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Nonlinear Analysis 49 (2002) 929–945

Nonlinear Analysis

www.elsevier.com/locate/na

On periodic solutions of first order linear functional differential equations

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Received 29 May 2000; accepted 28 June 2000

Keywords: Linear functional differential equations; Differential equations with deviating arguments; Periodic solutions; Dimension of periodic solution space

0. Introduction

Consider the functional differential equation

$$u'(t) = \ell(u)(t) + g(t), \tag{0.1}$$

where $\ell : C_\omega(R) \to L_\omega(R)$ is a linear bounded operator and $g \in L_\omega(R)$. An absolutely continuous function $u : R \to R$ is said to be an ω-periodic solution of Eq. (0.1) if u is periodic with the period $\omega > 0$, i.e.,

$$u(t + \omega) = u(t) \quad \text{for } t \in R,$$

and satisfies Eq. (0.1) almost everywhere in R.

In the present paper, new optimal sufficient conditions are established for the existence of a unique ω-periodic solution of Eq. (0.1). These conditions generalize and make the known results of analogous type more complete (see, e.g., [1–9]).

Along with Eq. (0.1) consider the important particular case, where (0.1) is the equation with deviating arguments:

$$u'(t) = \sum_{k=1}^{n} p_k(t)u(\tau_k(t)) + g(t). \tag{0.2}$$

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0362-546X/02/$ - see front matter
PII: S0362-546X(01)00147-X

Here $p_k \in L_\omega(R)$, $k=\overline{1,n}$, and $\tau_k : R \to R$, $k=\overline{1,n}$, are measurable functions such that

$$\tau_k(t+\omega) = \tau_k(t) + \omega h_k(t) \quad \text{for } t \in R, \quad k=1,\dots,n,$$

where the functions $h_k : R \to R$, $k=\overline{1,n}$, assume only integer values.

It is known (see, e.g., [6]) that Eq. (0.1), resp. Eq. (0.2), has a unique ω-periodic solution iff the corresponding homogeneous equation:

$$u'(t) = \ell(u)(t), \tag{0.1a}$$

resp.

$$u'(t) = \sum_{k=1}^{n} p_k(t) u(\tau_k(t)), \tag{0.2a}$$

has only a trivial ω-periodic solution. On the other hand (in spite of ordinary differential equations), for any natural m it is easy to construct a homogeneous equation of the type (0.1a), which has at least m linearly independent ω-periodic solutions. Therefore, there naturally arises the question on the dimension of ω-periodic solution space of homogeneous equation (0.1a). In the sequel, we also give sufficient conditions guaranteeing that the dimension of the above-mentioned space is not greater than one.

Throughout the paper the following notation will be used.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a, a+\omega]; R)$ is the Banach space of continuous functions $u : [a, a+\omega] \to R$ with the norm:

$$\|u\|_C = \max\{|u(t)| : a \leqslant t \leqslant a+\omega\}.$$

$C_\omega(R)$ is the Banach space of continuous ω-periodic functions $u : R \to R$ with the norm:

$$\|u\|_{C_\omega} = \max\{|u(t)| : 0 \leqslant t \leqslant \omega\}.$$

$C_\omega(R_+) = \{u \in C_\omega(R) : u(t) \geqslant 0 \text{ for } 0 \leqslant t \leqslant \omega\}$.

$\tilde{C}(I; D)$, where $I \subset R$, $D \subset R$, is the set of absolutely continuous functions $u : I \to D$.

$L(]a, a+\omega[; R)$ is the Banach space of Lebesgue integrable functions $p :]a, a+\omega[\to R$ with the norm:

$$\|p\|_L = \int_a^{a+\omega} |p(s)|\,\mathrm{d}s.$$

$L_\omega(R)$ is the Banach space of Lebesgue integrable, ω-periodic functions $p : R \to R$ with the norm:

$$\|p\|_{L_\omega} = \int_0^{\omega} |p(s)|\,\mathrm{d}s.$$

$L_\omega(R_+) = \{p \in L_\omega(R) : p(t) \geqslant 0 \text{ for } t \in R\}$.

$\mathscr{L}_\omega(R)$ is the set of linear bounded operators $\ell : C_\omega(R) \to L_\omega(R)$ such that

$$\sup\{|\ell(v)(\cdot)| : \|v\|_{C_\omega} = 1\} \in L_\omega(R_+).$$

$\mathscr{P}_\omega(R)$ is the set of linear operators $\ell \in \mathscr{L}_\omega(R)$ transforming $C_\omega(R_+)$ into $L_\omega(R_+)$.

$[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

Everywhere in what follows, we will assume that the operator $\ell \in \mathscr{L}_\omega(R)$ is nontrivial and admits the representation $\ell = \ell_1 - \ell_2$, where

$$\ell_1, \ell_2 \in \mathscr{P}_\omega(R).$$

It is obvious that for each $x \in [0, \omega[$ the operators ℓ_1 and ℓ_2 uniquely define the corresponding operators:

$$\tilde{\ell}_{ix} : \{u \in C([x, x+\omega]; R)\colon u(x) = u(x+\omega)\} \to L(]x, x+\omega[; R), \quad (i = 1, 2).$$

In the sequel, we will assume that the linear bounded operators

$$\ell_{ix} : C([x, x+\omega]; R) \to L(]x, x+\omega[; R), \quad (i = 1, 2)$$

are extensions of the operators $\tilde{\ell}_{1x}$ and $\tilde{\ell}_{2x}$, respectively. Furthermore, we will assume that ℓ_{1x} and ℓ_{2x} are nonnegative operators, i.e., they transform $C([x, x+\omega]; R_+)$ into $L(]x, x+\omega[; R_+)$.

In particular, if

$$\ell(v)(t) \overset{def}{=} \sum_{k=1}^{n} p_k(t) v(\tau_k(t)),$$

then we will assume that $p_k(t) \not\equiv 0$, $k = \overline{1, n}$, and

$$\ell_{1x}(v)(t) \overset{def}{=} \sum_{k=1}^{n} [p_k(t)]_+ v(\tau_{kx}(t)), \quad \ell_{2x}(v)(t) \overset{def}{=} \sum_{k=1}^{n} [p_k(t)]_- v(\tau_{kx}(t)),$$

where $\tau_{kx}(t) = \tau_k(t) - \eta_{kx}(t)\omega$ for $t \in]x, x+\omega[$, and $\eta_{kx}(t)$ is the integer part of the number $\frac{1}{\omega}(\tau_k(t) - x)$.

1. Main results

1.1. Existence and uniqueness theorems

Theorem 1.1. *Let $i, j \in \{1, 2\}$, $i \neq j$ and*

$$\|\ell_i(1)\|_{L_\omega} < 1, \tag{1.1}$$

$$\frac{\|\ell_i(1)\|_{L_\omega}}{1 - \|\ell_i(1)\|_{L_\omega}} < \|\ell_j(1)\|_{L_\omega}. \tag{1.2}$$

Let, moreover, one of the following items be fulfilled:

(a) *for every $x \in [0, \omega[$ there exists $\gamma_x \in \tilde{C}([x, x+\omega];]0, +\infty[)$ such that*

$$\gamma_x'(t) \geqslant \ell_{1x}(\gamma_x)(t) + \ell_2(1)(t) \quad \textit{for } t \in]x, x+\omega[, \tag{1.3}$$

$$\gamma_x(x+\omega) \leqslant 4; \tag{1.4}$$

(b) *for every* $x \in [0,\omega[$ *there exists* $\gamma_x \in \tilde{C}([x,x+\omega];]0,+\infty[)$ *such that*

$$-\gamma_x'(t) \geqslant \ell_{2x}(\gamma_x)(t) + \ell_1(1)(t) \quad \text{for } t \in]x,x+\omega[,$$

$$\gamma_x(x) \leqslant 4.$$

Then Eq. (0.1) *has a unique* ω*-periodic solution.*

Corollary 1.1. *Let either*

$$\int_0^\omega |p_1(s)|\,\mathrm{d}s < 1, \quad \frac{\int_0^\omega |p_1(s)|\,\mathrm{d}s}{1-\int_0^\omega |p_1(s)|\,\mathrm{d}s} < \sum_{k=2}^n \int_0^\omega |p_k(s)|\,\mathrm{d}s \tag{1.5}$$

or

$$\sum_{k=2}^n \int_0^\omega |p_k(s)|\,\mathrm{d}s < 1, \quad \frac{\sum_{k=2}^n \int_0^\omega |p_k(s)|\,\mathrm{d}s}{1-\sum_{k=2}^n \int_0^\omega |p_k(s)|\,\mathrm{d}s} < \int_0^\omega |p_1(s)|\,\mathrm{d}s. \tag{1.6}$$

Let, moreover, one of the following items be fulfilled:

(a) $p_k(t) \leqslant 0$, $k=\overline{2,n}$, $p_1(t) \geqslant 0$, $\tau_1(t)=t$ *for* $t \in R$,

$$\sum_{k=2}^n \int_0^\omega |p_k(s)| \exp\left(\int_s^\omega p_1(\xi)\,\mathrm{d}\xi\right) \mathrm{d}s < 4; \tag{1.7}$$

(b) $p_k(t) \leqslant 0$, $k=\overline{2,n}$, $p_1(t) \geqslant 0$ *for* $t \in R$, *and either*

$$\sum_{k=2}^n \int_0^\omega |p_k(s)|\,\mathrm{d}s < 4\left(1-\int_0^\omega |p_1(s)|\,\mathrm{d}s\right) \tag{1.8}$$

or

$$\int_0^\omega |p_1(s)|\,\mathrm{d}s < 4\left(1-\sum_{k=2}^n \int_0^\omega |p_k(s)|\,\mathrm{d}s\right); \tag{1.9}$$

(c) $p_k(t) \geqslant 0$, $k=\overline{2,n}$, $p_1(t) \leqslant 0$, $\tau_1(t)=t$ *for* $t \in R$,

$$\sum_{k=2}^n \int_0^\omega p_k(s) \exp\left(\int_0^s |p_1(\xi)|\,\mathrm{d}\xi\right) \mathrm{d}s < 4; \tag{1.10}$$

(d) $p_k(t) \geqslant 0$, $k=\overline{2,n}$, $p_1(t) \leqslant 0$ *for* $t \in R$, *and either* (1.8) *or* (1.9) *is fulfilled.*
Then Eq. (0.2) *has a unique* ω*-periodic solution.*

Theorem 1.2. *Let* $i,j \in \{1,2\}$, $i \neq j$, *and conditions* (1.1) *and* (1.2) *be fulfilled. Let, moreover, there exist* $\gamma \in \tilde{C}([0,\omega];]0,+\infty[)$ *such that one of the following items is fulfilled*:

(a)

$$\gamma'(t) \geqslant \ell_{10}(\gamma)(t) + \ell_2(1)(t) \quad \text{for } t \in]0,\omega[, \tag{1.11}$$

$$\gamma(\omega) \leqslant 1; \tag{1.12}$$

(b)

$$-\gamma(t)' \geqslant \ell_{20}(\gamma)(t) + \ell_1(1)(t) \quad \textit{for } t \in]0,\omega[, \tag{1.11$'$}$$

$$\gamma(0) \leqslant 1. \tag{1.12$'$}$$

Then Eq. (0.1) *has a unique ω-periodic solution.*

Remark 1.1. In the case where $\ell \in \mathscr{P}_\omega(R)$ $(-\ell \in \mathscr{P}_\omega(R))$ it is clear that $\ell_1 \equiv \ell$ and $\ell_2 \equiv 0$ $(\ell_1 \equiv 0,\ \ell_2 \equiv -\ell)$. Then condition (1.12) (condition (1.12′)) becomes unimportant and Theorem 1.2 coincides with the result obtained in [3].

Remark 1.2. As it will be clear from the proof, if assumptions of Theorem 1.2 are fulfilled and $g(t) \leqslant 0$ for $t \in R$, $g \not\equiv 0$, then the unique ω-periodic solution u of Eq. (0.1) satisfies the condition $u(0) > 0$.

Corollary 1.2. *Let either*

$$\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_+\,\mathrm{d}s < 1, \quad \frac{\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_+\,\mathrm{d}s}{1-\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_+\,\mathrm{d}s} < \sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_-\,\mathrm{d}s \tag{1.13}$$

or

$$\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_-\,\mathrm{d}s < 1, \quad \frac{\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_-\,\mathrm{d}s}{1-\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_-\,\mathrm{d}s} < \sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_+\,\mathrm{d}s. \tag{1.14}$$

Let, moreover, one of the following items be fulfilled:

(a)

$$\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_-\exp\left(\sum_{i=1}^{n}\int_s^{\omega}[p_i(\xi)]_+\,\mathrm{d}\xi\right)\mathrm{d}s < 1, \tag{1.15}$$

$$(t-\tau_{k0}(t))[p_k(t)]_+ \geqslant 0 \quad \textit{for } t \in]0,\omega[, \quad k = 1,\dots,n; \tag{1.16}$$

(b)

$$\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_+\exp\left(\sum_{i=1}^{n}\int_0^{s}[p_i(\xi)]_-\,\mathrm{d}\xi\right)\mathrm{d}s < 1,$$

$$(\tau_{k0}(t)-t)[p_k(t)]_- \geqslant 0 \quad \textit{for } t \in]0,\omega[, \quad k = 1,\dots,n;$$

(c) $p_k(t) \geqslant 0$, $k = \overline{1,n}$ *for* $t \in R$ *and*

$$\sum_{k=1}^{n}\int_t^{\tau_{i0}(t)} p_k(s)\,\mathrm{d}s \leqslant \frac{1}{e} \quad \textit{for } t \in]0,\omega[, \quad i = 1,\dots,n; \tag{1.17}$$

(d) $p_k(t) \leqslant 0$, $k = \overline{1,n}$ *for* $t \in R$ *and*

$$\sum_{k=1}^{n}\int_{\tau_{i0}(t)}^{t} |p_k(s)\,\mathrm{d}s \leqslant \frac{1}{e} \quad \textit{for } t \in]0,\omega[, \quad i = 1,\dots,n;$$

(e)

$$\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_-\,\mathrm{d}s+\alpha+\beta<1, \tag{1.18}$$

where

$$\alpha=\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_+\sum_{i=1}^{n}\int_0^{\tau_{k0}(s)}[p_i(\xi)]_-\,\mathrm{d}\xi\exp\left(\sum_{j=1}^{n}\int_s^{\omega}[p_j(\xi)]_+\,\mathrm{d}\xi\right)\mathrm{d}s, \tag{1.19}$$

$$\beta=\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_+\sigma_k(s)\sum_{i=1}^{n}\int_s^{\tau_{k0}(s)}[p_i(\xi)]_+\mathrm{d}\xi\exp\left(\sum_{j=1}^{n}\int_s^{\omega}[p_j(\xi)]_+\,\mathrm{d}\xi\right)\mathrm{d}s, \tag{1.20}$$

and $\sigma_k(t)=\frac{1}{2}(1+\operatorname{sgn}(\tau_{k0}(t)-t))$ *for* $t\in\,]0,\omega[,\ k=\overline{1,n}$;

(f)

$$\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_+\,\mathrm{d}s+\tilde{\alpha}+\tilde{\beta}<1,$$

where

$$\tilde{\alpha}=\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_-\sum_{i=1}^{n}\int_{\tau_k 0(s)}^{\omega}[p_i(\xi)]_+\mathrm{d}\xi\exp\left(\sum_{j=1}^{n}\int_0^{s}[p_j(\xi)]_-\,\mathrm{d}\xi\right)\mathrm{d}s, \tag{1.19$'$}$$

$$\tilde{\beta}=\sum_{k=1}^{n}\int_0^{\omega}[p_k(s)]_-\tilde{\sigma}_k(s)\sum_{i=1}^{n}\int_{\tau_{k0}(s)}^{s}[p_i(\xi)]_-\mathrm{d}\xi\exp\left(\sum_{j=1}^{n}\int_0^{s}[p_j(\xi)]_-\,\mathrm{d}\xi\right)\mathrm{d}s, \tag{1.20$'$}$$

and $\tilde{\sigma}_k(t)=\frac{1}{2}(1+\operatorname{sgn}(t-\tau_{k0}(t)))$ *for* $t\in\,]0,\omega[,\ k=\overline{1,n}$.

Then Eq. (0.2) *has a unique* ω*-periodic solution.*

Remark 1.3. According to [6], Eq. (0.1) has a unique ω-periodic solution iff corresponding homogeneous equation (0.1a) has only a trivial ω-periodic solution. Consequently, under the conditions of Theorems 1.1 and 1.2, the dimension of the space of ω-periodic solutions of Eq. (0.1a) is zero. In particular, under the conditions of Corollaries 1.1 and 1.2, the dimension of the space of ω-periodic solutions of Eq. (0.2a) is zero.

Remark 1.4. The conditions in Theorem 1.2 are optimal and they cannot be weakened as shown in the following example.

Let $\varepsilon_0 > 0$ be an arbitrarily fixed number. Choose an integer $n > 1$ and $\varepsilon \in]0, (n-1)/n(n+1)[$ such that

$$\frac{1}{n^2} < \frac{\varepsilon_0}{2}, \quad (n+1)\varepsilon < \frac{\varepsilon_0}{2}$$

and put:

$$\delta = \left(2 + n + \frac{1}{n^2\varepsilon} + \frac{1}{\varepsilon}\right)^{-1}\left(\varepsilon_0 - \frac{1}{n^2} - (n+1)\varepsilon\right), \quad t_1 = (n+1)\varepsilon,$$

$$t_2 = 1 + (2n+1)\varepsilon, \quad t_3 = 1 + (3n+1)\varepsilon + \frac{1}{n}, \quad \omega = 2 + (2n+1)\varepsilon,$$

$$c_1 = \left(1 - \frac{\delta}{\varepsilon}\right)t_1 + \delta, \quad c_2 = \left(\frac{\delta}{\varepsilon} - \frac{\delta + (n+1)\varepsilon}{n\varepsilon}\right)t_2 + c_1,$$

$$c_3 = \left(\frac{\delta + (n+1)\varepsilon}{n\varepsilon} - 1\right)t_3 + c_2.$$

Consider the equation:

$$u'(t) = p(t)u(\tau(t)). \tag{1.21}$$

Here

$$p(t) = \begin{cases} -1 & \text{for } t \in [\nu\omega, t_1 + \nu\omega[\cup [t_3 + \nu\omega, (\nu+1)\omega] \\ \dfrac{1}{\varepsilon} & \text{for } t \in [t_1 + \nu\omega, t_2 + \nu\omega[\\ \dfrac{1}{n\varepsilon} & \text{for } t \in [t_2 + \nu\omega, t_3 + \nu\omega[\end{cases},$$

$$\tau(t) = \begin{cases} t_2 & \text{for } t \in [\nu\omega, t_1 + \nu\omega[\cup [t_3 + \nu\omega, (\nu+1)\omega] \\ 0 & \text{for } t \in [t_1 + \nu\omega, t_2 + \nu\omega[\\ t_1 & \text{for } t \in [t_2 + \nu\omega, t_3 + \nu\omega[\end{cases},$$

where ν is an integer. Obviously, $\tau_0(t) = \tau(t)$ for $t \in]0, \omega[$ and

$$\frac{\int_0^\omega [p(s)]_- \, \mathrm{d}s}{1 - \int_0^\omega [p(s)]_- \, \mathrm{d}s} = n - 1, \quad \int_0^\omega [p(s)]_+ \, \mathrm{d}s = 1 + n + \frac{1}{\varepsilon} + \frac{1}{n^2\varepsilon} > n - 1.$$

Define the function γ as follows:

$$\gamma(t) = \begin{cases} t + \delta & \text{for } t \in [0, t_1[\\ \dfrac{\delta}{\varepsilon}t + c_1 & \text{for } t \in [t_1, t_2[\\ \dfrac{\delta + (n+1)\varepsilon}{n\varepsilon}t + c_2 & \text{for } t \in [t_2, t_3[\\ t + c_3 & \text{for } t \in [t_3, \omega] \end{cases}.$$

It is easy to verify that

$$\gamma'(t) = [p(t)]_+\gamma(\tau_0(t)) + [p(t)]_- \quad \text{for } t \in]0,\omega[$$

and

$$\gamma(\omega) = 1 + \varepsilon_0.$$

On the other hand, Eq. (1.21) has a nontrivial ω-periodic solution

$$u(t) = \begin{cases} t - \varepsilon - v\omega & \text{for } t \in [v\omega, t_1 + v\omega[\\ -t + (2n+1)\varepsilon - v\omega & \text{for } t \in [t_1 + v\omega, t_2 + v\omega[\\ t - 2 - (2n+1)\varepsilon - v\omega & \text{for } t \in [t_2 + v\omega, (v+1)\omega] \end{cases},$$

where v is an integer. This example shows that the inequality (1.12), resp. (1.12′) in Theorem 1.2 cannot be replaced by the inequality

$$\gamma(\omega) \leqslant 1 + \varepsilon, \quad \text{resp. } \gamma(0) \leqslant 1 + \varepsilon,$$

as small as $\varepsilon > 0$ will be.

1.2. On the dimension of ω-periodic solution space of Eq. (0.1a)

Theorem 1.3. *Let there exist $\gamma \in \tilde{C}([0,\omega];]0,+\infty[)$ such that one of the following items is fulfilled*:
(a) *inequality* (1.11) *holds and $\gamma(\omega) \leqslant 4$;*
(b) *inequality* (1.11′) *holds and $\gamma(0) \leqslant 4$.*
Then the space of ω-periodic solutions of Eq. (0.1a) *is no more than one-dimensional.*

Corollary 1.3. *Let one of the following items be fulfilled*:
(a)

$$\sum_{k=1}^{n} \int_0^{\omega} [p_k(s)]_- \exp\left(\sum_{i=1}^{n} \int_s^{\omega} [p_i(\xi)]_+ \,\mathrm{d}\xi \right) \mathrm{d}s < 4,$$

$$(t - \tau_{k0}(t))[p_k(t)]_+ \geq 0 \quad \textit{for } t \in]0,\omega[, \quad k = 1,\dots,n;$$

(b)

$$\sum_{k=0}^{n} \int_0^{\omega} [p_k(s)]_+ \exp\left(\sum_{i=1}^{n} \int_0^{s} [p_i(\xi)]_- \,\mathrm{d}\xi \right) \mathrm{d}s < 4,$$

$$(\tau_{k0}(t) - t)[p_k(t)]_- \geqslant 0 \quad \textit{for } t \in]0,\omega[, \quad k = 1,\dots,n;$$

(c)

$$\sum_{k=1}^{n} \int_0^{\omega} [p_k(s)]_- \,\mathrm{d}s + \alpha + 4\beta < 4,$$

where α and β are defined by (1.19) *and* (1.20) *with $\sigma_k(t) = \frac{1}{2}(1 + \operatorname{sgn}(\tau_{k0}(t) - t))$ for $t \in]0,\omega[$, $k = \overline{1,n}$;*

(d)

$$\sum_{k=1}^{n} \int_0^{\omega} [p_k(s)]_+ \, ds + \tilde{\alpha} + 4\tilde{\beta} < 4,$$

where $\tilde{\alpha}$ *and* $\tilde{\beta}$ *are defined by* (1.19′) *and* (1.20′) *with* $\tilde{\sigma}_k(t) = \frac{1}{2}(1 + \operatorname{sgn}(t - \tau_{k0}(t)))$ *for* $t \in]0, \omega[$, $k = \overline{1, n}$.

Then the space of ω*-periodic solutions of Eq.* (0.2a) *is no more than one-dimensional.*

Remark 1.5. The conditions in Theorem 1.3 are optimal and they cannot be weakened as shown in the following example.

Let $\varepsilon > 0$ be an arbitrarily fixed number. Consider the equation:

$$u'(t) = u(\tau(t)) - u(\mu(t)), \tag{1.22}$$

with $\tau(t) = 0$ for $t \in R$, and

$$\mu(t) = \begin{cases} 3 & \text{for } t \in [4v, 1 + 4v[\cup [3 + 4v, 4 + 4v] \\ 1 & \text{for } t \in [1 + 4v, 3 + 4v[\end{cases},$$

where v is an integer. Obviously, $\tau_0(t) = \tau(t)$ for $t \in]0, 4[$. Define the function γ as follows:

$$\gamma(t) = \left(\frac{\varepsilon}{5} + 1\right) t + \frac{\varepsilon}{5}.$$

It is easy to verify that

$$\gamma'(t) = \gamma(\tau_0(t)) + 1 \quad \text{for } t \in]0, \omega[$$

and

$$\gamma(\omega) = 4 + \varepsilon.$$

On the other hand, Eq. (1.22) has two linearly independent ω-periodic solutions $u_1(t) = 1$ for $t \in R$, and

$$u_2(t) = \begin{cases} t - 4v & \text{for } t \in [4v, 1 + 4v[\\ 2 - t - 4v & \text{for } t \in [1 + 4v, 3 + 4v[\\ t - 4 - 4v & \text{for } t \in [3 + 4v, 4 + 4v] \end{cases},$$

where v is an integer. This example shows that the inequality $\gamma(\omega) \leqslant 4$, resp. $\gamma(0) \leqslant 4$ in Theorem 1.3 cannot be replaced by the inequality:

$$\gamma(\omega) \leqslant 4 + \varepsilon, \quad \text{resp. } \gamma(0) \leqslant 4 + \varepsilon,$$

as small as $\varepsilon > 0$ will be.

2. Proofs

First we formulate, in a suitable for us form, the result proven in [2].

Lemma 2.1. *Let* $a \in [0, \omega[$, $\hat{\ell} : C([a, a+\omega]; R) \to L(]a, a+\omega[; R)$ *be a linear bounded operator mapping* $C([a, a+\omega]; R_+)$ *into* $L(]a, a+\omega[; R_+)$. *Let, moreover, there exist a function* $\gamma \in \tilde{C}([a, a+\omega];]0, +\infty[)$ *such that*

$$\gamma'(t) \geqslant \hat{\ell}(\gamma)(t) \quad \text{for } t \in]a, a+\omega[.$$

Then for any $g \in L(]a, a+\omega[; R)$ *the Cauchy problem*:

$$u'(t) = \hat{\ell}(u)(t) + g(t), \quad u(a) = 0$$

has a unique solution.[1] *Further, the inequalities*

$$v(t) \geqslant 0 \quad \text{for } t \in [a, a+\omega]$$

and

$$v'(t) \geqslant 0 \quad \text{for } t \in]a, a+\omega[$$

are fulfilled whenever the function $v \in \tilde{C}([a, a+\omega]; R)$ *satisfies the conditions*

$$v'(t) \geqslant \hat{\ell}(v)(t) \quad \text{for } t \in]a, a+\omega[, \quad v(a) \geqslant 0.$$

Remark 2.1. According to Lemma 2.1, the functions γ_x and γ in Theorems 1.1–1.3 are monotone.

Lemma 2.2. *Let* $i, j \in \{1, 2\}$, $i \neq j$, *and inequalities* (1.1) *and* (1.2) *be fulfilled. Then an arbitrary nontrivial* ω*-periodic solution of Eq.* (0.1a) *changes its sign.*

Proof. Assume the contrary. Let there exist a nontrivial ω-periodic solution u of Eq. (0.1a) and let this solution be still nonpositive or still nonnegative. Put

$$u_* = \min\{u(t) : 0 \leqslant t \leqslant \omega\}, \quad u^* = \max\{u(t) : 0 \leqslant t \leqslant \omega\}. \tag{2.1}$$

Without loss of generality we can assume that

$$u_* \geqslant 0, \quad u^* > 0. \tag{2.2}$$

Choose $t_* \in [0, \omega[$ and $t^* \in]t_*, t_* + \omega[$ such that

$$u(t_*) = u_*, \quad u(t^*) = u^*. \tag{2.3}$$

Integrating (0.1a) from t_* to t^* and from t^* to $t_* + \omega$ and taking into account (2.1)–(2.3), we obtain:

$$\begin{aligned} u^* - u_* &= \int_{t_*}^{t^*} [\ell_1(u)(s) - \ell_2(u)(s)]\,\mathrm{d}s \\ &\leqslant u^* \int_{t_*}^{t^*} \ell_1(1)(s)\,\mathrm{d}s \leqslant u^* \|\ell_1(1)\|_{L_\omega}, \end{aligned} \tag{2.4}$$

[1] Under a solution of this problem we understand a function $u \in \tilde{C}([a, a+\omega]; R)$ satisfying the corresponding equation almost everywhere in $]a, a+\omega[$ and the initial condition.

$$u^* - u_* = \int_{t^*}^{t_*+\omega} [\ell_2(u)(s) - \ell_1(u)(s)]\,ds$$

$$\leqslant u^* \int_{t^*}^{t_*+\omega} \ell_2(1)(s)\,ds \leqslant u^* \|\ell_2(1)\|_{L_\omega}. \tag{2.5}$$

On the other hand, the integration of (0.1a) from 0 to ω yields

$$\int_0^\omega \ell_1(u)(s)\,ds = \int_0^\omega \ell_2(u)(s)\,ds.$$

Hence by (2.1) and (2.2) it follows that

$$u_* \|\ell_2(1)\|_{L_\omega} \leqslant u^* \|\ell_1(1)\|_{L_\omega}, \tag{2.6}$$

$$u_* \|\ell_1(1)\|_{L_\omega} \leqslant u^* \|\ell_2(1)\|_{L_\omega}. \tag{2.7}$$

Thus for $i=1$ in view of (1.1), (1.2), (2.4), and (2.6), and for $i=2$ in view of (1.1), (1.2), (2.5), and (2.7), we successively get the contradictions

$$u^* \leqslant u^* \left(\|\ell_1(1)\|_{L_\omega} + \frac{\|\ell_1(1)\|_{L_\omega}}{\|\ell_2(1)\|_{L_\omega}} \right) < u^*,$$

$$u^* \leqslant u^* \left(\|\ell_2(1)\|_{L_\omega} + \frac{\|\ell_2(1)\|_{L_\omega}}{\|\ell_1(1)\|_{L_\omega}} \right) < u^*. \qquad \square$$

Proof of Theorem 1.1. Suppose that conditions (a) are fulfilled (the case where (b) are fulfilled can be proved analogously). According to Theorem 1.1 in [6], it is sufficient to show that homogeneous equation (0.1a) has only a trivial ω-periodic solution. Assume the contrary. Let there exist a nontrivial ω-periodic solution u of Eq. (0.1a). Put

$$m = -\min\{u(t): 0 \leqslant t \leqslant \omega\}, \quad M = \max\{u(t): 0 \leqslant t \leqslant \omega\}. \tag{2.8}$$

By virtue of conditions (1.1), (1.2), and Lemma 2.2, the function u changes its sign. Therefore,

$$m > 0, \quad M > 0, \tag{2.9}$$

and there exists $x \in [0,\omega[$ such that

$$u(x) = 0. \tag{2.10}$$

It is obvious that the function u satisfies also the equality

$$u'(t) = \ell_{1x}(u)(t) - \ell_{2x}(u)(t) \quad \text{for } t \in]x, x+\omega[. \tag{2.11}$$

According to condition (1.3) and Lemma 2.1, for any $g \in L(]x, x+\omega[; R)$ the problem:

$$y'(t) = \ell_{1x}(y)(t) + g(t), \quad y(x) = 0$$

has a unique solution. Denote by α and β respectively the solutions of the problems:

$$\alpha'(t) = \ell_{1x}(\alpha)(t) + \frac{1}{M}\ell_{2x}([u]_+)(t), \quad \alpha(x) = 0, \tag{2.12}$$

$$\beta'(t) = \ell_{1x}(\beta)(t) + \frac{1}{m}\ell_{2x}([u]_-)(t), \quad \beta(x) = 0. \tag{2.13}$$

Since the operators ℓ_{1x} and ℓ_{2x} are nonnegative, by (2.9), (2.12), (2.13), and Lemma 2.1 we have:

$$\alpha(t) \geqslant 0, \quad \beta(t) \geqslant 0 \quad \text{for } t \in [x, x+\omega], \tag{2.14}$$

$$\alpha'(t) \geqslant 0, \quad \beta'(t) \geqslant 0 \quad \text{for } t \in]x, x+\omega[. \tag{2.15}$$

From (1.3) and (2.11)–(2.13) together with the nonnegativeness of the operator ℓ_{2x} and the fact that

$$\frac{1}{M}[u(t)]_+ + \frac{1}{m}[u(t)]_- \leqslant 1 \quad \text{for } t \in [x, x+\omega],$$

it immediately follows that almost everywhere in $]x, x+\omega[$ the inequalities:

$$(m\beta(t) - u(t))' \geqslant \ell_{1x}(m\beta - u)(t), \quad (M\alpha(t) + u(t))' \geqslant \ell_{1x}(M\alpha + u)(t),$$

$$(\gamma_x(t) - \alpha(t) - \beta(t))' \geqslant \ell_{1x}(\gamma_x - \alpha - \beta)(t)$$

are fulfilled. The last inequalities according to Lemma 2.1 result in:

$$(\alpha(t) + \beta(t))' \leqslant \gamma_x'(t) \quad \text{for } t \in]x, x+\omega[, \tag{2.16}$$

$$(u(t) - m\beta(t))' \leqslant 0 \quad \text{for } t \in]x, x+\omega[, \tag{2.17}$$

$$(u(t) + M\alpha(t))' \geqslant 0 \quad \text{for } t \in]x, x+\omega[. \tag{2.18}$$

Choose $t_m,\ t_M \in]x, x+\omega[$ such that

$$u(t_m) = -m, \quad u(t_M) = M. \tag{2.19}$$

Suppose $t_m < t_M$ ($t_m > t_M$). Integrating (2.17) and (2.18) from t_m to t_M (from t_M to t_m) in view of (2.14), (2.15), and (2.19) we obtain:

$$M + m \leqslant m(\beta(t_M) - \beta(t_m)) \leqslant m\beta(x+\omega),$$

$$(M + m \leqslant M(\alpha(t_m) - \alpha(t_M)) \leqslant M\alpha(x+\omega)). \tag{2.20}$$

On the other hand, if we integrate (2.18) and (2.17) from x to t_m and from t_M to $x+\omega$ (from x to t_M and from t_m to $x+\omega$), then we get:

$$m + u(x) \leqslant M\alpha(t_m), \quad M - u(x+\omega) \leqslant M(\alpha(x+\omega) - \alpha(t_M)),$$

$$(M - u(x) \leqslant m\beta(t_M), \quad m + u(x+\omega) \leqslant m(\beta(x+\omega) - \beta(t_m))).$$

Summing the last two inequalities and taking into account (2.15), we find:

$$M + m \leqslant M\alpha(x+\omega) \quad (M + m \leqslant m\beta(x+\omega)). \tag{2.21}$$

Thus from (2.20) and (2.21) it immediately follows:

$$4 \leqslant 2 + \frac{m}{M} + \frac{M}{m} \leqslant \alpha(x+\omega) + \beta(x+\omega). \tag{2.22}$$

However, according to (1.4) and (2.16):

$$\alpha(x+\omega) + \beta(x+\omega) \leqslant \int_x^{x+\omega} \gamma_x'(s)\,\mathrm{d}s < \gamma_x(x+\omega) \leqslant 4,$$

whence in view of (2.22) we obtain a contradiction. □

Proof of Corollary 1.1. (a) According to (1.7) we can choose $\varepsilon > 0$ such that

$$\sum_{k=2}^{n} \int_0^\omega |p_k(s)| \exp\left(\int_s^\omega p_1(\xi)\,\mathrm{d}\xi\right) \mathrm{d}s \leqslant 4 - \varepsilon \exp\left(\int_0^\omega p_1(\xi)\,\mathrm{d}\xi\right).$$

Let $x \in [0,\omega[$ and put for $t \in [x, x+\omega]$

$$\gamma_x(t) = \varepsilon \exp\left(\int_x^t p_1(s)\,\mathrm{d}s\right) + \sum_{k=2}^{n} \int_x^t |p_k(s)| \exp\left(\int_s^t p_1(\xi)\,\mathrm{d}\xi\right) \mathrm{d}s.$$

It is obvious that

$$\gamma_x'(t) = p_1(t)\gamma_x(t) + \sum_{k=2}^{n} |p_k(t)| \quad \text{for } t \in\,]x, x+\omega[$$

and $\gamma_x(x+\omega) \leqslant 4$. Consequently, the conditions (a) of Theorem 1.1 are fulfilled with:

$$\ell_1(v)(t) \stackrel{def}{=} p_1(t)v(t), \quad \ell_2(v)(t) \stackrel{def}{=} \sum_{k=2}^{n} |p_k(t)| v(\tau_k(t))$$

(b) According to (1.8) and (1.9), we can choose $\varepsilon > 0$ such that

$$\varepsilon \leqslant 4\left(1 - \int_0^\omega p_1(s)\,\mathrm{d}s\right) - \sum_{k=2}^{n} \int_0^\omega |p_k(s)|\,\mathrm{d}s$$

$$\left(\varepsilon \leqslant 4\left(1 - \sum_{k=2}^{n} \int_0^\omega |p_k(s)|\,\mathrm{d}s\right) - \int_0^\omega p_1(s)\,\mathrm{d}s\right).$$

Let $x \in [0,\omega[$ and put:

$$\gamma_x(t) = \varepsilon + 4\int_x^t p_1(s)\,\mathrm{d}s + \sum_{k=2}^{n} \int_x^t |p_k(s)|\,\mathrm{d}s \quad \text{for } t \in [x, x+\omega],$$

$$\left(\gamma_x(t) = \varepsilon + 4\sum_{k=2}^{n} \int_t^{x+\omega} |p_k(s)|\,\mathrm{d}s + \int_t^{x+\omega} p_1(s)\,\mathrm{d}s \quad \text{for } t \in [x, x+\omega]\right).$$

It is clear that $\gamma_x(x+\omega) \leqslant 4$ ($\gamma_x(x) \leqslant 4$) and

$$\gamma_x'(t) = 4p_1(t) + \sum_{k=2}^{n} |p_k(t)| \quad \text{for } t \in]x, x+\omega[$$

$$\left(\gamma_x'(t) = -4\sum_{k=2}^{n} |p_k(t)| - p_1(t) \quad \text{for } t \in]x, x+\omega[\right).$$

Since γ_x is nondecreasing (nonincreasing), from the last equality we obtain:

$$\gamma_x'(t) \geqslant p_1(t)\gamma_x(\tau_{1x}(t)) + \sum_{k=2}^{n} |p_k(t)| \quad \text{for } t \in]x, x+\omega[$$

$$\left(-\gamma_x'(t) \geqslant \sum_{k=2}^{n} |p_k(t)|\gamma_x(\tau_{kx}(t)) + p_1(t) \quad \text{for } t \in]x, x+\omega[\right).$$

Consequently, conditions (a) (conditions (b)) of Theorem 1.1 are fulfilled with:

$$\ell_1(v)(t) \overset{def}{=} p_1(t)v(\tau_1(t)), \quad \ell_2(v)(t) \overset{def}{=} \sum_{k=2}^{n} |p_k(t)|v(\tau_k(t)).$$

(c) According to (1.10), we can choose $\varepsilon > 0$ such that

$$\sum_{k=2}^{n} \int_0^{\omega} p_k(s) \exp\left(\int_0^s |p_1(\xi)| \, d\xi \right) ds \leqslant 4 - \varepsilon \exp\left(\int_0^{\omega} |p_1(\xi)| \, d\xi \right).$$

Let $x \in [0, \omega[$ and put for $t \in [x, x+\omega]$:

$$\gamma_x(t) = \varepsilon \exp\left(\int_t^{x+\omega} |p_1(s)| \, ds \right) + \sum_{k=2}^{n} \int_t^{x+\omega} p_k(s) \exp\left(\int_t^s |p_1(\xi)| \, d\xi \right) ds.$$

Obviously, $\gamma_x(x) \leqslant 4$ and

$$\gamma_x'(t) = -|p_1(t)|\gamma_x(t) - \sum_{k=2}^{n} p_k(t) \quad \text{for } t \in]x, x+\omega[.$$

Consequently, conditions (b) of Theorem 1.1 are fulfilled with

$$\ell_1(v)(t) \overset{def}{=} \sum_{k=2}^{n} p_k(t)v(\tau_k(t)), \quad \ell_2(v)(t) \overset{def}{=} |p_1(t)|v(t).$$

The case (d) can be proved analogously to (b). □

Proof of Theorem 1.2. Let conditions (a) of Theorem 1.2 be fulfilled. By Theorem 1.1 in [6], it is sufficient to show that homogeneous equation (0.1a) has only a trivial ω-periodic solution. Assume the contrary. Let there exist a nontrivial ω-periodic solution u of Eq. (0.1a). Without loss of generality we can assume that

$$u(0) \leqslant 0. \tag{2.23}$$

It is evident that the function u satisfies also the equality:

$$u'(t) = \ell_{10}(u)(t) - \ell_{20}(u)(t) \quad \text{for } t \in]0, \omega[. \tag{2.24}$$

Define M and m by equalities (2.8) and choose t_m, $t_M \in [0, \omega[$ such that equalities (2.19) are fulfilled. By virtue of conditions (1.1), (1.2), and Lemma 2.2, inequalities

(2.9) hold. From (1.11) and (2.24) in view of (2.8), (2.9), and the nonnegativeness of the operator ℓ_{20} it immediately follows that

$$
\begin{aligned}
(m\gamma(t) - u(t))' &\geqslant \ell_{10}(m\gamma - u)(t) + \ell_{20}(m+u)(t) \\
&\geqslant \ell_{10}(m\gamma - u)(t) \quad \text{for } t \in \,]0,\omega[.
\end{aligned}
$$

Hence, according to Lemma 2.1 and conditions (2.9) and (2.23), we get:

$$u'(t) \leqslant m\gamma'(t) \quad \text{for } t \in \,]0,\omega[. \tag{2.25}$$

Suppose that $t_m < t_M$. Integrating (2.25) from t_m to t_M and taking into account the fact that the function γ is nondecreasing (see Remark 2.1) and satisfies condition (1.12), we obtain the contradiction:

$$M + m \leqslant m(\gamma(t_M) - \gamma(t_m)) \leqslant m(\gamma(\omega) - \gamma(0)) < m. \tag{2.26}$$

Suppose now that $t_m > t_M$. Then the integration of (2.25) from 0 to t_M and from t_m to ω, respectively, yields:

$$M - u(0) \leqslant m(\gamma(t_M) - \gamma(0)), \quad m + u(\omega) \leqslant m(\gamma(\omega) - \gamma(t_m)).$$

Summing the last two inequalities and taking into account the fact that the function γ is nondecreasing and satisfies condition (1.12), we obtain contradiction (2.26). □

Proof of Corollary 1.2. Put

$$
\begin{aligned}
&\ell(v)(t) \overset{def}{=} \sum_{k=1}^{n} p_k(t) v(\tau_k(t)), \\
&\ell_{10}(v)(t) \overset{def}{=} \sum_{k=1}^{n} [p_k(t)]_+ v(\tau_{k0}(t)), \quad \ell_{20}(v)(t) \overset{def}{=} \sum_{k=1}^{n} [p_k(t)]_- v(\tau_{k0}(t)).
\end{aligned}
$$

(a) Due to (1.15), we can choose $\varepsilon > 0$ such that

$$
\begin{aligned}
&\sum_{k=1}^{n} \int_0^{\omega} [p_k(s)]_- \exp\left(\sum_{i=1}^{n} \int_s^{\omega} [p_i(\xi)]_+ \, d\xi \right) ds \\
&\qquad \leqslant 1 - \varepsilon \exp\left(\sum_{k=1}^{n} \int_0^{\omega} [p_k(\xi)]_+ \, d\xi \right).
\end{aligned}
$$

Put for $t \in [0,\omega]$:

$$\gamma(t) = \varepsilon \exp\left(\sum_{k=1}^{n} \int_0^{t} [p_k(s)]_+ \, ds \right) + \sum_{k=1}^{n} \int_0^{t} [p_k(s)]_- \exp\left(\sum_{i=1}^{n} \int_s^{t} [p_i(\xi)]_+ \, d\xi \right) ds.$$

It is obvious that $\gamma(\omega) \leqslant 1$ and

$$\gamma'(t) = \sum_{k=1}^{n} [p_k(t)]_+ \gamma(t) + \sum_{k=1}^{n} [p_k(t)]_- \quad \text{for } t \in \,]0,\omega[.$$

Since γ is nondecreasing, from the last inequality in view of (1.16) it immediately follows that inequality (1.11) is fulfilled.

(c) Put

$$\gamma(t) = \varepsilon \exp\left(e\sum_{j=1}^{n} \int_0^t p_j(s)\,\mathrm{d}s \right) \quad \text{for } t \in [0,\omega],$$

where $\varepsilon > 0$ is such that $\gamma(\omega) \leqslant 1$. Clearly,

$$\gamma'(t) = e\sum_{k=1}^{n} p_k(t)\gamma(t) = \sum_{k=1}^{n} p_k(t)\gamma(\tau_{k0}(t)) \exp\left(1 + e\sum_{j=1}^{n} \int_{\tau_{k0}(t)}^t p_j(s)\,\mathrm{d}s \right).$$

Hence together with (1.17) we get

$$\gamma'(t) \geqslant \sum_{k=1}^{n} p_k(t)\gamma(\tau_{k0}(t)) \quad \text{for } t \in]0,\omega[.$$

Consequently, conditions (a) of Theorem 1.2 are fulfilled.

(e) By (1.18), we have $\beta < 1$, where β is defined by (1.20). In view of this and according to Corollary 1.1 (iii) in [2], for any $g \in L(]0,\omega[;R)$ and $c \in R$ the problem:

$$u'(t) = \sum_{k=1}^{n} p_k(t)u(\tau_{k0}(t)) + g(t), \quad u(0) = c$$

is uniquely solvable and, moreover, the inequalities:

$$v(t) \geqslant 0, \quad v'(t) \geqslant 0 \quad \text{for } t \in]0,\omega[$$

hold whenever the function $v \in \tilde{C}([0,\omega];R)$ satisfies the inequalities:

$$v'(t) \geqslant \sum_{k=1}^{n} [p_k(t)]_+ v(\tau_{k0}(t)) \quad \text{for } t \in]0,\omega[, \quad v(0) \geqslant 0.$$

Choose $\delta > 0$ and $\varepsilon > 0$ such that

$$(1-\beta)^{-1}\left(\sum_{k=1}^{n} \int_0^{\omega} [p_k(s)]_-\,\mathrm{d}s + \alpha \right) \leqslant 1 - \delta, \tag{2.27}$$

$$\varepsilon \leqslant \delta(1-\beta)\exp\left(-\sum_{k=1}^{n} \int_0^{\omega} [p_k(s)]_+\,\mathrm{d}s \right). \tag{2.28}$$

Denote by γ the solution of the Cauchy problem:

$$u'(t) = \sum_{k=1}^{n} [p_k(t)]_+ u(\tau_{k0}(t)) + \sum_{k=1}^{n} [p_k(t)]_-, \quad u(0) = \varepsilon.$$

As said above, the function γ is nondecreasing. Obviously, γ is also a solution of the equation:

$$u'(t) = \sum_{k=1}^{n} [p_k(t)]_+ u(t) + \sum_{k=1}^{n} [p_k(t)]_+ \sum_{i=1}^{n} \int_t^{\tau_{k0}(t)} [p_i(s)]_+ \gamma(\tau_{i0}(s))\,\mathrm{d}s$$

$$+ \sum_{k=1}^{n} [p_k(t)]_+ \sum_{i=1}^{n} \int_t^{\tau_{k0}(t)} [p_i(s)]_-\,\mathrm{d}s + \sum_{k=1}^{n} [p_k(t)]_-.$$

Therefore, according to the Cauchy formula:

$$\gamma(\omega) \leqslant \beta\gamma(\omega) + \sum_{k=1}^{n} \int_0^{\omega} [p_k(s)]_- \, ds + \alpha + \varepsilon \exp\left(\sum_{k=1}^{n} \int_0^{\omega} [p_k(s)]_- \, ds \right).$$

The last inequality together with (2.27) and (2.28) results in $\gamma(\omega) \leqslant 1$. Consequently, conditions (a) of Theorem 1.2 are fulfilled.

The cases (b), (d), and (f) can be proved analogously. □

Proof of Theorem 1.3. Assume the contrary. Let the dimension of ω-periodic solution space of Eq. (0.1a) is greater than one. Then there exists a nontrivial ω-periodic solution u of Eq. (0.1a) such that $u(0) = u(\omega) = 0$. By virtue of Lemma 2.1, it is clear that u changes sign. In the same way as in the proof of Theorem 1.1 (for the case $x = 0$), we obtain the contradiction $u \equiv 0$. □

The proof of Corollary 1.3 is analogous to that of Corollary 1.2.

Acknowledgements

This work was supported by the Grant No. 201/99/0295 of the Grant Agency of the Czech Republic.

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