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On a two-point boundary value problem for the second order ordinary differential equations with singularities

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Introduction

Throughout the paper the following notation will be used. R is the set of all real numbers, $R_+ = [0, +\infty[$.

$\tilde{C}_{loc}(]a,b[;R)$ is the set of functions $u :]a,b[\to R$ which are absolutely continuous on every compact interval contained in $]a,b[$.

$\tilde{C}'_{loc}(I;R)$, where $I \subset]a,b[$, is the set of functions $u : I \to R$ which are absolutely continuous together with their first derivative on every compact interval contained in I.

$L(]a,b[;D)$, where $D \subset R$, is the set of Lebesgue integrable functions $p :]a,b[\to D$.

$L_{loc}(]a,b[;D)$, where $D \subset R$, is the set of functions $p :]a,b[\to D$, which are Lebesgue integrable on every compact interval contained in $]a,b[$.

$K(]a,b[\times R^2;R)$ is the class of Caratheòdory.

$K_{loc}(]a,b[\times R^2;R)$ is the set of functions $f :]a,b[\times R^2 \to R$ such that $f \in K(]a+\varepsilon, b-\varepsilon[\times R^2;R)$ for any small $\varepsilon > 0$.

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$\gamma : L_{\mathrm{loc}}(]a,b[;R) \to L_{\mathrm{loc}}(]a,b[;R_+)$ is the operator defined by

$$\gamma(p)(t) = \exp\left[\left|\int_{\frac{a+b}{2}}^{t} p(s)\,\mathrm{d}s\right|\right] \quad \text{for } a < t < b.$$

If $\gamma(p) \in L(]a,b[;R_+)$, then

$$\gamma_{ab}(p)(t) = \frac{1}{\gamma(p)(t)} \int_a^t \gamma(p)(s)\,\mathrm{d}s \int_t^b \gamma(p)(s)\,\mathrm{d}s \quad \text{for } a < t < b.$$

$u(s+)$ and $u(s-)$ are the right-hand side and left-hand side limits of the function u at the point s.

Consider the boundary value problem

$$u'' = f(t,u,u'), \tag{0.1}$$

$$u(a+) = 0, \quad u(b-) = 0, \tag{0.2}$$

where $f \in K_{\mathrm{loc}}(]a,b[\times R^2; R)$. By a solution of problem (0.1), (0.2) we understand a function $u \in \tilde{C}'_{\mathrm{loc}}(]a,b[;R)$ satisfying (0.1) almost everywhere in $]a,b[$ and also conditions (0.2).

The investigation of the question on the solvability of (0.1), (0.2)'s type problems originates from the works of Picard [10], Tonelli [11], Epheser [4], Bernstein [1], and Nagumo [9]. Nowadays there exists a sufficiently developed theory on singular problems of the type (0.1), (0.2) (see [6]). In this paper we give new sufficient conditions for the solvability of problem (0.1), (0.2) which make the results obtained in [6] more complete. However (as in [6]) we do not exclude the possibility for the function f having nonintegrable singularities with respect to the first argument at the points $t = a$ and $t = b$.

Before we formulate the main results, we introduce some definitions.

Definition 0.1. A function $\gamma \in \tilde{C}_{\mathrm{loc}}(]a,b[;R)$ is said to be a lower (an upper) function of Eq. (0.1) if γ' admits the representation

$$\gamma'(t) = \gamma_0(t) + \gamma_1(t) \quad \text{for } a < t < b,$$

where $\gamma_0 \in \tilde{C}_{\mathrm{loc}}(]a,b[;R)$ and $\gamma_1 :]a,b[\to R$ is nondecreasing (nonincreasing) function such that $\gamma_1'(t) = 0$ almost everywhere in $]a,b[$, and, moreover, the inequality

$$\gamma''(t) \geqslant f(t,\gamma(t),\gamma'(t)) \quad (\gamma''(t) \leqslant f(t,\gamma(t),\gamma'(t)))$$

holds almost everywhere in $]a,b[$.

Definition 0.2. We say that the vector function $(p,g) :]a,b[\to R_+^2$ belongs to the set $V_1(]a,b[)$ if

$$\gamma(g) \in L(]a,b[;R_+), \quad \gamma_{ab}(g)p \in L(]a,b[;R_+) \tag{0.3}$$

and for any $a_1 \in [a,b[$, $b_1 \in]a_1,b]$ and measurable function $q :]a,b[\to R$ satisfying the inequality $|q(t)| \leqslant g(t)$ for $a < t < b$, the problem

$$u'' = -p(t)u + q(t)u', \quad u(a_1+) = 0, \; u(b_1-) = 0$$

has only the trivial solution.

Definition 0.3. Let $(p,g) :]a,b[\to R_+^2$, $p(t) > 0$ for $a < t < b$, $\mu \in]0,1[$, and

$$\gamma(g) \in L(]a,b[; R_+), \quad \gamma_{ab}^{\mu}(g)p \in L(]a,b[; R_+). \tag{0.4}$$

Let, moreover, u_1 and u_2 be solutions of the equation

$$u'' = -p(t)|u|^{\mu}|u'|^{1-\mu} \operatorname{sgn} u - g(t)|u'| \tag{0.5}$$

satisfying the conditions

$$\operatorname{mes}\{t \in]a,b[: u_i'(t) = 0\} = 0, \quad i = 1,2, \tag{0.6}$$

$$u_1(a+) = 0, \quad \lim_{t \to a+} \frac{u_1'(t)}{\gamma(g)(t)} = 1, \tag{0.7}$$

$$u_2(b-) = 0, \quad \lim_{t \to b-} \frac{u_2'(t)}{\gamma(g)(t)} = -1. \tag{0.8}$$

We say that the vector function (p,g) belongs to the set $V_{\mu}(]a,b[)$ if at least one of the following three conditions is fulfilled:

$$u_1(t) > 0 \quad \text{for } a < t < b, \quad \int_a^b \left|\frac{u_1'(s)}{u_1(s)}\right|^{\mu} \operatorname{sgn} u_1'(s)\, \mathrm{d}s = 0, \tag{0.9}$$

$$u_2(t) > 0 \quad \text{for } a < t < b, \quad \int_a^b \left|\frac{u_2'(s)}{u_2(s)}\right|^{\mu} \operatorname{sgn} u_2'(s)\, \mathrm{d}s = 0, \tag{0.10}$$

$$u_i(t) > 0 \quad \text{for } a < t < b, \quad (-1)^i \int_a^b \left|\frac{u_i'(s)}{u_i(s)}\right|^{\mu} \operatorname{sgn} u_i'(s)\, \mathrm{d}s < 0, \quad i = 1,2. \tag{0.11}$$

Remark 0.1. Condition (0.3) (resp. (0.4)) is fulfilled, e.g., if for $a < t < b$,

$$g(t) \leqslant \frac{\lambda}{(t-a)(b-t)} + h_0(t),$$

$$p(t) \leqslant \frac{h_1(t)}{(t-a)(b-t)} \quad \left(p(t) \leqslant \frac{h_1(t)}{[(t-a)(b-t)]^{\mu}}\right),$$

where $\lambda \in]0, b-a[$ and $h_0, h_1 \in L(]a,b[; R_+)$.

Remark 0.2. From Lemma 2.1 and Remark 2.4 in [2] it immediately follows that condition (0.4) guarantees the existence of solutions of problems (0.5), (0.7) and (0.5), (0.8) defined on the whole segment $[a,b]$ and satisfying conditions (0.6).

Remark 0.3. The effective sufficient conditions for the vector function (p, g) to belong to the sets $V_1(]a,b[)$ and $V_\mu(]a,b[)$ can be found in [2,3,5–8].

1. Main results

Theorem 1.1. *Let α and β be, respectively, lower and upper functions of the equation* (0.1), $\alpha(t) \leqslant \beta(t)$ *for* $a < t < b$, *and*

$$\alpha(a+) = 0, \quad \alpha(b-) = 0, \quad \beta(a+) = 0, \quad \beta(b-) = 0. \tag{1.1}$$

Let, moreover,

$$f(t,x,y) \geqslant -h_1(t) - h_2(t)|y| - h_0 y^2 \quad \text{for } a < t < b, \alpha(t) \leqslant x \leqslant \beta(t), \quad y \in R \tag{1.2}$$

$$(f(t,x,y) \leqslant h_1(t) + h_2(t)|y| + h_0 y^2 \quad \text{for } a < t < b, \ \alpha(t) \leqslant x \leqslant \beta(t), \quad y \in R) \tag{1.3}$$

where $h_0 \in R_+$ *and* $h_i \in L_{\mathrm{loc}}(]a,b[; R_+)$, $i = 1, 2$, *satisfy the conditions*

$$\gamma(h_2) \in L(]a,b[; R_+), \quad \gamma_{ab}(h_2)h_1 \in L(]a,b[; R_+). \tag{1.4}$$

Then problem (0.1), (0.2) *has at least one solution* u *such that*

$$\alpha(t) \leqslant u(t) \leqslant \beta(t) \quad \text{for } a \leqslant t \leqslant b. \tag{1.5}$$

Corollary 1.1. *Let*

$$f(t,x,y) \geqslant -h_1(t) - h_2(t)|y| - h_0 y^2 \quad \text{for } a < t < b, \quad x, y \in R, \tag{1.6}$$

$$f(t,x,y) \leqslant p_1(t) + p_2(t)|y| + p_0|y|^{\lambda+1} \quad \text{for } a < t < b, \ x < 0, \ y \in R, \tag{1.7}$$

$$(f(t,x,y) \leqslant h_1(t) + h_2(t)|y| + h_0 y^2 \quad \text{for } a < t < b, \ x, y \in R, \tag{1.8}$$

$$f(t,x,y) \geqslant -p_1(t) - p_2(t)|y| - p_0|y|^{\lambda+1} \quad \text{for } a < t < b, \ x > 0, \ y \in R), \tag{1.9}$$

where $\lambda \geqslant 1$, $h_0, p_0 \in R_+$, *and*

$$(h_0 h_1, h_2) \in V_1(]a,b[), \quad (\lambda^{(\lambda+1)/\lambda} p_0^{1/\lambda} p_1, \lambda p_2) \in V_{1/\lambda}(]a,b[). \tag{1.10}$$

Then problem (0.1), (0.2) *has at least one solution.*

Remark 1.1. Both of conditions (1.10) are essential and they cannot be omitted (see On Remark 1.1 below).

Corollary 1.2. *Let inequalities* (1.6) *and* (1.7), ((1.8) *and* (1.9)) *be fulfilled, with* $h_0 > 0,\ p_0 > 0$ *and*

$$h_1(t) = \frac{l_{11}h_0^{-1}}{(t-a)^{2v}}, \quad h_2(t) = \frac{l_{12}}{(t-a)^v} + \frac{v}{t-a} \quad \text{for } a < t < \frac{a+b}{2},$$
$$h_1(t) = \frac{l_{11}h_0^{-1}}{(b-t)^{2v}}, \quad h_2(t) = \frac{l_{12}}{(b-t)^v} + \frac{v}{b-t} \quad \text{for } \frac{a+b}{2} < t < b, \tag{1.11}$$

$$p_1(t) = \frac{l_{21}\lambda^{-(\lambda+1)/\lambda}p_0^{-1/\lambda}}{(t-a)^{\mu(1+\lambda)}}, \quad p_2(t) = \frac{l_{22}\lambda^{-1}}{(t-a)^{\mu\lambda}} + \frac{\mu}{t-a} \quad \text{for } a < t < \frac{a+b}{2},$$
$$p_1(t) = \frac{l_{21}\lambda^{-(\lambda+1)/\lambda}p_0^{-1/\lambda}}{(b-t)^{\mu(1+\lambda)}}, \quad p_2(t) = \frac{l_{22}\lambda^{-1}}{(b-t)^{\mu\lambda}} + \frac{\mu}{b-t} \quad \text{for } \frac{a+b}{2} < t < b. \tag{1.12}$$

where $v \in [0,1[,\ \lambda > 1,\ \mu \in [0,\frac{1}{\lambda}[$ *and* $l_{ij} \in]0,+\infty[,\ i,j = 1,2.$

Let, moreover,

$$\int_0^{+\infty} \frac{ds}{l_{11} + l_{12}s + s^2} > \frac{1}{1-v}\left(\frac{b-a}{2}\right)^{1-v}, \tag{1.13}$$
$$\int_0^{+\infty} \frac{ds}{l_{21} + l_{22}s + s^{1+\lambda}} \geqslant \frac{1}{\lambda(1-\lambda\mu)}\left(\frac{b-a}{2}\right)^{1-\lambda\mu}. \tag{1.14}$$

Then problem (0.1), (0.2) *has at least one solution.*

2. Auxiliary propositions

Lemma 2.1. *Let* $r_0, h_0 \in R_+$, *and* $h_i \in L_{\text{loc}}(]a,b[;R_+)$, $i = 1,2$, *satisfy conditions* (1.4). *Then for any* $a_1 \in]a,(a+b)/2[, b_1 \in](a+b)/2,b[$ *and* $u \in \tilde{C}'_{\text{loc}}(]a_1,b_1[;R)$ *satisfying the inequalities*

$$|u(t)| \leqslant r_0 \quad \text{for } a_1 \leqslant t \leqslant b_1, \tag{2.1}$$

$$u''(t) \geqslant -h_1(t) - h_2(t)|u'(t)| - h_0[u'(t)]^2 \quad \text{for } a_1 < t < b_1 \tag{2.2}$$

$$u''(t) \leqslant h_1(t) + h_2(t)|u'(t)| + h_0[u'(t)]^2 \quad \text{for } a_1 < t < b_1, \tag{2.3}$$

the estimate

$$|u'(t)| \leqslant \varphi(t,a_1,b_1,r_0) \quad \text{for } a_1 < t < b_1 \tag{2.4}$$

holds, where

$$\varphi(t,x,y,z) = e^{10h_0 z}\left(2z\int_a^b \gamma(h_2)(s)\,ds + \int_a^b \gamma_{ab}(h_2)(s)h_1(s)\,ds\right)\gamma(h_2)(t)$$
$$\left(\int_x^t [\gamma(h_2)(s)]^{-1}\,ds \int_t^y [\gamma(h_2)(s)]^{-1}\,ds\right)^{-1} \quad \text{for } x < t < y.$$

Proof. We will prove the lemma in the case where condition (2.2) is fulfilled. The case where (2.3) is fulfilled can be proved analogously.

Let $u \in \tilde{C}'_{loc}(]a_1, b_1[; R)$ satisfy conditions (2.1) and (2.2). Put

$$g_0(t) = -h_0 u'(t) - h_2(t) \operatorname{sgn} u'(t) \quad \text{for } a_1 < t < b_1, \tag{2.5}$$

$$g_1(t) = u''(t) + h_2(t)|u'(t)| + h_0[u'(t)]^2 \quad \text{for } a_1 < t < b_1. \tag{2.6}$$

It is clear that

$$u''(t) = g_0(t)u'(t) + g_1(t) \quad \text{for } a_1 < t < b_1. \tag{2.7}$$

According to (2.2) and (2.6), we have

$$g_1(t) \geqslant -h_1(t) \quad \text{for } a_1 < t < b_1. \tag{2.8}$$

Put

$$\sigma(g_0)(t) = \exp\left(\int_{(a+b)/2}^{t} g_0(s)\,\mathrm{d}s\right),$$
$$\sigma_{a_1 b_1}(g_0)(t) = \frac{1}{\sigma(g_0)(t)} \int_{a_1}^{t} \sigma(g_0)(s)\,\mathrm{d}s \int_{t}^{b_1} \sigma(g_0)(s)\,\mathrm{d}s,$$
$$\sigma_{a_1}(g_0)(t) = \frac{1}{\sigma(g_0)(t)} \int_{a_1}^{t} \sigma(g_0)(s)\,\mathrm{d}s,$$
$$\sigma_{b_1}(g_0)(t) = \frac{1}{\sigma(g_0)(t)} \int_{t}^{b_1} \sigma(g_0)(s)\,\mathrm{d}s.$$

Let $t_1 \in]a_1, b_1[$ be an arbitrary point such that $u'(t_1) \neq 0$. Then either

$$u'(t_1) > 0 \tag{2.9}$$

or

$$u'(t_1) < 0. \tag{2.10}$$

Suppose that condition (2.9) (condition (2.10)) is fulfilled. Multiplying both sides of (2.7) by $\sigma_{b_1}(g_0)(t)$ (by $\sigma_{a_1}(g_0)(t)$) and integrating from t_1 to b_1 (from a_1 to t_1), we obtain

$$u'(t_1)\sigma_{b_1}(g_0)(t_1) = u(b_1) - u(t_1) - \int_{t_1}^{b_1} \sigma_{b_1}(g_0)(s)g_1(s)\,\mathrm{d}s$$

$$(-u'(t_1)\sigma_{a_1}(g_0)(t_1) = u(a_1) - u(t_1) - \int_{a_1}^{t_1} \sigma_{a_1}(g_0)(s)g_1(s)\,\mathrm{d}s).$$

Hence, in view of (2.1), (2.8) and (2.9) ((2.10)), we get

$$|u'(t_1)|\sigma_{b_1}(g_0)(t_1) \leqslant 2r_0 + \int_{t_1}^{b_1} \sigma_{b_1}(g_0)(s)h_1(s)\,\mathrm{d}s$$

$$(|u'(t_1)|\sigma_{a_1}(g_0)(t_1) \leqslant 2r_0 + \int_{a_1}^{t_1} \sigma_{a_1}(g_0)(s)h_1(s)\,\mathrm{d}s).$$

Multiplying the last inequality by $\int_{a_1}^{t_1} \sigma(g_0)(s)\,\mathrm{d}s$ (by $\int_{t_1}^{b_1} \sigma(g_0)(s)\,\mathrm{d}s$), we easily find that for any $t \in \,]a_1, b_1[$,

$$|u'(t)|\sigma_{a_1 b_1}(g_0)(t) \leqslant 2r_0 \int_{a_1}^{b_1} \sigma(g_0)(s)\,\mathrm{d}s + \int_{a_1}^{b_1} \sigma_{a_1 b_1}(g_0)(s) h_1(s)\,\mathrm{d}s. \tag{2.11}$$

In view of (2.1) and (2.5) it is not difficult to verify that

$$\sigma_{a_1 b_1}(g_0)(t) \leqslant \gamma_{ab}(h_2)(t)\exp(4r_0 h_0) \quad \text{for } a_1 < t < b_1,$$

$$[\gamma(h_2)(t)\exp(2r_0 h_0)]^{-1} \leqslant \sigma(g_0)(t) \leqslant \gamma(h_2)(t)\exp(2r_0 h_0) \quad \text{for } a_1 < t < b_1,$$

$$\sigma_{a_1 b_1}(g_0)(t) \geqslant \frac{\exp(-6r_0 h_0)}{\gamma(h_2)(t)} \int_{a_1}^{t} [\gamma(h_2)(s)]^{-1}\,\mathrm{d}s \int_{t}^{b_1} [\gamma(h_2)(s)]^{-1}\,\mathrm{d}s$$
$$\text{for } a_1 < t < b_1.$$

Taking into account these estimates, from (2.11) it can be easily seen that estimate (2.4) holds. □

Lemma 2.2. *Let* $h_0 \in R_+$, $h_i \in L_{\mathrm{loc}}(]a, b[; R_+)$, $i = 1, 2$, *and*

$$(h_0 h_1, h_2) \in V_1(]a, b[). \tag{2.12}$$

Then there exists $\delta \in \tilde{C}'_{\mathrm{loc}}(]a, b[; R)$ *such that*

$$\delta(t) > 0 \quad \text{for } a < t < b, \quad \delta(a+) = 0, \quad \delta(b-) = 0, \tag{2.13}$$

and

$$\delta''(t) = -h_1(t) - h_2(t)|\delta'(t)| - h_0[\delta'(t)]^2 \quad \text{for } a < t < b. \tag{2.14}$$

Proof. Let $h_0 > 0$. According to (2.12) and Theorem 2.2 in [6], the problem

$$u'' = -h_0 h_1(t)|u| - h_2(t)|u'| - h_0 h_1(t), \quad u(a+) = 0, \quad u(b-) = 0$$

has at least one solution u. Since $u''(t) < 0$ for $a < t < b$, we have $u(t) > 0$ for $a < t < b$. It is easy to verify that the function

$$\delta(t) = \frac{1}{h_0}\ln(1 + u(t)) \quad \text{for } a < t < b$$

satisfies conditions (2.13) and (2.14).

Suppose now that $h_0 = 0$. Then according to Theorem 2.2. in [6], the problem

$$u'' = -h_1(t) - h_2(t)|u'|, \quad u(a+) = 0, \quad u(b-) = 0$$

has at least one solution u. Since $u''(t) < 0$ for $a < t < b$, we have $u(t) > 0$ for $a < t < b$. It is obvious that $\delta(t) = u(t)$ for $a < t < b$ satisfies conditions (2.13) and (2.14). □

Lemma 2.3. *Let* $\mu \in]0,1[$, $p,g \in L_{\rm loc}(]a,b[;R_+)$, *and* $(p,g) \in V_\mu(]a,b[)$. *Then there exist* $c \in]a,b[$ *and* $\gamma \in \tilde{C}'_{\rm loc}(]a,c[\,\cup\,]c,b[;R)$ *such that*

$$\gamma(t) > 0 \quad \textit{for } a < t < b, \quad \gamma(a+) = 0, \quad \gamma(b-) = 0, \tag{2.15}$$

$$-\infty < \gamma'(c+) \leqslant \gamma'(c-) < +\infty \tag{2.16}$$

and

$$\gamma''(t) = -\mu p(t) - \mu g(t)|\gamma'(t)| - \mu|\gamma'(t)|^{(\mu+1)/\mu} \quad \textit{for } a < t < b. \tag{2.17}$$

Proof. Let u_1 and u_2 be solutions of problems (0.5), (0.7) and (0.5), (0.8), respectively, satisfying conditions (0.6). Put

$$\rho_1(t) = \left|\frac{u_1'(t)}{u_1(t)}\right|^\mu \operatorname{sgn} u_1'(t), \quad \rho_2(t) = \left|\frac{u_2'(t)}{u_2(t)}\right|^\mu \operatorname{sgn} u_2'(t) \quad \text{for } a < t < b.$$

It is clear that

$$\rho_1'(t) = -\mu p(t) - \mu g(t)|\rho_1(t)| - \mu|\rho_1(t)|^{(\mu+1)/\mu} \quad \text{for } a < t < b, \tag{2.18}$$

$$\rho_2'(t) = -\mu p(t) - \mu g(t)|\rho_2(t)| - \mu|\rho_2(t)|^{(\mu+1)/\mu} \quad \text{for } a < t < b. \tag{2.19}$$

Show now that for any $b_1 \in]a,b[$,

$$\rho_1 \in L(]a,b_1[;R_+). \tag{2.20}$$

Since

$$\lim_{t\to a+} \frac{u_1'(t)}{\gamma(g)(t)} \frac{\int_a^t \gamma(g)(s)\,{\rm d}s}{u_1(t)} = 1,$$

there exists $M_{b_1} > 0$ such that

$$|\rho_1(t)| \leqslant M_{b_1}\left(\frac{\gamma(g)(t)}{\int_a^t \gamma(g)(s)\,{\rm d}s}\right)^\mu \quad \text{for } a < t < b_1. \tag{2.21}$$

However, since $\mu \in]0,1[$, we have $(\gamma(g)(t))^\mu \leqslant \gamma(g)(t)$ for $a < t < b$, which together with (2.21) results in (2.20). Analogously we can find that $\rho_2 \in L(]a_1,b[;R)$ for any $a_1 \in]a,b[$.

Suppose that conditions (0.11) are fulfilled. Choose $c \in]a,b[$ such that

$$\int_a^c \rho_1(s)\,{\rm d}s = -\int_c^b \rho_2(s)\,{\rm d}s \tag{2.22}$$

and put

$$\gamma(t) = \begin{cases} \displaystyle\int_a^t \rho_1(s)\,{\rm d}s & \text{for } a \leqslant t \leqslant c, \\ \displaystyle -\int_t^b \rho_2(s)\,{\rm d}s & \text{for } c < t \leqslant b. \end{cases}$$

In view of (0.11), (2.18), (2.19) and (2.22) we can easily verify that γ satisfies conditions (2.15)–(2.17).

Suppose now that condition (0.9), (0.10)) holds. Put $c = (a+b)/2$ and

$$\gamma(t) = \int_a^t \rho_1(s)\,\mathrm{d}s \quad \left(\gamma(t) = \int_t^b \rho_2(s)\,\mathrm{d}s\right) \quad \text{for } a < t < b.$$

In view of (0.9) and (2.18) ((0.10) and (2.19)) we can easily verify that γ satisfies the conditions (2.15)–(217). □

Finally, to make the reference more convenient we will formulate a lemma from [2] (see [2, Lemma 2.5]).

Lemma 2.4. *Let u be a nontrivial solution of the equation*

$$u'' = p(t)|u|^{\mu}|u'|^{1-\mu}\operatorname{sgn} u$$

satisfying the condition

$$u(a+) = 0, \quad (u(b-) = 0)$$

and let $v \in \tilde{C}'_{\mathrm{loc}}(]a,c];R)$ $(v \in \tilde{C}'_{\mathrm{loc}}([c,b[;R))$, where $c \in]a,b[$, have a finite limit $v(a+) \geqslant 0$ $(v(b-) \geqslant 0)$ and satisfy the conditions

$$v'(t) > 0 \quad \text{for } a < t \leqslant c \quad (v'(t) < 0 \quad \text{for } c \leqslant t < b),$$

$$v''(t) \leqslant p(t)|v(t)|^{\mu}|v'(t)|^{1-\mu} \quad \text{for } a < t < c \quad (\text{for } c < t < b).$$

Then $u'(t) \neq 0$ for $a < t \leqslant c$ (for $c \leqslant t < b$).

3. Proofs

Proof of Theorem 1.1. Choose sequences $(t_{ik})_{k=1}^{+\infty}$ and $(s_{ik})_{k=1}^{+\infty}$, $i = 1,2$ such that

$$a < t_{1k+1} < t_{1k} < \frac{a+b}{2} < t_{2k} < t_{2k+1} < b, \quad k = 1,2,\dots,$$

$$t_{1k+1} < s_{1k+1} < t_{1k}, \quad t_{2k} < s_{2k+1} < t_{2k+1}, \quad k = 1,2,\dots,$$

$$t_{11} < s_{11} < s_{21} < t_{21}, \quad \lim_{k\to+\infty} t_{1k} = a, \quad \lim_{k\to+\infty} t_{2k} = b.$$

Put

$$r_0 = \sup\{|\alpha(t)| + |\beta(t)|:\ a < t < b\} \tag{3.1}$$

and let φ be the function appearing in Lemma 2.1. Let, moreover,

$$\psi(t) = \begin{cases} \varphi(t, t_{11}, t_{21}, r_0) & \text{for } t \in]s_{11}, s_{21}[, \\ \varphi(t, t_{1k+1}, t_{2k+1}, r_0) & \text{for } t \in]s_{1k+1}, s_{1k}[\cup]s_{2k}, s_{2k+1}[, \end{cases} \tag{3.2}$$

$$\rho(t) = |\alpha'(t)| + |\beta'(t)| + \psi(t) \quad \text{for } a < t < b, \tag{3.3}$$

$$\tilde{f}(t,x,y) = \begin{cases} f(t,x,y) & \text{for } |y| \leqslant \rho(t), \\ \left(2 - \dfrac{|y|}{\rho(t)}\right) f(t,x,y) & \text{for } \rho(t) < |y| < 2\rho(t), \\ 0 & \text{for } |y| \geqslant 2\rho(t), \end{cases} \tag{3.4}$$

$$f_0(t,x,y) = \begin{cases} \tilde{f}(t,\alpha(t),y) & \text{for } x \leqslant \alpha(t), \\ \tilde{f}(t,x,y) & \text{for } \alpha(t) < x < \beta(t), \\ \tilde{f}(t,\beta(t),y) & \text{for } x \geqslant \beta(t). \end{cases} \tag{3.5}$$

Consider the boundary value problem

$$u'' = f_0(t,u,u'), \tag{3.6}$$

$$u(t_{1k}) = \alpha(t_{1k}), \quad u(t_{2k}) = \alpha(t_{2k}). \tag{3.7}$$

In view of (3.1) and (3.3)–(3.5), it can be easily seen that $f_0 \in K_{\text{loc}}(]a,b[\times R^2;R)$ and for any natural k there exists $q_k \in L(]t_{1k},t_{2k}[;R_+)$ such that $|f_0(t,x,y)| \leqslant q_k(t)$ for $t_{1k} < t < t_{2k}$, $x,y \in R$. It is also obvious that α and β are lower and upper functions of Eq. (3.6). Therefore, according to Scorza–Dragoni theorem (see, e.g., [6, Lemma 3.7]), problem (3.6), (3.7) has at least one solution u_k such that

$$\alpha(t) \leqslant u_k(t) \leqslant \beta(t) \quad \text{for } t_{1k} \leqslant t \leqslant t_{2k}. \tag{3.8}$$

Taking into account (3.5), (3.6) and (3.8), we obtain

$$u_k''(t) = \tilde{f}(t,u_k(t),u_k'(t)) \quad \text{for } t_{1k} < t < t_{2k}. \tag{3.9}$$

Hence, due to (3.1), (3.4), (3.8) and (1.2) ((1.13)), we conclude that the function $u(t) = u_k(t)$ for $t_{1k} \leqslant t \leqslant t_{2k}$ satisfies the conditions of Lemma 2.1 with $a_1 = t_{1k}$, $b_1 = t_{2k}$, and consequently, the estimate

$$|u_k'(t)| \leqslant \varphi(t,t_{1k},t_{2k},r_0) \quad \textit{for } t_{1k} < t < t_{2k} \tag{3.10}$$

holds. The latter inequality, according to (3.2)–(3.4) and (3.9), yields

$$u_k'(t) = f(t,u_k(t),u_k'(t)) \quad \text{for } s_{1k} < t < s_{2k}. \tag{3.11}$$

From (3.8), (3.10) and (3.11) it follows that the sequences $(u_k)_{k=1}^{+\infty}$ and $(u_k')_{k=1}^{+\infty}$ are uniformly bounded and equicontinuous in $]a,b[$ (i.e., on every compact interval contained in $]a,b[$). Therefore, according to Arzelà–Ascoli lemma, without loss of generality we can assume that

$$\lim_{k\to+\infty} u_k(t) = u_0(t), \quad \lim_{k\to+\infty} u_k'(t) = u_0'(t)$$

uniformly in $]a,b[$. It is also evident that $u_0 \in \tilde{C}'_{\text{loc}}(]a,b[;R)$. Due to (3.11) we have that u_0 is a solution of Eq. (0.1), and by (1.1), (3.7) and (3.8) we find that u_0 satisfies also the boundary conditions (0.2). □

Proof of Corollary 1.1. We will prove the corollary in the case where $\lambda > 1$. The case $\lambda = 1$ can be proved analogously.

Since $(h_0 h_1, h_2) \in V_1(]a,b[)$, according to Lemma 2.2 there exists $\delta \in \tilde{C}'_{\text{loc}}(]a,b[;R)$ satisfying conditions (2.13) and (2.14). In view of (2.14) and (1.6) ((1.8)) it is clear that $\beta(t)=\delta(t)$ for $a<t<b$ ($\alpha(t)=-\delta(t)$ for $a<t<b$) is an upper (a lower) function of Eq. (0.1).

On the other hand, since

$$(\lambda^{(\lambda+1)/\lambda} p_0^{1/\lambda} p_1, \lambda p_2) \in V_{1/\lambda}(]a,b[),$$

according to Lemma 2.3 there exist $c \in]a,b[$ and $\gamma \in \tilde{C}'_{\text{loc}}(]a,c[\cup]c,b[;R)$ satisfying conditions (2.15)–(2.17) with

$$\mu = \frac{1}{\lambda}, \quad p(t) \equiv \lambda^{(\lambda+1)/\lambda} p_0^{1/\lambda} p_1(t), \quad \text{and} \quad g(t) \equiv \lambda p_2(t).$$

Hence by (1.7) ((1.9)) it is obvious that

$$\alpha(t) = -(\lambda p_0)^{-1/\lambda}\gamma(t) \quad \text{for } a<t<b \quad (\beta(t) = (\lambda p_0)^{-1/\lambda}\gamma(t) \quad \text{for } a<t<b)$$

is a lower (an upper) function of Eq. (0.1). Consequently, the assumptions of Theorem 1.1 are fulfilled. □

On Remark 1.1: Suppose $h_1 \in L(]a,b[;R_+)$ and $(h_1,0) \notin V_1(]a,b[)$. Then there exist $a_1 \in [a,b[$ and $b_1 \in]a_1,b]$ such that the problem

$$\begin{aligned} &v'' = -h_1(t)v, \\ &v(a_1+) = 0, \quad v(b_1-) = 0 \end{aligned} \tag{3.12}$$

has a nontrivial solution. Let $f(t,x,y) = -h_1(t) - y^2$ for $a<t<b$. Let, moreover, $p_1 \in L(]a,b[;]0,+\infty[)$ be such that for some $\lambda > 1$, the inclusion $(\lambda^{1/\lambda} p_1, 0) \in V_{1/\lambda}(]a,b[)$ holds (for the existence of such a function see [3, Theorem 3.2]). Then, obviously, conditions (1.6) and (1.7) with $h_2 \equiv 0$, $p_2 \equiv 0$ are fulfilled. We will show that in that case problem (0.1), (0.2) has no solution. Assume the contrary that u is a solution of problem (0.1), (0.2). It can be easily verified that the function $v(t) = \exp(u(t))$ for $a \leqslant t \leqslant b$ satisfies Eq. (3.12). It is also clear that $v(t) > 0$ for $a \leqslant t \leqslant b$. But this contradicts Sturm's separation theorem.

Now let $p_1 \in L(]a,b[;]0,+\infty[)$ be such that a solution u_1 of the problem

$$u'' = -p_1(t)|u|^{\mu}|u'|^{1-\mu}\operatorname{sgn} u, \quad u(a+) = 0, \quad \lim_{t\to a+} u'(t) = 1,$$

where $\mu \in]0,1[$, satisfies the conditions $u_1(t) > 0$ for $a<t<b$, $\operatorname{mes}\{t \in]a,b[: u_1'(t) = 0\} = 0$ and

$$\int_a^b \left|\frac{u_1'(s)}{u_1(s)}\right|^{\mu} \operatorname{sgn} u_1'(s)\,\mathrm{d}s < 0.$$

Then, clearly,

$$(\lambda^{(\lambda+1)/\lambda} p_0^{1/\lambda} p_1, \lambda p_2) \notin V_{1/\lambda}(]a,b[),$$

where $p_2 \equiv 0$, $\lambda = 1/\mu$, $p_0 = \mu$. Put $f(t,x,y) = \mu p_1(t) + \mu|y|^{(\mu+1)/\mu}$ for $a < t < b$, $x, y \in R$. We will show that in that case problem (0.1), (0.2) has no solution. Assume the contrary. Let u be a solution of the problem

$$u'' = \mu p_1(t) + \mu|u'|^{(\mu+1)/\mu}, \quad u(a+) = 0, \quad u(b-) = 0.$$

Define the functions ρ_1 and ρ by

$$\rho_1(t) = \left|\frac{u_1'(t)}{u_1(t)}\right|^{\mu} \operatorname{sgn} u_1'(t), \quad \rho(t) = -u'(t) \quad \text{for } a < t < b.$$

It is clear that

$$\rho_1'(t) = -\mu p_1(t) - \mu|\rho_1(t)|^{(\mu+1)/\mu},$$
$$\rho'(t) = -\mu p_1(t) - \mu|\rho(t)|^{(\mu+1)/\mu} \quad \text{for } a < t < b, \tag{3.13}$$

$$\int_a^b \rho_1(s)\,\mathrm{d}s < 0, \quad \int_a^b \rho(s)\,\mathrm{d}s = 0. \tag{3.14}$$

Show now that

$$\rho_1(t) \leqslant \rho(t) \quad \text{for } a < t < b. \tag{3.15}$$

Assume the contrary that $\rho_1(t_1) > \rho(t_1)$ for some $t_1 \in]a,b[$. Then in view of (3.14) there exist $t_0 \in]a,b[$ and $\varepsilon > 0$ such that either

$$0 > \rho_1(t) > \rho(t) \quad \text{for } t_0 - \varepsilon < t < t_0, \;\; \rho_1(t_0) = \rho(t_0) \tag{3.16}$$

or

$$0 < \rho(t) < \rho_1(t) \quad \text{for } t_0 < t < t_0 + \varepsilon, \quad \rho_1(t_0) = \rho(t_0). \tag{3.17}$$

Let condition (3.16), ((3.17)) be fulfilled. Then from (3.13) we get

$$\rho_1(t) = \rho_1(t_0) + \mu\int_t^{t_0} p_1(s)\,\mathrm{d}s + \mu\int_t^{t_0} |\rho_1(s)|^{(\mu+1)/\mu}\,\mathrm{d}s$$
$$< \rho(t_0) + \mu\int_t^{t_0} p_1(s)\,\mathrm{d}s + \mu\int_t^{t_0} |\rho(s)|^{(\mu+1)/\mu}\,\mathrm{d}s = \rho(t) \quad \text{for } t_0 - \varepsilon < t < t_0$$
$$(\rho_1(t) = \rho_1(t_0) - \mu\int_{t_0}^{t} p_1(s)\,\mathrm{d}s - \mu\int_{t_0}^{t} |\rho_1(s)|^{(\mu+1)/\mu}\,\mathrm{d}s$$
$$< \rho(t_0) - \mu\int_{t_0}^{t} p_1(s)\,\mathrm{d}s - \mu\int_{t_0}^{t} |\rho(s)|^{(\mu+1)/\mu}\,\mathrm{d}s = \rho(t) \quad \text{for } t_0 < t < t_0 + \varepsilon),$$

which contradicts (3.16), ((3.17)) Therefore, (3.15) is fulfilled.

Since ρ is nonincreasing, according to (3.15) there exists a finite limit $\rho(b-)$ and, consequently, $|\rho|^{1/\mu} \in L(]a_1,b[;R_+)$ for any $a_1 \in]a,b[$.

Put

$$v(t) = \exp\left(-\int_t^b |\rho(s)|^{1/\mu} \operatorname{sgn} \rho(s)\, ds \right) \quad \text{for } a < t < b.$$

It can be easily verified that $v \in \tilde{C}'_{\text{loc}}(]a,b[; R_+)$ and

$$v''(t) = -p_1(t)|v(t)|^{\mu}|v'(t)|^{1-\mu}, \quad \operatorname{sgn} v'(t) = \operatorname{sgn} \rho(t) \quad \text{for } a < t < b.$$

According to (3.15), there exists $a_0 \in\,]a,b[$ such that

$$\int_t^{a_0} |\rho(s)|^{1/\mu} \operatorname{sgn} \rho(s)\, ds > \int_t^{a_0} |\rho_1(s)|^{1/\mu} \operatorname{sgn} \rho_1(s)\, ds = \ln \frac{u(a_0)}{u(t)} \quad \text{for } a < t < a_0.$$

Hence we have $v(a+) = 0$. Therefore, according to Lemma 2.4 it is easy to see that there exists $c \in\,]a,b[$ such that

$$\rho_1(c) = 0, \quad \rho(c) = 0.$$

Thus the functions ρ_1 and ρ are solutions of the Cauchy problem

$$w' = -\mu p_1(t) - \mu|w|^{(\mu+1)/\mu}, \quad w(c) = 0.$$

However, this problem is uniquely solvable. Consequently, $\rho_1 \equiv \rho$, which contradicts condition (3.14).

Proof of Corollary 1.2. According to (1.11), (1.13) and Theorem 4.4 in [5] we have that $(h_0 h_1, h_2) \in V_1(]a,b[)$. We will show that conditions (1.12) and (1.14) guarantee the inclusion $(\lambda^{(\lambda+1)/\lambda} p_0^{1/\lambda} p_1, \lambda p_2) \in V_{1/\lambda}(]a,b[)$.

Without loss of generality, we can assume that

$$\int_0^{+\infty} \frac{ds}{l_{21} + l_{22}s + s^{1+\lambda}} = \frac{1}{\lambda(1-\lambda\mu)} \left(\frac{b-a}{2} \right)^{1-\lambda\mu}.$$

Define the functions ρ_1 and ρ_2 by the equalities

$$\int_{\rho_1(t)}^{+\infty} \frac{ds}{l_{21} + l_{22}s + s^{1+\lambda}} = \frac{1}{\lambda(1-\lambda\mu)} (t-a)^{1-\lambda\mu} \quad \text{for } a < t \leqslant \frac{a+b}{2},$$

$$\int_{\rho_2(t)}^{+\infty} \frac{ds}{l_{21} + l_{22}s + s^{1+\lambda}} = \frac{1}{\lambda(1-\lambda\mu)} (b-t)^{1-\lambda\mu} \quad \text{for } \frac{a+b}{2} \leqslant t < b.$$

Obviously,

$$\rho_1(t) > 0 \quad \text{for } a < t < \frac{a+b}{2}, \quad \rho_2(t) > 0 \quad \text{for } \frac{a+b}{2} < t < b,$$

$$\rho_1(a+) = +\infty, \quad \rho_1\left(\frac{a+b}{2} \right) = 0, \quad \rho_2\left(\frac{a+b}{2} \right) = 0, \quad \rho_2(b-) = +\infty,$$

$$\rho_1(a+b-t) = \rho_2(t) \quad \text{for } \frac{a+b}{2} < t < b. \tag{3.18}$$

Let

$$v(t)=\begin{cases}\exp\left(-\displaystyle\int_t^{(a+b)/2}\rho_1^{\lambda}(s)(s-a)^{-\lambda\mu}\,\mathrm{d}s\right) & \text{for } a<t\leqslant\dfrac{a+b}{2},\\ \exp\left(-\displaystyle\int_{(a+b)/2}^{t}\rho_2^{\lambda}(s)(b-s)^{-\lambda\mu}\,\mathrm{d}s\right) & \text{for } \dfrac{a+b}{2}<t<b.\end{cases}$$

Then

$$v(t)>0\quad\text{for } a<t<b,\quad v'(t)(t)>0\quad\text{for } t\in\left]a,\frac{a+b}{2}\right[\cup\left]\frac{a+b}{2},b\right[,$$

$$v(a+)=0,\quad v(b-)=0,\quad v'\left(\frac{a+b}{2}\right)=0,$$

$$v''(t)=-\lambda^{(\lambda+1)/\lambda}p_0^{1/\lambda}p_1(t)|v(t)|^{1/\lambda}|v'(t)|^{1-1/\lambda}-\lambda p_2(t)|v'(t)|\quad\text{for } a<t<b,$$

and

$$\left|\frac{v'(t)}{v(t)}\right|^{1/\lambda}\operatorname{sgn}v'(t)=\begin{cases}\rho_1(t)(t-a)^{-\mu} & \text{for } a<t\leqslant\dfrac{a+b}{2},\\ -\rho_2(t)(b-t)^{-\mu} & \text{for } \dfrac{a+b}{2}<t<b.\end{cases}$$

In view of (3.18) we have

$$\int_a^b\left|\frac{v'(s)}{v(s)}\right|^{1/\lambda}\operatorname{sgn}v'(s)\,\mathrm{d}s=0.$$

Hence $(\lambda^{(\lambda+1)/\lambda}p_0^{1/\lambda}p_1,\lambda p_2)\in V_{1/\lambda}(]a,b[)$ and the assumptions of Corollary 1.1 are fulfilled. □

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