E. Bravyi; Robert Hakl; Alexander Lomtatidze

Optimal conditions for unique solvability of the Cauchy problem for first order linear functional differential equations

Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 3, 513--530

Persistent URL: `http://dml.cz/dmlcz/127739`

Czechoslovak Mathematical Journal, 52 (127) (2002), 513–530

OPTIMAL CONDITIONS FOR UNIQUE SOLVABILITY OF THE CAUCHY PROBLEM FOR FIRST ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

E. Bravyi, Perm, R. Hakl and A. Lomtatidze, Brno

(Received July 12, 1999)

Abstract. Nonimprovable, in a sense sufficient conditions guaranteeing the unique solvability of the problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) = c,$$

where $\ell\colon C(I,\mathbb{R}) \to L(I,\mathbb{R})$ is a linear bounded operator, $q \in L(I,\mathbb{R})$, and $c \in \mathbb{R}$, are established.

Keywords: linear functional differential equations, Cauchy problem, existence and uniqueness, differential inequalities

MSC 2000: 34K06, 34K10

1. Statement of the problem and formulation of the main results

On the segment $I = [a, b]$ we will consider the functional differential equation

$$u'(t) = \ell(u)(t) + q(t) \tag{1.1}$$

and its particular case

$$u'(t) = \sum_{k=1}^{m} p_k(t)u(\tau_k(t)) + q(t) \tag{1.1'}$$

This work was supported by Grant No. 201/99/0295 of the Grant Agency of the Czech Republic and by Grants Nos. 96-15-96195, 99-01-01278 of the RFBR and the Competition Centre of the FNS.

with the initial condition

$$u(a) = c. \tag{1.2}$$

Here $\ell\colon C(I,\mathbb{R}) \to L(I,\mathbb{R})$ is a linear bounded operator, $c \in \mathbb{R}$, $p_k \in L(I,\mathbb{R})$ $(k = 1,\dots,m)$, $q \in L(I,\mathbb{R})$, and $\tau_k\colon I \to I$ $(k = 1,\dots,m)$ are measurable functions.

In this paper, optimal conditions for the unique solvability of the problems (1.1), (1.1′) and (1.2) are established which are different from the previous results (see [1, 2, 4, 5] and references therein). More precisely, they are interesting especially in the case where the equations (1.1) and (1.1′) are not evolutional.

Throughout this paper the following notation and terms are used:

$\mathbb{R}$ is the set of real numbers;

$\mathbb{R}_+$ is the set of nonnegative real numbers;

$$[x]_+ = \frac{|x| + x}{2}, \qquad [x]_- = \frac{|x| - x}{2};$$

$C(I,\mathbb{R})$ is the Banach space of continuous functions $u\colon I \to \mathbb{R}$ with the norm

$$\|u\|_C = \max\{|u(t)|\colon t \in I\};$$

$C(I,\mathbb{R}_+) = \{u \in C(I,\mathbb{R})\colon u(t) \geqslant 0 \text{ for } t \in I\}$;

$L(I,\mathbb{R})$ is the Banach space of Lebesgue integrable functions $u\colon I \to \mathbb{R}$ with the norm

$$\|u\|_L = \int_a^b |u(t)|\,\mathrm{d}t;$$

$L(I,\mathbb{R}_+) = \{u \in L(I,\mathbb{R})\colon u(t) \geqslant 0 \text{ for almost all } t \in I\}$;

$\mathcal{L}_I$ is the set of linear bounded operators $\ell\colon C(I,\mathbb{R}) \to L(I,\mathbb{R})$ such that the function

$$t \longmapsto \sup\{|\ell(u)(t)|\colon \|u\|_C = 1\}$$

belongs to $L(I,\mathbb{R})$.

$\mathcal{P}_I$ is the set of linear operators $\ell\colon C(I,\mathbb{R}) \to L(I,\mathbb{R})$ mapping $C(I,\mathbb{R}_+)$ into $L(I,\mathbb{R}_+)$.

An absolutely continuous function $u\colon I \to \mathbb{R}$ is said to be a solution of the equation (1.1) if it satisfies this equation almost everywhere on I.

1.1. Theorem on differential inequalities.

First we introduce

Definiton 1.1. We will say that an operator $\ell \in \mathcal{L}_{\mathcal{I}}$ belongs to the set $\mathcal{S}_{\mathcal{I}}$ if the homogeneous problem

$$u'(t) = \ell(u)(t), \qquad u(a) = 0 \tag{1.3}$$

has only the trivial solution and for arbitrary $q \in L(I, \mathbb{R}_+)$ and $c \in \mathbb{R}_+$, the solution of (1.1), (1.2) is a nonnegative function.

Remark. If $\ell \in \mathcal{P}_{\mathcal{I}}$, then the inclusion $\ell \in \mathcal{S}_{\mathcal{I}}$ holds if and only if the problem

$$u'(t) \leqslant \ell(u)(t), \qquad u(a) = 0 \tag{1.4}$$

has no nontrivial nonnegative solution.

Remark 1.2. From Definition 1.1 it follows immediately that the inclusion

$$\ell \in \mathcal{S}_{\mathcal{I}}$$

holds if and only if for the equation (1.1) the classical theorem on differential inequalities holds (see e.g. [3]), i.e. for any absolutely continuous functions u_1 and $u_2\colon I \to \mathbb{R}$ such that

$$u_1'(t) \leqslant \ell(u_1)(t) + q(t), \qquad u_2'(t) \geqslant \ell(u_2)(t) + q(t) \qquad \text{a.e. on } I$$

and

$$u_1(a) \leqslant u_2(a),$$

the inequality

$$u_1(t) \leqslant u_2(t) \qquad \text{for } t \in I$$

is fulfilled. So, Theorem 1.1 formulated below is in fact a theorem on differential inequalities. On the other hand, due to the Fredholm property of the problem (1.1), (1.2) (see [4, 5]), it is clear that if $\ell \in \mathcal{S}_{\mathcal{I}}$ then this problem is uniquely solvable for any $c \in \mathbb{R}$ and $q \in L(I, \mathbb{R})$.

Theorem 1.1. *Let one of the following conditions be fulfilled:*

(i) $\ell \in \mathcal{P}_{\mathcal{I}}$ *and there exist a nonnegative integer* k, *a natural number* $m > k$ *and a constant* $\alpha \in \,]0, 1[$ *such that*

$$\ell_m(t) \leqslant \alpha \ell_k(t) \qquad \textit{for } t \in I, \tag{1.5}$$

where

$$\ell_0(t) \equiv 1, \qquad \ell_i(t) = \int_a^t \ell(\ell_{i-1})(s)\,\mathrm{d}s \qquad (i = 1, 2, \dots);$$

(ii) $\ell \in \mathcal{P}_{\mathcal{I}}$ *and there exists an absolutely continuous function* $\gamma\colon I \to]0, +\infty[$ *such that*

$$\gamma'(t) \geqslant \ell(\gamma)(t) \qquad \text{a.e. on } I; \tag{1.6}$$

(iii) $\ell \in \mathcal{P}_{\mathcal{I}}$ *and there exists* $\bar{\ell} \in \mathcal{P}_{\mathcal{I}}$ *such that for any* $v \in C(I, \mathbb{R}_+)$, *the inequalities*

$$\ell(\varphi(v))(t) - \ell(1)(t)\varphi(v)(t) \leqslant \bar{\ell}(v)(t) \qquad \text{a.e. on } I \tag{1.7}$$

and

$$\int_a^b \bar{\ell}(1)(s) \exp\left(\int_s^b \ell(1)(\xi)\,\mathrm{d}\xi\right) \mathrm{d}s < 1 \tag{1.8}$$

are fulfilled, where

$$\varphi(v)(t) = \int_a^t \ell(v)(s)\,\mathrm{d}s \qquad \text{for } t \in I;$$

(iv) ℓ *is a Volterra's type operator,* $-\ell \in \mathcal{P}_{\mathcal{I}}$ *and there exists an absolutely continuous function* $\gamma\colon I \to]0, +\infty[$ *such that*

$$\gamma'(t) \leqslant \ell(\gamma)(t) \qquad \text{a.e. on } I. \tag{1.9}$$

Then $\ell \in \mathcal{S}_{\mathcal{I}}$.

Corollary 1.1. *Let one of the following conditions be fulfilled:*

(i) $p_i(t) \geqslant 0$ *almost everywhere on* I $(i = 1, \dots, m)$ *and*

$$\sum_{i,k=1}^{m} \int_a^t p_k(s)\left(\int_a^{\tau_k(s)} p_i(\xi)\,\mathrm{d}\xi\right) \mathrm{d}s \leqslant \alpha \sum_{i=1}^{m} \int_a^t p_i(s)\,\mathrm{d}s \quad \text{for } t \in I, \tag{1.10}$$

where $\alpha \in]0, 1[$;

(ii) $p_i(t) \geqslant 0$ *almost everywhere on* I $(i = 1, \dots, m)$ *and*

$$\sum_{i=1}^{m} \int_t^{\tau_k(t)} p_i(s)\,\mathrm{d}s \leqslant \frac{1}{e} \qquad \text{for } t \in I \quad (k = 1, \dots, m); \tag{1.11}$$

(iii) $p_i(t) \geqslant 0$ *almost everywhere on* I $(i = 1, \dots, m)$ *and*

$$\int_a^b \sum_{k=1}^m p_k(s)\sigma_k(s) \int_s^{\tau_k(s)} \sum_{i=1}^m p_i(\xi)\,\mathrm{d}\xi \exp\biggl(\int_s^b \sum_{j=1}^m p_j(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}s < 1, \tag{1.12}$$

where $\sigma_k(t) = \frac{1}{2}\bigl(1 + \operatorname{sgn}(\tau_k(t) - t)\bigr)$ *for almost all* $t \in I$ $(k = 1, \dots, m)$;

(iv) $p_i(t) \leqslant 0$ *almost everywhere on* I $(i = 1, \dots, m)$ *and*

$$\sum_{i=1}^m \int_{\tau_k(t)}^t |p_i(s)|\,\mathrm{d}s \leqslant \frac{1}{e} \qquad \text{for } t \in I \quad (k = 1, \dots, m), \tag{1.13}$$

where $\tau_k(t) \leqslant t$ *for almost all* $t \in I$ $(k = 1, \dots, m)$. *Then the operator*

$$\ell(v)(t) \overset{\text{def}}{=} \sum_{k=1}^m p_k(t) v(\tau_k(t))$$

belongs to the set $\mathcal{S}_\mathcal{I}$.

1.2. Existence and uniqueness theorems.

Theorem 1.2. *Let one of the following conditions be fulfilled:*

(i) *there exist* $\ell_i \in \mathcal{P}_\mathcal{I}$ $(i = 0, 1)$ *and an absolutely continuous function* $\gamma\colon I \to \,]0, +\infty[$ *such that* $\ell = \ell_0 - \ell_1$ *and*

$$\gamma'(t) \geqslant \ell_0(\gamma)(t) + \ell_1(1)(t) \qquad \text{a.e. on } I, \tag{1.14}$$

$$\gamma(b) \leqslant 3; \tag{1.15}$$

(ii) $-\ell \in \mathcal{P}_\mathcal{I}$ *and there exists an absolutely continuous function* $\gamma\colon I \to \,]0, +\infty[$ *such that* (1.9) *is satisfied.*

Then the problem (1.1), (1.2) *has a unique solution.*

Corollary 1.2. *Let*

$$\int_a^b \sum_{k=1}^m [p_k(s)]_- \exp\biggl(\int_s^b \sum_{i=1}^m [p_i(\xi)]_+\,\mathrm{d}\xi\biggr)\,\mathrm{d}s < 3, \tag{1.16}$$

$$\bigl(t - \tau_k(t)\bigr)[p_k(t)]_+ \geqslant 0 \qquad \text{a.e. on } I \quad (k = 1, \dots, m). \tag{1.17}$$

Then the problem (1.1′), (1.2) *has a unique solution.*

Corollary 1.3. *Let*

$$\int_a^b \sum_{k=1}^m [p_k(s)]_- \,\mathrm{d}s + \alpha + 3\beta < 3, \tag{1.18}$$

where

$$\alpha = \int_a^b \sum_{k=1}^m [p_k(s)]_+ \bigg(\int_a^{\tau_k(s)} \sum_{i=1}^m [p_i(\xi)]_- \,\mathrm{d}\xi \bigg) \exp\bigg(\int_s^b \sum_{j=1}^m [p_j(\xi)]_+ \,\mathrm{d}\xi \bigg) \,\mathrm{d}s$$

and

$$\beta = \int_a^b \sum_{k=1}^m [p_k(s)]_+ \sigma_k(s) \bigg(\int_s^{\tau_k(s)} \sum_{i=1}^m [p_i(\xi)]_+ \,\mathrm{d}\xi \bigg) \exp\bigg(\int_s^b \sum_{j=1}^m [p_j(\xi)]_+ \,\mathrm{d}\xi \bigg) \,\mathrm{d}s,$$

where $\sigma_k(t) = \frac{1}{2}\big(1 + \operatorname{sgn}(\tau_k(t) - t)\big)$ *for almost all* $t \in I$ $(k = 1, \dots, m)$. *Then the problem* (1.1′), (1.2) *has a unique solution.*

Theorem 1.3. *Let* $\ell = \ell_0 - \ell_1$*, where* $\ell_i \in \mathcal{P}_{\mathcal{I}}$ $(i = 0, 1)$,

$$\int_a^b \ell_0(1)(s) \,\mathrm{d}s < 1 \tag{1.19}$$

and

$$\int_a^b \ell_1(1)(s) \,\mathrm{d}s < 1 + 2\bigg(1 - \int_a^b \ell_0(1)(s) \,\mathrm{d}s\bigg)^{\frac{1}{2}}. \tag{1.20}$$

Then the problem (1.1), (1.2) *has a unique solution.*

Remark 1.3. The conditions in Theorems 1.2 and 1.3, in general, do not guarantee $\ell \in \mathcal{S}_{\mathcal{I}}$. Let

$$\ell(u)(t) = -\varepsilon \int_t^b u(s) \,\mathrm{d}s, \qquad \varepsilon > 0.$$

It is easy to verify that for a sufficiently small ε, the conditions of Theorems 1.2 and 1.3 are fulfilled. Suppose that $\ell \in \mathcal{S}_{\mathcal{I}}$. Let u_0 be the solution of the problem (1.1), (1.2), with $c = 0$ and

$$q(t) = \begin{cases} 0 & \text{for } a \leqslant t < \frac{1}{2}(a+b), \\ 1 & \text{for } \frac{1}{2}(a+b) \leqslant t \leqslant b. \end{cases} \tag{1.21}$$

Then

$$u_0(t) \geqslant 0, \qquad u_0(t) \not\equiv 0 \qquad \text{for } t \in I. \tag{1.22}$$

Therefore we can find $a_1 \in \,]a, \frac{1}{2}(a+b)[$ such that $\ell(u_0)(t) < 0$ for $t \in [a, a_1[$. It follows from (1.1) and (1.21) that $u_0'(t) < 0$ for $t \in [a, a_1[$, which together with $u(a) = 0$ contradicts (1.22). Consequently, $\ell \notin \mathcal{S}_{\mathcal{I}}$.

Remark 1.4. If $\ell_1 \equiv 0$, then the condition (1.20) becomes unimportant and for the solvability of (1.1), (1.2) we get the condition (1.19), which corresponds to the result obtained in [4].

At the end of this section let us present an example verifying the optimality of the above formulated conditions in existence and uniqueness theorems.

Example 1.1. Let $a = 0$, $b = 3$, $\varepsilon > 0$,

$$\tau(t) = \begin{cases} 3 & \text{for } 0 \leqslant t \leqslant 1, \\ 1 & \text{for } 1 < t \leqslant 3, \end{cases} \qquad \ell(v)(t) = -v(\tau(t)). \tag{1.23}$$

It is clear that $\ell_0 \equiv 0$, $\ell_1(v)(t) = v(\tau(t))$, the function $\gamma(t) = t + \varepsilon$ satisfies the inequality (1.14) and $\gamma(3) = 3 + \varepsilon$. On the other hand, the problem (1.3) has the nontrivial solution

$$u(t) = \begin{cases} t & \text{for } 0 \leqslant t \leqslant 1, \\ 2 - t & \text{for } 1 < t \leqslant 3. \end{cases}$$

Consequently, the condition (1.15) cannot be replaced by the condition

$$\gamma(b) \leqslant 3 + \varepsilon$$

for an arbitrarily small $\varepsilon > 0$. This example shows also that the strict inequalities in (1.16), (1.18) and (1.20) cannot be replaced by nonstrict ones.

Remark 1.5. Let the operator ℓ be defined by (1.23). Then for an arbitrarily small $\varepsilon > 0$, the function $\gamma(t) \equiv \varepsilon$ satisfies the inequality

$$\gamma'(t) \leqslant \ell(\gamma)(t) + \varepsilon. \tag{1.24}$$

On the other hand, as shown above, the problem (1.3) has a nontrivial solution. Consequently, the inequality (1.9) cannot be replaced by the inequality (1.24) for an arbitrarily small ε.

2. Proofs of the main results

P r o o f of Theorem 1.1. (i) Let $u\colon I \to \mathbb{R}_+$ be an absolutely continuous function which satisfies (1.4). According to Remark 1.1, it is sufficient to show that $u(t) \equiv 0$. Denote by $\widetilde{\ell}_i$ $(i = 0, 1, \ldots)$ the operators defined by

$$\widetilde{\ell}_0(u)(t) = u(t), \qquad \widetilde{\ell}_i(u)(t) = \int_a^t \ell\big(\widetilde{\ell}_{i-1}(u)\big)(s)\,\mathrm{d}s \quad (i = 1, 2, \ldots).$$

Then we have

$$\widetilde{\ell}_i(1)(t) = \ell_i(t) \qquad (i = 0, 1, \ldots)$$

and

$$\widetilde{\ell}_m(u)(t) = \widetilde{\ell}_{m-k}(\widetilde{\ell}_k(u))(t). \tag{2.1}$$

Now (1.4) and the nonnegativeness of ℓ result in

$$u(t) \leqslant \widetilde{\ell}_i(u)(t) \qquad (i = 0, 1, \ldots) \tag{2.2}$$

and

$$u(t) \leqslant \|u\|_C \cdot \widetilde{\ell}_k(1)(t) = \|u\|_C \cdot \ell_k(t). \tag{2.3}$$

Let

$$v(t) = \begin{cases} 0 & \text{if } \ell_k(t) = 0, \\ \dfrac{u(t)}{\ell_k(t)} & \text{if } \ell_k(t) \neq 0. \end{cases} \tag{2.4}$$

Then (2.3) implies

$$\varrho = \operatorname{ess\,sup}\{v(t)\colon\ t \in I\} < +\infty$$

and

$$u(t) \leqslant \varrho \ell_k(t) = \varrho \widetilde{\ell}_k(1)(t).$$

Hence by (2.1), (2.2) and (1.5) we find

$$u(t) \leqslant \widetilde{\ell}_{m-k}(u)(t) \leqslant \varrho \widetilde{\ell}_{m-k}(\widetilde{\ell}_k(1))(t) = \varrho \widetilde{\ell}_m(1)(t) = \varrho \ell_m(t) \leqslant \alpha \varrho \ell_k(t),$$

whence in view of (2.4) we obtain

$$v(t) \leqslant \alpha\varrho$$

and, consequently,

$$\varrho \leqslant \alpha\varrho.$$

Since $\alpha \in \,]0,1[$, we have $\varrho = 0$, which implies $u(t) \equiv 0$.

(ii) Let u be a nontrivial solution of the problem (1.3). Due to $\ell \in \mathcal{P}_{\mathcal{I}}$ and (1.3), we obtain

$$|u(t)|' = \ell(u)(t) \operatorname{sgn} u(t) \leqslant \ell(|u|)(t) \qquad \text{a.e. on } I. \tag{2.5}$$

We can find $t_* \in \,]a,b]$ such that

$$\frac{|u(t_*)|}{\gamma(t_*)} = \lambda_*,$$

where

$$\lambda_* = \max\left\{ \frac{|u(t)|}{\gamma(t)} : \ t \in I \right\}.$$

Put $v(t) = \lambda_*\gamma(t) - |u(t)|$ for $t \in I$. It is obvious that

$$v(t) \geqslant 0 \quad \text{for } t \in I, \quad v(a) = \lambda_*\gamma(a) > 0, \quad v(t_*) = 0. \tag{2.6}$$

By (1.6), (2.5) and (2.6) we have

$$v'(t) \geqslant \lambda_*\ell(\gamma)(t) - \ell(|u|)(t) = \ell(v)(t) \geqslant 0 \qquad \text{a.e. on } I,$$

which contradicts (2.6). Consequently, the problem (1.3) has only the trivial solution.

Now let u_0 be the solution of the problem (1.1), (1.2) with $c \geqslant 0$ and $q \in L(I, \mathbb{R}_+)$. Suppose that $[u_0(t)]_- \not\equiv 0$. By virtue of $\ell \in \mathcal{P}_{\mathcal{I}}$ and (1.1) we find

$$\begin{aligned} [u_0(t)]'_- &= \frac{1}{2}\big(\ell(u_0)(t) \operatorname{sgn} u_0(t) - \ell(u_0)(t)\big) + \frac{1}{2}q(t)\big(\operatorname{sgn} u_0(t) - 1\big) \\ &\leqslant \ell([u_0]_-)(t) \qquad \text{a.e. on } I. \end{aligned} \tag{2.7}$$

We can choose $t_0 \in \,]a,b]$ such that

$$\frac{[u_0(t_0)]_-}{\gamma(t_0)} = \lambda_0,$$

where

$$\lambda_0 = \max\left\{ \frac{[u_0(t)]_-}{\gamma(t)} : \ t \in I \right\}.$$

Put $v_0(t) = \lambda_0\gamma(t) - [u_0(t)]_-$ for $t \in I$. It is clear that

$$v_0(t) \geqslant 0 \quad \text{for } t \in I, \quad v_0(a) = \lambda_0\gamma(a) > 0, \quad v_0(t_0) = 0. \tag{2.8}$$

By (1.6), (2.7) and (2.8) we have $v_0'(t) \geqslant \ell(v_0)(t) \geqslant 0$ almost everywhere on I, which contradicts (2.8). The contradiction obtained proves that $[u_0(t)]_- \equiv 0$. Consequently, $u_0(t) \geqslant 0$ for $t \in I$.

(iii) According to Remark 1.1, it is sufficient to show that the problem (1.4) has no nontrivial nonnegative solution. Assume to the contrary that there exists an absolutely continuous function $u\colon I \to \mathbb{R}_+$ such that $u(t) \not\equiv 0$ and (1.4) is fulfilled. Clearly,

$$u'(t) = \ell(u)(t) - q(t), \tag{2.9}$$

where $q(t) = \ell(u)(t) - u'(t) \geqslant 0$ almost everywhere on I. In view of (2.9) we find that u satisfies also the inequality

$$\begin{aligned} u'(t) = \ell(1)(t)u(t) &+ \big(\ell(\varphi(u))(t) - \ell(1)(t)\varphi(u)(t)\big) \\ &+ \big(\ell(1)(t)Q(t) - \ell(Q)(t) - q(t)\big) \qquad \text{a.e. on } I, \end{aligned}$$

where

$$Q(t) = \int_a^t q(s)\,\mathrm{d}s \geqslant 0 \qquad \text{for } t \in I.$$

From the last inequality according to the Cauchy formula we get

$$\begin{aligned} u(t) = \int_a^t &\big[\ell(\varphi(u))(s) - \ell(1)(s)\varphi(u)(s) \\ &+ H(s)\big] \exp\Big(\int_s^t \ell(1)(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}s \quad \text{for } t \in I, \end{aligned} \tag{2.10}$$

where

$$H(t) = \ell(1)(t)Q(t) - \ell(Q)(t) - q(t) \qquad \text{a.e. on } I.$$

It is evident that

$$\int_a^t \big(\ell(1)(s)Q(s) - q(s)\big)\exp\Big(-\int_a^s \ell(1)(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}s = -Q(t)\exp\Big(-\int_a^t \ell(1)(s)\,\mathrm{d}s\Big).$$

By virtue of this equality and (1.7), we obtain from (2.10) that

$$u(t) \leqslant \int_a^t \bar{\ell}(u)(s)\exp\Big(\int_s^t \ell(1)(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}s \qquad \text{for } t \in I.$$

Hence by (1.8) we get the contradiction $\|u\|_C < \|u\|_C$.

(iv) It is well-known (see e.g. Theorem 1.2′ in [4]) that if ℓ is a Volterra operator, then the problem (1.1), (1.2) has a unique solution. Let u_0 be a solution of (1.1), (1.2), where $q \in L(I, \mathbb{R}_+)$ and $c \geqslant 0$. It remains to show that

$$u_0(t) \geqslant 0 \qquad \text{for } t \in I.$$

First note that if $c = 0$ and $\|q\|_L \neq 0$, then u_0 must assume positive values, since (1.1) and $-\ell \in \mathcal{P}_{\mathcal{I}}$ would yield the contradiction $u_0'(t) \geqslant 0$. Consequently,

$$\max\{u_0(t)\colon\ t \in I\} > 0. \tag{2.11}$$

Let

$$c_0 = \max\left\{\frac{u_0(t)}{\gamma(t)}\colon\ t \in I\right\}. \tag{2.12}$$

Then the inequalities

$$c_0 > 0, \qquad c_0\gamma(t) - u_0(t) \geqslant 0 \qquad \text{for } t \in I \tag{2.13}$$

are fulfilled. Moreover, there exists $t_1 \in I$ such that

$$c_0\gamma(t_1) - u_0(t_1) = 0. \tag{2.14}$$

Due to the nonpositiveness of ℓ we have

$$\big(c_0\gamma(t) - u_0(t)\big)' \leqslant \ell(c_0\gamma - u_0)(t) - q(t) \leqslant 0.$$

Hence

$$u_0(t) > 0 \qquad \text{for } t \in [t_1, b] \tag{2.15}$$

and, consequently, $u_0(b) > 0$.

Now let $b_1 \in \,]a, b]$ be an arbitrarily but fixed point. Denote by ℓ_1, u_{01}, γ_1 and q_1 the restrictions of ℓ, u_0, γ and q to the interval $[a, b_1]$. Since ℓ is a Volterra operator, we have

$$\gamma_1'(t) \leqslant \ell_1(\gamma_1)(t) \qquad \text{a.e. on } [a, b_1]$$

and

$$u_{01}'(t) = \ell_1(u_{01})(t) + q_1(t) \qquad \text{a.e. on } [a, b_1].$$

From the above it immediately follows that either $u_{01}(t) \equiv 0$ or $u_{01}(b_1) > 0$. The arbitrariness of $b_1 \in \,]a, b]$ and (1.2) result in $u_0(t) \geqslant 0$ for $t \in I$. □

P r o o f of Corollary 1.1. Put

$$\ell(u)(t)=\sum_{i=1}^{m}p_i(t)u(\tau_i(t)) \tag{2.16}$$

and

$$\bar\ell(u)(t)=\sum_{k=1}^{m}p_k(t)\sigma_k(t)\int_t^{\tau_k(t)}\sum_{i=1}^{m}p_i(\xi)u(\tau_i(\xi))\,\mathrm{d}\xi, \tag{2.17}$$

where $\sigma_k(t)=\frac{1}{2}\big(1+\operatorname{sgn}\big(\tau_k(t)-t\big)\big)$ for almost all $t\in I$ $(k=1,\dots,m)$.

(i) By (2.16) and (1.10) we have

$$\ell_2(t)\leqslant\alpha\ell_1(t)\qquad\text{for } t\in I,$$

where

$$\ell_1(t)=\int_a^t\ell(1)(s)\,\mathrm{d}s,\qquad \ell_2(t)=\int_a^t\ell(\ell_1)(s)\,\mathrm{d}s$$

and the assumptions of Theorem 1.1 are fulfilled.

(ii) Define a function

$$\gamma(t)=\exp\biggl(e\sum_{k=1}^{m}\int_a^t p_k(s)\,\mathrm{d}s\biggr).$$

Then by (2.16) and (1.11) we get

$$\begin{aligned}\ell(\gamma)(t)&=\gamma(t)\sum_{i=1}^{m}p_i(t)\exp\biggl(e\sum_{k=1}^{m}\int_t^{\tau_i(t)}p_k(s)\,\mathrm{d}s\biggr)\\&\leqslant e\gamma(t)\sum_{i=1}^{m}p_i(t)=\gamma'(t)\qquad\text{a.e. on } I\end{aligned}$$

and the assumptions of Theorem 1.1 are fulfilled.

(iii) By (2.16), (2.17) and (1.12), for any $u\in C(I,\mathbb{R}_+)$ we have

$$\ell(\varphi(u))(t)-\ell(1)(t)\varphi(u)(t)=\sum_{k=1}^{m}p_k(t)\int_t^{\tau_k(t)}\sum_{i=1}^{m}p_i(\xi)u(\tau_i(\xi))\,\mathrm{d}\xi\leqslant\bar\ell(u)(t)$$

and the assumptions of Theorem 1.1 are fulfilled.

(iv) Define a function

$$\gamma(t)=\exp\biggl(-e\sum_{k=1}^{m}\int_a^t|p_k(s)|\,\mathrm{d}s\biggr).$$

Then by (2.16) and (1.13) we get

$$\ell(\gamma)(t) = \gamma(t)\sum_{i=1}^{m} p_i(t)\exp\biggl(e\sum_{k=1}^{m}\int_{\tau_i(t)}^{t}|p_k(s)|\,\mathrm{d}s\biggr)$$
$$\geqslant e\gamma(t)\sum_{i=1}^{m} p_i(t) = \gamma'(t) \qquad \text{a.e. on } I$$

and the assumptions of Theorem 1.1 are fulfilled. □

Remark 2.1. As has been said above, the problem (1.1), (1.2) has the Fredholm property. Therefore to prove Theorems 1.2 and 1.3 it is sufficient to show that the corresponding homogeneous problem (1.3) has only the trivial solution.

Proof of Theorem 1.2. (i) We shall show that the homogeneous problem (1.3) has only the trivial solution. Assume on the contrary that there exists a nontrivial solution u. Put

(2.18) $$m = -\min\{u(t)\colon\ t\in I\}, \qquad M = \max\{u(t)\colon\ t\in I\}.$$

From (1.14) by Theorem 1.1 (ii) we have $\ell_0 \in \mathcal{S}_{\mathcal{I}}$. Hence by Definition 1.1 we find that u must change sign on I, i.e.

(2.19) $$m > 0, \qquad M > 0.$$

Denote by γ_i $(i=0,1)$ the solutions of the problems

(2.20) $$\gamma_0'(t) = \ell_0(\gamma_0)(t) + \frac{1}{M}\,\ell_1([u]_+)(t), \qquad \gamma_0(a) = 0,$$

(2.21) $$\gamma_1'(t) = \ell_0(\gamma_1)(t) + \frac{1}{m}\,\ell_1([u]_-)(t), \qquad \gamma_1(a) = 0.$$

Due to $\ell_0 \in \mathcal{S}_{\mathcal{I}}$, by (2.20) and (2.21) we have

(2.22) $$\gamma_0(t) \geqslant 0, \qquad \gamma_1(t) \geqslant 0 \qquad \text{for } t\in I,$$

(2.23) $$\gamma_0'(t) \geqslant 0, \qquad \gamma_1'(t) \geqslant 0 \qquad \text{a.e. on } I.$$

It is clear that

$$(2.24)\qquad \big(M\gamma_0(t)+u(t)\big)'=\ell_0(M\gamma_0+u)(t)+\ell_1([u]_-)(t),$$
$$M\gamma_0(a)+u(a)=0,$$
$$(2.25)\qquad \big(m\gamma_1(t)-u(t)\big)'=\ell_0(m\gamma_1-u)(t)+\ell_1([u]_+)(t),$$
$$m\gamma_1(a)-u(a)=0,$$
$$\big(\gamma(t)-\gamma_0(t)-\gamma_1(t)\big)'=\ell_0(\gamma-\gamma_0-\gamma_1)(t)+h(t),$$
$$\gamma(a)-\gamma_0(a)-\gamma_1(a)>0,$$

where

$$h(t)=\ell_1\bigg(1-\frac{[u]_+}{M}-\frac{[u]_-}{m}\bigg)(t)\qquad \text{a.e. on } I.$$

Hence, taking into account that $\ell_0\in\mathcal{S}_{\mathcal{I}}$ and

$$\frac{[u(t)]_+}{M}+\frac{[u(t)]_-}{m}\leqslant 1\qquad \text{for } t\in I,$$

we have

$$(2.26)\qquad -M\gamma_0(t)\leqslant u(t)\leqslant m\gamma_1(t)\qquad \text{for } t\in I,$$
$$(2.27)\qquad \gamma_0(t)+\gamma_1(t)<\gamma(t)\qquad \text{for } t\in I.$$

From (2.24), (2.25), together with $\ell_i\in\mathcal{P}_{\mathcal{I}}$ $(i=0,1)$ and (2.26), we get

$$(2.28)\qquad \big(M\gamma_0(t)+u(t)\big)'\geqslant 0,\quad \big(m\gamma_1(t)-u(t)\big)'\geqslant 0\quad \text{a.e. on } I.$$

We can choose $t_1\in\,]a,b]$ and $t_2\in\,]a,b]$ such that

$$(2.29)\qquad u(t_1)=M,\qquad u(t_2)=-m.$$

Now we suppose that $t_1<t_2$ $(t_2<t_1)$. Integrating the first (the second) inequality of (2.28) from t_1 to t_2 (from t_2 to t_1), by (2.22) and (2.23) we get

$$(2.30)\qquad M+m\leqslant M\big(\gamma_0(t_2)-\gamma_0(t_1)\big)\leqslant M\gamma_0(b),$$
$$(2.31)\qquad \big(M+m\leqslant m\big(\gamma_1(t_1)-\gamma_1(t_2)\big)\leqslant m\gamma_1(b)\big).$$

On the other hand, from (2.26) together with (2.22), (2.23) and (2.29) we have

$$(2.32)\qquad m\leqslant M\gamma_0(t_2)\leqslant M\gamma_0(b),\qquad M\leqslant m\gamma_1(t_1)\leqslant m\gamma_1(b).$$

Now by (2.27), (2.30), (2.31) and (2.32) it is clear that

$$3 \leqslant 1 + \frac{M}{m} + \frac{m}{M} \leqslant \gamma_0(b) + \gamma_1(b) < \gamma(b),$$

which contradicts (1.15).

(ii) It is sufficient to show that the homogeneous problem (1.3) has only the trivial solution.

Assume on the contrary that there exists a nontrivial solution u_0 of the problem (1.3). Note that u_0 must change its sign and, consequently, (2.11) holds. Let c_0 be the number defined by (2.12). Then (2.13) holds and

$$\big(c_0\gamma(t) - u_0(t)\big)' \leqslant \ell(c_0\gamma - u_0)(t).$$

Hence, by (2.13) and the fact that ℓ is nonpositive, we find that $c_0\gamma - u_0$ is a nonincreasing function and for some $t_1 \in I$ the inequality (2.14) holds. Then (2.15) holds and, consequently,

$$u_0(b) > 0. \tag{2.33}$$

Now, if we put

$$v(t) = -u_0(t) \tag{2.34}$$

then we have that v is a solution of (1.3). Consequently, we can show as above that $v(b) > 0$, which is a contradiction to (2.33) and (2.34). □

Proof of Corollary 1.2. It follows from (1.16) that we can find $\varepsilon > 0$ such that

$$\int_a^b \sum_{k=1}^m [p_k(s)]_- \exp\bigg(\int_s^b \sum_{i=1}^m [p_i(\xi)]_+ \,\mathrm{d}\xi\bigg)\,\mathrm{d}s \leqslant 3 - \varepsilon \exp\bigg(\int_a^b \sum_{i=1}^m [p_i(\xi)]_+ \,\mathrm{d}\xi\bigg).$$

Put

$$\gamma(t) = \varepsilon \exp\bigg(\int_a^t \sum_{i=1}^m [p_i(\xi)]_+ \,\mathrm{d}\xi\bigg) + \int_a^t \sum_{k=1}^m [p_k(s)]_- \exp\bigg(\int_s^t \sum_{i=1}^m [p_i(\xi)]_+ \,\mathrm{d}\xi\bigg)\,\mathrm{d}s$$

for $t \in I$. Clearly, (1.15) is fulfilled and

$$\gamma'(t) = \sum_{i=1}^m [p_i(t)]_+ \gamma(t) + \sum_{k=1}^m [p_k(t)]_- \qquad \text{a.e. on } I. \tag{2.35}$$

Since γ is nondecreasing, from (1.17) we have

$$\big(\gamma(t) - \gamma(\tau_k(t))\big)[p_k(t)]_+ \geqslant 0 \qquad \text{a.e on } I \quad (k = 1, \dots, m).$$

Therefore (2.35) implies (1.14), where

$$\ell_0(v)(t) = \sum_{k=1}^m [p_k(t)]_+ v(\tau_k(t)), \qquad \ell_1(v)(t) = \sum_{k=1}^m [p_k(t)]_- v(\tau_k(t)).$$

□

Proof of Corollary 1.3. From (1.18) we have $\beta < 1$. Consequently, by Corollary 1.1 we find that $\ell_0 \in \mathcal{S}_{\mathcal{I}}$, where

$$\ell_0(v)(t) = \sum_{k=1}^m [p_k(t)]_+ v(\tau_k(t)).$$

Choose $\delta > 0$ and $\varepsilon > 0$ such that

$$(1-\beta)^{-1}\bigg(\int_a^b \sum_{k=1}^m [p_k(s)]_- \,\mathrm{d}s + \alpha\bigg) < 3 - \delta, \tag{2.36}$$

$$\varepsilon < \delta(1-\beta)\exp\bigg(-\int_a^b \sum_{k=1}^m [p_k(s)]_+ \,\mathrm{d}s\bigg). \tag{2.37}$$

Denote by γ the solution of the problem

$$u'(t) = \sum_{k=1}^m [p_k(t)]_+ u(\tau_k(t)) + \sum_{k=1}^m [p_k(t)]_-, \qquad u(a) = \varepsilon.$$

Due to $\ell_0 \in \mathcal{S}_{\mathcal{I}}$ we find that γ is a nondecreasing function. It is also clear that γ is a solution of the equation

$$\begin{aligned} u'(t) = {} & \sum_{k=1}^m [p_k(t)]_+ u(t) + \sum_{k=1}^m [p_k(t)]_+ \int_t^{\tau_k(t)} \sum_{i=1}^m [p_i(s)]_+ \gamma(\tau_i(s))\,\mathrm{d}s \\ & + \sum_{k=1}^m [p_k(t)]_+ \int_t^{\tau_k(t)} \sum_{i=1}^m [p_i(s)]_- \,\mathrm{d}s + \sum_{k=1}^m [p_k(t)]_-. \end{aligned}$$

Hence the Cauchy formula yields

$$\gamma(b) \leqslant \beta\gamma(b) + \bigg(\int_a^b \sum_{k=1}^m [p_k(s)]_- \,\mathrm{d}s + \alpha\bigg) + \varepsilon \exp\bigg(\int_a^b \sum_{i=1}^m [p_i(s)]_+ \,\mathrm{d}s\bigg).$$

This inequality and (2.36), (2.37) result in $\gamma(b) < 3$. Consequently, the assumptions of Theorem 1.2 are fulfilled. □

P r o o f of Theorem 1.3. We shall show that the homogeneous problem (1.3) has only the trivial solution. Assume on the contrary that there exists a nontrivial solution u. Let m and M be numbers defined by (2.18). Taking into account that $\ell_i \in \mathcal{P}_{\mathcal{I}}$ $(i = 0, 1)$ and (1.19), from Theorem 1.1 (i) we find that (2.19) holds. Choose t_1, $t_2 \in \,]a, b]$ such that (2.29) is fulfilled. Without loss of generality we can assume that $t_1 < t_2$. If we integrate the equation (1.3) from a to t_1 and from t_1 to t_2, then in view of (2.18), (2.19), (2.29) and $\ell_i \in \mathcal{P}_{\mathcal{I}}$ $(i = 0, 1)$ we get

$$M \leqslant M \cdot C + m \cdot A, \qquad M + m \leqslant M \cdot B + m \cdot D, \tag{2.38}$$

where

$$A = \int_a^{t_1} \ell_1(1)(s)\,\mathrm{d}s, \quad B = \int_{t_1}^{t_2} \ell_1(1)(s)\,\mathrm{d}s, \tag{2.39}$$

$$C = \int_a^{t_1} \ell_0(1)(s)\,\mathrm{d}s, \quad D = \int_{t_1}^{t_2} \ell_0(1)(s)\,\mathrm{d}s. \tag{2.40}$$

From (1.19), (2.38), (2.39) and (2.40) we have

$$\begin{gathered} C < 1, \qquad D < 1, \qquad B > 1, \\ M \leqslant \frac{A}{1 - C}\, m, \qquad m \leqslant \frac{B - 1}{1 - D}\, M. \end{gathered}$$

These inequalities imply

$$(1 - C)(1 - D) \leqslant A(B - 1). \tag{2.41}$$

On the other hand, we have

$$(1 - C)(1 - D) \geqslant 1 - (C + D) \geqslant 1 - \int_a^b \ell_0(1)(s)\,\mathrm{d}s$$

and

$$4A(B - 1) \leqslant (A + B - 1)^2 \leqslant \left(\int_a^b \ell_1(1)(s)\,\mathrm{d}s - 1\right)^2.$$

Hence by (2.41) we get

$$\int_a^b \ell_1(1)(s)\,\mathrm{d}s \geqslant 1 + 2\left(1 - \int_a^b \ell_0(1)(s)\,\mathrm{d}s\right)^{\frac{1}{2}},$$

which contradicts (1.20). □

References

[1] *N. V. Azbelev, V. P. Maksimov and L. F. Rakhmatullina*: Introduction to the Theory of Functional Differential Equations. Nauka, Moscow, 1991. (In Russian.)

[2] *Sh. Gelashvili and I. Kiguradze*: On multi-point boundary value problems for systems of functional differential and difference equations. Mem. Differential Equations Math. Phys. *5* (1995), 1–113.

[3] *P. Hartman*: Ordinary Differential Equations. John Wiley, New York, 1964.

[4] *I. Kiguradze and B. Půža*: On boundary value problems for systems of linear functional differential equations. Czechoslovak Math. J. *47* (1997), 341–373.

[5] *Š. Schwabik, M. Tvrdý and O. Vejvoda*: Differential and integral equations: boundary value problems and adjoints. Academia, Praha, 1979.

Authors' addresses: E. Bravyi, Department of Mathematical Analysis, Perm State Technical University, Komsomolsky str. 29a, 614000 Perm, Russia, e-mail: `bravyi@pi.ccl.ru`; R. Hakl, A. Lomtatidze, Department of Mathematical Analysis, Masaryk University, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic, e-mails: `{hakl,bacho}@math.muni.cz`.