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ON CAUCHY PROBLEM FOR FIRST ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NON-VOLTERRA'S TYPE

E. Bravyi, Perm, R. Hakl and A. Lomtatidze, Brno

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Abstract. On the segment $I=[a,b]$ consider the problem

$$u'(t)=f(u)(t), \quad u(a)=c,$$

where $f\colon C(I,\mathbb{R}) \to L(I,\mathbb{R})$ is a continuous, in general nonlinear operator satisfying Carathéodory condition, and $c\in\mathbb{R}$. The effective sufficient conditions guaranteeing the solvability and unique solvability of the considered problem are established. Examples verifying the optimality of obtained results are given, as well.

Keywords: nonlinear functional differential equation, initial value problem, non–Volterra's type operator

MSC 2000: 34K10

1. Statement of the problem and formulation of the main results

On the segment $I=[a,b]$ we will consider the functional differential equation

$$u'(t)=f(u)(t) \tag{1.1}$$

with the initial condition

$$u(a)=c, \tag{1.2}$$

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where $f\colon C(I,\mathbb{R})\to L(I,\mathbb{R})$ is a continuous operator and $c\in\mathbb{R}$. In the case when f is a Volterra operator, the problem (1.1), (1.2) has already been sufficiently studied (see [1]–[3], [5], [6], [9]–[24] and references therein). There is also a lot of interesting results on solvability and unique solvability of this problem even in the case when f is not a Volterra operator (see, e.g., [1], [2], [7]–[9], [23]). However, in that case the theory on the problem (1.1), (1.2) is not still completed. In the present paper, we try to fill this gap in a certain way. More precisely, nonimprovable in some sense conditions are found guaranteeing the existence and uniqueness of a solution of the problem (1.1), (1.2).

Along with (1.1) we will consider an important special case when (1.1) is the equation with deviated arguments, i.e.,

$$u'(t)=g\bigl(t,u(t),u(\tau_1(t)),\dots,u(\tau_m(t))\bigr), \tag{1.1'}$$

where $g\colon I\times\mathbb{R}^{m+1}\to\mathbb{R}$ is a function satisfying the local Carathéodory conditions and $\tau_k\colon I\to I$ $(k=1,\dots,m)$ are measurable functions.

Throughout this paper, the following notation and terms will be used:

$\mathbb{R}$ is the set of all real numbers;

$\mathbb{R}_+$ is the set of all nonnegative real numbers;

$$[x]_+=\frac{|x|+x}{2},\quad [x]_-=\frac{|x|-x}{2};$$

$C(I,\mathbb{R})$ is the Banach space of continuous functions $u\colon I\to\mathbb{R}$ with the norm

$$\|u\|_C=\max\{|u(t)|\colon t\in I\};$$

$C(I,\mathbb{R}_+)=\{u\in C(I,\mathbb{R})\colon u(t)\geqslant 0 \text{ for } t\in I\}$;

$L(I,\mathbb{R})$ is the Banach space of Lebesgue integrable functions $u\colon I\to\mathbb{R}$ with the norm

$$\|u\|_L=\int_a^b|u(t)|\,\mathrm{d}t;$$

$L(I,\mathbb{R}_+)=\{u\in L(I,\mathbb{R})\colon u(t)\geqslant 0 \text{ for almost all } t\in I\}$;

$\mathcal{L}_\mathcal{I}$ is the set of linear operators $\ell\colon C(I,\mathbb{R})\to L(I,\mathbb{R})$ such that

$$\sup\{|\ell(u)(\cdot)|\colon \|u\|_C=1\}\in L(I,\mathbb{R}_+);$$

$\mathcal{P}_\mathcal{I}$ is the set of linear operators $\ell\colon C(I,\mathbb{R})\to L(I,\mathbb{R})$ mapping $C(I,\mathbb{R})_+$ into $L(I,\mathbb{R}_+)$.

We will say that an operator $f\colon C(I,\mathbb{R})\to L(I,\mathbb{R})$ satisfies the local Carathéodory conditions if it is continuous and

$$f_r^*(\cdot)=\sup\{|f(u)(\cdot)|\colon \|u\|_C\leqslant r\}\in L(I,\mathbb{R}_+)$$

for an arbitrary $r\in\mathbb{R}_+$.

We will say that a function $g\colon I\times\mathbb{R}^{m+1}\to\mathbb{R}$ satisfies the local Carathéodory conditions if $g(\cdot,x_0,x_1,\dots,x_m)\colon I\to\mathbb{R}$ is measurable for all $(x_0,x_1,\dots,x_m)\in\mathbb{R}^{m+1}$, $g(t,\cdot,\dots,\cdot)\colon \mathbb{R}^{m+1}\to\mathbb{R}$ is continuous for almost all $t\in I$ and

$$g_r^*(\cdot)=\sup\{|g(\cdot,x_0,x_1,\dots,x_m)|\colon |x_i|\leqslant r\quad (i=0,1,\dots,m)\}\in L(I,\mathbb{R}_+)$$

for an arbitrary $r\in\mathbb{R}_+$.

An absolutely continuous function $u\colon I\to\mathbb{R}$ is said to be a solution of the equation (1.1) if it satisfies this equation almost everywhere on I.

Below we will always assume that the operator $f\colon C(I,\mathbb{R})\to L(I,\mathbb{R})$ and the function $g\colon I\times\mathbb{R}^{m+1}\to\mathbb{R}$ satisfy the local Carathéodory conditions.

Definition 1.1. We will say that an operator $\ell_0\in\mathcal{L}_\mathcal{I}$ belongs to the set $\mathcal{S}_\mathcal{I}$ if the homogeneous problem

$$u'(t)=\ell_0(u)(t),\quad u(a)=0 \tag{1.3}$$

has only the trivial solution and for any $h\in L(I,\mathbb{R}_+)$ and $c\in\mathbb{R}_+$, the solution of the equation

$$u'(t)=\ell_0(u)(t)+h(t) \tag{1.4}$$

satisfying (1.2) is a nonnegative function.

Effective conditions guaranteeing $\ell_0\in\mathcal{S}_\mathcal{I}$ can be found in [4].

Theorem 1.1. *Let there exist $\ell_0\in\mathcal{S}_\mathcal{I}\cap\mathcal{P}_\mathcal{I}$ and $h\in L(I,\mathbb{R}_+)$ such that for any $u\in C(I,\mathbb{R})$ the inequality*

$$f(u)(t)\operatorname{sgn}u(t)\leqslant\ell_0(|u|)(t)+h(t)\quad \text{a.e on } I \tag{1.5}$$

is fulfilled. Then the problem (1.1), (1.2) *has at least one solution.*

Remark 1.1. An analogous result follows from Theorem 1.1 in [7].

Theorem 1.2. *Let for any $u \in C(I, \mathbb{R})$ the inequality*

$$[f(u)(t) + \ell_1(u)(t) - \ell_0(u)(t)] \operatorname{sgn} u(t) \leqslant h(t) \quad \text{a.e. on } I \tag{1.6}$$

be fulfilled, where $\ell_i \in \mathcal{P}_{\mathcal{I}}$ $(i = 0, 1)$ and $h \in L(I, \mathbb{R}_+)$. If, moreover, either

$$\Big(\int_a^b \ell_1(1)(t)\,\mathrm{d}t\Big)^2 < 4\Big(1 - \int_a^b \ell_0(1)(t)\,\mathrm{d}t\Big) \tag{1.7}$$

or there exists an absolutely continuous function $\gamma\colon I \to\,]0, +\infty[$ such that

$$\gamma'(t) \geqslant \ell_0(\gamma)(t) + \ell_1(1)(t) \quad \text{a.e. on } I, \tag{1.8}$$

$$\gamma(b) \leqslant 2, \tag{1.9}$$

then the problem (1.1), (1.2) *has at least one solution.*

Corollary 1.1. *Let the inequality*

$$g(t, x_0, x_1, \ldots, x_m) \operatorname{sgn} x_0 \leqslant \sum_{i=0}^{m} p_i(t)|x_i| + h(t) \tag{1.10}$$

hold on the set $I \times \mathbb{R}^{m+1}$, where $p_i \in L(I, \mathbb{R}_+)$ $(i = 0, 1, \ldots, m)$ and $h \in L(I, \mathbb{R}_+)$. Let, moreover, one of the following three conditions be fulfilled:

$$\sum_{i=0}^{m} \int_t^{\tau_k(t)} p_i(s)\,\mathrm{d}s \leqslant \frac{1}{e} \quad \text{for } t \in I \quad (k = 1, \ldots, m); \tag{1.11}$$

$$\int_a^b \sum_{k=1}^{m} p_k(s)\sigma_k(s) \int_s^{\tau_k(s)} \sum_{i=0}^{m} p_i(\xi)\,\mathrm{d}\xi \exp\Big(\int_s^b \sum_{j=0}^{m} p_j(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}s < 1; \tag{1.12}$$

$$\sum_{i,k=0}^{m} \int_a^t p_k(s)\Big(\int_a^{\tau_k(s)} p_i(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}s \leqslant \alpha \sum_{i=0}^{m} \int_a^t p_i(s)\,\mathrm{d}s \quad \text{for } t \in I, \tag{1.13}$$

where $\tau_0(t) \equiv t$, $\sigma_k(t) = \frac{1}{2}(1 + \operatorname{sgn}(\tau_k(t) - t))$ for almost all $t \in I$ $(k = 1, \ldots, m)$ and $\alpha \in\,]0, 1[$. Then the problem (1.1′), (1.2) *has at least one solution.*

Corollary 1.2. *Let the inequality*

$$\Big[g(t, x_0, x_1, \ldots, x_m) + \sum_{i=0}^{m} p_i(t)x_i\Big] \operatorname{sgn} x_0 \leqslant h(t) \tag{1.14}$$

be fulfilled on the set $I \times \mathbb{R}^{m+1}$, *where* $p_i \in L(I, \mathbb{R})$ $(i = 0, 1, \ldots, m)$ *and* $h \in L(I, \mathbb{R}_+)$. *Let, moreover,*

$$(t - \tau_k(t))[p_k(t)]_- \geqslant 0 \quad \text{a.e. on } I \quad (k = 1, \ldots, m) \tag{1.15}$$

and

$$\sum_{k=0}^{m} \int_a^b [p_k(s)]_+ \exp\left(\sum_{i=0}^{m} \int_s^b [p_i(\xi)]_- \, \mathrm{d}\xi\right) \mathrm{d}s < 2. \tag{1.16}$$

Then the problem (1.1′), (1.2) *has at least one solution.*

Theorem 1.3. *Let there exist* $\ell_0 \in \mathcal{S}_{\mathcal{I}} \cap \mathcal{P}_{\mathcal{I}}$ *such that for any* $u_k \in C(I, \mathbb{R})$ $(k = 1, 2)$ *the inequality*

$$[f(u_1)(t) - f(u_2)(t)] \operatorname{sgn}(u_1(t) - u_2(t)) \leqslant \ell_0(|u_1 - u_2|)(t) \quad \text{a.e. on } I \tag{1.17}$$

is fulfilled. Then the problem (1.1), (1.2) *has a unique solution.*

Theorem 1.4. *Let for any* $u_k \in C(I, \mathbb{R})$ $(k = 1, 2)$ *the inequality*

$$\begin{aligned} &[f(u_1)(t) - f(u_2)(t) + \ell_1(u_1 - u_2)(t) - \ell_0(u_1 - u_2)(t)] \\ &\qquad \times \operatorname{sgn}(u_1(t) - u_2(t)) \leqslant 0 \quad \text{a.e. on } I \end{aligned} \tag{1.18}$$

be fulfilled, where $\ell_i \in \mathcal{P}_{\mathcal{I}}$ $(i = 0, 1)$. *Let, moreover, either the condition* (1.7) *be satisfied or there exist an absolutely continuous function* $\gamma\colon I \to \,]0, +\infty[$ *satisfying conditions* (1.8), (1.9). *Then the problem* (1.1), (1.2) *has a unique solution.*

Corollary 1.3. *Let the inequality*

$$[g(t, x_0, x_1, \ldots, x_m) - g(t, y_0, y_1, \ldots, y_m)] \operatorname{sgn}(x_0 - y_0) \leqslant \sum_{i=0}^{m} p_i(t)|x_i - y_i| \tag{1.19}$$

be fulfilled on the set $I \times \mathbb{R}^{m+1}$, *where* $p_i \in L(I, \mathbb{R}_+)$ $(i = 0, 1, \ldots, m)$. *Let, moreover, one of the conditions* (1.11), (1.12) *and* (1.13) *hold, where* $\tau_0(t) \equiv t$, $\sigma_k(t) = \frac{1}{2}(1 + \operatorname{sgn}(\tau_k(t) - t))$ *for almost all* $t \in I$ $(k = 1, \ldots, m)$ *and* $\alpha \in \,]0, 1[$. *Then the problem* (1.1′), (1.2) *has a unique solution.*

Corollary 1.4. *Let the inequality*

$$\left[g(t, x_0, x_1, \ldots, x_m) - g(t, y_0, y_1, \ldots, y_m) + \sum_{i=0}^{m} p_i(t)(x_i - y_i)\right] \operatorname{sgn}(x_0 - y_0) \leqslant 0 \tag{1.20}$$

be fulfilled on the set $I \times \mathbb{R}^{m+1}$, *where* $p_i \in L(I, \mathbb{R})$ $(i = 0, 1, \dots, m)$. *Let, moreover, the conditions* (1.15) *and* (1.16) *be satisfied. Then the problem* (1.1′), (1.2) *has a unique solution.*

At the end of this section, we introduce examples verifying the optimality of the above formulated conditions in the existence and uniqueness theorems.

Example 1.1. Consider the differential equation

$$u'(t) = \sum_{i=0}^{m} p_i(t)[|u(\tau_i(t))| + 1], \tag{1.21}$$

where $p_k \in L(I, \mathbb{R}_+)$ $(k = 0, 1, \dots, m)$, $\tau_0(t) \equiv t$ and $\tau_i\colon I \to I$ $(i = 1, \dots, m)$ are measurable functions. If the condition (1.13) is fulfilled, where $\alpha \in]0, 1[$, then by Corollary 1.3 the problem (1.21), (1.2) has a unique solution.

Let us show that if

$$\sum_{i,k=0}^{m} \int_a^t p_k(s)\left(\int_a^{\tau_k(s)} p_i(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s \geqslant \sum_{i=0}^{m} \int_a^t p_i(s)\,\mathrm{d}s > 0 \quad \text{for } t \in]a, b], \tag{1.22}$$

then for any $c \in \mathbb{R}_+$ the problem (1.21), (1.2) has no solution[1]. Assume on the contrary that for some $c \in \mathbb{R}_+$ this problem has a solution u. Then we find

$$u(t) = c + \sum_{i=0}^{m} \int_a^t p_i(s)|u(\tau_i(s))|\,\mathrm{d}s + u_0(t), \tag{1.23}$$

where

$$u_0(t) = \sum_{i=0}^{m} \int_a^t p_i(s)\,\mathrm{d}s.$$

If we put

$$\varrho = \inf\left\{\frac{u(t)}{u_0(t)}\colon\ a < t \leqslant b\right\},$$

then in view of (1.22) and (1.23) we get

$$\begin{aligned}\varrho &\geqslant \inf\left\{\frac{1}{u_0(t)} \sum_{i=0}^{m} \int_a^t p_i(s)|u(\tau_i(s))|\,\mathrm{d}s\colon\ t \in I\right\} + 1 \\ &\geqslant \varrho \inf\left\{\frac{1}{u_0(t)} \sum_{i=0}^{m} \int_a^t p_i(s) u_0(\tau_i(s))\,\mathrm{d}s\colon\ t \in I\right\} + 1 \geqslant \varrho + 1.\end{aligned}$$

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[1] (1.22) is fulfilled, e.g., if $p_0(t) \equiv 0$, $\tau_k(t) \equiv b_k \in]a, b]$, $p_k(t) \equiv \alpha_k/(b_k - a) \geqslant 0$ $(k = 1, \dots, m)$ and $\sum\limits_{k=1}^{m} \alpha_k = 1$.

The contradiction obtained proves the nonsolvability of the problem (1.21), (1.2).

This example shows that in Corollaries 1.1 and 1.3, the assumption $\alpha \in]0,1[$ in the inequality (1.13) cannot be replaced by the assumption $\alpha \in]0,1]$.

Example 1.2. Let $\varepsilon \in]0,1[$, $a=0$, $b=3$,

$$p(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant 1, \\ 0 & \text{for } 1 < t < 2 - \frac{\varepsilon}{2}, \\ 1 & \text{for } 2 - \frac{\varepsilon}{2} \leqslant t \leqslant 3, \end{cases}$$

$$\overline{p}(t) = \begin{cases} 0 & \text{for } 0 \leqslant t \leqslant 1, \\ \frac{1}{2-t} & \text{for } 1 < t < 2 - \frac{\varepsilon}{2}, \\ 0 & \text{for } 2 - \frac{\varepsilon}{2} \leqslant t \leqslant 3, \end{cases}$$

$$\tau(t) = \begin{cases} 3 & \text{for } 0 \leqslant t < 2 - \frac{\varepsilon}{2}, \\ 1 & \text{for } 2 - \frac{\varepsilon}{2} \leqslant t \leqslant 3. \end{cases}$$

Consider the differential equation

$$u'(t) = -p(t)u(\tau(t)) - \overline{p}(t)u(t) + q(t), \tag{1.24}$$

where $q \in L(I,\mathbb{R})$. It is clear that the operator

$$f(u)(t) = -p(t)u(\tau(t)) - \overline{p}(t)u(t) + q(t)$$

satisfies the conditions (1.6) and (1.18), where $\ell_1(u)(t) = p(t)u(\tau(t))$, $\ell_0(u)(t) \equiv 0$ and $h(t) = |q(t)|$. Moreover, the function

$$\gamma(t) = \delta + \int_0^t p(s)\,\mathrm{d}s,$$

where $\delta \in]0, \frac{\varepsilon}{2}]$, satisfies the inequalities (1.8) and

$$\gamma(b) \leqslant 2 + \varepsilon. \tag{1.25}$$

On the other hand, the homogeneous problem

$$u'(t) = -p(t)u(\tau(t)) - \overline{p}(t)u(t), \quad u(a) = 0$$

has the nontrivial solution

$$u_0(t) = \begin{cases} t & \text{for } 0 \leqslant t \leqslant 1, \\ 2-t & \text{for } 1 < t \leqslant 3. \end{cases}$$

Consequently, we can find $c \in \mathbb{R}$ and $q \in L(I,\mathbb{R})$ such that the problem (1.24), (1.2) has no solution.

This example shows that in Theorems 1.2 and 1.4 the inequalities (1.9) and (1.7) cannot be replaced by the inequalities (1.25) and

$$\int_a^b \ell_1(1)(t)\,\mathrm{d}t < (2+\varepsilon)\Big(1 - \int_a^b \ell_0(1)(t)\,\mathrm{d}t\Big)^{\frac{1}{2}},$$

respectively, for an arbitrarily small $\varepsilon > 0$.

The same example shows as well that in Corollaries 1.2 and 1.4, the inequality (1.16) cannot be replaced by the inequality

$$\sum_{k=0}^{m} \int_a^b [p_k(s)]_+ \exp\Big(\sum_{i=0}^{m} \int_s^b [p_i(\xi)]_- \,\mathrm{d}\xi\Big)\,\mathrm{d}s < 2+\varepsilon$$

for an arbitrarily small $\varepsilon > 0$.

2. Auxiliary propositions

2.1. Lemmas on solvability of problem (1.1), (1.2).

From Corollary 2 of [9] we get

Lemma 2.1. *Let there exist a positive number ϱ and an operator $\ell \in \mathcal{L}_{\mathcal{I}}$ such that the homogeneous problem*

$$u'(t) + \ell(u)(t) = 0, \quad u(a) = 0 \tag{2.1}$$

has only the trivial solution and for every $\lambda \in \,]0,1[$ an arbitrary solution of the problem

$$u'(t) + \ell(u)(t) = \lambda[f(u)(t) + \ell(u)(t)], \quad u(a) = \lambda c \tag{2.2}$$

admits the estimate

$$\|u\|_C \leqslant \varrho. \tag{2.3}$$

Then the problem (1.1), (1.2) *has at least one solution.*

Definition 2.1. We will say that a pair of operators (ℓ, ℓ_0) belongs to the set $\mathcal{A}_{\mathcal{I}}$ if $\ell \in \mathcal{L}_{\mathcal{I}}$, $\ell_0 \in \mathcal{P}_{\mathcal{I}}$ and there exists a positive number r such that for an arbitrary $h \in L(I, \mathbb{R}_+)$, any absolutely continuous function u satisfying the inequality

$$[u'(t) + \ell(u)(t)]\operatorname{sgn} u(t) \leqslant \ell_0(|u|)(t) + h(t) \quad \text{a.e. on } I \tag{2.4}$$

admits the estimate

$$\|u\|_C \leqslant r(|u(a)| + \|h\|_L). \tag{2.5}$$

Lemma 2.2. *Let there exist* $(\ell, \ell_0) \in \mathcal{A}_\mathcal{I}$ *and* $h \in L(I, \mathbb{R}_+)$ *such that for any* $u \in C(I, \mathbb{R})$ *the inequality*

$$[f(u)(t) + \ell(u)(t)] \operatorname{sgn} u(t) \leqslant \ell_0(|u|)(t) + h(t) \quad \text{a.e. on } I \tag{2.6}$$

is fulfilled. Then the problem (1.1), (1.2) *has at least one solution.*

P r o o f. First note that due to the condition $(\ell, \ell_0) \in \mathcal{A}_\mathcal{I}$, the homogeneous problem (2.1) has only the trivial solution.

Let r be a number from Definition 2.1. Put

$$\varrho = r(|c| + \|h\|_L).$$

Assume now that u is a solution of the problem (2.2) for some $\lambda \in\,]0, 1[$. Then according to (2.6) it satisfies the differential inequality (2.4). Hence, in view of the condition $(\ell, \ell_0) \in \mathcal{A}_\mathcal{I}$ and the fact how ϱ is defined, we get the estimate (2.3).

Since ϱ depends neither on u nor on λ, from Lemma 2.1 it follows that the estimate (2.3) guarantees the solvability of the problem (1.1), (1.2). □

Lema 2.3. *Let there exist* $(\ell, \ell_0) \in \mathcal{A}_\mathcal{I}$ *such that for any* $u_1, u_2 \in C(I, \mathbb{R})$ *the inequality*

$$\begin{aligned}[f(u_1)(t) - f(u_2)(t) + \ell(u_1 - u_2)(t)] \operatorname{sgn}(u_1(t) - u_2(t)) \\ \leqslant \ell_0(|u_1 - u_2|)(t) \quad \text{a.e. on } I\end{aligned} \tag{2.7}$$

is fulfilled. Then the problem (1.1), (1.2) *has a unique solution.*

P r o o f. (2.7) implies that the operator f for any $u \in C(I, \mathbb{R})$ satisfies the inequality (2.6), where $h(t) = |f(0)(t)|$. Hence by Lemma 2.2 the problem (1.1), (1.2) is solvable. It remains to show that this problem has not more than one solution.

Let u_1 and u_2 be arbitrary solutions of the problem (1.1), (1.2). Put $u(t) = u_1(t) - u_2(t)$. Then by (2.7) we get

$$[u'(t) + \ell(u)(t)] \operatorname{sgn} u(t) \leqslant \ell_0(|u|)(t) \quad \text{a.e. on } I, \quad u(a) = 0.$$

This inequality and the condition $(\ell, \ell_0) \in \mathcal{A}_\mathcal{I}$ result in that $u(t) \equiv 0$. Consequently, $u_1(t) \equiv u_2(t)$. □

2.2 Lemmas on a priori estimates.

Lemma 2.4. *Let $\ell_0 \in \mathcal{L}_{\mathcal{I}}$ and let the homogeneous problem* (1.3) *have only the trivial solution. Then there exists a positive number r_0 such that for any $h \in L(I,\mathbb{R})$, an arbitrary solution of the equation* (1.4) *admits the estimate*

$$\|u\|_C \leqslant r_0(|u(a)| + \|h\|_L). \tag{2.8}$$

P r o o f. Denote by

$$\mathbb{R} \times L(I,\mathbb{R}) = \{(c,h)\colon\ c \in \mathbb{R},\ \ h \in L(I,\mathbb{R})\}$$

the Banach space with the norm

$$\|(c,h)\|_{\mathbb{R}\times L} = |c| + \|h\|_L$$

and by V the operator mapping every $(c,h) \in \mathbb{R} \times L(I,\mathbb{R})$ to the solution v of the problem (1.4), (1.2). According to Theorem 1.4 of [8], $V\colon\ \mathbb{R} \times L(I,\mathbb{R}) \to C(I,\mathbb{R})$ is a linear bounded operator. Denote by r_0 the norm of V. Then, clearly, for any $(c,h) \in \mathbb{R} \times L(I,\mathbb{R})$ the inequality

$$\|V(c,h)\|_C \leqslant r_0(|c| + \|h\|_L)$$

holds. Consequently, an arbitrary solution u of the equation (1.4) admits the estimate (2.8). □

From the definition of the set $\mathcal{S}_{\mathcal{I}}$ we immediately obtain

Lemma 2.5. *Let $\ell_0 \in \mathcal{S}_{\mathcal{I}}$, $h \in L(I,\mathbb{R})$ and $v_i\colon\ I \to \mathbb{R}$ $(i = 1,2)$ be absolutely continuous functions satisfying the inequalities*

$$v_1'(t) \leqslant \ell_0(v_1)(t) + h(t), \quad v_2'(t) \geqslant \ell_0(v_2)(t) + h(t) \quad \text{a.e. on } I,$$

and

$$v_1(a) \leqslant v_2(a).$$

Then

$$v_1(t) \leqslant v_2(t) \quad \text{for } t \in I.$$

Lemma 2.6. *Let* $\ell_0 \in \mathcal{P}_{\mathcal{I}}$. *If either*

$$\int_a^b \ell_0(1)(t)\,\mathrm{d}t < 1$$

or there exists an absolutely continuous function $\gamma\colon I \to]0,+\infty[$ *such that*

$$\gamma'(t) \geqslant \ell_0(\gamma)(t) \quad \text{a.e. on } I,$$

then $\ell_0 \in \mathcal{S}_{\mathcal{I}}$.

Lemma 2.6 is a corollary of Theorem 1.1 in [4].

Lemma 2.7. *If* $\ell_0 \in \mathcal{S}_{\mathcal{I}} \cap \mathcal{P}_{\mathcal{I}}$, *then*

$$(0,\ell_0) \in \mathcal{A}_{\mathcal{I}}.$$

Proof. Let r_0 be the number appearing in Lemma 2.4 and $u\colon I \to \mathbb{R}$ an arbitrary absolutely continuous function satisfying the differential inequality

$$u'(t)\operatorname{sgn} u(t) \leqslant \ell_0(|u|)(t) + h(t) \quad \text{a.e. on } I,$$

i.e.

$$|u(t)|' \leqslant \ell_0(|u|)(t) + h(t) \quad \text{a.e. on } I. \tag{2.9}$$

Then by Lemma 2.5 it follows that

$$|u(t)| \leqslant \overline{u}(t) \quad \text{for } t \in I, \tag{2.10}$$

where $\overline{u}(t)$ is a solution of (1.4) satisfying the initial condition

$$\overline{u}(a) = |u(a)|. \tag{2.11}$$

According to Lemma 2.4 we have

$$\|\overline{u}\|_C \leqslant r_0(|u(a)| + \|h\|_L). \tag{2.12}$$

(2.10) and (2.12) yield the estimate (2.5), where $r = r_0$ is a number which depends neither on u nor on h. Consequently, $(0,\ell_0) \in \mathcal{A}_{\mathcal{I}}$. $\square$

Lemma 2.8. *If operators* $\ell_i \in \mathcal{P}_{\mathcal{I}}$ $(i=0,1)$ *satisfy the inequality* (1.7) *then*

$$(\ell_1 - \ell_0, 0) \in \mathcal{A}_I. \tag{2.13}$$

Proof. First note that due to (1.7) and Lemma 2.6 we have $\ell_0 \in \mathcal{S}_{\mathcal{I}}$ and, consequently, the homogeneous problem (1.3) has only the trivial solution. Let r_0 be the number appearing in Lemma 2.4. According to Definition 2.1, it is sufficient to show that there exists a positive number r such that for any $h \in L(I, \mathbb{R}_+)$, an arbitrary absolutely continuous function $u\colon I \to \mathbb{R}$ satisfying the differential inequality

$$[u'(t) + \ell_1(u)(t) - \ell_0(u)(t)] \operatorname{sgn} u(t) \leqslant h(t) \tag{2.14}$$

admits the estimate (2.5). If u does not change sign then (2.14) implies (2.9). From (2.9) by Lemma 2.5 we have the estimate (2.10), where $\overline{u}$ is a solution of the equation (1.4) satisfying the initial condition (2.11). On the other hand, due to Lemma 2.4, the function $\overline{u}$ admits the estimate (2.12). Consequently,

$$\|u\|_C \leqslant r_0(|u(a)| + \|h\|_L). \tag{2.15}$$

Suppose now that u changes sign. Then

$$\mu_i = \max\{(-1)^i u(t)\colon\ t \in I\} > 0 \quad (i=0,1). \tag{2.16}$$

Moreover, there exist numbers $a_i \in [a,b]$ and $b_i \in [a_i, b]$ $(i=0,1)$ such that

$$[a_0, b_0] \cap [a_1, b_1] = \emptyset, \tag{2.17}$$

$$0 \leqslant (-1)^i u(a_i) \leqslant |u(a)|, \quad \mu_i = (-1)^i u(b_i) \quad (i=0,1) \tag{2.18}$$

and for every $i \in \{0,1\}$ either $a_i = b_i$ or $a_i < b_i$ and $(-1)^i u(t) > 0$ for $a_i < t < b_i$. Therefore, from (2.14) we find for every $i \in \{0,1\}$ that

$$(-1)^i u'(t) \leqslant (-1)^i[\ell_0(u)(t) - \ell_1(u)(t)] + h(t) \quad \text{a.e. on } [a_i, b_i]. \tag{2.19$_i$}$$

If we integrate the inequality (2.19_0) from a_0 to b_0 and the inequality (2.19_1) from a_1 to b_1, then in view of (2.18) we get

$$\mu_0 \leqslant |u(a)| + \int_{a_0}^{b_0} \ell_0(u)(t)\,\mathrm{d}t - \int_{a_0}^{b_0} \ell_1(u)(t)\,\mathrm{d}t + \int_{a_0}^{b_0} h(t)\,\mathrm{d}t,$$
$$\mu_1 \leqslant |u(a)| + \int_{a_1}^{b_1} \ell_1(u)(t)\,\mathrm{d}t - \int_{a_1}^{b_1} \ell_0(u)(t)\,\mathrm{d}t + \int_{a_1}^{b_1} h(t)\,\mathrm{d}t.$$

Hence by (2.16) we obtain

$$(1-\eta_{00})\mu_0 \leqslant \eta_{10}\mu_1 + w(u,h), \tag{2.20}$$

$$(1-\eta_{01})\mu_1 \leqslant \eta_{11}\mu_0 + w(u,h), \tag{2.21}$$

where

$$\eta_{ik} = \int_{a_k}^{b_k} \ell_i(1)(t)\,\mathrm{d}t \quad (i,k=0,1)$$

and

$$w(u,h) = |u(a)| + \int_a^b h(t)\,\mathrm{d}t. \tag{2.22}$$

Moreover, on account of (1.7) and (2.17) we have

$$\delta = 1 - \int_a^b \ell_0(1)(t)\,\mathrm{d}t - \frac{1}{4}\Big(\int_a^b \ell_1(1)(t)\,\mathrm{d}t\Big)^2 > 0,$$

$$\eta_{00} + \eta_{01} \leqslant \int_a^b \ell_0(1)(t)\,\mathrm{d}t < 1,$$

$$(1-\eta_{00})(1-\eta_{01}) \geqslant 1 - (\eta_{00}+\eta_{01}) \geqslant 1 - \int_a^b \ell_0(1)(t)\,\mathrm{d}t$$

and

$$\begin{aligned}\eta_{10}\eta_{11} &\leqslant \frac{1}{4}(\eta_{10}+\eta_{11})^2 \leqslant \frac{1}{4}\Big(\int_a^b \ell_1(1)(t)\,\mathrm{d}t\Big)^2 \\ &= 1 - \int_a^b \ell_0(1)(t)\,\mathrm{d}t - \delta \leqslant (1-\eta_{00})(1-\eta_{01}) - \delta.\end{aligned} \tag{2.23}$$

Put

$$r = r_0 + \frac{1}{\delta}\Big(1 + \int_a^b \ell_1(1)(t)\,\mathrm{d}t\Big).$$

Then, according to (2.23) and the fact that $1-\eta_{0i}$ $(i=0,1)$ are positive numbers, (2.20) and (2.21) yield

$$\begin{aligned}(1-\eta_{00})(1-\eta_{01})\mu_0 &\leqslant \eta_{10}(1-\eta_{01})\mu_1 + (1-\eta_{01})w(u,h) \\ &\leqslant \eta_{10}\eta_{11}\mu_0 + (\eta_{10}+1)w(u,h) \\ &\leqslant [(1-\eta_{00})(1-\eta_{01}) - \delta]\mu_0 + r\delta w(u,h)\end{aligned}$$

and

$$\begin{aligned}(1-\eta_{00})(1-\eta_{01})\mu_1 &\leqslant \eta_{11}(1-\eta_{00})\mu_0 + (1-\eta_{00})w(u,h) \\ &\leqslant \eta_{10}\eta_{11}\mu_1 + (\eta_{11}+1)w(u,h) \\ &\leqslant \big[(1-\eta_{00})(1-\eta_{01}) - \delta\big]\mu_1 + r\delta w(u,h).\end{aligned}$$

Therefore,

$$\mu_i \leqslant r w(u,h) \quad (i=0,1),$$

whence, by (2.16) and (2.22), we have the estimate (2.5). In the case when u does not change sign, the estimate (2.5) follows from (2.15). Therefore, since r depends neither on u nor on h, the lemma is valid. □

Lema 2.9. *Let* $\ell_i \in \mathcal{P}_{\mathcal{I}}$ $(i=0,1)$ *and let there exist an absolutely continuous function* $\gamma\colon I \to \,]0,+\infty[$ *satisfying the inequalities* (1.8) *and* (1.9). *Then the condition* (2.13) *holds.*

Proof. Due to (1.8) and Lemma 2.6 we have $\ell_0 \in \mathcal{S}_{\mathcal{I}}$ and, consequently, the assumptions of Lemma 2.4 are fulfilled. Let r_0 be the number appearing in Lemma 2.4 and

$$r = r_0 + 4(1+\gamma(b))\big(4-(\gamma(b)-\gamma(a))^2\big)^{-1} r_0. \tag{2.24}$$

Let $h \in L(I,\mathbb{R})$ and let $u\colon I \to \mathbb{R}$ be an arbitrary absolutely continuous function satisfying the inequality (2.14). To prove the lemma it is sufficient to show that u satisfies the estimate (2.5) as well.

First assume that u does not change sign. Then from (2.14) we find (2.9). Hence by Lemma 2.5 it follows that (2.10) holds, where $\overline{u}$ is a solution of the equation (1.4) satisfying the initial condition (2.11). According to Lemma 2.4, the function $\overline{u}$ admits the estimate (2.12). Hence by (2.10) and (2.24) we get the estimate (2.5).

Suppose now that u changes sign. Then the inequalities (2.16) are fulfilled.

Denote by γ_i $(i=0,1)$ the solutions of the problems

$$\gamma_0'(t) = \ell_0(\gamma_0)(t) + \frac{1}{\mu_0}\ell_1([u]_+)(t), \quad \gamma_0(a)=0, \tag{2.25}$$

$$\gamma_1'(t) = \ell_0(\gamma_1)(t) + \frac{1}{\mu_1}\ell_1([u]_-)(t), \quad \gamma_1(a)=0. \tag{2.26}$$

Then

$$\begin{aligned}
(\mu_0\gamma_0(t)+\overline{u}(t))' &= \ell_0(\mu_0\gamma_0+\overline{u})(t) + h_0(t) \quad \text{a.e. on } I,\\
(\mu_1\gamma_1(t)+\overline{u}(t))' &= \ell_0(\mu_1\gamma_1+\overline{u})(t) + h_1(t) \quad \text{a.e. on } I,
\end{aligned}$$

where

$$h_0(t) = \ell_1([u]_+)(t) + h(t), \quad h_1(t) = \ell_1([u]_-)(t) + h(t).$$

On the other hand, from (2.14) we get

$$[u(t)]'_+ \leqslant \ell_0([u]_+)(t) + h_1(t) \quad \text{a.e. on } I,$$
$$[u(t)]'_- \leqslant \ell_0([u]_-)(t) + h_0(t) \quad \text{a.e. on } I.$$

Moreover,

$$[u(a)]_+ \leqslant \mu_1\gamma_1(a) + \overline{u}(a), \quad [u(a)]_- \leqslant \mu_0\gamma_0(a) + \overline{u}(a).$$

Hence by Lemma 2.5,

$$[u(t)]_+ \leqslant \mu_1\gamma_1(t) + \overline{u}(t) \quad \text{for } t \in I, \tag{2.27}$$
$$[u(t)]_- \leqslant \mu_0\gamma_0(t) + \overline{u}(t) \quad \text{for } t \in I. \tag{2.28}$$

Equations (2.25) and (2.26) immediately imply

$$(\gamma_0(t) + \gamma_1(t))' \leqslant \ell_0(\gamma_0 + \gamma_1)(t) + \ell_1(1)(t) \quad \text{a.e. on } I,$$
$$\gamma_0(a) + \gamma_1(a) = 0 < \gamma(a),$$

whence by (1.8) and Lemma 2.5 we find

$$\gamma_0(t) + \gamma_1(t) \leqslant \gamma(t) \quad \text{for } t \in I.$$

Due to this inequality and the fact that ℓ_0 is a nonnegative operator, we have

$$(\gamma_0(t) + \gamma_1(t))' \leqslant \gamma'(t).$$

Now, if we integrate the last inequality from a to b, we get

$$\gamma_0(b) + \gamma_1(b) \leqslant \gamma(b) - \gamma(a).$$

Taking into account the monotonicity of γ_i $(i = 0, 1)$, from (2.16), (2.27) and (2.28) we obtain

$$\mu_0 \leqslant \mu_1\gamma_1(b) + \|\overline{u}\|_C, \quad \mu_1 \leqslant \mu_0\gamma_0(b) + \|\overline{u}\|_C,$$
$$\mu_0 \leqslant \gamma_1(b)\gamma_0(b)\mu_0 + (1 + \gamma_1(b))\|\overline{u}\|_C \leqslant \frac{(\gamma(b) - \gamma(a))^2}{4}\,\mu_0 + (1 + \gamma(b))\|\overline{u}\|_C$$

and

$$\mu_1 \leqslant \frac{(\gamma(b) - \gamma(a))^2}{4}\,\mu_1 + (1 + \gamma(b))\|\overline{u}\|_C.$$

By (1.9) the last two inequalities result in

$$\mu_i \leqslant 4(1 + \gamma(b))\bigl(4 - (\gamma(b) - \gamma(a))^2\bigr)^{-1}\|\overline{u}\|_C \quad (i = 0, 1).$$

Hence in view of (2.12) and (2.24) we get the estimate (2.5). □

Theorem 1.1 follows from Lemmas 2.2 and 2.7, Theorem 1.2 follows from Lemmas 2.2, 2.8 and 2.9, Theorem 1.3 follows from Lemmas 2.3 and 2.7, and Theorem 1.4 follows from Lemmas 2.3, 2.8 and 2.9.

P r o o f of Corollaries 1.1 and 1.3. Put

$$f(u)=g\big(t,u(t),u(\tau_1(t)),\dots,u(\tau_m(t))\big) \tag{3.1}$$

and

$$\ell_0(u)(t)=\sum_{i=0}^{m}p_i(t)u(\tau_i(t)),\quad \text{where } \tau_0(t)\equiv t.$$

Then the equation (1.1′) and the conditions (1.10) and (1.19) can be written as (1.1), (1.5) and (1.17). According to Theorems 1.1 and 1.3, to prove Corollaries 1.1 and 1.3 it is sufficient to show that

$$\ell_0\in\mathcal{S}_{\mathcal{I}}.$$

But this inclusion follows from Corollary 1.1 in [4]. □

P r o o f of Corollaries 1.2 and 1.4. Let f be an operator defined by (3.1). Define operators

$$\ell_1(u)(t)=\sum_{i=0}^{m}[p_i(t)]_+u(\tau_i(t)),\quad \ell_0(u)(t)=\sum_{i=0}^{m}[p_i(t)]_-u(\tau_i(t)), \tag{3.2}$$

where $\tau_0(t)\equiv t$. Then the condition (1.14) (the condition (1.20)) can be written as (1.6) (as (1.18)). By virtue of Theorem 1.2 (Theorem 1.4), to prove Corollary 1.2 (Corollary 1.4) it is sufficient to show that the function

$$\gamma(t)=\varepsilon\exp\bigg(\sum_{k=0}^{m}\int_a^t[p_k(\xi)]_-\,\mathrm{d}\xi\bigg)+\sum_{i=0}^{m}\int_a^t[p_i(s)]_+\exp\bigg(\sum_{k=0}^{m}\int_s^t[p_k(\xi)]_-\,\mathrm{d}\xi\bigg)\,\mathrm{d}s,$$

where $\varepsilon>0$ is such that

$$\sum_{i=0}^{m}\int_a^b[p_i(s)]_+\exp\bigg(\sum_{k=0}^{m}\int_s^b[p_k(\xi)]_-\,\mathrm{d}\xi\bigg)\,\mathrm{d}s\leqslant 2-\varepsilon\exp\bigg(\sum_{k=0}^{m}\int_a^b[p_k(\xi)]_-\,\mathrm{d}\xi\bigg),$$

satisfies the inequalities (1.8) and (1.9).

First note that the function γ is nondecreasing since

$$\gamma'(t)=\sum_{i=0}^{m}[p_i(t)]_++\sum_{k=0}^{m}[p_k(t)]_-\gamma(t)\quad\text{a.e. on } I. \tag{3.3}$$

Therefore (1.15) implies

$$[p_k(t)]_-\gamma(t) \geqslant [p_k(t)]_-\gamma(\tau_k(t)) \quad \text{a.e. on } I \quad (k=1,\dots,m).$$

By the last inequalities and (3.2), the inequality (1.8) follows from (3.3). The inequality (1.9) follows from (1.16). □

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Authors' addresses: E. Bravyi, Department of Mathematical Analysis, Perm State Technical University, Komsomolsky str. 29a, 614000 Perm, Russia, e-mail: `bravyi@pi.ccl.ru`; R. Hakl, Mathematical Institute, Academy of Sciences of the Czech Republic, Žižkova 22, 616 62 Brno, Czech Republic, e-mail: `hakl@ipm.cz`; A. Lomtatidze, Mathematical Institute, Academy of Sciences of the Czech Republic, Žižkova 22, 616 62 Brno, Czech Republic, and Department of Mathematical Analysis, Masaryk University, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic, e-mail: `bacho@math.muni.cz`.