \frac{\left[Q_1(s)\int_{-\infty}^{s} F_1(\xi)\,d\xi\right]^2}{F_1(s)}\,ds < +\infty\,, \tag{6}$$

$$\int_{-\infty} \frac{Q_2^2(s)}{F_2(s)}\,ds < +\infty, \qquad \int^{+\infty} \frac{\left[Q_2(s)\int_{s}^{+\infty} F_2(\xi)\,d\xi\right]^2}{F_2(s)}\,ds < +\infty\,. \tag{7}$$

Theorem 2. *Let the conditions* (5), (6), *and*

$$\sup\left\{\int_{-\infty}^{t} F_1(s)\,ds \int_{t}^{+\infty} \frac{Q_1^2(s)}{F_1(s)}\,ds + \right. \tag{8}$$
$$\left. +\frac{1}{\int_{-\infty}^{t} F_1(s)\,ds} \int_{-\infty}^{t} \frac{\left[Q_1(s)\int_{-\infty}^{s} F_1(\xi)\,d\xi\right]^2}{F_1(s)}\,ds \;:\; t\in R\right\} > 1$$

be fulfilled. Then the equation (1) *is conjugate.*

Theorem 2′. *Let the conditions* (5), (7), *and*

$$\sup\left\{\int_{t}^{+\infty} F_2(s)\,ds \int_{-\infty}^{t} \frac{Q_2^2(s)}{F_2(s)}\,ds + \right.$$
$$\left. +\frac{1}{\int_{t}^{+\infty} F_2(s)\,ds} \int_{t}^{+\infty} \frac{\left[Q_2(s)\int_{s}^{+\infty} F_2(\xi)\,d\xi\right]^2}{F_2(s)}\,ds \;:\; t\in R\right\} > 1$$

be fulfilled. Then the equation (1) *is conjugate.*

Finally we formulate disconjugacy theorems.

Theorem 3. *Let the conditions* (5) *and* (6) *hold. Let, moreover, at least one of the following three inequalities be fulfilled:*

$$\sup\left\{\int_{-\infty}^{t} F_1(s)\,ds \int_{t}^{+\infty} \frac{Q_1^2(s)}{F_1(s)}\,ds \;:\; t\in R\right\} \le \frac{1}{4}\,; \tag{9}$$

$$\sup\left\{\frac{1}{\int_{-\infty}^{t} F_1(s)\,ds} \int_{-\infty}^{t} \frac{\left[Q_1(s)\int_{-\infty}^{s} F_1(\xi)\,d\xi\right]^2}{F_1(s)}\,ds \;:\; t\in R\right\} \le \frac{1}{4}\,; \tag{10}$$

$$\sup\left\{\int_{-\infty}^{t} F_1(s)\,ds \int_{t}^{+\infty} \frac{Q_1^2(s)}{F_1(s)}\,ds + \right. \tag{11}$$
$$\left. +\frac{1}{\int_{-\infty}^{t} F_1(s)\,ds} \int_{-\infty}^{t} \frac{\left[Q_1(s)\int_{-\infty}^{s} F_1(\xi)\,d\xi\right]^2}{F_1(s)}\,ds \;:\; t\in R\right\} \le \sqrt{2}-1\,.$$

Then the equation (1) *is disconjugate.*

Theorem 3′. *Let the conditions* (5) *and* (7) *hold. Let, moreover, at least one of the following three inequalities be fulfilled:*

$$\sup\left\{\int\limits_t^{+\infty} F_2(s)\,ds \int\limits_{-\infty}^{t} \frac{Q_2^2(s)}{F_2(s)}\,ds \;:\; t\in R\right\} \le \frac{1}{4}\,;$$

$$\sup\left\{\frac{1}{\int_t^{+\infty} F_2(s)\,ds}\int\limits_t^{+\infty} \frac{\left[Q_2(s)\int_s^{+\infty} F_2(\xi)\,d\xi\right]^2}{F_2(s)}\,ds \;:\; t\in R\right\} \le \frac{1}{4}\,;$$

$$\sup\left\{\int\limits_t^{+\infty} F_2(s)\,ds \int\limits_{-\infty}^{t} \frac{Q_2^2(s)}{F_2(s)}\,ds + \right.$$

$$\left. +\frac{1}{\int_t^{+\infty} F_2(s)\,ds}\int\limits_t^{+\infty} \frac{\left[Q_2(s)\int_s^{+\infty} F_2(\xi)\,d\xi\right]^2}{F_2(s)}\,ds \;:\; t\in R\right\} \le \sqrt{2}-1\,.$$

Then the equation (1) *is disconjugate.*

2. Auxiliary Propositions

Lemma 1. *Let the equation* (1) *be disconjugate. Then it has a solution not having a zero.*

Proof. Let $\widetilde{v}_n$ and $\widetilde{w}_n$, where $n\in N$, be solutions of the equation (1) satisfying the initial conditions

$$\widetilde{v}_n(-n)=0,\qquad \widetilde{v}_n'(-n)=1\,,$$
$$\widetilde{w}_n(n)=0,\qquad \widetilde{w}_n'(n)=-1\,.$$

Evidently, $\widetilde{v}_n(0)\ne 0$ and $\widetilde{w}_n(0)\ne 0$ for $n\in N$. Put

$$v_n(t)=\frac{\widetilde{v}_n(t)}{\widetilde{v}_n(0)}\,,\qquad w_n(t)=\frac{\widetilde{w}_n(t)}{\widetilde{w}_n(0)}\qquad \text{for }\; t\in R,\quad n\in N\,.$$

It is clear that for any $n\in N$, v_n and w_n are solutions of the equation (1) and

$$\begin{aligned} &v_n(t)\,\mathrm{sgn}(t+n)>0 \qquad \text{for }\; t\in R\setminus\{-n\}\,,\\ &w_n(t)\,\mathrm{sgn}(t-n)<0 \qquad \text{for }\; t\in R\setminus\{n\}\,,\end{aligned} \tag{12}$$

$$v_n(0)=1,\qquad w_n(0)=1\,. \tag{13}$$

Since the equation (1) is disconjugate, we can easily verify that for $t\in R\setminus\{0\}$ and $n,m\in N$,

$$(v_{n+1}(t)-v_n(t))\,\mathrm{sgn}\,t<0\,,\qquad (w_{n+1}(t)-w_n(t))\,\mathrm{sgn}\,t>0\,, \tag{14}$$

$$(v_n(t)-w_m(t))\,\mathrm{sgn}\,t>0\,. \tag{15}$$

(14) and (15) result in

$$v_{n+1}'(0)<v_n'(0)\,,\qquad w_{n+1}'(0)>w_n'(0)\qquad \text{for }\; n\in N\,, \tag{16}$$

$$v_n'(0) > w_m'(0) \qquad \text{for } n, m \in N\,. \tag{17}$$

Put $\lambda = |w_1'(0)| + |v_1'(0)|$ and

$$M_a = \max\{w_1(t) : -a \le t \le 0\} + \max\{v_1(t) : 0 \le t \le a\}\,,$$

where $a > 0$. Then by (14), (16), and (17) we have

$$|v_n'(0)| \le \lambda, \qquad |w_n'(0)| \le \lambda \qquad \text{for } n \in N\,, \tag{18}$$

$$|v_n(t)| \le M_a, \qquad |w_n(t)| \le M_a \qquad \text{for } -a \le t \le a, \quad n > a. \tag{19}$$

Since v_n and w_n are solutions of the equation (1), in view of (18) and (19), we obtain

$$|v_n'(t)| \le \lambda_a\,, \qquad |w_n'(t)| \le \lambda_a \qquad \text{for } -a \le t \le a\,, \quad n > a\,, \tag{20}$$

where $\lambda_a = \lambda + M_a \int_{-a}^{a} |p(s)|\,ds$.

By (19) and (20), it is clear that the sequences $(v_k^{(i)})_{k=1}^{+\infty}$ and $(w_k^{(i)})_{k=1}^{+\infty}$, $i = 0, 1$, are uniformly bounded and equicontinuous in R (i.e., on every segment contained in R). Therefore, according to the Arzelà–Ascoli lemma, without loss of generality we can assume that

$$\lim_{k\to+\infty} v_k^{(i)}(t) = v^{(i)}(t)\,, \qquad \lim_{k\to+\infty} w_k^{(i)}(t) = w^{(i)}(t) \qquad , i = 0, 1$$

uniformly in R. It is easy to see that the functions v and w are solutions of the equation (1).

In view of (12), (14), and (15) we have

$$v(t) > 0 \qquad \text{for } t < 0, \qquad w(t) > 0 \qquad \text{for } t > 0\,,$$
$$(v(t) - w(t))\operatorname{sgn} t \ge 0 \qquad \text{for } t \ne 0\,.$$

Consequently, $v(t) > 0$ for $t \ne 0$. However, according to (13), $v(0) = 1$. Thus v has no zero. □

Lemma 2. *Let a solution u of the equation* (1) *have no zero in the interval* $[a, +\infty[$*, resp. in the interval* $]-\infty, a]$*. Then*

$$\int_a^{+\infty} \rho^2(s)\,ds < +\infty\,, \quad \textit{resp.} \quad \int_{-\infty}^{a} \rho^2(s)\,ds < +\infty\,,$$

and the equality

$$\rho(t) = c(+\infty) - \int_0^t p(s)\,ds + \int_t^{+\infty} \rho^2(s)\,ds \qquad \textit{for } t \ge a\,, \quad \textit{resp.}$$

$$\rho(t) = -c(-\infty) + \int_t^0 p(s)\,ds - \int_{-\infty}^{t} \rho^2(s)\,ds \qquad \textit{for } t \le a$$

holds, where $\rho(t) = \frac{u'(t)}{u(t)}$ for $t \ge a$, resp. for $t \le a$.

For the proof of this lemma see [7], p. 365, Lemma 7.1.

Lemma 3. *Let the function* $g :]a,b[\to [0,+\infty[$ *be integrable on every segment contained in* $]a,b]$, *and*

$$\int_a^b (s-a)^2 g(s)\, ds = +\infty\,.$$

Then every nontrivial solution of the equation $v'' + g(x)v = 0$ *has a sequence of zeros tending to* a.

Lemma 3 immediately follows from Theorem 1.1 in [3].

3. Proofs

Proof of Proposition 2. Put

$$\varphi(t) = \int_0^t F_1(s)\, ds \qquad \text{for } t \in R\,. \tag{21}$$

According to (2) there exists a finite limit

$$a \stackrel{def}{=} \lim_{t\to-\infty} \varphi(t)\,, \tag{22}$$

and $a < 0$.

Define the function g by the equalities

$$g(x) = \frac{Q_1^2(t)}{F_1^2(t)}, \qquad x = \varphi(t) \qquad \text{for } t \in R\,, \tag{23}$$

and in the interval $]a, 0[$ consider the equation

$$v'' + g(x)v = 0\,. \tag{24}$$

In view of (23), from (3) we get

$$\int_a^0 (x-a)^2 g(x)dx = +\infty\,.$$

Consequently, by Lemma 3, every nontrivial solution of the equation (24) has a sequence of zeros tending to a. It can be dirrectly verify that if v is a solution of the equation (24), then the function u defined by the equality

$$u(t) = v(\varphi^{-1}(t)) \exp\left(-\int_0^t Q_1(s)\, ds \right) \qquad \text{for } t \in R \tag{25}$$

is a solution of the equation (1). Obviously, u has a sequence of zeros tending to $-\infty$. □

Proof of Theorem 1. According to Lemma 1, to prove the theorem it is sufficient to show that the equation (1) has no solution without zeros. Assume the contrary.

Suppose u is a solution of the equation (1) with no zero in R. Then, by virtue of Lemma 2, the equalities

$$\rho(t) = c(+\infty) - \int_0^t p(s)\,ds + \int_t^{+\infty} \rho^2(s)\,ds \qquad \text{for } t \geq 0\,,$$

$$\rho(t) = -c(-\infty) + \int_t^0 p(s)\,ds - \int_{-\infty}^t \rho^2(s)\,ds \qquad \text{for } t \leq 0$$

are fulfilled, where $\rho(t) = \frac{u'(t)}{u(t)}$ for $t \in R$. Set in the latter equalities $t = 0$. If we subtract the second equality from the first one, we obtain

$$c(+\infty) + c(-\infty) = -\int_{-\infty}^{+\infty} \rho^2(s)\,ds\,. \tag{26}$$

Since $p \not\equiv 0$, then $\rho \not\equiv 0$ and, consequently, $\int_{-\infty}^{+\infty} \rho^2(s)\,ds > 0$. Hence, on account of (4) and (26), we get the contradiction $0 < 0$. □

Proof of Theorem 2. According to Theorem 1.1 in [4], if

$$\limsup_{t\to+\infty} \frac{t}{\ln t}\Big(c(+\infty) - c(t)\Big) > \frac{1}{4}\,,$$

then the equation (1) is oscillatory in the neighbourhood of $+\infty$. Therefore we assume that

$$\limsup_{t\to+\infty} \frac{t}{\ln t}\Big(c(+\infty) - c(t)\Big) \leq \frac{1}{4}\,. \tag{27}$$

Define the function φ by (21). Due to (27), it is easy to see that

$$\lim_{t\to+\infty} \varphi(t) = +\infty\,. \tag{28}$$

According to (5) we have

$$\lim_{t\to-\infty} \frac{|t|}{\ln|t|}\Big(-c(+\infty) - c(t)\Big) = +\infty\,. \tag{29}$$

Therefore there exists a finite limit (22). Consequently, the function φ monotonicly transforms R into $]a, +\infty[$.

Define the function g by (23), and in the interval $]a, +\infty[$ consider the equation (24). It is clear that if v is a solution of the equation (24), then the function u defined by (25) is a solution of the equation (1). Therefore, according to Lemma 1, to prove the theorem it is sufficient to show that the equation (24) has no solution without zeros in $]a, +\infty[$. Assume the contrary, let v be a solution of the equation (24) with no zero in $]a, +\infty[$. Put $\sigma(x) = \frac{v'(x)}{v(x)}$ for $x > a$. Taking into account (6) and Lemma 2, we find

$$\sigma(x)=\int_x^{+\infty} g(\tau)d\tau+\int_x^{+\infty}\sigma^2(\tau)d\tau \qquad \text{for } x>a\,. \tag{30}$$

Evidently, $\sigma'(x)=-g(x)-\sigma^2(x)$ for $x>a$. Multiplying both sides of the last equality by $x-\lambda$, where $\lambda>a$, and integrating from λ to x, we get

$$\begin{aligned}\sigma(x)&=-\frac{1}{(x-\lambda)^2}\int_\lambda^x(\tau-\lambda)^2g(\tau)d\tau+\frac{1}{(x-\lambda)^2}\int_\lambda^x(\tau-\lambda)\sigma(\tau)(2-(\tau-\lambda)\sigma(\tau))d\tau\\&\le-\frac{1}{(x-\lambda)^2}\int_\lambda^x(\tau-\lambda)^2g(\tau)d\tau+\frac{1}{x-\lambda}\qquad \text{for } x>\lambda\,.\end{aligned}$$

Hence, on account of (6) and (23), it immediately follows

$$\sigma(x)\le-\frac{1}{(x-a)^2}\int_a^x(\tau-a)^2g(\tau)d\tau+\frac{1}{x-a}\qquad \text{for } x>a\,. \tag{31}$$

From (30) and (31) we have

$$(x-a)\int_x^{+\infty}g(\tau)d\tau+\frac{1}{(x-a)^2}\int_a^x(\tau-a)^2g(\tau)d\tau\le 1-(x-a)\int_x^{+\infty}\sigma^2(\tau)d\tau \quad \text{for } x>a\,.$$

Consequently,

$$\sup\left\{(x-a)\int_x^{+\infty}g(\tau)d\tau+\frac{1}{(x-a)^2}\int_a^x(\tau-a)^2g(\tau)d\tau : x>a\right\}\le 1\,.$$

Taking now into account (8) and (23), we obtain the contradiction $1<1$. □

The proof of Theorem 2′ is analogous to that of Theorem 2.

Proof of Theorem 3. Define the function φ by (21) and show that (28) is fulfilled. Indeed, if there exists $M>0$ such that

$$\varphi(t)\le M \qquad \text{for } t>0\,, \tag{32}$$

then according to (6) and the Hölder inequality we have

$$\left|\int_0^t Q_1(s)\,ds\right|\le\sqrt{\varphi(t)}\sqrt{\int_0^t\frac{Q_1^2(s)}{F_1(s)}\,ds}\le\sqrt{M}\sqrt{\int_0^{+\infty}\frac{Q_1^2(s)}{F_1(s)}\,ds}<+\infty \qquad \text{for } t>0\,,$$

which contradicts (32). Consequently, (28) is fulfilled.

In view of (5) we have (29). Therefore there exists a finite limit (22). Thus the function φ monotonicly transforms R into $]a,+\infty[$.

Define the function g by (23), and in the interval $]a,+\infty[$ consider the equation (24). It is clear that if v is a solution of the equation (24), then the function u

defined by (25) is a solution of the equation (1). Therefore, to prove the theorem it is sufficient to show that the equation (24) has a solution with no zero in $]a,+\infty[$. However, to prove this fact it is sufficient to show that there exists a function $\rho :]a,+\infty[\to R$ which is absolutely continuous on every segment contained in $]a,+\infty[$ and satisfies the inequality

$$\rho'(x) \le -g(x) - \rho^2(x) \qquad \text{for } x > a\,. \tag{33}$$

Suppose that (9) is fulfilled. On account of (23), from (9) we obtain

$$(x-a)\int\limits_x^{+\infty} g(s)\,ds \le \frac{1}{4} \qquad \text{for } x > a\,.$$

It can be easily verified that the function

$$\rho(x) = \frac{1}{4(x-a)} + \int\limits_x^{+\infty} g(s)\,ds \qquad \text{for } x > a$$

satisfies the inequality (33).

Now suppose that (10) is fulfilled. Then in view of (23) we have

$$\frac{1}{x-a}\int\limits_a^x (s-a)^2 g(s)\,ds \le \frac{1}{4} \qquad \text{for } x > a.$$

It is easy to verify that the function

$$\rho(x) = \frac{3}{4(x-a)} - \frac{1}{(x-a)^2}\int\limits_a^x (s-a)^2 g(s)\,ds \qquad \text{for } x > a$$

satisfies the inequality (33).

Finally suppose that (11) is fulfilled. Then by (23) we get

$$(x-a)\int\limits_x^{+\infty} g(s)\,ds + \frac{1}{x-a}\int\limits_a^x (s-a)^2 g(s)\,ds \le \sqrt{2}-1 \qquad \text{for } x > a\,.$$

We can easily verify that the function

$$\rho(x) = \frac{1}{2(x-a)} + \frac{1}{2}\left(\int\limits_x^{+\infty} g(s)\,ds - \frac{1}{(x-a)^2}\int\limits_a^x (s-a)^2 g(s)\,ds\right) \qquad \text{for } x > a$$

satisfies the inequality (33). □

Theorem 3′ can be proved analogously.

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T. Chantladze, D. Ugulava
N. Muskhelishvili Institute of Computational Mathematics
Georgian Academy of Sciences
8, Akuri St., 380093 Tbilisi, GEORGIA

Alexander Lomtatidze
Department of Mathematical Analysis
Faculty of Sciences, Masaryk University
Janáčkovo nám. 2a, 662 95 Brno, CZECH REPUBLIC
E-mail: bacho@math.muni.cz