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On a periodic-type boundary value problem for first-order nonlinear functional differential equations

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1. Introduction

The following notation is used throughou.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b];R)$ is the Banach space of continuous functions $u:[a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)|: a \leqslant t \leqslant b\}$.

$C([a,b];R_+) = \{u \in C([a,b];R): u(t) \geqslant 0 \text{ for } t \in [a,b]\}$.

$\tilde{C}([a,b];R)$ is the set of absolutely continuous functions $u:[a,b] \to R$.

$B^i_{\lambda c}([a,b];R) = \{u \in C([a,b];R) : (-1)^{i+1}(u(a) - \lambda u(b))\,\mathrm{sgn}((2-i)u(a) + (i-1)u(b)) \leqslant c\}$, where $c \in R$, $i = 1, 2$.

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p:[a,b] \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|\,\mathrm{d}s$.

$L([a,b];R_+) = \{p \in L([a,b];R): p(t) \geqslant 0 \text{ for almost all } t \in [a,b]\}$.

$\mathscr{M}_{ab}$ is the set of measurable functions $\tau:[a,b] \to [a,b]$.

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$\tilde{\mathscr{L}}_{ab}$ is the set of linear operators $\ell : C([a,b];R) \to L([a,b];R)$ for which there is a function $\eta \in L([a,b];R_+)$ such that

$$|\ell(v)(t)| \leqslant \eta(t)\|v\|_C \quad \text{for } t \in [a,b], \ v \in C([a,b];R).$$

$\mathscr{P}_{ab}$ is the set of linear operators $\ell \in \tilde{\mathscr{L}}_{ab}$ transforming the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$.

K_{ab} is the set of continuous operators $F : C([a,b];R) \to L([a,b];R)$ satisfying the Carathèodory conditions, i.e., for every $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|F(v)(t)| \leqslant q_r(t) \quad \text{for } t \in [a,b], \quad \|v\|_C \leqslant r.$$

$K([a,b] \times A; B)$, where $A \subseteq R^2$, $B \subseteq R$, is the set of functions $f : [a,b] \times A \to B$ satisfying the Carathèodory conditions, i.e., $f(\cdot,x) : [a,b] \to B$ is a measurable function for all $x \in A$, $f(t,\cdot) : A \to B$ is a continuous function for almost all $t \in [a,b]$, and for every $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|f(t,x)| \leqslant q_r(t) \quad \text{for } t \in [a,b], \ x \in A, \ \|x\| \leqslant r.$$

$[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

By a solution of the equation

$$u'(t) = F(u)(t), \tag{1}$$

where $F \in K_{ab}$, we understand a function $u \in \tilde{C}([a,b];R)$ satisfying Eq. (1) almost everywhere in $[a,b]$.

Consider the problem on the existence and uniqueness of a solution of (1) satisfying the boundary condition

$$u(a) - \lambda u(b) = h(u), \tag{2}$$

where $\lambda \in R_+$, and $h : C([a,b];R) \to R$ is a continuous functional.

The general boundary value problems for functional differential equations have been studied very intensively. There are a lot of interesting general results (see, e.g., [1–27] and the references therein), but still only a few effective criteria for the solvability of special boundary value problems for functional differential equations are known even in the linear case. In the present paper, we try to fill to some extent the existing gap. More precisely, in Section 2 there are established non-improvable effective sufficient conditions for the solvability and unique solvability of the problem (1), (2). Sections 3, 4 and 5 are devoted, respectively, to the auxiliary propositions, the proofs of the main results and the examples verifying their optimality.

All results will be concretized for the differential equation with deviating arguments of the form

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + f(t, u(t), u(\nu(t))), \tag{3}$$

where $p, g \in L([a,b];R_+)$, $\tau, \mu, \nu \in \mathscr{M}_{ab}$, and $f \in K([a,b] \times R^2; R)$.

The special cases of the discussed boundary value problem are the Cauchy problem (for $\lambda = 0$ and $h \equiv$ Const.) and the periodic boundary value problem (for $\lambda = 1$ and $h \equiv$ Const.). In these cases, the theorems below coincide with the results obtained in [5,10]. The antiperiodic-type boundary value problem (i.e., the case $\lambda < 0$) for the linear equation and for the nonlinear one is studied, respectively, in [13] and [15].

From the general theory of linear boundary value problems for functional differential equations we need the following well-known result (see, e.g., [3,19,27]).

Theorem 1.1. *Let $\ell \in \tilde{\mathscr{L}}_{ab}$. Then the problem*

$$u'(t) = \ell(u)(t) + q_0(t), \quad u(a) - \lambda u(b) = c_0, \tag{4}$$

where $q_0 \in L([a,b];R)$, $c_0 \in R$, is uniquely solvable if and only if the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \tag{1_0}$$

$$u(a) - \lambda u(b) = 0 \tag{2_0}$$

has only the trivial solution.

Remark 1.1. From the Riesz–Schauder theory it follows that if $\ell \in \tilde{\mathscr{L}}_{ab}$ and problems (1_0) and (2_0) has a non-trivial solution, then there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that problem (4) has no solution.

2. Main results

Throughout the paper we assume that $q \in K([a,b] \times R_+; R_+)$ is non-decreasing in the second argument, and satisfies

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)\,\mathrm{d}s = 0. \tag{5}$$

Theorem 2.1. *Let $\lambda \in]0,1]$, $c \in R_+$,*

$$h(v)\operatorname{sgn} v(a) \leqslant c \quad \textit{for } v \in C([a,b];R) \tag{6}$$

and let there exist

$$\ell_0, \ell_1 \in \mathscr{P}_{ab} \tag{7}$$

such that on the set $B^1_{\lambda c}([a,b);R)$ the inequality

$$[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)]\operatorname{sgn} v(t) \leqslant q(t, \|v\|_C) \quad \textit{for } t \in [a,b] \tag{8}$$

holds. If, moreover,

$$\|\ell_0(1)\|_L < 1, \tag{9}$$

$$\frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{1-\lambda}{\lambda} < \|\ell_1(1)\|_L < 2\sqrt{1 - \|\ell_0(1)\|_L}, \tag{10}$$

then problem (1), (2) has at least one solution.

Remark 2.1. Theorem 1.1 is non-improvable in a certain sense. More precisely, the first inequality in (10) cannot be replaced by the non-strict one, and the second inequality in (10) cannot be replaced by

$$\|\ell_1(1)\|_L \leqslant \varepsilon + 2\sqrt{1 - \|\ell_0(1)\|_L}$$

no matter how small $\varepsilon > 0$ would be (see Examples 5.1–5.3).

Theorem 2.2. *Let* $\lambda \in]0,1]$, $c \in R_+$,

$$h(v)\operatorname{sgn} v(b) \geqslant -c \quad \textit{for } v \in C([a,b];R), \tag{11}$$

and let there exist $\ell_0, \ell_1 \in \mathscr{P}_{ab}$ *such that on the set* $B^2_{\lambda c}([a,b];R)$ *the inequality*

$$[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)]\operatorname{sgn} v(t) \geqslant -q(t, \|v\|_C) \quad \textit{for } t \in [a,b] \tag{12}$$

holds. If, moreover,

$$\|\ell_1(1)\|_L < \lambda, \tag{13}$$

$$\frac{1}{\lambda - \|\ell_1(1)\|_L} - 1 < \|\ell_0(1)\|_L < 2\sqrt{\lambda - \|\ell_1(1)\|_L}, \tag{14}$$

then problem (1), (2) *has at least one solution.*

Remark 2.2. Theorem 2.2 is non-improvable in a certain sense. More precisely, the first inequality in (14) cannot be replaced by the non-strict one, and the second inequality in (14) cannot be replaced by

$$\|\ell_0(1)\|_L \leqslant \varepsilon + 2\sqrt{\lambda - \|\ell_1(1)\|_L}$$

no matter how small $\varepsilon > 0$ would be (see Examples 5.4–5.6).

Remark 2.3. Let $\lambda \in [1, +\infty[$. Define operator $\psi : L([a,b];R) \to L([a,b];R)$ by

$$\psi(w)(t) \stackrel{\text{def}}{=} w(a+b-t) \quad \text{for } t \in [a,b].$$

Let φ be a restriction of ψ to the space $C([a,b];R)$. Put $\vartheta = 1/\lambda$, and

$$\hat{F}(w)(t) \stackrel{\text{def}}{=} -\psi(F(\varphi(w)))(t), \quad \hat{h}(w) \stackrel{\text{def}}{=} -\vartheta h(\varphi(w)).$$

It is clear that if u is a solution of problem (1), (2), then the function $v \stackrel{\text{def}}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \hat{F}(v)(t), \quad v(a) - \vartheta v(b) = \hat{h}(v) \tag{15}$$

and vice versa, if v is a solution of problem (15), then the function $u \stackrel{\text{def}}{=} \varphi(v)$ is a solution of problem (1), (2).

Therefore, from Theorems 2.1 and 2.2 it immediately follows

Theorem 2.3. *Let* $\lambda \in [1, +\infty[$, $c \in R_+$, *condition* (11) *be fulfilled, and there exist* $\ell_0, \ell_1 \in \mathscr{P}_{ab}$ *such that on the set* $B^2_{\lambda c}([a,b];R)$ *inequality* (12) *holds. Let, moreover,*

$$\|\ell_1(1)\|_L < 1, \tag{16}$$

$$\frac{\|\ell_1(1)\|_L}{1 - \|\ell_1(1)\|_L} + 1 - \lambda < \|\ell_0(1)\|_L < 2\sqrt{1 - \|\ell_1(1)\|_L}. \tag{17}$$

Then, problem (1), (2) *has at least one solution.*

Theorem 2.4. *Let* $\lambda \in [1, +\infty[$, $c \in R_+$, *condition* (6) *be fulfilled, and there exist* $\ell_0, \ell_1 \in \mathscr{P}_{ab}$ *such that on the set* $B^1_{\lambda c}([a,b];R)$ *inequality* (8) *holds. Let, moreover,*

$$\|\ell_0(1)\|_L < \frac{1}{\lambda}, \tag{18}$$

$$\frac{1}{1/\lambda - \|\ell_0(1)\|_L} - 1 < \|\ell_1(1)\|_L < 2\sqrt{\frac{1}{\lambda} - \|\ell_0(1)\|_L}. \tag{19}$$

Then, problem (1), (2) *has at least one solution.*

Remark 2.4. On account of Remarks 2.1–2.3 it is clear that Theorems 2.3 and 2.4 are also non-improvable.

Next we establish theorems on the unique solvability of problem (1), (2).

Theorem 2.5. *Let* $\lambda \in]0,1]$,

$$[h(v) - h(w)]\operatorname{sgn}(v(a) - w(a)) \leqslant 0 \quad \textit{for } v, w \in C([a,b];R) \tag{20}$$

and let there exist $\ell_0, \ell_1 \in \mathscr{P}_{ab}$ *such that on the set* $B^1_{\lambda c}([a,b];R)$, *where* $c = |h(0)|$, *the inequality*

$$[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)]\operatorname{sgn}(v(t) - w(t)) \leqslant 0 \tag{21}$$

holds. Let, moreover, (9) and (10) *be fulfilled. Then, problem* (1), (2) *is uniquely solvable.*

Theorem 2.6. *Let* $\lambda \in]0,1]$,

$$[h(v) - h(w)]\operatorname{sgn}(v(b) - w(b)) \geqslant 0 \quad \textit{for } v, w \in C([a,b];R) \tag{22}$$

and let there exist $\ell_0, \ell_1 \in \mathscr{P}_{ab}$ *such that on the set* $B^2_{\lambda c}([a,b];R)$, *where* $c = |h(0)|$, *the inequality*

$$[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)]\operatorname{sgn}(v(t) - w(t)) \geqslant 0 \tag{23}$$

holds. Let, moreover, (13) *and* (14) *be fulfilled. Then, problem* (1), (2) *is uniquely solvable.*

According to Remark 2.3, from Theorems 2.5 and 2.6 we have

Theorem 2.7. *Let $\lambda \in [1, +\infty[$, condition* (22) *be satisfied, and there exist $\ell_0, \ell_1 \in \mathscr{P}_{ab}$ such that on the set $B^2_{\lambda c}([a,b]; R)$, where $c = |h(0)|$, inequality* (23) *holds. Let, moreover,* (16) *and* (17) *be fulfilled. Then, problem* (1), (2) *is uniquely solvable.*

Theorem 2.8. *Let $\lambda \in [1, +\infty[$, condition* (20) *be satisfied, and there exist $\ell_0, \ell_1 \in \mathscr{P}_{ab}$ such that on the set $B^1_{\lambda c}([a,b]; R)$, where $c = |h(0)|$, inequality* (21) *holds. Let, moreover,* (18) *and* (19) *be fulfilled. Then, problem* (1), (2) *is uniquely solvable.*

Remark 2.5. Theorems 2.5–2.8 are non-improvable in a certain sense (see Examples 5.1–5.6).

For the equation of type (3) from Theorems 2.1–2.8 we get the following assertions.

Corollary 2.1. *Let $\lambda \in]0,1]$, $c \in R_+$, condition* (6) *be fulfilled, and*

$$f(t,x,y)\,\mathrm{sgn}\,x \leqslant q(t) \quad \mathit{for}\ t \in [a,b],\ \ x,y \in R, \tag{24}$$

where $q \in L([a,b]; R_+)$. Let, moreover,

$$\int_a^b p(s)\,\mathrm{d}s < 1, \tag{25}$$

$$\frac{\int_a^b p(s)\,\mathrm{d}s}{1 - \int_a^b p(s)\,\mathrm{d}s} - \frac{1-\lambda}{\lambda} < \int_a^b g(s)\,\mathrm{d}s < 2\sqrt{1 - \int_a^b p(s)\,\mathrm{d}s}. \tag{26}$$

Then, problem (3), (2) *has at least one solution.*

Corollary 2.2. *Let $\lambda \in]0,1]$, $c \in R_+$, condition* (11) *be fulfilled, and*

$$f(t,x,y)\,\mathrm{sgn}\,x \geqslant -q(t) \quad \mathit{for}\ t \in [a,b],\ \ x,y \in R, \tag{27}$$

where $q \in L([a,b]; R_+)$. Let, moreover,

$$\int_a^b g(s)\,\mathrm{d}s < \lambda, \tag{28}$$

$$\frac{1}{\lambda - \int_a^b g(s)\,\mathrm{d}s} - 1 < \int_a^b p(s)\,\mathrm{d}s < 2\sqrt{\lambda - \int_a^b g(s)\,\mathrm{d}s}. \tag{29}$$

Then, problem (3), (2) *has at least one solution.*

Corollary 2.3. *Let $\lambda \in [1, +\infty[$, $c \in R_+$, and conditions* (11) *and* (27), *where $q \in L([a,b]; R_+)$, be fulfilled. Let, moreover,*

$$\int_a^b g(s)\,\mathrm{d}s < 1, \tag{30}$$

$$\frac{\int_a^b g(s)\,\mathrm{d}s}{1-\int_a^b g(s)\,\mathrm{d}s}+1-\lambda<\int_a^b p(s)\,\mathrm{d}s<2\sqrt{1-\int_a^b g(s)\,\mathrm{d}s}. \tag{31}$$

Then, problem (3), (2) *has at least one solution.*

Corollary 2.4. *Let* $\lambda\in[1,+\infty[$, $c\in R_+$, *and conditions* (6) *and* (24), *where* $q\in L([a,b];R_+)$, *be fulfilled. Let, moreover,*

$$\int_a^b p(s)\,\mathrm{d}s<\frac{1}{\lambda}, \tag{32}$$

$$\frac{1}{1/\lambda-\int_a^b p(s)\,\mathrm{d}s}-1<\int_a^b g(s)\,\mathrm{d}s<2\sqrt{\frac{1}{\lambda}-\int_a^b p(s)\,\mathrm{d}s}. \tag{33}$$

Then, problem (3), (2) *has at least one solution.*

Corollary 2.5. *Let* $\lambda\in]0,1]$, *conditions* (20) *and*

$$[f(t,x_1,y_1)-f(t,x_2,y_2)]\,\mathrm{sgn}\,(x_1-x_2)\leqslant 0 \quad \text{for } t\in[a,b],\ x_1,x_2,y_1,y_2\in R \tag{34}$$

be fulfilled. Let, moreover, (25) *and* (26) *hold. Then, problem* (3), (2) *is uniquely solvable.*

Corollary 2.6. *Let* $\lambda\in]0,1]$, *conditions* (22) *and*

$$[f(t,x_1,y_1)-f(t,x_2,y_2)]\,\mathrm{sgn}\,(x_1-x_2)\geqslant 0 \quad \text{for } t\in[a,b],\ x_1,x_2,y_1,y_2\in R \tag{35}$$

be fullfiled. Let, moreover, (28) *and* (29) *hold. Then, problem* (3), (2) *is uniquely solvable.*

Corollary 2.7. *Let* $\lambda\in[1,+\infty[$, *and conditions* (22), (30), (31) *and* (35) *hold. Then, problem* (3), (2) *is uniquely solvable.*

Corollary 2.8. *Let* $\lambda\in[1,+\infty[$, *and conditions* (20), (32), (33), *and* (34) *hold. Then, problem* (3), (2) *is uniquely solvable.*

3. Auxiliary propositions

First, we formulate the result from [22, Theorem 1] in a suitable form for us.

Lemma 3.1. *Let there exist a positive number* ρ *and an operator* $\ell\in\tilde{\mathscr{L}}_{ab}$ *such that homogeneous problem* (1_0), (2_0) *has only the trivial solution, and let for every*

$\delta \in]0,1[$ and for an arbitrary function $u \in \tilde{C}([a,b];R)$ satisfying

$$u'(t) = \ell(u)(t) + \delta[F(u)(t) - \ell(u)(t)], \quad u(a) - \lambda u(b) = \delta h(u), \tag{36}$$

the estimate

$$\|u\|_C \leqslant \rho \tag{37}$$

hold. Then, problem (1), (2) *has at least one solution.*

Definition 3.1. We say that the operator $\ell \in \tilde{\mathscr{L}}_{ab}$ belongs to the set $U_i(\lambda)$, $i \in \{1,2\}$, if there exists a positive number r such that for any $q^* \in L([a,b];R_+)$ and $c \in R_+$, every function $u \in \tilde{C}([a,b];R)$, satisfying the inequalities

$$(-1)^{i+1}[u(a) - \lambda u(b)] \operatorname{sgn}((2-i)u(a) + (i-1)u(b)) \leqslant c, \tag{38}$$

$$(-1)^{i+1}[u'(t) - \ell(u)(t)] \operatorname{sgn} u(t) \leqslant q^*(t) \quad \text{for } t \in [a,b], \tag{39}$$

admits the estimate

$$\|u\|_C \leqslant r(c + \|q^*\|_L). \tag{40}$$

Lemma 3.2. *Let $i \in \{1,2\}$, $c \in R_+$,*

$$(-1)^{i+1} h(v) \operatorname{sgn}((2-i)v(a) + (i-1)v(b)) \leqslant c \quad \textit{for } v \in C([a,b];R) \tag{41}$$

and let there exist $\ell \in U_i(\lambda)$ such that on the set $B^i_{\lambda c}([a,b];R)$ the inequality

$$(-1)^{i+1}[F(v)(t) - \ell(v)(t)] \operatorname{sgn} v(t) \leqslant q(t, \|v\|_C) \quad \textit{for } t \in [a,b] \tag{42}$$

is fulfilled. Then, problem (1), (2) *has at least one solution.*

Proof. First note that due to the condition $\ell \in U_i(\lambda)$, homogeneous problem (1_0), (2_0) has only the trivial solution.

Let r be the number appearing in Definition 3.1. According to (5) there exists $\rho > 2rc$ such that

$$\frac{1}{x} \int_a^b q(s,x)\,\mathrm{d}s < \frac{1}{2r} \quad \text{for } x > \rho.$$

Now assume that a function $u \in \tilde{C}([a,b];R)$ satisfies (36) for some $\delta \in]0,1[$. Then according to (41), u satisfies inequality (38), i.e., $u \in B^i_{\lambda c}([a,b];R)$. By (42) we obtain that inequality (39) is fulfilled for $q^*(t) = q(t, \|u\|_C)$. Hence by the condition $\ell \in U_i(\lambda)$ and the definition of the number ρ we get estimate (37).

Since ρ depends neither on u nor on δ, from Lemma 3.1 it follows that problem (1), (2) has at least one solution. □

Lemma 3.3. *Let $i \in \{1,2\}$,*

$$\begin{aligned}(-1)^{i+1}&[h(u_1) - h(u_2)] \operatorname{sgn}((2-i)(u_1(a) - u_2(a)) \\ &+ (i-1)(u_1(b) - u_2(b))) \leqslant 0 \quad \textit{for } u_1, u_2 \in C([a,b];R)\end{aligned} \tag{43}$$

and let there exist $\ell \in U_i(\lambda)$ *such that on the set* $B^i_{\lambda c}([a,b];R)$, *where* $c = |h(0)|$, *the inequality*

$$(-1)^{i+1}[F(u_1)(t) - F(u_2)(t) - \ell(u_1 - u_2)(t)]\,\mathrm{sgn}(u_1(t) - u_2(t)) \leqslant 0 \tag{44}$$

holds. Then problem (1), (2) *is uniquely solvable.*

Proof. From (43) it follows that condition (41) is fulfilled, where $c = |h(0)|$. By (44) we get that on the set $B^i_{\lambda c}([a,b];R)$ inequality (42) holds, where $q \equiv |F(0)|$. Consequently, all the assumptions of Lemma 3.2 are fulfilled and this guarantees that problem (1), (2) has at least one solution. It remains to show that problem (1), (2) has at most one solution.

Let u_1, u_2 be arbitrary solutions of problem (1), (2). Put $u(t) = u_1(t) - u_2(t)$ for $t \in [a,b]$. Then, by (43) and (44) we get

$$(-1)^{i+1}[u(a) - \lambda u(b)]\,\mathrm{sgn}((2-i)u(a) + (i-1)u(b)) \leqslant 0,$$

$$(-1)^{i+1}[u'(t) - \ell(u)(t)]\,\mathrm{sgn}\,u(t) \leqslant 0, \quad \text{for } t \in [a,b].$$

This together with the condition $\ell \in U_i(\lambda)$ results in $u \equiv 0$. Consequently, $u_1 \equiv u_2$. □

Lemma 3.4. *Let* $\lambda \in]0,1]$, *the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$, *where* ℓ_0 *and* ℓ_1 *satisfy conditions* (7), (9) *and* (10). *Then* ℓ *belongs to the set* $U_1(\lambda)$.

Proof. Let $q^* \in L([a,b];R_+)$, $c \in R_+$ and $u \in \tilde{C}([a,b];R)$ satisfy (38) and (39) for $i = 1$. We show that (40) holds, where

$$r = \frac{\lambda\|\ell_1(1)\|_L + 2 - \lambda}{(1 - \|\ell_0(1)\|_L)(\lambda\|\ell_1(1)\|_L + 1 - \lambda) - \lambda\|\ell_0(1)\|_L} + \frac{\|\ell_1(1)\|_L + 1}{1 - \|\ell_0(1)\|_L - \frac{1}{4}\|\ell_1(1)\|_L^2}. \tag{45}$$

It is clear that

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \tilde{q}(t), \tag{46}$$

where

$$\tilde{q}(t) = u'(t) - \ell(u)(t) \quad \text{for } t \in [a,b]. \tag{47}$$

Obviously,

$$\tilde{q}(t)\,\mathrm{sgn}\,u(t) \leqslant q^*(t) \quad \text{for } t \in [a,b], \tag{48}$$

and

$$[u(a) - \lambda u(b)]\,\mathrm{sgn}\,u(a) \leqslant c. \tag{49}$$

First, suppose that u does not change its sign. According to (49) and the assumption $\lambda \in]0,1]$, we obtain

$$|u(a)| - |u(b)| \leqslant c, \tag{50}$$

$$|u(a)| - |u(b)| \leqslant |u(a)| \frac{\lambda - 1}{\lambda} + \frac{c}{\lambda}. \tag{51}$$

Put

$$\bar{M} = \max\{|u(t)|: t \in [a,b]\}, \quad \bar{m} = \min\{|u(t)|: t \in [a,b]\} \tag{52}$$

and choose $t_1, t_2 \in [a,b]$ such that $t_1 \neq t_2$ and

$$|u(t_1)| = \bar{M}, \quad |u(t_2)| = \bar{m}. \tag{53}$$

Obviously, $\bar{M} \geqslant 0$, $\bar{m} \geqslant 0$, and either

$$t_1 < t_2 \tag{54}$$

or

$$t_1 > t_2. \tag{55}$$

Due to (7), (48) and (52), (46) implies

$$|u(t)|' \leqslant \bar{M}\ell_0(1)(t) - \bar{m}\ell_1(1)(t) + q^*(t) \quad \text{for } t \in [a,b]. \tag{56}$$

If (54) holds, then the integration of (56) from a to t_1 and from t_2 to b, in view of (7) and (53), results in

$$\bar{M} - |u(a)| \leqslant \bar{M} \int_a^{t_1} \ell_0(1)(s)\,\mathrm{d}s + \int_a^{t_1} q^*(s)\,\mathrm{d}s,$$

$$|u(b)| - \bar{m} \leqslant \bar{M} \int_{t_2}^{b} \ell_0(1)(s)\,\mathrm{d}s + \int_{t_2}^{b} q^*(s)\,\mathrm{d}s.$$

Summing the last two inequalities and taking into account (7) and (50), we obtain

$$\bar{M} - \bar{m} - c \leqslant \bar{M} - \bar{m} + |u(b)| - |u(a)| \leqslant \bar{M}\|\ell_0(1)\|_L + \|q^*\|_L.$$

If (55) is fulfilled, then the integration of (56) from t_2 to t_1, on account of (7) and (53), yields

$$\bar{M} - \bar{m} - c \leqslant \bar{M} - \bar{m} = \bar{M} \int_{t_2}^{t_1} \ell_0(1)(s)\,\mathrm{d}s + \int_{t_2}^{t_1} q^*(s)\,\mathrm{d}s \leqslant \bar{M}\|\ell_0(1)\|_L + \|q^*\|_L.$$

Therefore, in both cases (54) and (55), the inequality

$$\bar{M} - \bar{m} - c \leqslant \bar{M}\|\ell_0(1)\|_L + \|q^*\|_L \tag{57}$$

holds.

On the other hand, the integration of (56) from a to b, yields

$$|u(b)| - |u(a)| \leqslant \bar{M}\|\ell_0(1)\|_L - \bar{m}\|\ell_1(1)\|_L + \|q^*\|_L.$$

Hence, by (51) and the assumption $\lambda \in]0,1]$,

$$\begin{aligned} \bar{m}\|\ell_1(1)\|_L &\leqslant \bar{M}\|\ell_0(1)\|_L + |u(a)|\frac{\lambda - 1}{\lambda} + \frac{c}{\lambda} + \|q^*\|_L \\ &\leqslant \bar{M}\|\ell_0(1)\|_L + \bar{m}\frac{\lambda - 1}{\lambda} + \|q^*\| + \frac{c}{\lambda}. \end{aligned}$$

From the last inequality and (57), in view of (9) and the assumption $\lambda \in]0,1]$, it follows that

$$\bar{m}\left(\|\ell_1(1)\|_L + \frac{1-\lambda}{\lambda}\right) \leqslant \bar{M}\|\ell_0(1)\|_L + \|q^*\|_L + \frac{c}{\lambda},$$

$$\bar{M}(1 - \|\ell_0(1)\|_L) \leqslant \bar{m} + \|q^*\|_L + c.$$

Thus, on account of (10),

$$\|u\|_C = \bar{M} \leqslant r_0(\lambda\|\ell_1(1)\|_L + 2 - \lambda)(c + \|q^*\|_L),$$

where $r_0 = [(1 - \|\ell_0(1)\|_L)(\lambda\|\ell_1(1)\|_L + 1 - \lambda) - \lambda\|\ell_0(1)\|_L]^{-1}$. Therefore, estimate (40) holds.

Now suppose that u changes its sign. Put

$$M = \max\{u(t) : t \in [a,b]\}, \quad m = -\min\{u(t) : t \in [a,b]\}, \tag{58}$$

and choose $t_M,\ t_m \in [a,b]$ such that

$$u(t_M) = M, \quad u(t_m) = -m. \tag{59}$$

Obviously, $M > 0$, $m > 0$, and either

$$t_m < t_M \tag{60}$$

or

$$t_m > t_M. \tag{61}$$

First suppose that (60) is fulfilled. It is clear that there exists $\alpha_2 \in]t_m, t_M[$ such that

$$u(t) > 0 \quad \text{for } \alpha_2 < t \leqslant t_M, \quad u(\alpha_2) = 0. \tag{62}$$

Let

$$\alpha_1 = \inf\{t \in [a, t_m] : u(s) < 0 \quad \text{for } t \leqslant s \leqslant t_m\}.$$

Obviously,

$$u(t) < 0 \quad \text{for } \alpha_1 < t \leqslant t_m \quad \text{and} \quad u(\alpha_1) = 0 \quad \text{if } \alpha_1 > a. \tag{63}$$

Put

$$\alpha_3 = \begin{cases} b & \text{if } u(b) \geqslant 0, \\ \inf\{t \in]t_M, b] : u(s) < 0 \text{ for } t \leqslant s \leqslant b\} & \text{if } u(b) < 0. \end{cases}$$

It is clear that if $\alpha_3 < b$, then

$$u(t) < 0 \quad \text{for } \alpha_3 < t \leqslant b, \quad u(\alpha_3) = 0. \tag{64}$$

The integration of (46) from α_1 to t_m, from α_2 to t_M and from α_3 to b, in view of (7), (48), (58), (59), (62), (63) and (64), yields

$$u(\alpha_1) + m \leqslant M\int_{\alpha_1}^{t_m} \ell_1(1)(s)\,\mathrm{d}s + m\int_{\alpha_1}^{t_m} \ell_0(1)(s)\,\mathrm{d}s + \int_{\alpha_1}^{t_m} q^*(s)\,\mathrm{d}s, \tag{65}$$

$$M \leqslant M\int_{\alpha_2}^{t_M} \ell_0(1)(s)\,\mathrm{d}s + m\int_{\alpha_2}^{t_M} \ell_1(1)(s)\,\mathrm{d}s + \int_{\alpha_2}^{t_M} q^*(s)\,\mathrm{d}s, \tag{66}$$

$$u(\alpha_3) - u(b) \leqslant M\int_{\alpha_3}^{b} \ell_1(1)(s)\,\mathrm{d}s + m\int_{\alpha_3}^{b} \ell_0(1)(s)\,\mathrm{d}s + \int_{\alpha_3}^{b} q^*(s)\,\mathrm{d}s. \tag{67}$$

If $u(b) \geqslant 0$ or $u(a) > 0$, then according to (49), (63) and the assumption $\lambda > 0$, we obtain $u(\alpha_1) \geqslant -c$ and from (65) we find

$$-c + m \leqslant M\int_I \ell_1(1)(s)\,\mathrm{d}s + m\int_I \ell_0(1)(s)\,\mathrm{d}s + \int_I q^*(s)\,\mathrm{d}s, \tag{68}$$

where $I = [\alpha_1, t_m] \cup [\alpha_3, b]$.

If $u(b) < 0$ and $u(a) \leqslant 0$, then multiplying both sides of (67) by λ and taking into account (7), (58), (64) and the assumption $\lambda \in]0,1]$, we get

$$-\lambda u(b) \leqslant M\int_{\alpha_3}^{b} \ell_1(1)(s)\,\mathrm{d}s + m\int_{\alpha_3}^{b} \ell_0(1)(s)\,\mathrm{d}s + \int_{\alpha_3}^{b} q^*(s)\,\mathrm{d}s.$$

Summing the last inequality and (65), according to (63) and the condition

$$u(a) - \lambda u(b) \geqslant -(u(a) - \lambda u(b))\,\mathrm{sgn}\,u(a) \geqslant -c,$$

we obtain that inequality (68), where $I = [\alpha_1, t_m] \cup [\alpha_3, b]$, holds.

From (66) and (68) we have

$$M(1 - C_1) \leqslant mA_1 + \|q^*\|_L + c, \quad m(1 - D_1) \leqslant MB_1 + \|q^*\|_L + c, \tag{69}$$

where

$$A_1 = \int_{\alpha_2}^{t_M} \ell_1(1)(s)\,\mathrm{d}s, \quad B_1 = \int_I \ell_1(1)(s)\,\mathrm{d}s,$$

$$C_1 = \int_{\alpha_2}^{t_M} \ell_0(1)(s)\,\mathrm{d}s, \quad D_1 = \int_I \ell_0(1)(s)\,\mathrm{d}s.$$

Due to (9), $C_1 < 1$, $D_1 < 1$. Consequently, (69) implies

$$\begin{aligned} 0 < M(1 - C_1)(1 - D_1) &\leqslant A_1(MB_1 + \|q^*\|_L + c) + \|q^*\|_L + c \\ &\leqslant MA_1B_1 + (\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1), \end{aligned}$$

$$\begin{aligned} 0 < m(1 - C_1)(1 - D_1) &\leqslant B_1(mA_1 + \|q^*\|_L + c) + \|q^*\|_L + c \\ &\leqslant mA_1B_1 + (\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1). \end{aligned} \tag{70}$$

Obviously,

$$(1 - C_1)(1 - D_1) \geqslant 1 - (C_1 + D_1) \geqslant 1 - \|\ell_0(1)\|_L > 0,$$

$$4A_1B_1 \leqslant (A_1 + B_1)^2 \leqslant \|\ell_1(1)\|_L^2.$$

By the last inequalities and (10), from (70) we get

$$M \leqslant r_1(\|\ell_1(1)\|_L + 1)(c + \|q^*\|_L),$$
$$m \leqslant r_1(\|\ell_1(1)\|_L + 1)(c + \|q^*\|_L), \tag{71}$$

where

$$r_1 = \left(1 - \|\ell_0(1)\|_L - \tfrac{1}{4}\|\ell_1(1)\|_L^2\right)^{-1}. \tag{72}$$

Therefore, estimate (40) holds.

If (61) holds, the validity of estimate (40) can be proved analogously. □

Lemma 3.5. *Let $\lambda \in]0,1]$, the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 satisfy conditions* (7), (13) *and* (14). *Then, ℓ belongs to the set $U_2(\lambda)$.*

Proof. Let $q^* \in L([a,b];R_+)$, $c \in R_+$ and $u \in \tilde{C}([a,b];R)$ satisfy (38) and (39) for $i=2$. We show that (40) holds, where

$$r = \frac{\|\ell_0(1)\|_L + 1}{(\lambda - \|\ell_1(1)\|_L)\|\ell_0(1)\|_L - \|\ell_1(1)\|_L - 1 + \lambda}$$
$$+ \frac{\|\ell_0(1)\|_L + 1}{\lambda - \|\ell_1(1)\|_L - \frac{1}{4}\|\ell_0(1)\|_L^2}. \tag{73}$$

Obviously, u satisfies (46), where $\tilde{q}$ is defined by (47). Clearly,

$$-\tilde{q}(t)\operatorname{sgn} u(t) \leqslant q^*(t) \quad \text{for } t \in [a,b] \tag{74}$$

and

$$-(u(a) - \lambda u(b))\operatorname{sgn} u(b) \leqslant c. \tag{75}$$

First suppose that u does not change its sign. According to (75) and the assumption $\lambda \in]0,1]$, we obtain

$$\lambda|u(b)| - |u(a)| \leqslant c, \tag{76}$$

$$|u(b)| - |u(a)| \leqslant |u(b)|(1-\lambda) + c. \tag{77}$$

Define numbers $\bar{M}$ and $\bar{m}$ by (52) and choose $t_1, t_2 \in [a,b]$ such that $t_1 \neq t_2$ and (53) is fulfilled. Obviously, $\bar{M} \geqslant 0$, $\bar{m} \geqslant 0$, and either (54) or (55) holds. Due to (7), (52) and (74), (46) implies

$$-|u(t)|' \leqslant \bar{M}\ell_1(1)(t) - \bar{m}\ell_0(1)(t) + q^*(t) \quad \text{for } t \in [a,b]. \tag{78}$$

If (55) holds, then the integration of (78) from a to t_2 and from t_1 to b, in view of (7) and (53), results in

$$|u(a)| - \bar{m} \leqslant \bar{M}\int_a^{t_2} \ell_1(1)(s)\,ds + \int_a^{t_2} q^*(s)\,ds, \tag{79}$$

$$\bar{M} - |u(b)| \leqslant \bar{M} \int_{t_1}^{b} \ell_1(1)(s)\,\mathrm{d}s + \int_{t_1}^{b} q^*(s)\,\mathrm{d}s. \tag{80}$$

Multiplying both sides of (80) by λ and taking into account (7) and the condition $\lambda \in]0,1]$, we obtain

$$\lambda\bar{M} - \lambda|u(b)| \leqslant \bar{M} \int_{t_1}^{b} \ell_1(1)(s)\,\mathrm{d}s + \int_{t_1}^{b} q^*(s)\,\mathrm{d}s.$$

Summing the last inequality and (79), in view of (76), we get

$$\lambda\bar{M} - \bar{m} - c \leqslant \lambda\bar{M} - \bar{m} + |u(a)| - \lambda|u(b)| \leqslant \bar{M}\|\ell_1(1)\|_L + \|q^*\|_L.$$

If (54) is fulfilled, then the integration of (78) from t_1 to t_2, on account of (7) and (53), yields

$$\lambda\bar{M} - \bar{m} - c \leqslant \bar{M} - \bar{m} \leqslant \bar{M} \int_{t_1}^{t_2} \ell_1(1)(s)\,\mathrm{d}s + \int_{t_1}^{t_2} q^*(s)\,\mathrm{d}s \leqslant \bar{M}\|\ell_1(1)\|_L + \|q^*\|_L.$$

Therefore, in both cases (54) and (55), the inequality

$$\lambda\bar{M} - \bar{m} - c \leqslant \bar{M}\|\ell_1(1)\|_L + \|q^*\|_L \tag{81}$$

holds.

On the other hand, the integration of (78) from a to b implies

$$|u(a)| - |u(b)| \leqslant \bar{M}\|\ell_1(1)\|_L - \bar{m}\|\ell_0(1)\|_L + \|q^*\|_L.$$

Hence, by (77) and the assumption $\lambda \in]0,1]$, we get

$$\begin{aligned} \bar{m}\|\ell_0(1)\|_L &\leqslant \bar{M}\|\ell_1(1)\|_L + |u(b)|(1-\lambda) + c + \|q^*\|_L \\ &\leqslant \bar{M}\|\ell_1(1)\|_L + \bar{M}(1-\lambda) + \|q^*\| + c. \end{aligned}$$

From the last inequality and (81), in view of (13), it follows that

$$\bar{m}\|\ell_0(1)\|_L \leqslant \bar{M}(\|\ell_1(1)\|_L + 1 - \lambda) + \|q^*\|_L + c,$$

$$\bar{M}(\lambda - \|\ell_1(1)\|_L) \leqslant \bar{m} + \|q^*\|_L + c.$$

Thus, on account of (14),

$$\|u\|_C = \bar{M} \leqslant r_0(\|\ell_0(1)\|_L + 1)(c + \|q^*\|_L),$$

where $r_0 = [(\lambda - \|\ell_1(1)\|_L)\|\ell_0(1)\|_L - \|\ell_1(1)\|_L - 1 + \lambda]^{-1}$. Therefore, estimate (40) holds.

Now suppose that u changes its sign. Define numbers M and m by (58) and choose $t_M,\ t_m \in [a,b]$ such that (59) is fulfilled. Obviously, $M > 0$, $m > 0$, and either (60) or (61) holds.

First suppose that (60) is fulfilled. It is clear that there exists $\alpha_1 \in]t_m, t_M[$ such that

$$u(t) < 0 \quad \text{for } t_m \leqslant t < \alpha_1, \quad u(\alpha_1) = 0. \tag{82}$$

Let

$$\alpha_2 = \sup\{t \in [t_M, b] : u(s) > 0 \text{ for } t_M \leqslant s \leqslant t\}.$$

Obviously,

$$u(t) > 0 \quad \text{for } t_M \leqslant t < \alpha_2 \quad \text{and} \quad u(\alpha_2) = 0 \quad \text{if } \alpha_2 < b. \tag{83}$$

Put

$$\alpha_3 = \begin{cases} a & \text{if } u(a) \leqslant 0, \\ \sup\{t \in [a, t_m[: u(s) > 0 \text{ for } a \leqslant s \leqslant t\} & \text{if } u(a) > 0. \end{cases}$$

It is clear that if $\alpha_3 > a$, then

$$u(t) > 0 \quad \text{for } a \leqslant t < \alpha_3, \quad u(\alpha_3) = 0. \tag{84}$$

The integration of (46) from t_m to α_1, from t_M to α_2 and from a to α_3, in view of (7), (58), (59), (74), (82), (83) and (84), yields

$$m \leqslant M \int_{t_m}^{\alpha_1} \ell_0(1)(s)\,\mathrm{d}s + m \int_{t_m}^{\alpha_1} \ell_1(1)(s)\,\mathrm{d}s + \int_{t_m}^{\alpha_1} q^*(s)\,\mathrm{d}s, \tag{85}$$

$$M - u(\alpha_2) \leqslant M \int_{t_M}^{\alpha_2} \ell_1(1)(s)\,\mathrm{d}s + m \int_{t_M}^{\alpha_2} \ell_0(1)(s)\,\mathrm{d}s + \int_{t_M}^{\alpha_2} q^*(s)\,\mathrm{d}s, \tag{86}$$

$$u(a) - u(\alpha_3) \leqslant M \int_{a}^{\alpha_3} \ell_1(1)(s)\,\mathrm{d}s + m \int_{a}^{\alpha_3} \ell_0(1)(s)\,\mathrm{d}s + \int_{a}^{\alpha_3} q^*(s)\,\mathrm{d}s. \tag{87}$$

If $u(b) < 0$ or $u(a) \leqslant 0$, then according to (75) and (83), we obtain $u(\alpha_2) \leqslant c/\lambda$, and from (86), in view of (7), (58) and the assumption $\lambda \in]0,1]$, it follows that

$$\lambda M - c \leqslant M \int_I \ell_1(1)(s)\,\mathrm{d}s + m \int_I \ell_0(1)(s)\,\mathrm{d}s + \int_I q^*(s)\,\mathrm{d}s, \tag{88}$$

where $I = [a, \alpha_3] \cup [t_M, \alpha_2]$.

If $u(b) \geqslant 0$ and $u(a) > 0$, then multiplying both sides of (86) by λ and taking into account (7), (58), (83) and the assumption $\lambda \in]0,1]$, we get

$$\lambda M - \lambda u(b) \leqslant M \int_{t_M}^{\alpha_2} \ell_1(1)(s)\,\mathrm{d}s + m \int_{t_M}^{\alpha_2} \ell_0(1)(s)\,\mathrm{d}s + \int_{t_M}^{\alpha_2} q^*(s)\,\mathrm{d}s.$$

Summing the last inequality and (87), according to (84) and the condition

$$u(a) - \lambda u(b) \geqslant (u(a) - \lambda u(b))\,\mathrm{sgn}\,u(b) \geqslant -c,$$

we obtain that the inequality (88), where $I = [a, \alpha_3] \cup [t_M, \alpha_2]$, holds.

From (85) and (88) we find

$$m(1 - A_1) \leqslant MC_1 + \|q^*\|_L + c, \quad M(\lambda - B_1) \leqslant mD_1 + \|q^*\|_L + c, \tag{89}$$

where

$$A_1 = \int_{t_m}^{\alpha_1} \ell_1(1)(s)\,\mathrm{d}s, \quad B_1 = \int_I \ell_1(1)(s)\,\mathrm{d}s,$$

$$C_1 = \int_{t_m}^{\alpha_1} \ell_0(1)(s)\,\mathrm{d}s, \quad D_1 = \int_I \ell_0(1)(s)\,\mathrm{d}s.$$

Due to (13), $A_1 < \lambda$, $B_1 < \lambda$. Consequently, (89) implies

$$\begin{aligned} 0 < m(1 - A_1)(\lambda - B_1) &\leqslant C_1(mD_1 + \|q^*\|_L + c) + \|q^*\|_L + c \\ &\leqslant mC_1D_1 + (\|q^*\|_L + c)(\|\ell_0(1)\|_L + 1), \\ 0 < M(1 - A_1)(\lambda - B_1) &\leqslant D_1(MC_1 + \|q^*\|_L + c) + \|q^*\|_L + c \\ &\leqslant MC_1D_1 + (\|q^*\|_L + c)(\|\ell_0(1)\|_L + 1). \end{aligned} \tag{90}$$

Obviously, in view of the assumption $\lambda \in]0, 1]$,

$$(1 - A_1)(\lambda - B_1) \geqslant \lambda - \lambda A_1 - B_1 \geqslant \lambda - (A_1 + B_1) \geqslant \lambda - \|\ell_1(1)\|_L > 0,$$

$$4C_1D_1 \leqslant (C_1 + D_1)^2 \leqslant \|\ell_0(1)\|_L^2.$$

By the last inequalities, from (90) we get

$$\begin{aligned} &M \leqslant r_1(\|\ell_0(1)\|_L + 1)(c + \|q^*\|_L), \\ &m \leqslant r_1(\|\ell_0(1)\|_L + 1)(c + \|q^*\|_L), \end{aligned} \tag{91}$$

where

$$r_1 = (\lambda - \|\ell_1(1)\|_L - \tfrac{1}{4}\|\ell_0(1)\|_L^2)^{-1}. \tag{92}$$

Therefore, estimate (40) holds.

If (61) holds, the validity of estimate (40) can be proved analogously. □

4. Proofs of the main results

Theorem 2.1 follows from Lemmas 3.2 and 3.4, Theorem 2.2 follows from Lemmas 3.2 and 3.5, Theorem 2.5 follows from Lemmas 3.3 and 3.4, and Theorem 2.6 follows from Lemmas 3.3 and 3.5.

Proof of Corollary 2.1. Obviously, conditions (24), (25) and (26) yield conditions (8), (9) and (10), where

$$F(v)(t) \stackrel{\text{def}}{=} p(t)v(\tau(t)) - g(t)v(\mu(t)) + f(t, v(t), v(v(t))),$$
$$\ell_0(v)(t) \stackrel{\text{def}}{=} p(t)v(\tau(t)), \quad \ell_1(v)(t) \stackrel{\text{def}}{=} g(t)v(\mu(t)). \tag{93}$$

Consequently, all the assumptions of Theorem 2.5 are fulfilled. □

Proof of Corollary 2.5. Obviously, conditions (34), (25) and (26) yield conditions (21), (9) and (10), where F, ℓ_0 and ℓ_1 are defined by (93). Consequently, all the assumptions of Theorem 2.5 are fulfilled. □

Corollaries 2.2–2.4 and 2.6–2.8 can be proved analogously.

5. On Remarks 2.1 and 2.2

On Remark 2.1. Let $\lambda \in]0,1]$ (for the case $\lambda = 0$, see [5]). Denote by G the set of pairs $(x, y) \in R_+ \times R_+$ such that

$$x < 1, \quad \frac{x}{1-x} - \frac{1-\lambda}{\lambda} < y < 2\sqrt{1-x}.$$

According to Theorem 2.1, if (6) is fulfilled and there exist $\ell_0, \ell_1 \in \mathscr{P}_{ab}$ such that $(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in G$, and on the set $B^1_{\lambda c}([a,b];R)$ inequality (8) holds, then problem (1), (2) is solvable.

Put $G_0 = \{(x, y) \in R \times R\colon 0 \leqslant x < x_1, y = 2\sqrt{1-x}\}$, where $x_1 \in]0,1[$ is such that

$$\frac{x_1}{1-x_1} - \frac{1-\lambda}{\lambda} = 2\sqrt{1-x_1}.$$

Below we give the examples which show that for any pair $(x_0, y_0) \notin G \cup G_0$, $x_0 \geqslant 0$, $y_0 \geqslant 0$, there exist functions $p_0 \in L([a,b];R)$, $-p_1 \in L([a,b];R_+)$, and $\tau \in \mathscr{M}_{ab}$ such that

$$\int_a^b [p_0(s)]_+ \, \mathrm{d}s = x_0, \qquad \int_a^b [p_0(s)]_- \, \mathrm{d}s = y_0 \tag{94}$$

and the problem

$$u'(t) = p_0(t)u(\tau(t)) + p_1(t)u(t), \quad u(a) - \lambda u(b) = 0 \tag{95}$$

has a non-trivial solution. Then by Remark 1.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that problem (1), (2), where

$$F(v)(t) \stackrel{\text{def}}{=} p_0(t)v(\tau(t)) + p_1(t)v(t) + q_0(t), \quad h(v) \stackrel{\text{def}}{=} c_0, \tag{96}$$

has no solution, while conditions (6) and (8) are fulfilled, where $\ell_0(v)(t) \stackrel{\text{def}}{=} [p_0(t)]_+ v(\tau(t))$, $\ell_1(v)(t) \stackrel{\text{def}}{=} [p_0(t)]_- v(\tau(t))$, $q \equiv |q_0|$, and $c = |c_0|$.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin G \cup G_0$, then (x_0, y_0) belongs to at least one of the following sets:

$$G_1 = \{(x, y) \in R_+ \times R_+ \colon 1 \leqslant x,\ 0 \leqslant y\},$$

$$G_2 = \left\{(x, y) \in R_+ \times R_+ \colon 1 - \lambda \leqslant x < 1,\ y \leqslant \frac{x}{1-x} - \frac{1-\lambda}{\lambda}\right\},$$

$$G_3 = \{(x, y) \in R_+ \times R_+ \colon 0 \leqslant x < 1,\ 2\sqrt{1-x} < y\}.$$

Example 5.1. Let $(x_0, y_0) \in G_1$. Put $a = 0,\ b = 3,\ \varepsilon = \lambda/(1 + y_0)$,

$$p_0(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[, \\ x_0 & \text{for } t \in [1,2[, \\ 0 & \text{for } t \in [2,3], \end{cases} \qquad p_1(t) = \begin{cases} 0 & \text{for } t \in [0,2[, \\ -\dfrac{x_0 + \varepsilon - 1}{1 - (x_0 + \varepsilon - 1)(t-3)} & \text{for } t \in [2,3], \end{cases}$$

$$\tau(t) = \begin{cases} 1 & \text{for } t \in [0,1[, \\ 3 & \text{for } t \in [1,3]. \end{cases}$$

Then (94) holds, and problem (95) has the non-trivial solution

$$u(t) = \begin{cases} (\lambda - \varepsilon)t - \lambda & \text{for } t \in [0,1[, \\ -x_0(t-1) - \varepsilon & \text{for } t \in [1,2[, \\ (x_0 + \varepsilon - 1)(t-3) - 1 & \text{for } t \in [2,3]. \end{cases}$$

Example 5.2. Let $(x_0, y_0) \in G_2$. Put $a = 0,\ b = 3$,

$$p_0(t) = \begin{cases} x_0 & \text{for } t \in [0,1[, \\ -y_0 & \text{for } t \in [1,2[, \\ 0 & \text{for } t \in [2,3], \end{cases} \qquad \tau(t) = \begin{cases} 1 & \text{for } t \in [0,1[, \\ 0 & \text{for } t \in [1,3], \end{cases}$$

$$p_1(t) = \begin{cases} 0 & \text{for } t \in [0,2[, \\ -\dfrac{(\lambda - 1 + x_0 - \lambda y_0(1-x_0))}{1 - x_0 - (\lambda - 1 + x_0 - \lambda y_0(1-x_0))(t-3)} & \text{for } t \in [2,3]. \end{cases}$$

Then (94) holds, and problem (95) has the non-trivial solution

$$u(t) = \begin{cases} -\dfrac{\lambda x_0}{1-x_0} t - \lambda & \text{for } t \in [0,1[, \\ \lambda y_0(t-1) - \dfrac{\lambda}{1-x_0} & \text{for } t \in [1,2[, \\ \left(\dfrac{\lambda - 1 + x_0}{1-x_0} - \lambda y_0\right)(t-3) - 1 & \text{for } t \in [2,3]. \end{cases}$$

Example 5.3. Let $(x_0, y_0) \in G_3$, and choose $\varepsilon > 0$ such that $\lambda - \varepsilon > 0$, $2\sqrt{1-x_0} + 2\varepsilon \leqslant y_0$. Put $a = 0$, $b = 6$, $t_0 = \varepsilon/(\sqrt{1-x_0} + \varepsilon) + 1$,

$$p_0(t) = \begin{cases} 0 & \text{for } t \in [0,1[, \\ -(\sqrt{1-x_0} + \varepsilon) & \text{for } t \in [1,2[, \\ 0 & \text{for } t \in [2,3[, \\ -(\sqrt{1-x_0} + \varepsilon) & \text{for } t \in [3,4[, \\ x_0 & \text{for } t \in [4,5[, \\ 2\sqrt{1-x_0} + 2\varepsilon - y_0 & \text{for } t \in [5,6], \end{cases} \qquad \tau(t) = \begin{cases} 5 & \text{for } t \in [0,2[, \\ 2 & \text{for } t \in [2,4[, \\ 5 & \text{for } t \in [4,5[, \\ t_0 & \text{for } t \in [5,6], \end{cases}$$

$$p_1(t) = \begin{cases} -\dfrac{\lambda - \varepsilon}{\lambda - (\lambda - \varepsilon)t} & \text{for } t \in [0,1[, \\ 0 & \text{for } t \in [1,2[, \\ -\dfrac{(1-\varepsilon)}{(1-\varepsilon)(2-t) + 1} & \text{for } t \in [2,3[, \\ 0 & \text{for } t \in [3,6]. \end{cases}$$

Then (94) holds, and problem (95) has the non-trivial solution

$$u(t) = \begin{cases} (\lambda - \varepsilon)t - \lambda & \text{for } t \in [0,1[, \\ (\varepsilon + \sqrt{1-x_0})(t-1) - \varepsilon & \text{for } t \in [1,2[, \\ (1-\varepsilon)\sqrt{1-x_0}(2-t) + \sqrt{1-x_0} & \text{for } t \in [2,3[, \\ (1 - x_0 + \varepsilon\sqrt{1-x_0})(3-t) + \varepsilon\sqrt{1-x_0} & \text{for } t \in [3,4[, \\ x_0(5-t) - 1 & \text{for } t \in [4,5[, \\ -1 & \text{for } t \in [5,6]. \end{cases}$$

On Remark 2.2. Let $\lambda \in]0,1]$. Denote by H the set of pairs $(x, y) \in R_+ \times R_+$ such that

$$y < \lambda, \quad \frac{1}{\lambda - y} - 1 < x < 2\sqrt{\lambda - y}.$$

According to Theorem 2.2, if (11) is fulfilled and there exist $\ell_0, \ell_1 \in \mathscr{P}_{ab}$ such that $(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in H$, and on the set $B^2_{\lambda c}([a,b]; R)$ inequality (12) holds, then problem (1), (2) is solvable.

Put $H_0 = \{(x, y) \in R \times R\colon 0 \leqslant y < y_1,\ x = 2\sqrt{\lambda - y}\}$, where $y_1 \in]0, \lambda[$ is such that

$$\frac{1}{\lambda - y_1} - 1 = 2\sqrt{\lambda - y_1}.$$

Below, we give the examples which show that for any pair $(x_0, y_0) \notin H \cup H_0$, $x_0 \geqslant 0$, $y_0 \geqslant 0$, there exist functions $p_0 \in L([a,b]; R)$, $p_1 \in L([a,b]; R_+)$, and $\tau \in \mathscr{M}_{ab}$ such that

(94) is fulfilled and problem (95) has a non-trivial solution. Then by Remark 1.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that problem (1), (2), where F and h are defined by (96), has no solution, while conditions (11) and (12) are fulfilled with $\ell_0(v)(t) \stackrel{\text{def}}{=} [p_0(t)]_+ v(\tau(t))$, $\ell_1(v)(t) \stackrel{\text{def}}{=} [p_0(t)]_- v(\tau(t))$, $q \equiv |q_0|$, and $c = |c_0|$.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin H \cup H_0$, then (x_0, y_0) belongs to at least one of the following sets:

$$H_1 = \{(x,y) \in R_+ \times R_+ \colon \lambda \leqslant y,\ 0 \leqslant x\},$$

$$H_2 = \left\{(x,y) \in R_+ \times R_+ \colon 0 \leqslant y < \lambda,\ x \leqslant \frac{1}{\lambda - y} - 1\right\},$$

$$H_3 = \{(x,y) \in R_+ \times R_+ \colon 0 \leqslant y < \lambda,\ 2\sqrt{\lambda - y} < x\}.$$

Example 5.4. Let $(x_0, y_0) \in H_1$. Put $a = 0,\ b = 3,\ \varepsilon = 1/(1 + x_0)$,

$$p_0(t) = \begin{cases} 0 & \text{for } t \in [0,1[, \\ -y_0 & \text{for } t \in [1,2[, \\ x_0 & \text{for } t \in [2,3], \end{cases} \qquad p_1(t) = \begin{cases} \dfrac{y_0 + \varepsilon - \lambda}{(y_0 + \varepsilon - \lambda)t + \lambda} & \text{for } t \in [0,1[, \\ 0 & \text{for } t \in [1,3], \end{cases}$$

$$\tau(t) = \begin{cases} 1 & \text{for } t \in [0,1[, \\ 3 & \text{for } t \in [1,2[, \\ 2 & \text{for } t \in [2,3]. \end{cases}$$

Then (94) holds, and problem (95) has the non-trivial solution

$$u(t) = \begin{cases} (y_0 + \varepsilon - \lambda)t + \lambda & \text{for } t \in [0,1[, \\ y_0(2 - t) + \varepsilon & \text{for } t \in [1,2[, \\ (1 - \varepsilon)(t - 3) + 1 & \text{for } t \in [2,3]. \end{cases}$$

Example 5.5. Let $(x_0, y_0) \in H_2$. Put $a = 0,\ b = 3$,

$$p_0(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[, \\ x_0 & \text{for } t \in [1,2[, \\ 0 & \text{for } t \in [2,3], \end{cases} \qquad \tau(t) = \begin{cases} 3 & \text{for } t \in [0,1[, \\ 1 & \text{for } t \in [1,3], \end{cases}$$

$$p_1(t) = \begin{cases} 0 & \text{for } t \in [0,2[, \\ \dfrac{1 - (\lambda - y_0)(1 + x_0)}{(1 - (\lambda - y_0)(1 + x_0))(t - 3) + 1} & \text{for } t \in [2,3]. \end{cases}$$

Then (94) holds, and problem (95) has the non-trivial solution

$$u(t)=\begin{cases} -y_0t+\lambda & \text{for } t\in[0,1[,\\ x_0(\lambda-y_0)(t-1)+\lambda-y_0 & \text{for } t\in[1,2[,\\ (1-(\lambda-y_0)(1+x_0))(t-3)+1 & \text{for } t\in[2,3]. \end{cases}$$

Example 5.6. Let $(x_0,y_0)\in H_3$, and choose $\varepsilon>0$ such that $1-\varepsilon>0$, $2\sqrt{\lambda-y_0}+2\varepsilon\leqslant x_0$. Put $a=0$, $b=6$, $t_0=4-\varepsilon/(\sqrt{\lambda-y_0}+\varepsilon)$,

$$p_0(t)=\begin{cases} -y_0 & \text{for } t\in[0,1[,\\ \sqrt{\lambda-y_0}+\varepsilon & \text{for } t\in[1,2[,\\ 0 & \text{for } t\in[2,3[,\\ \sqrt{\lambda-y_0}+\varepsilon & \text{for } t\in[3,4[,\\ 0 & \text{for } t\in[4,5[,\\ x_0-2\sqrt{\lambda-y_0}-2\varepsilon & \text{for } t\in[5,6], \end{cases} \qquad \tau(t)=\begin{cases} 6 & \text{for } t\in[0,1[,\\ 3 & \text{for } t\in[1,3[,\\ 5 & \text{for } t\in[3,5[,\\ t_0 & \text{for } t\in[5,6], \end{cases}$$

$$p_1(t)=\begin{cases} 0 & \text{for } t\in[0,2[,\\ \dfrac{1-\varepsilon}{1-(1-\varepsilon)(3-t)} & \text{for } t\in[2,3[,\\ 0 & \text{for } t\in[3,4[,\\ \dfrac{1-\varepsilon}{(1-\varepsilon)(t-5)+1} & \text{for } t\in[4,5[,\\ 0 & \text{for } t\in[5,6]. \end{cases}$$

Then (94) holds, and problem (95) has the non-trivial solution

$$u(t)=\begin{cases} -y_0t+\lambda & \text{for } t\in[0,1[,\\ (\lambda-y_0+\varepsilon\sqrt{\lambda-y_0})(1-t)+\lambda-y_0 & \text{for } t\in[1,2[,\\ (1-\varepsilon)\sqrt{\lambda-y_0}(3-t)-\sqrt{\lambda-y_0} & \text{for } t\in[2,3[,\\ (\sqrt{\lambda-y_0}+\varepsilon)(t-4)+\varepsilon & \text{for } t\in[3,4[,\\ (1-\varepsilon)(t-5)+1 & \text{for } t\in[4,5[,\\ 1 & \text{for } t\in[5,6]. \end{cases}$$

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