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ORDINARY DIFFERENTIAL EQUATIONS

Oscillation Conditions for a Third-Order Linear Equation

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1. INTRODUCTION

Consider the equation

$$u''' + p(t)u = 0, \tag{1.1}$$

where $p : [0, +\infty[\ \to\ ]-\infty, +\infty[$ is a locally integrable function. A *solution* of Eq. (1.1) is defined as a function $u : [0, +\infty[\ \to\ ]-\infty, +\infty[$ locally absolutely continuous together with its first- and second-order derivatives and satisfying the equation almost everywhere.

A nontrivial solution of Eq. (1.1) is said to be *oscillating* if it has infinitely many zeros and *nonoscillating* otherwise. Equation (1.1) is oscillating if it has at least one oscillating solution and nonoscillating otherwise.

In the present paper, we prove integral oscillation criteria for Eq. (1.1), which generalize and supplement some results of [1–6]. Throughout the following, we assume that p is of constant sign, i.e., either

$$p(t) \le 0 \quad \text{for} \quad t > 0, \tag{1.2}$$

or

$$p(t) \ge 0 \quad \text{for} \quad t > 0. \tag{1.3}$$

We set

$$h(t) = t^2 \left[|p(t)| - \frac{2\sqrt{3}}{9t^3} \right], \qquad H(t) = \frac{1}{\ln^2 t} \int_1^t \frac{1}{s} \int_1^s \frac{1}{\xi} \int_1^\xi h(\eta) d\eta\, d\xi\, ds \qquad \text{for} \quad t > 1.$$

Theorem 1.1. *If*

$$\limsup_{t \to +\infty} \int_1^t h(s) ds = +\infty, \tag{1.4}$$

then Eq. (1.1) *is oscillating.*

This theorem generalizes Theorem 1 in [5].

Theorem 1.2. *Suppose that either* $\lim_{t\to+\infty} H(t) = +\infty$ *or* $-\infty < \liminf_{t\to+\infty} H(t) < \limsup_{t\to+\infty} H(t)$. *Then Eq.* (1.1) *is oscillating.*

By this theorem, for further investigation of the oscillation of Eq. (1.1), it suffices to consider the case in which either $\liminf_{t\to+\infty} H(t) = -\infty$ or there exists a finite limit

$$\lim_{t \to +\infty} H(t) = C. \tag{1.5}$$

Theorem 1.3. *Suppose that condition* (1.5) *is satisfied and*

$$\limsup_{t\to+\infty} \frac{\ln t}{\ln\ln t}(C - H(t)) > \frac{\sqrt{3}}{2}. \tag{1.6}$$

Then Eq. (1.1) *is oscillating.*

Remark 1.1. Let $p(t) = 2\sqrt{3}/(9t^3) + \varepsilon/(t^3 \ln^2 t)$ for $t > 1$. By Theorem 1.3, Eq. (1.1) is oscillating for $\varepsilon > \sqrt{3}/2$. We can readily see that this equation satisfies neither the assumptions of Theorem 2.11 in [4, p. 57] nor those of Theorem 1.3(a) in [1].

Corollary 1.1. Suppose that condition (1.5) is satisfied and

$$\liminf_{t\to+\infty} \ln t \left[2C - \int_1^t h(s)ds \right] > \frac{\sqrt{3}}{2}.$$

Then Eq. (1.1) is oscillating.

Corollary 1.2. Suppose that the integral $\int_1^{+\infty} h(s)ds$ converges and

$$\liminf_{t\to+\infty} \ln t \int_t^{+\infty} h(s)ds > \frac{\sqrt{3}}{2}.$$

Then Eq. (1.1) is oscillating.

2. AUXILIARY ASSERTIONS

We introduce the following notation: $x_0 = 1 - 2/\sqrt{3}$, $x_1 = 1 - 1/\sqrt{3}$, $x_2 = 1 + 1/\sqrt{3}$, and

$$H(\tau, t) = \int_\tau^t \frac{1}{s} \int_\tau^s \frac{1}{\xi} \int_\tau^\xi h(\eta)d\eta\, d\xi\, ds \qquad \text{for} \quad t, \tau > 0.$$

In addition, for an arbitrary solution u of Eq. (1.1) nonvanishing on some interval $[t_0, +\infty[$, we set

$$\varrho(t) = u'(t)/u(t), \qquad \sigma(t) = u''(t)/u(t) \qquad \text{for} \quad t > t_0,$$

$$\alpha_0(t) = 3t\varrho(t) - (3/2)(t\varrho(t))^2, \qquad \alpha(t) = t^2\sigma(t) - 2t\varrho(t) + (1/2)(t\varrho(t))^2 \qquad \text{for} \quad t > t_0,$$

$$G(\tau, t) = \int_\tau^t \frac{(s\varrho(s) - x_2)^2 (s\varrho(s) - x_0)}{s} ds \qquad \text{for} \quad t_0 < \tau < t \le +\infty.$$

Lemma 2 in [2] implies the following assertion.

Lemma 2.1. *Suppose that condition* (1.3) *is satisfied,* $t_0 > 0$*, and* u *is a solution of Eq.* (1.1) *satisfying the inequalities*

$$u(t) > 0, \qquad u'(t) > 0, \qquad u''(t) > 0 \qquad \textit{for} \quad t > t_0. \tag{2.1}$$

Then

$$\limsup_{t\to+\infty} \frac{tu'(t)}{u(t)} < +\infty, \qquad \limsup_{t\to+\infty} \frac{t^2u''(t)}{u(t)} < +\infty. \tag{2.2}$$

Lemma 2.2. *If the assumptions of Lemma* 2.1 *are valid, then*

$$t^2\varrho'(t) = -\int_\tau^t h(s)ds - G(\tau,t) + 2t\varrho(t) - \frac{3}{2}(t\varrho(t))^2 + \alpha(\tau) \qquad \textit{for} \quad t > \tau > t_0, \tag{2.3}$$

$$\int_\tau^t \frac{s\varrho(s)-x_2}{s}ds = -H(\tau,t) - \int_\tau^t \frac{1}{s}\int_\tau^s \frac{G(\tau,\xi)-\alpha_0(\xi)}{\xi}d\xi\, ds + \frac{\alpha(\tau)\ln^2 t}{2} + \delta_0(\tau)\ln t + \delta_1(\tau) \qquad \textit{for} \quad t > \tau > t_0, \tag{2.4}$$

where $\delta_0(\tau) = \tau\varrho(\tau) - \alpha(\tau)\ln\tau - x_2$ *and* $\delta_1(\tau) = -(1/2)\alpha(\tau)\ln^2\tau - \delta_0(\tau)\ln\tau$ *for* $\tau > t_0$.

Proof. Obviously,

$$\varrho'(t) = \sigma(t) - \varrho^2(t) \quad \text{for} \quad t > t_0 \tag{2.5}$$

and $\sigma'(t) = -p(t) - \sigma(t)\varrho(t)$ for $t > t_0$. Consequently, $\sigma'(t) = -p(t) - \varrho(t)\left(\varrho'(t) + \varrho^2(t)\right)$ for $t > t_0$. Multiplying both sides of the last relation by t^2 and integrating the resulting expressions from τ to t, we obtain

$$t^2\sigma(t) = -\int_\tau^t s^2 p(s)ds + \int_\tau^t \frac{s\varrho(s)(s\varrho(s)-1)(2-s\varrho(s))}{s}ds + 2t\varrho(t) - \frac{1}{2}(t\varrho(t))^2 + \alpha(\tau) \qquad \text{for} \quad t > \tau > t_0.$$

This, together with (2.5), implies that

$$t^2\varrho'(t) = -\int_\tau^t h(s)ds + \int_\tau^t \frac{s\varrho(s)(s\varrho(s)-1)(2-s\varrho(s)) - 2\sqrt{3}/9}{s}ds + 2t\varrho(t) - \frac{1}{2}(t\varrho(t))^2 + \alpha(\tau) \qquad \text{for} \quad t > \tau > t_0.$$

Now, since $x(x-1)(2-x) - 2\sqrt{3}/9 = -\left(x-x_2\right)^2\left(x-x_0\right)$, we see that relation (2.3) is valid.

Multiplying both sides of (2.3) by t^{-1} and integrating the resulting expressions from τ to t, we obtain

$$t\varrho(t) - x_2 = -\int_\tau^t \frac{1}{s}\int_\tau^s h(\eta)d\eta\, ds - \int_\tau^t \frac{G(\tau,s)-\alpha_0(s)}{s}ds + \alpha(\tau)\ln t + \delta_0(\tau) \qquad \text{for} \quad t > \tau > t_0.$$

Multiplying both sides of the resulting relation once more by t^{-1} and integrating from τ to t, we obtain (2.4). The proof of the lemma is complete.

Lemma 2.3. *Suppose that condition* (1.3) *is satisfied,* $\liminf_{t\to+\infty} H(t) > -\infty$, $t_0 > 0$, *and* u *is a solution of Eq.* (1.1) *satisfying inequalities* (2.1). *Then*

$$\int_{t_0}^{+\infty} \frac{1}{s}\left(s\varrho(s) - x_2\right)^2 ds < +\infty. \tag{2.6}$$

Proof. Suppose the contrary; namely, let inequality (2.6) fail. Then, by (2.1),

$$t\varrho(t) - x_0 > |x_0| \qquad \text{for} \quad t > t_0 \tag{2.7}$$

and $G(\tau,+\infty)=+\infty$ for $\tau>t_0$. Therefore,

$$\lim_{t\to+\infty}\frac{1}{\ln^2 t}\int_\tau^t\frac{1}{s}\int_\tau^s\frac{G(\tau,\xi)}{\xi}d\xi\,ds=+\infty\qquad\text{for}\quad t>\tau>t_0.\tag{2.8}$$

By Lemma 2.1, the function α_0 is bounded on the interval $[t_0,+\infty[$. Consequently, there exists an $r>0$ such that

$$\frac{1}{\ln^2 t}\left|\int_\tau^t\frac{1}{s}\int_\tau^s\frac{\alpha_0(\xi)}{\xi}d\xi\,ds\right|<r\qquad\text{for}\quad t>\tau>t_0.\tag{2.9}$$

Taking into account relations (2.8) and (2.9) and the lower boundedness of the function H, from (2.4), we obtain the inequality

$$\int_{t_0}^t\frac{s\varrho(s)-x_2}{s}ds<-\frac{1}{2}\int_{t_0}^t\frac{1}{s}\int_{t_0}^s\frac{G(t_0,\xi)}{\xi}d\xi\,ds\qquad\text{for}\quad t>t_1\tag{2.10}$$

for some $t_1>t_0$. By the Hölder inequality,

$$\left|\int_{t_0}^t(s\varrho(s)-x_2)\,s^{-1}ds\right|\le\sqrt{\ln t}\left(\int_{t_0}^t(s\varrho(s)-x_2)^2\,s^{-1}ds\right)^{1/2}\qquad\text{for}\quad t>t_1.$$

If, in addition, we use inequality (2.7), then from (2.10), we obtain

$$t\ln t\,(tv'(t))'>\lambda v^2(t)\qquad\text{for}\quad t>t_1,\tag{2.11}$$

where

$$\lambda=\frac{x_0^2}{4},\qquad v(t)=\int_{t_0}^t\frac{1}{s}\int_{t_0}^s\frac{1}{\xi}\int_{t_0}^\xi\frac{(\eta\varrho(\eta)-x_2)^2}{\eta}d\eta\,d\xi\,ds.$$

In addition, by the above assumption, $\lim_{t\to+\infty}v(t)/\ln^2 t=+\infty$. Therefore, without loss of generality, we can assume that $v(t)>\ln^2 t$ for $t>t_1$; then from (2.11), we obtain

$$tv'(t)\,(tv'(t))'>\lambda v^{3/2}(t)v'(t)\qquad\text{for}\quad t>t_1.$$

The integration of the last inequality from t_1 to t gives

$$(tv'(t))^2>(4\lambda/5)\left[v^{5/2}(t)-v^{5/2}(t_1)\right]+\left[t_2v'(t_1)\right]^2\qquad\text{for}\quad t>t_1.$$

Since v is unbounded, it follows that there exists a $t_2>t_1$ such that $v^{-5/4}(t)v'(t)>2\lambda/(5t)$ for $t>t_2$. Integrating this inequality from t_2 to t, we arrive at a contradiction; namely,

$$v^{-1/4}(t_2)>(\lambda/10)\ln(t/t_2)\qquad\text{for}\quad t\ge t_2.$$

This contradiction completes the proof of the lemma.

Lemma 2.4. *Let condition* (1.3) *be satisfied, and let Eq.* (1.1) *have a solution u satisfying conditions* (2.1) *and* (2.6). *Then there exists a finite limit* (1.5).

Proof. By Lemma 2.1 and condition (2.6), we have

$$G(\tau, +\infty) < +\infty, \qquad \int_{\tau}^{+\infty} \frac{\left(s\varrho(s) - x_2\right)^2 \left(s\varrho(s) - x_1\right)^2}{s} ds < +\infty \qquad \text{for} \quad \tau > t_0. \tag{2.12}$$

On the other hand, by Lemma 2.2, relation (2.4) is valid. This, together with (2.6) and the relation $(3/2)x^2 - 3x + 1 = (3/2)\left(x - x_1\right)\left(x - x_2\right)$, implies that

$$\begin{aligned} \int_{\tau}^{t} \frac{s\varrho(s) - x_2}{s} ds = -H(\tau, t) + I_1(\tau, t) + I_2(\tau, t) + \gamma(\tau, t) + \frac{1}{2}\alpha(\tau) \ln^2 t \\ + \delta_0(\tau) \ln t + \delta_1(\tau) \qquad \text{for} \quad t > \tau > t_0, \end{aligned} \tag{2.13}$$

where

$$I_1(\tau, t) = \int_{\tau}^{t} \frac{1}{s} \int_{\tau}^{s} \frac{G(\xi, +\infty)}{\xi} d\xi \, ds, \qquad \gamma(\tau, t) = (1/2) \ln^2(t/\tau)(1 - G(\tau, t)) \qquad \text{for} \quad t > \tau > t_0,$$

$$I_2(\tau, t) = -\frac{1}{2} \int_{\tau}^{t} \frac{1}{s} \int_{\tau}^{s} \frac{\left(\eta\varrho(\eta) - x_1\right)\left(\eta\varrho(\eta) - x_2\right)}{\eta} d\eta \, ds \qquad \text{for} \quad t > \tau > t_0.$$

By virtue of the Hölder inequality, we obtain

$$\frac{1}{\ln^2 t} \left| \int_{\tau}^{t} \frac{s\varrho(s) - x_2}{s} ds \right| \le \frac{1}{\ln^{3/2} t} \left(\int_{\tau}^{t} \frac{\left(s\varrho(s) - x_2\right)^2}{s} ds \right)^{1/2} \qquad \text{for} \quad t > \tau > t_0,$$

$$\left| \int_{\tau}^{s} \frac{(\eta\varrho(\eta) - x_1)(\eta\varrho(\eta) - x_2)}{\eta} d\eta \right| \le \sqrt{\ln s} \left(\int_{\tau}^{s} \frac{(\eta\varrho(\eta) - x_2)^2(\eta\varrho(\eta) - x_1)^2}{\eta} d\eta \right)^{1/2} \qquad \text{for} \quad s > \tau > t_0.$$

Therefore, from (2.12), we obtain

$$\lim_{t \to +\infty} \frac{1}{\ln^2 t} \int_{\tau}^{t} \frac{s\varrho(s) - x_2}{s} ds = 0, \qquad \lim_{t \to +\infty} \frac{1}{\ln^2 t} I_i(\tau, t) = 0 \qquad \text{for} \quad \tau > t_0 \qquad (i = 1, 2).$$

This, together with (2.13), means that there exists a finite limit

$$\lim_{t \to +\infty} \left(H(\tau, t)/\ln^2 t\right) = [\alpha(\tau) + 1 - G(\tau, +\infty)]/2 \qquad \text{for} \quad \tau > t_0.$$

If we now use the relation

$$H(\tau, t) = H(t) \ln^2 t - H(\tau) \ln^2 \tau - \ln \frac{t}{\tau} \int_{1}^{\tau} \frac{1}{s} \int_{1}^{s} h(\eta) d\eta \, ds - \frac{1}{2} \ln^2 \frac{t}{\tau} \int_{1}^{\tau} h(s) ds \qquad \text{for} \quad t > \tau > 0,$$

then we observe the existence of the finite limit

$$\lim_{t \to +\infty} H(t) = \frac{1}{2} \left[\alpha(\tau) + \int_{1}^{\tau} h(s) ds + 1 - G(\tau, +\infty) \right] \qquad \text{for} \quad \tau > t_0.$$

The proof of the lemma is complete.

Lemmas 2.3 and 2.4 readily imply the following assertion.

Lemma 2.5. *Suppose that conditions* (1.3) *and* (1.5) *are satisfied*, $t_0 > 0$, *and* u *is a solution of Eq.* (1.1) *satisfying condition* (2.1). *Then*

$$t^2\varrho'(t) + \frac{3}{2}(t\varrho(t))^2 - 2t\varrho(t) + 1 = 2C - \int_1^t h(s)ds + G(t,+\infty) \qquad \text{for} \quad t > t_0. \tag{2.14}$$

From Lemma 2.8′ and Theorem 1.3 in [4], we obtain the following assertions.

Lemma 2.6. *Let condition* (1.3) *be satisfied. Equation* (1.1) *is nonoscillating if and only if it has a solution* u *satisfying relation* (2.1) *for some* $t_0 > 0$.

Lemma 2.7. *Let condition* (1.2) *be satisfied. Equation* (1.1) *is oscillating if and only if so is the equation* $u''' - p(t)u = 0$.

Lemma 2.8. *Let*

$$F(x,y) = y\left[3x^2 - 6x + 2 - y\left(x(x-1)(x-2) + 2\sqrt{3}/9\right)\right] \qquad \text{for} \quad x, y > 0. \tag{2.15}$$

Then

$$\limsup_{y\to+\infty} F(x,y) \le \sqrt{3} \qquad \text{for} \quad x \ge 0. \tag{2.16}$$

Proof. We choose $y_0 > 3\sqrt{3}$ large enough to ensure that

$$\lambda_1(y) \stackrel{\text{def}}{=} 1 + 1/y - \sqrt{1/3 + 1/y^2} < 1, \qquad \lambda_2(y) \stackrel{\text{def}}{=} 1 + 1/y + \sqrt{1/3 + 1/y^2} < 2 \tag{2.17}$$

for $y > y_0$. We can readily see that $\partial F(x,y)/\partial x = -3y^2\left(x - \lambda_1(y)\right)\left(x - \lambda_2(y)\right)$.

Since $F(0,y) < 0$ and $F(1,y) < 0$ for $y > y_0$, it follows from (2.17) that

$$\begin{aligned} F(x,y) \le F\left(\lambda_2(y), y\right) = y\left(\lambda_2(y) - x_2\right)\Big[3\left(\lambda_2(y) - x_1\right) \\ - y\left(\lambda_2(y) - x_2\right)\left(\lambda_2(y) - x_0\right)\Big] \qquad \text{for} \quad x \ge 0, \quad y \ge y_0. \end{aligned} \tag{2.18}$$

Obviously,

$$\lim_{y\to+\infty} y\left(\lambda_2(y) - x_2\right) = 1, \qquad \lim_{y\to+\infty}\left(\lambda_2(y) - x_1\right) = 2/\sqrt{3}, \qquad \lim_{y\to+\infty}\left(\lambda_2(y) - x_0\right) = \sqrt{3}.$$

This, together with (2.18), implies (2.16). The proof of the lemma is complete.

3. PROOF OF THE MAIN RESULTS

By Lemma 2.7, it suffices to prove all assertions for the case in which condition (1.3) is satisfied.

Proof of Theorem 1.1. Suppose the contrary: Eq. (1.1) is nonoscillating. Then, by Lemmas 2.1, 2.2, and 2.6, it has a solution u satisfying conditions (2.1)–(2.3). If, in addition, we use the relation $t^2\varrho'(t) = t^2\sigma(t) - (t\varrho(t))^2$, then from (2.3), we obtain

$$\limsup_{t\to+\infty} \int_1^t h(s)ds < +\infty,$$

which contradicts condition (1.4). The proof of the theorem is complete.

Proof of Theorem 1.2. Suppose the contrary. Then, by Lemma 2.6, Eq. (1.1) has a solution u satisfying condition (2.1). Since the function H is bounded below, it follows from Lemma 2.3 that relation (2.6) is valid; therefore, by Lemma 2.4, there exists a finite limit (1.5). This contradicts the assumptions of the theorem and completes the proof.

Proof of Theorem 1.3. Suppose the contrary. Then Eq. (1.1) is nonoscillating. By Lemma 2.6, it has a solution u satisfying condition (2.1). Relation (1.5) and Lemma 2.5 imply (2.14). Multiplying both sides of this relation by t^{-1} and integrating from τ to t, we obtain

$$\begin{aligned} t\varrho(t) &+ \int_\tau^t \frac{1-\alpha_0(s)}{s} ds \\ &= 2C\ln t - \int_1^t \frac{1}{s}\int_1^s h(\eta)d\eta\, ds + \int_\tau^t \frac{G(s,+\infty)}{s} ds + \beta_0(\tau) \qquad \text{for} \quad t>\tau>t_0, \end{aligned} \tag{3.1}$$

where $\beta_0(\tau) = \int_1^\tau h(s)ds + \tau\varrho(\tau) + 2C\ln\tau$ for $\tau > t_0$. On the other hand, multiplying both sides of (2.14) by $t^{-1}\ln t$ and integrating the resulting expressions from τ to t, we obtain

$$\begin{aligned} t\varrho(t) &+ \frac{1}{\ln t}\int_\tau^t \frac{(1-\alpha_0(s))\ln s}{s} ds - \frac{1}{\ln t}\int_\tau^t \varrho(s)ds \\ &= C\ln t - \frac{1}{\ln t}\int_\tau^t \frac{\ln s}{s}\int_1^s h(\eta)d\eta\, ds + \frac{1}{\ln t}\int_\tau^t \frac{G(s,+\infty)\ln s}{s} ds + \frac{1}{\ln t}\beta_1(\tau) \qquad \text{for} \quad t>\tau>t_0, \end{aligned} \tag{3.2}$$

where

$$\beta_1(\tau) = \tau\varrho(\tau)\ln\tau - C\ln^2\tau + \int_1^\tau \frac{\ln s}{s}\int_1^s h(\eta)d\eta\, ds \qquad \text{for} \quad \tau > t_0.$$

Subtracting (3.2) from (3.1) and using the relations

$$\int_1^t \frac{\ln(t/s)}{s}\int_1^s h(\eta)d\eta\, ds = H(t)\ln^2 t \qquad \text{for} \quad t>0,$$

$$\frac{1}{\ln t}\int_\tau^t \frac{\ln(t/s)}{s} G(s,+\infty)ds = \int_\tau^t \frac{1}{s\ln^2 s}\int_\tau^s \frac{G(\xi,+\infty)\ln\xi}{\xi} d\xi\, ds \qquad \text{for} \quad t>\tau>t_0,$$

$$\frac{1}{\ln t}\int_\tau^t \frac{(1-\alpha_0(s))\ln(t/s)}{s} ds = \int_\tau^t \frac{1}{s\ln^2 s}\int_\tau^t \frac{(1-\alpha_0(\eta))\ln\eta}{\eta} d\eta\, ds \qquad \text{for} \quad t>\tau>t_0,$$

we obtain

$$\ln t(C-H(t)) + \beta_0(\tau) - \frac{\beta_1(\tau)}{\ln t} = \frac{1}{\ln t}\int_\tau^t \varrho(s)ds + I_0(\tau,t) \qquad \text{for} \quad t>\tau>t_0, \tag{3.3}$$

where

$$I_0(\tau,t) = \int_\tau^t \frac{Q(\tau,s)}{s\ln^2 s} ds, \qquad Q(\tau,t) = \int_\tau^t \frac{[1-\alpha_0(s)-G(s,+\infty)]\ln s}{s} ds \qquad \text{for} \quad t>\tau>t_0. \tag{3.4}$$

Obviously,

$$2\int\limits_{\tau}^{t}\frac{G(s,+\infty)\ln s}{s}ds = G(t,+\infty)\ln^2 t + \int\limits_{\tau}^{t}\frac{(s\varrho(s)-x_2)^2\,(s\varrho(s)-x_0)\ln^2 s}{s}ds - \beta_2(\tau) \qquad \text{for} \quad t>\tau>t_0,$$

where $\beta_2(\tau)=G(\tau,+\infty)\ln^2\tau$ for $\tau>t_0$. Therefore, from (3.4), we obtain

$$Q(\tau,t)\le\frac{1}{2}\int\limits_{\tau}^{t}\frac{1}{s}F(s\varrho(s),\ln s)ds+\beta_2(\tau) \qquad \text{for} \quad t>\tau>t_0, \tag{3.5}$$

where F is the function given by (2.15).

Using relation (2.1) and Lemma 2.8, from (3.5), we find that, for each $\varepsilon>0$, there exists a $\tau_\varepsilon>t_0$ such that $Q(\tau,t)<\left(\sqrt{3}/2+\varepsilon\right)\ln t+\beta_2(\tau)$ for $t>\tau>\tau_\varepsilon$. This, together with (3.4), implies that

$$I_0(\tau,t)<\left(\sqrt{3}/2+\varepsilon\right)\ln\ln t+\beta_2(\tau) \qquad \text{for} \quad t>\tau>\tau_\varepsilon. \tag{3.6}$$

By (2.2), the first term on the right-hand side in (3.3) is bounded. This, together with (3.6) and (3.3), implies that

$$\limsup_{t\to+\infty}\frac{\ln t}{\ln\ln t}(C-H(t))\le\sqrt{3}/2.$$

We have arrived at a contradiction with condition (1.6). The proof of the theorem is complete.

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