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ON A BOUNDARY-VALUE PROBLEM OF PERIODIC TYPE FOR FIRST-ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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We establish unimprovable sufficient conditions for the unique solvability of the boundary-value problem

$$u'(t) = \ell(u)(t) + q(t), \quad u(a) = \lambda u(b) + c$$

and for the nonnegativity of its solution; here, $\ell\colon C([a,b];R) \to L([a,b];R)$ is a linear bounded operator, $q \in L([a,b];R)$, $\lambda \in R_+$, and $c \in R$.

Introduction

The following notation is used throughout the paper:

R is the set of all real numbers, $R_+ = [0, +\infty[$, and $R_- =]-\infty, 0]$;

$C([a,b];R)$ is the Banach space of continuous functions $u\colon [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)|\colon a \le t \le b\}$;

$C([a,b];R_+) = \{u \in C([a,b];R)\colon u(t) \ge 0 \text{ for } t \in [a,b]\}$;

$\widetilde{C}([a,b];R)$ is the set of absolutely continuous functions $u\colon [a,b] \to R$;

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p\colon [a,b] \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$;

$L([a,b];D) = \{p \in L([a,b];R)\colon p\colon [a,b] \to D\}$, where $D \subseteq R$;

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau\colon [a,b] \to [a,b]$;

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell\colon C([a,b];R) \to L([a,b];R)$;

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$;

$[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

A solution of the equation

$$u'(t) = \ell(u)(t) + q(t), \tag{0.1}$$

where $\ell \in \mathcal{L}_{ab}$ and $q \in L([a,b];R)$, is understood as a function $u \in \widetilde{C}([a,b];R)$ satisfying Eq. (0.1) almost everywhere in $[a,b]$.

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Consider the problem on the existence and uniqueness of a solution of (0.1) satisfying the boundary condition

$$u(a) = \lambda u(b) + c, \tag{0.2}$$

where $\lambda \in R_+$ and $c \in R$.

General boundary-value problems for functional differential equations were studied very extensively. Numerous general results were obtained (see, e.g., [1–27]), but only a few efficient criteria for the solvability of special boundary-value problems for functional differential equations are known even in the linear case. In the present paper, we try to fill the existing gap to a certain extent. More precisely, in Sec. 1, we give unimprovable efficient sufficient conditions for the unique solvability of problem (0.1), (0.2) and for the nonnegativity of the solution of this problem. Sections 2 and 3 are devoted, respectively, to the proofs of the main results and examples verifying their optimality.

All results are concretized for a differential equation with deviating arguments, i.e., for the case where Eq. (0.1) has the form

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + q(t), \tag{0.3}$$

where $p, g \in L([a,b]; R_+)$, $q \in L([a,b]; R)$, and $\tau, \mu \in \mathcal{M}_{ab}$.

Special cases of the boundary-value problem considered are a Cauchy problem (for $\lambda = 0$) and a periodic boundary-value problem (for $\lambda = 1$). In these cases, the theorems presented below coincide with the results obtained in [4] and [10].

Along with problem (0.1), (0.2), we consider the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \tag{0.1_0}$$

$$u(a) = \lambda u(b). \tag{0.2_0}$$

The following result is known from the general theory of linear boundary-value problems for functional differential equations (see, e.g., [3, 19, 27]):

Theorem 0.1. *Problem (0.1), (0.2) is uniquely solvable if and only if the corresponding homogeneous problem (0.1_0), (0.2_0) has only the trivial solution.*

1. Main Results

Theorem 1.1. *Suppose that $\lambda \in]0,1]$, the operator ℓ admits the representation $\ell = \ell_0 - \ell_1$, where*

$$\ell_0, \ell_1 \in \mathcal{P}_{ab}, \tag{1.1}$$

and either

$$\|\ell_0(1)\|_L < 1, \tag{1.2}$$

$$\frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{1-\lambda}{\lambda} < \|\ell_1(1)\|_L < 1 + \lambda + 2\sqrt{1 - \|\ell_0(1)\|_L} \tag{1.3}$$

or

$$\|\ell_1(1)\|_L < \lambda, \tag{1.4}$$

$$\frac{1}{\lambda - \|\ell_1(1)\|_L} - 1 < \|\ell_0(1)\|_L < 2 + 2\sqrt{\lambda - \|\ell_1(1)\|_L}. \tag{1.5}$$

Then problem (0.1), (0.2) has a unique solution.

Remark 1.1. For $\lambda = 0,$ the first inequality in (1.3) becomes unimportant. Consequently, Theorem 1.3 in [3] can be understood as the limit case of Theorem 1.3 as λ tends to zero.

Remark 1.2. Let $\lambda \in [1, +\infty[$ and $\ell = \ell_0 - \ell_1,$ $\ell_0, \ell_1 \in \mathcal{P}_{ab}.$ We define an operator $\psi\colon L([a,b];R) \to L([a,b];R)$ according to the formula

$$\psi(w)(t) \stackrel{\text{df}}{=} w(a+b-t) \quad \text{for } t \in [a,b].$$

Let φ be a restriction of ψ to the space $C([a,b];R).$ We set $\mu = \dfrac{1}{\lambda}$ and

$$\widehat{\ell}_0(w)(t) \stackrel{\text{df}}{=} \psi(\ell_0(\varphi(w)))(t), \qquad \widehat{\ell}_1(w)(t) \stackrel{\text{df}}{=} \psi(\ell_1(\varphi(w)))(t) \quad \text{for } t \in [a,b].$$

It is clear that if u is a solution of problem (0.1_0), (0.2_0), then the function $v \stackrel{\text{df}}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{\ell}_1(v)(t) - \widehat{\ell}_0(v)(t), \qquad v(a) = \mu v(b) \tag{1.6}$$

and, vice versa, if v is a solution of problem (1.6), then the function $u \stackrel{\text{df}}{=} \varphi(v)$ is a solution of problem (0.1_0), (0.2_0), .

It is also clear that

$$\|\widehat{\ell}_0(1)\|_L = \|\ell_0(1)\|_L, \qquad \|\widehat{\ell}_1(1)\|_L = \|\ell_1(1)\|_L.$$

Therefore, Theorem 1.1 immediately yields the following statement:

Theorem 1.2. *Suppose that $\lambda \in [1, +\infty[,$ the operator ℓ admits the representation $\ell = \ell_0 - \ell_1,$ where ℓ_0 and ℓ_1 satisfy condition (1.1), and either*

$$\|\ell_1(1)\|_L < 1,$$

$$\frac{\|\ell_1(1)\|_L}{1 - \|\ell_1(1)\|_L} + 1 - \lambda < \|\ell_0(1)\|_L < 1 + \frac{1}{\lambda} + 2\sqrt{1 - \|\ell_1(1)\|_L}$$

or

$$\|\ell_0(1)\|_L < \frac{1}{\lambda},$$

$$\frac{1}{\dfrac{1}{\lambda} - \|\ell_0(1)\|_L} - 1 < \|\ell_1(1)\|_L < 2 + 2\sqrt{\frac{1}{\lambda} - \|\ell_0(1)\|_L}.$$

Then problem (0.1), (0.2) has a unique solution.

Remark 1.3. In Sec. 3, we give examples (see Examples 3.1–3.6) showing that none of the strict inequalities (1.2) – (1.5) can be replaced by a nonstrict one. According to Remark 1.2 and the above reasoning, the conditions of Theorem 1.2 are also unimprovable.

Theorem 1.3. *Suppose that* $\lambda \in]0,1]$, $q \in L([a,b]; R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$, *and the operator* ℓ *admits the representation* $\ell = \ell_0 - \ell_1$, *where* ℓ_0 *and* ℓ_1 *satisfy condition (1.1). Also assume that*

$$\|\ell_0(1)\|_L < 1, \qquad \|\ell_1(1)\|_L < \lambda \quad (\textit{resp.},\ \|\ell_1(1)\|_L \leq \lambda) \tag{1.7}$$

and

$$\frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{1-\lambda}{\lambda} < \|\ell_1(1)\|_L. \tag{1.8}$$

Then problem (0.1), (0.2) has a unique solution, and this solution is positive (resp., nonnegative).

Theorem 1.4. *Suppose that* $\lambda \in]0,1]$, $q \in L([a,b]; R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$, *and the operator* ℓ *admits the representation* $\ell = \ell_0 - \ell_1$, *where* ℓ_0 *and* ℓ_1 *satisfy condition (1.1). Also assume that*

$$\|\ell_0(1)\|_L < 1 \quad (\textit{resp.},\ \|\ell_0(1)\|_L \leq 1), \qquad \|\ell_1(1)\|_L < \lambda \tag{1.9}$$

and

$$\frac{1}{\lambda - \|\ell_1(1)\|_L} - 1 < \|\ell_0(1)\|_L. \tag{1.10}$$

Then problem (0.1), (0.2) has a unique solution, and this solution is negative (resp., nonpositive).

According to Remark 1.2, Theorems 1.3 and 1.4 yield the following statement:

Theorem 1.5. *Suppose that* $\lambda \in [1, +\infty[$, $q \in L([a,b]; R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$, *and the operator* ℓ *admits the representation* $\ell = \ell_0 - \ell_1$, *where* ℓ_0 *and* ℓ_1 *satisfy condition (1.1). If, in addition,*

$$\|\ell_1(1)\|_L < 1, \qquad \|\ell_0(1)\|_L < \frac{1}{\lambda} \quad \left(\textit{resp.},\ \|\ell_0(1)\|_L \leq \frac{1}{\lambda}\right)$$

and

$$\frac{\|\ell_1(1)\|_L}{1 - \|\ell_1(1)\|_L} + 1 - \lambda < \|\ell_0(1)\|_L,$$

then problem (0.1), (0.2) has a unique solution, and this solution is negative (resp., nonpositive).

Theorem 1.6. *Suppose that* $\lambda \in [1, +\infty[$, $q \in L([a,b]; R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$, *and the operator* ℓ *admits the representation* $\ell = \ell_0 - \ell_1$, *where* ℓ_0 *and* ℓ_1 *satisfy condition (1.1). If, in addition,*

$$\|\ell_1(1)\|_L < 1 \quad (\textit{resp.,}\ \ \|\ell_1(1)\|_L \leq 1), \qquad \|\ell_0(1)\|_L < \frac{1}{\lambda}$$

and

$$\frac{1}{\dfrac{1}{\lambda} - \|\ell_0(1)\|_L} - 1 < \|\ell_1(1)\|_L,$$

then problem (0.1), (0.2) has a unique solution, and this solution is positive (resp., nonnegative).

Remark 1.4. In Sec. 3, we give examples (see Examples 3.7 and 3.8) showing that none of the inequalities (1.7) – (1.10) can be weakened. According to Remark 1.2 and the above reasoning, the conditions of Theorems 1.5 and 1.6 are also unimprovable.

For equations of the type (0.3), Theorems 1.1–1.6 yield the following assertions:

Corollary 1.1. *Let* $\lambda \in]0, 1]$, $p, g \in L([a,b]; R_+)$, *and either*

$$\int_a^b p(s)ds < 1, \qquad \frac{\int_a^b p(s)ds}{1 - \int_a^b p(s)ds} - \frac{1-\lambda}{\lambda} < \int_a^b g(s)ds < 1 + \lambda + 2\sqrt{1 - \int_a^b p(s)ds}$$

or

$$\int_a^b g(s)ds < \lambda, \qquad \frac{1}{\lambda - \int_a^b g(s)ds} - 1 < \int_a^b p(s)ds < 2 + 2\sqrt{\lambda - \int_a^b g(s)ds}.$$

Then problem (0.3), (0.2) has a unique solution.

Corollary 1.2. *Let* $\lambda \in [1, +\infty[$, $p, g \in L([a,b]; R_+)$, *and either*

$$\int_a^b g(s)ds < 1, \qquad \frac{\int_a^b g(s)ds}{1 - \int_a^b g(s)ds} + 1 - \lambda < \int_a^b p(s)ds < 1 + \frac{1}{\lambda} + 2\sqrt{1 - \int_a^b g(s)ds}$$

or

$$\int_a^b p(s)ds < \frac{1}{\lambda}, \qquad \frac{1}{\dfrac{1}{\lambda} - \displaystyle\int_a^b p(s)ds} - 1 < \int_a^b g(s)ds < 2 + 2\sqrt{\frac{1}{\lambda} - \int_a^b p(s)ds}.$$

Then problem (0.3), (0.2) has a unique solution.

Corollary 1.3. *Let* $\lambda \in]0,1]$, $p, g, q \in L([a,b];R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$,

$$\int_a^b p(s)ds < 1, \qquad \int_a^b g(s)ds < \lambda \qquad \left(\textit{resp.,} \ \int_a^b g(s)ds \leq \lambda\right),$$

and

$$\frac{\int_a^b p(s)ds}{1 - \int_a^b p(s)ds} - \frac{1-\lambda}{\lambda} < \int_a^b g(s)ds.$$

Then problem (0.3), (0.2) has a unique solution, and this solution is positive (resp., nonnegative).

Corollary 1.4. *Let* $\lambda \in]0,1]$, $p, g, q \in L([a,b];R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$,

$$\int_a^b p(s)ds < 1 \qquad \left(\textit{resp.,} \ \int_a^b p(s)ds \leq 1\right), \qquad \int_a^b g(s)ds < \lambda,$$

and

$$\frac{1}{\lambda - \int_a^b g(s)ds} - 1 < \int_a^b p(s)ds.$$

Then problem (0.3), (0.2) has a unique solution, and this solution is negative (resp., nonpositive).

Corollary 1.5. *Let* $\lambda \in [1,+\infty[$, $p, g, q \in L([a,b];R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$,

$$\int_a^b p(s)ds < \frac{1}{\lambda} \qquad \left(\textit{resp.,} \ \int_a^b p(s)ds \leq \frac{1}{\lambda}\right), \qquad \int_a^b g(s)ds < 1,$$

and

$$\frac{\int_a^b g(s)ds}{1 - \int_a^b g(s)ds} + 1 - \lambda < \int_a^b p(s)ds.$$

Then problem (0.3), (0.2) has a unique solution, and this solution is negative (resp., nonpositive).

Corollary 1.6. *Let* $\lambda \in [1, +\infty[,\ \ p, g, q \in L([a, b]; R_+),\ \ c \in R_+,\ \ \|q\|_L + c \neq 0,$

$$\int_a^b p(s)ds < \frac{1}{\lambda}, \qquad \int_a^b g(s)ds < 1 \qquad \left(\textit{resp.,}\ \ \int_a^b g(s)ds \leq 1\right),$$

and

$$\frac{1}{\frac{1}{\lambda} - \int_a^b p(s)ds} - 1 < \int_a^b g(s)ds.$$

Then problem (0.3), (0.2) has a unique solution, and this solution is positive (resp., nonnegative).

2. Proofs

To prove Theorems 1.1, 1.3, and 1.4, we need the following lemmas:

Lemma 2.1. *Suppose that* $\lambda \in]0, 1],\ \ q \in L([a, b]; R_-),\ \ c \in R_-,$ *the operator* ℓ *admits the representation* $\ell = \ell_0 - \ell_1,$ *where* ℓ_0 *and* ℓ_1 *satisfy condition (1.1), and*

$$\|\ell_0(1)\|_L < 1, \qquad \frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{1-\lambda}{\lambda} < \|\ell_1(1)\|_L. \tag{2.1}$$

Then problem (0.1), (0.2) does not have a nontrivial solution u *satisfying the inequality*

$$u(t) \geq 0 \quad \textit{for} \ \ t \in [a, b]. \tag{2.2}$$

Proof. Assume the contrary, i.e., assume that problem (0.1), (0.2) has a nontrivial solution u satisfying condition (2.2). We set

$$M = \max\{u(t)\colon t \in [a, b]\}, \qquad m = \min\{u(t)\colon t \in [a, b]\} \tag{2.3}$$

and choose $t_M, t_m \in [a, b]$ such that

$$u(t_M) = M, \qquad u(t_m) = m. \tag{2.4}$$

Obviously, $M > 0,\ \ m \geq 0,$ and either

$$t_M > t_m, \tag{2.5}$$

or

$$t_M < t_m. \tag{2.6}$$

First assume that (2.5) holds. The integration of (0.1) from t_m to t_M with regard for relations (1.1), (2.3), and (2.4) and the assumption that $q \in L([a,b]; R_-)$ yields

$$M - m = \int_{t_m}^{t_M} [\ell_0(u)(s) - \ell_1(u)(s) + q(s)]ds \le M \int_{t_m}^{t_M} \ell_0(1)(s)ds \le M\|\ell_0(1)\|_L.$$

Now assume that (2.6) is satisfied. The integration of (0.1) from a to t_M and from t_m to b with regard for relations (1.1), (2.3), and (2.4) and the assumption that $q \in L([a,b]; R_-)$ yields

$$M - u(a) \le M \int_a^{t_M} \ell_0(1)(s)ds, \qquad u(b) - m \le M \int_{t_m}^{b} \ell_0(1)(s)ds.$$

Summing up the last two inequalities and taking into account the condition

$$u(b) - u(a) \ge \lambda u(b) - u(a) = -c \ge 0,$$

we obtain

$$M(1 - \|\ell_0(1)\|_L) \le m. \tag{2.7}$$

Therefore, in both cases (2.5) and (2.6), inequality (2.7) is valid.

On the other hand, the integration of (0.1) from a to b with regard for relations (1.1) and (2.3) and the assumption that $q \in L([a,b]; R_-)$ yields

$$u(b) - u(a) = \int_a^b [\ell_0(u)(s) - \ell_1(u)(s) + q(s)]ds \le M\|\ell_0(1)\|_L - m\|\ell_1(1)\|_L.$$

Hence, by virtue of relations (2.3) and (0.2) and the conditions $\lambda \in]0,1]$ and $c \in R_-$, we get

$$m\|\ell_1(1)\|_L \le M\|\ell_0(1)\|_L + u(a)\left(1 - \frac{1}{\lambda}\right) + \frac{1}{\lambda}c \le M\|\ell_0(1)\|_L + m\left(1 - \frac{1}{\lambda}\right).$$

Thus,

$$m\left(\|\ell_1(1)\|_L + \frac{1-\lambda}{\lambda}\right) \le M\|\ell_0(1)\|_L.$$

This inequality, together with (2.7), yields

$$\|\ell_1(1)\|_L \le \frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{1-\lambda}{\lambda},$$

which contradicts the second inequality in (2.1).

Lemma 2.2. *Suppose that* $\lambda \in]0,1]$, $q \in L([a,b];R_+)$, $c \in R_+$, *the operator* ℓ *admits the representation* $\ell = \ell_0 - \ell_1$, *where* ℓ_0 *and* ℓ_1 *satisfy condition (1.1), and*

$$\|\ell_1(1)\|_L < \lambda, \qquad \frac{1}{\lambda - \|\ell_1(1)\|_L} - 1 < \|\ell_0(1)\|_L. \tag{2.8}$$

Then problem (0.1), (0.2) does not have a nontrivial solution u *satisfying inequality (2.2).*

Proof. Assume the contrary, i.e., assume that problem (0.1), (0.2) has a nontrivial solution u satisfying condition (2.2). We define numbers M and m by (2.3) and choose $t_M, t_m \in [a,b]$ such that (2.4) is satisfied. Obviously, $M > 0$, $m \geq 0$, and either (2.5) or (2.6) is valid.

First assume that (2.6) holds. The integration of (0.1) from t_M to t_m with regard for relations (1.1), (2.3), and (2.4) and the conditions $\lambda \in]0,1]$ and $q \in L([a,b];R_+)$ yields

$$\lambda M - m \leq M - m = \int_{t_M}^{t_m} [\ell_1(u)(s) - \ell_0(u)(s) - q(s)]ds \leq M \int_{t_M}^{t_m} \ell_1(1)(s)ds \leq M\|\ell_1(1)\|_L.$$

Now assume that (2.5) is satisfied. The integration of (0.1) from a to t_m and from t_M to b with regard for relations (1.1), (2.3), and (2.4) and the conditions $\lambda \in]0,1]$ and $q \in L([a,b];R_+)$ yields

$$u(a) - m \leq M \int_a^{t_m} \ell_1(1)(s)ds, \qquad \lambda(M - u(b)) \leq M - u(b) \leq M \int_{t_M}^{b} \ell_1(1)(s)ds.$$

Summing up the last two inequalities and taking into account the condition

$$u(a) - \lambda u(b) = c \geq 0,$$

we obtain

$$M(\lambda - \|\ell_1(1)\|_L) \leq m. \tag{2.9}$$

Therefore, in both cases (2.5) and (2.6), inequality (2.9) is valid.

On the other hand, the integration of (0.1) from a to b with regard for relations (1.1) and (2.3) and the assumption that $q \in L([a,b];R_+)$ yields

$$u(a) - u(b) = \int_a^b [\ell_1(u)(s) - \ell_0(u)(s) - q(s)]ds \leq M\|\ell_1(1)\|_L - m\|\ell_0(1)\|_L.$$

Hence, by virtue of relations (2.3) and (0.2) and the conditions $\lambda \in]0,1]$ and $c \in R_+$, we get

$$m\|\ell_0(1)\|_L \leq M\|\ell_1(1)\|_L + u(b)\,(1-\lambda) - c \leq M\|\ell_1(1)\|_L + M\,(1-\lambda)\,.$$

Thus,

$$m\|\ell_0(1)\|_L \leq M\,(\|\ell_1(1)\|_L - \lambda + 1)\,.$$

This inequality, together with (2.9), yields

$$\|\ell_0(1)\|_L \le \frac{1}{\lambda - \|\ell_1(1)\|_L} - 1,$$

which contradicts the second inequality in (2.8).

Proof of Theorem 1.1. According to Theorem 0.1, it is sufficient to show that the homogeneous problem (0.1_0), (0.2_0) does not have a nontrivial solution.

First assume that (1.2) and (1.3) hold. Assume the contrary, i.e., let problem (0.1_0), (0.2_0) have a nontrivial solution u. According to Lemma 2.1, u must change its sign. We set

$$M = \max\{u(t) : t \in [a,b]\}, \qquad m = -\min\{u(t) : t \in [a,b]\} \tag{2.10}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{2.11}$$

It is obvious that $M > 0$ and $m > 0$. Without loss of generality, we can assume that $t_m < t_M$. The integration of (0.1_0) from a to t_m, from t_m to t_M, and from t_M to b with regard for (2.10), (2.11), and (1.1) yields

$$u(a) + m = \int\limits_a^{t_m} [\ell_1(u)(s) - \ell_0(u)(s)]ds \le M \int\limits_a^{t_m} \ell_1(1)(s)ds + m \int\limits_a^{t_m} \ell_0(1)(s)ds, \tag{2.12}$$

$$M + m = \int\limits_{t_m}^{t_M} [\ell_0(u)(s) - \ell_1(u)(s)]ds \le M \int\limits_{t_m}^{t_M} \ell_0(1)(s)ds + m \int\limits_{t_m}^{t_M} \ell_1(1)(s)ds, \tag{2.13}$$

$$M - u(b) = \int\limits_{t_M}^{b} [\ell_1(u)(s) - \ell_0(u)(s)]ds \le M \int\limits_{t_M}^{b} \ell_1(1)(s)ds + m \int\limits_{t_M}^{b} \ell_0(1)(s)ds. \tag{2.14}$$

Multiplying both sides of (2.14) by λ and taking into account (2.10) and the assumption that $\lambda \in]0,1]$, we get

$$\lambda M - \lambda u(b) \le M \int\limits_{t_M}^{b} \ell_1(1)(s)ds + m \int\limits_{t_M}^{b} \ell_0(1)(s)ds.$$

Summing up the last inequality and (2.13), by virtue of (0.2_0) we obtain

$$\lambda M + m \le M \int\limits_J \ell_1(1)(s)ds + m \int\limits_J \ell_0(1)(s)ds, \tag{2.15}$$

where $J = [a, t_m] \cup [t_M, b]$. It follows from (2.13) and (2.15) that

$$M(1 - D) \le m(B - 1), \qquad m(1 - C) \le M(A - \lambda), \tag{2.16}$$

where

$$A = \int_J \ell_1(1)(s)ds, \qquad B = \int_{t_m}^{t_M} \ell_1(1)(s)ds,$$
$$C = \int_J \ell_0(1)(s)ds, \qquad D = \int_{t_m}^{t_M} \ell_0(1)(s)ds. \tag{2.17}$$

Due to (1.2), we have $C < 1$ and $D < 1$. Consequently, relation (2.16) yields $A > \lambda$, $B > 1$, and

$$0 < (1 - C)(1 - D) \le (A - \lambda)(B - 1). \tag{2.18}$$

Obviously,

$$(1 - C)(1 - D) \ge 1 - (C + D) = 1 - \|\ell_0(1)\|_L > 0,$$

$$4(A - \lambda)(B - 1) \le [A + B - (1 + \lambda)]^2 = [\|\ell_1(1)\|_L - (1 + \lambda)]^2.$$

By virtue of the last inequalities, relation (2.18) yields

$$0 < 4(1 - \|\ell_0(1)\|_L) \le [\|\ell_1(1)\|_L - (1 + \lambda)]^2,$$

which contradicts the second inequality in (1.3).

Now assume that (1.4) and (1.5) are satisfied. Assume the contrary, i.e., let problem (0.1_0), (0.2_0) have a nontrivial solution u. According to Lemma 2.2, u must change its sign. We define M and m by (2.10) and choose $t_M, t_m \in [a, b]$ such that (2.11) is satisfied. Without loss of generality, we can assume that $t_m < t_M$. As above, one can show that inequalities (2.12) – (2.15), where $J = [a, t_m] \cup [t_M, b]$, are true. It follows from (2.13) and (2.15) that

$$m(1 - B) \le M(D - 1), \qquad M(\lambda - A) \le m(C - 1), \tag{2.19}$$

where A, B, C, and D are defined by (2.17). According to (1.4), $A < \lambda$ and $B < \lambda \le 1$. Consequently, relation (2.19) yields $C > 1$, $D > 1$, and

$$0 < (\lambda - A)(1 - B) \le (C - 1)(D - 1). \tag{2.20}$$

Obviously,

$$(\lambda - A)(1 - B) \geq \lambda - (A + B) = \lambda - \|\ell_1(1)\|_L > 0,$$

$$4(C-1)(D-1) \leq (C+D-2)^2 = (\|\ell_0(1)\|_L - 2)^2.$$

By virtue of the last inequalities, relation (2.20) yields

$$0 < 4(\lambda - \|\ell_1(1)\|_L) \leq (\|\ell_0(1)\|_L - 2)^2,$$

which contradicts the second inequality in (1.5).

Proof of Theorem 1.3. According to Theorem 1.1 and conditions (1.7) and (1.8), problem (0.1), (0.2) has a unique solution u.

Let us show that u has no zero (resp., does not change its sign). Assume the contrary, i.e., assume that there exists $t_1 \in [a,b]$ $\big($resp., $t_2, t_3 \in [a,b]\big)$ such that

$$u(t_1) = 0 \quad (\text{resp., } u(t_2)u(t_3) < 0). \tag{2.21}$$

We define numbers M and m by (2.10) and choose $t_M, t_m \in [a,b]$ such that (2.11) is satisfied. Obviously,

$$M \geq 0, \quad m \geq 0, \quad M + m > 0 \tag{2.22}$$

$$\big(\text{resp., } M > 0, \ m > 0\big), \tag{2.23}$$

and either (2.5) or (2.6) is valid.

First assume that (2.5) holds. The integration of (0.1) from a to t_m and from t_M to b with regard for relations (1.1), (2.10), and (2.11) and the conditions $\lambda \in]0,1]$ and $q \in L([a,b]; R_+)$ yields

$$m + u(a) \leq M \int_a^{t_m} \ell_1(1)(s)ds + m \int_a^{t_m} \ell_0(1)(s)ds, \tag{2.24}$$

$$\lambda(M - u(b)) \leq M - u(b) \leq M \int_{t_M}^{b} \ell_1(1)(s)ds + m \int_{t_M}^{b} \ell_0(1)(s)ds. \tag{2.25}$$

Summing up the last two inequalities and taking into account the condition

$$u(a) - \lambda u(b) = c \geq 0,$$

we obtain

$$\lambda M + m \leq M\|\ell_1(1)\|_L + m\|\ell_0(1)\|_L, \tag{2.26}$$

whence, by virtue of (1.7), we arrive at a contradiction: $\lambda M + m < \lambda M + m$ (resp., $m < m$).

Now assume that (2.6) is satisfied. The integration of (0.1) from t_M to t_m with regard for relations (1.1), (2.10), and (2.11) and the assumption that $q \in L([a,b];R_+)$ yields

$$M + m = \int_{t_M}^{t_m} [\ell_1(u)(s) - \ell_0(u)(s) - q(s)]ds \le M\|\ell_1(1)\|_L + m\|\ell_0(1)\|_L. \tag{2.27}$$

Hence, by virtue of (1.7) and the assumption that $\lambda \in]0,1]$, we arrive at a contradiction $(M + m < M + m)$. Thus, u has no zero (resp., does not change its sign). Therefore, according to Lemma 2.1, u is positive (resp., nonnegative).

Proof of Theorem 1.4. According to Theorem 1.1 and conditions (1.9) and (1.10), problem (0.1), (0.2) has a unique solution u.

Let us show that u has no zero (resp., does not change its sign). Assume the contrary, i.e., assume that there exists $t_1 \in [a,b]$ $\big($resp., $t_2, t_3 \in [a,b]\big)$ such that (2.21) is satisfied. We define numbers M and m by (2.10) and choose $t_M, t_m \in [a,b]$ such that (2.11) is satisfied. It is obvious that (2.22) [resp., (2.23)] is satisfied and either (2.5) or (2.6) is valid.

By analogy with the proof of Theorem 1.3, one can show that assumption (2.5) leads to the contradiction $\lambda M + m < \lambda M + m$ (resp., $M < M$), and assumption (2.6) leads to the contradiction $M + m < M + m$. Thus, u has no zero (resp., does not change its sign) and, therefore, according to Lemma 2.2, u is negative (resp., nonpositive).

3. On Remarks 1.3 and 1.4

On Remark 1.3. Let $\lambda \in]0,1[$ (the cases $\lambda = 0$ and $\lambda = 1$ were studied in [4] and [10], respectively; there one can also find examples that verify the optimality of the results obtained). Denote by H^+ and H^- the sets of pairs $(x,y) \in R_+ \times R_+$ such that

$$x < 1, \qquad \frac{x}{1-x} - \frac{1-\lambda}{\lambda} < y < 1 + \lambda + 2\sqrt{1-x},$$

and

$$y < \lambda, \qquad \frac{1}{\lambda - y} - 1 < x < 2 + 2\sqrt{\lambda - y},$$

respectively. According to Theorem 1.1, if $(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in H^+ \cup H^-$, then problem (0.1), (0.2) has a unique solution. $\Big($Also note that, for $\lambda \le \frac{1}{4}$, we have $H^- = \emptyset$.$\Big)$

Below, we give examples that show that, for any pair $(x_0, y_0) \notin H^+ \cup H^-$, $x_0 \ge 0$, $y_0 \ge 0$, there exist functions $h \in L([a,b];R)$ and $\tau \in \mathcal{M}_{ab}$ such that

$$\int_a^b [h(s)]_+ ds = x_0, \qquad \int_a^b [h(s)]_- ds = y_0, \tag{3.1}$$

and the problem

$$u'(t) = h(t)u(\tau(t)), \qquad u(a) = \lambda u(b) \tag{3.2}$$

has a nontrivial solution. Then, by virtue of Theorem 0.1, there exist $q \in L([a,b];R)$ and $c \in R$ such that problem (0.1), (0.2), where $\ell = \ell_0 - \ell_1$,

$$\ell_0(w)(t) \overset{\text{df}}{=} [h(t)]_+ w(\tau(t)), \qquad \ell_1(w)(t) \overset{\text{df}}{=} [h(t)]_- w(\tau(t)), \tag{3.3}$$

either does not have a solution or has an infinite set of solutions.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0,y_0) \notin H^+ \cup H^-$, then (x_0,y_0) belongs to at least one of the following sets:

$$H_1 = \{(x,y) \in R \times R\colon 1 \le x,\ \lambda \le y\},$$

$$H_2 = \left\{(x,y) \in R \times R\colon 0 \le x < 1,\ 1 + \lambda + 2\sqrt{1-x} \le y\right\},$$

$$H_3 = \left\{(x,y) \in R \times R\colon 0 \le y < \lambda,\ 2 + 2\sqrt{\lambda - y} \le x\right\},$$

$$H_4 = \left\{(x,y) \in R \times R\colon 0 \le y < \lambda,\ y + 1 - \lambda \le x \le \frac{y+1-\lambda}{\lambda - y}\right\},$$

$$H_5 = \left\{(x,y) \in R \times R\colon 1 - \lambda < x < 1,\ \frac{x}{\lambda} + 1 - \frac{1}{\lambda} \le y \le \frac{x+\lambda-1}{\lambda(1-x)}\right\},$$

$$H_6 = \left\{(x,y) \in R \times R\colon 1 - \lambda < x < 1,\ x - 1 + \lambda \le y \le \frac{x}{\lambda} + 1 - \frac{1}{\lambda}\right\}.$$

Example 3.1. Let $(x_0,y_0) \in H_1$. We set $a = 0$, $b = 4$, and

$$h(t) = \begin{cases} -\lambda & \text{for } t \in [0,1[, \\ x_0 - 1 & \text{for } t \in [1,2[, \\ \lambda - y_0 & \text{for } t \in [2,3[, \\ 1 & \text{for } t \in [3,4], \end{cases} \qquad \tau(t) = \begin{cases} 4 & \text{for } t \in [0,1[\cup[3,4], \\ 1 & \text{for } t \in [1,3[. \end{cases}$$

Then relations (3.1) hold and problem (3.2) has the nontrivial solution

$$u(t) = \begin{cases} \lambda(1-t) & \text{for } t \in [0,1[, \\ 0 & \text{for } t \in [1,3[, \\ t - 3 & \text{for } t \in [3,4]. \end{cases}$$

Example 3.2. Let $(x_0, y_0) \in H_2$. We set $a = 0$, $b = 6$, $\alpha = \sqrt{1 - x_0}$, $\beta = y_0 - 1 - \lambda - 2\alpha$, and

$$h(t) = \begin{cases} -\lambda & \text{for } t \in [0,1[, \\ -\beta & \text{for } t \in [1,2[, \\ -\alpha & \text{for } t \in [2,4[, \\ -1 & \text{for } t \in [4,5[, \\ x_0 & \text{for } t \in [5,6], \end{cases} \qquad \tau(t) = \begin{cases} 6 & \text{for } t \in [0,1[\cup[2,3[\cup[5,6], \\ 1 & \text{for } t \in [1,2[, \\ 3 & \text{for } t \in [3,5[. \end{cases}$$

Then relations (3.1) hold and problem (3.2) has the nontrivial solution

$$u(t) = \begin{cases} \lambda(1-t) & \text{for } t \in [0,1[, \\ 0 & \text{for } t \in [1,2[, \\ \alpha(2-t) & \text{for } t \in [2,3[, \\ \alpha^2(t-3) - \alpha & \text{for } t \in [3,4[, \\ \alpha(t-5) + \alpha^2 & \text{for } t \in [4,5[, \\ x_0(t-6) + 1 & \text{for } t \in [5,6]. \end{cases}$$

Example 3.3. Let $(x_0, y_0) \in H_3$. We set $a = 0$, $b = 6$, $\alpha = \sqrt{\lambda - y_0}$, $\beta = x_0 - 2 - 2\alpha$, and

$$h(t) = \begin{cases} \alpha & \text{for } t \in [0,1[, \\ -y_0 & \text{for } t \in [1,2[, \\ \beta & \text{for } t \in [2,3[, \\ 1 & \text{for } t \in [3,4[, \\ \alpha & \text{for } t \in [4,5[, \\ 1 & \text{for } t \in [5,6], \end{cases} \qquad \tau(t) = \begin{cases} 4 & \text{for } t \in [0,1[\cup[3,4[, \\ 6 & \text{for } t \in [1,2[\cup[4,6], \\ 2 & \text{for } t \in [2,3[. \end{cases}$$

Then relations (3.1) hold and problem (3.2) has the nontrivial solution

$$u(t) = \begin{cases} -\alpha^2 t + \lambda & \text{for } t \in [0,1[, \\ y_0(2-t) & \text{for } t \in [1,2[, \\ 0 & \text{for } t \in [2,3[, \\ \alpha(3-t) & \text{for } t \in [3,4[, \\ \alpha(t-5) & \text{for } t \in [4,5[, \\ t-5 & \text{for } t \in [5,6]. \end{cases}$$

Example 3.4. Let $(x_0, y_0) \in H_4$. We set $a = 0$, $b = 2$, $\alpha = 1 - \lambda + y_0$, $t_0 = \dfrac{1}{x_0} - \dfrac{1}{\alpha} + 2$, and

$$h(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[, \\ x_0 & \text{for } t \in [1,2], \end{cases} \qquad \tau(t) = \begin{cases} 2 & \text{for } t \in [0,1[, \\ t_0 & \text{for } t \in [1,2]. \end{cases}$$

Then relations (3.1) hold and problem (3.2) has the nontrivial solution

$$u(t)=\begin{cases}-y_0t+\lambda & \text{for } t\in[0,1[,\\ \alpha(t-2)+1 & \text{for } t\in[1,2].\end{cases}$$

Example 3.5. Let $(x_0,y_0)\in H_5$. We set $a=0$, $b=2$, $\alpha=\dfrac{\lambda+x_0-1}{1-x_0}$, $\beta=\dfrac{\lambda x_0}{1-x_0}$, $t_0=\left(\dfrac{\alpha}{y_0}-\lambda\right)\dfrac{1}{\beta}$, and

$$h(t)=\begin{cases}x_0 & \text{for } t\in[0,1[,\\ -y_0 & \text{for } t\in[1,2],\end{cases}\qquad \tau(t)=\begin{cases}1 & \text{for } t\in[0,1[,\\ t_0 & \text{for } t\in[1,2].\end{cases}$$

Then relations (3.1) hold and problem (3.2) has the nontrivial solution

$$u(t)=\begin{cases}\beta t+\lambda & \text{for } t\in[0,1[,\\ \alpha(2-t)+1 & \text{for } t\in[1,2].\end{cases}$$

Example 3.6. Let $(x_0,y_0)\in H_6$. We set $a=0$, $b=2$, $\alpha=\lambda+x_0-1$, $t_0=\dfrac{\alpha-y_0}{x_0y_0}+2$, and

$$h(t)=\begin{cases}-y_0 & \text{for } t\in[0,1[,\\ x_0 & \text{for } t\in[1,2],\end{cases}\qquad \tau(t)=\begin{cases}t_0 & \text{for } t\in[0,1[,\\ 2 & \text{for } t\in[1,2].\end{cases}$$

Then relations (3.1) hold and problem (3.2) has the nontrivial solution

$$u(t)=\begin{cases}-\alpha t+\lambda & \text{for } t\in[0,1[,\\ x_0(t-2)+1 & \text{for } t\in[1,2].\end{cases}$$

On Remark 1.4. Let $\lambda\in]0,1]$ (the case $\lambda=0$ was studied in [4]). Denote by G^+ and G^- the sets of pairs $(x,y)\in R_+\times R_+$ such that

$$x<1,\qquad \frac{x}{1-x}-\frac{1-\lambda}{\lambda}<y<\lambda,$$

and

$$y<\lambda,\qquad \frac{1}{\lambda-y}-1<x<1,$$

respectively. It is clear that $G^+\subset H^+$ and $G^-\subset H^-$. (Also note that, for $\lambda\le\dfrac{1}{2}$, we have $G^-=\emptyset$.)

By virtue of Theorem 1.3 (resp., Theorem 1.4), if

$$(\|\ell_0(1)\|_L,\|\ell_1(1)\|_L)\in G^+\qquad \big(\text{resp.,}\quad (\|\ell_0(1)\|_L,\|\ell_1(1)\|_L)\in G^-\big),$$

then problem (0.1), (0.2), where $q \in L([a,b];R_+)$, $c \in R_+$, and $\|q\|_L + c \neq 0$, has a unique solution, and this solution is positive (resp., negative).

Below, we give examples that show that, for any pair $(x_0, y_0) \in H^+ \setminus G^+$ $\big($resp., $(x_0, y_0) \in H^- \setminus G^-\big)$, there exist functions $h \in L([a,b];R)$, $q \in L([a,b];R_+)$, and $\tau \in \mathcal{M}_{ab}$ such that $q \not\equiv 0$, relations (3.1) are satisfied, and the problem

$$u'(t) = h(t)u(\tau(t)) + q(t), \qquad u(a) = \lambda u(b), \tag{3.4}$$

or, equivalently, problem (0.1), (0.2_0), where $\ell = \ell_0 - \ell_1$ and ℓ_0 and ℓ_1 are defined by (3.3), has a solution that is not positive (resp., negative).

It also follows from Example 3.7 (resp., Example 3.8) that, in Theorem 1.3 (resp., Theorem 1.4), the inequality $\|\ell_1(1)\|_L \leq \lambda$ $\big($resp., $\|\ell_0(1)\|_L \leq 1\big)$ in condition (1.7) [resp., (1.9)] cannot be replaced by the inequality $\|\ell_1(1)\|_L \leq \lambda + \varepsilon$ $\big($resp., $\|\ell_0(1)\|_L \leq 1 + \varepsilon\big)$ for arbitrarily small $\varepsilon > 0$.

Example 3.7. Let $(x_0, y_0) \in H^+ \setminus G^+$. We set $a = 0$, $b = 2$, $\alpha = y_0 - x_0 - \lambda + 1$, $\beta = 1 + y_0 - \lambda$, $\tau \equiv 2$, and

$$h(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[, \\ x_0 & \text{for } t \in [1,2], \end{cases} \qquad q(t) = \begin{cases} 0 & \text{for } t \in [0,1[, \\ \alpha & \text{for } t \in [1,2]. \end{cases}$$

Then relations (3.1) hold and problem (3.4) has the solution

$$u(t) = \begin{cases} -y_0 t + \lambda & \text{for } t \in [0,1[, \\ \beta(t-2) + 1 & \text{for } t \in [1,2] \end{cases}$$

with $u(1) = \lambda - y_0 \leq 0$.

Example 3.8. Let $(x_0, y_0) \in H^- \setminus G^-$. We set $a = 0$, $b = 2$, $\alpha = x_0 - y_0 + \lambda - 1$, $\beta = x_0 + \lambda - 1$, $\tau \equiv 2$, and

$$h(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[, \\ x_0 & \text{for } t \in [1,2], \end{cases} \qquad q(t) = \begin{cases} \alpha & \text{for } t \in [0,1[, \\ 0 & \text{for } t \in [1,2]. \end{cases}$$

Then relations (3.1) hold and problem (3.4) has the solution

$$u(t) = \begin{cases} \beta t - \lambda & \text{for } t \in [0,1[, \\ x_0(2-t) - 1 & \text{for } t \in [1,2] \end{cases}$$

with $u(1) = x_0 - 1 \geq 0$.

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