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On a boundary value problem for first-order scalar functional differential equations

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Abstract

Nonimprovable effective sufficient conditions for solvability and unique solvability of the boundary value problem

$$u'(t) = F(u)(t), \qquad u(a) = h(u),$$

where $F : C([a,b];R) \to L([a,b];R)$ is a continuous operator satisfying the Caratheodory condition and $h : C([a,b];R) \to R$ is a continuous functional, are established.

Keywords: Boundary value problems; First-order scalar functional differential equations; Existence and uniqueness of a solution

1. Introduction

The following notation is used throughout. R is the set of all real numbers, $R_+ = [0, +\infty[$. $C([a,b];R)$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)|: a \leqslant t \leqslant b\}$.

$$C([a,b];R_+) = \{u \in C([a,b];R): u(t) \geqslant 0 \text{ for } t \in [a,b]\},$$

$$C_a([a,b];R) = \{u \in C([a,b];R): u(a) = 0\}.$$

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$\tilde{C}([a,b];R)$ is the set of absolutely continuous functions u: $[a,b] \to R$. $B_c([a,b];R) = \{u \in C([a,b];R)\colon\ |u(a)| \leqslant c\}$, where $c \in R_+$. $L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|\,\mathrm{d}s$.

$L([a,b];R_+)p = \{p \in L([a,b];R)\colon\ p(t) \geqslant 0$ for almost all $t \in [a,b]\}$.

$\mathscr{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$\tilde{\mathscr{L}}_{ab}$ is the set of linear operators $\ell : C([a,b];R) \to L([a,b];R)$ for which there is a function $\eta \in L([a,b];R_+)$ such that

$$|\ell(v)(t)| \leqslant \eta(t)\|v\|_C \quad \text{for } t \in [a,b], \quad v \in C([a,b];R).$$

$\mathscr{P}_{ab}$ is the set of linear operators $\ell \in \tilde{\mathscr{L}}_{ab}$ transforming the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$. $\mathscr{K}_{ab}$ is the set of continuous operators $F : C([a,b];R) \to L([a,b];R)$ satisfying the Carathèodory conditions, i.e., for each $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|F(v)(t)| \leqslant q_r(t) \quad \text{for } t \in [a,b], \quad \|v\|_C \leqslant r.$$

$K([a,b] \times A;B)$, where $A \subseteq R^2$, $B \subseteq R$, is the set of functions $f : [a,b] \times A \to B$ satisfying the Carathèodory conditions, i.e., $f(\cdot,x) : [a,b] \to B$ is a measurable function for all $x \in A$, $f(t,\cdot) : A \to B$ is a continuous function for almost all $t \in [a,b]$, and for each $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|f(t,x)| \leqslant q_r(t) \quad \text{for } t \in, \ x \in A, \ \|x\| \leqslant r.$$

$[x]_+ = \frac{1}{2}(|x| + x), [x]_- = \frac{1}{2}(|x| - x)$.

By a solution of the equation

$$u'(t) = F(u)(t), \tag{1}$$

where $F \in \mathscr{K}_{ab}$, we understand a function $u \in \tilde{C}([a,b];R)$ satisfying Eq. (1) almost everywhere in $[a,b]$.

Consider the problem on the existence and uniqueness of a solution of (1) satisfying the boundary condition

$$u(a) = h(u), \tag{2}$$

where $h : C([a,b];R) \to R$ is a continuous functional.

For ordinary differential equations, i.e., when the operator F is so-called Nemitsky's operator, problem (1), (2) and analogous problems for systems of linear and nonlinear ordinary differential equations have been studied in details (see [8,16–19] and the references therein). Foundation of the theory of general boundary value problems for functional differential equations was laid in monographs [30,1] (see also [2,3,9,15,20–29,31]). In spite of a large number of papers devoted to the boundary value problems for functional differential equations, nowadays only a few effective sufficient conditions for the solvability are known even for the linear case

$$u'(t) = \ell(u)(t) + q_0(t), \tag{3}$$

$$u(a) = c_0, \tag{4}$$

where $\ell \in \tilde{\mathscr{L}}_{ab}$, $q_0 \in L([a,b];R)$ and $c_0 \in R$ (see [5–7,9–14,20–28,31]). In the present paper, we try to fill this gap in a certain way. More precisely, in Sections 1 and 2 there are established nonimprovable effective sufficient conditions for the solvability and unique solvability of problems (3), (4) and (1), (2), respectively. These results make theorems in [5,6] more complete. Section 3 is devoted to the examples verifying the optimality of the main results.

All results will be concretized for the differential equation with deviating arguments of the form

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + q_0(t) \tag{5}$$

and

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + f(t, u(t), u(\nu(t))), \tag{6}$$

where $p, g \in L([a,b];R_+)$, $q_0 \in L([a,b];R)$, $\tau, \mu, \nu \in \mathscr{M}_{ab}$, and $f \in K([a,b] \times R^2; R)$.

2. Linear problem

From the general theory of linear boundary value problems for functional differential equations we need the following well-known result (see, e.g., [4,20,30]).

Theorem 2.1. *Problem (3), (4) is uniquely solvable iff the corresponding homogeneous problem*

$$u'(t) = \ell(u)(t), \tag{3_0}$$

$$u(a) = 0 \tag{4_0}$$

has only the trivial solution.

Remark 2.1. From the Riesz–Schauder theory it follows that if $\ell \in \tilde{\mathscr{L}}_{ab}$ and the problem (3_0), (4_0) has a nontrivial solution, then there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that problem (3), (4) has no solution.

Definition 2.1. We will say that an operator $\ell \in \tilde{\mathscr{L}}_{ab}$ belongs to the set $\mathscr{S}_{ab}(a)$ if the homogeneous problem (3_0), (4_0) has only the trivial solution, and for arbitrary $q_0 \in L([a,b];R_+)$ and $c_0 \in R_+$, the solution of problem (3), (4) is nonnegative.

Remark 2.2. From Definition 2.1 it immediately follows that $\ell \in \mathscr{S}_{ab}(a)$ iff for Eq. (3) the classical theorem on differential inequalities holds (see, e.g., [16]), i.e., whenever $u, v \in \tilde{C}([a,b];R)$ satisfy the inequalities

$$u'(t) \leqslant \ell(u)(t) + q_0(t), \quad v'(t) \geqslant \ell(v)(t) + q_0(t) \quad \text{for } t \in [a,b],$$

$$u(a) \leqslant v(a),$$

then

$$u(t) \leqslant v(t) \quad \text{for } t \in [a,b].$$

Theorem 2.2. *Let there exist $\ell_0, \ell_1 \in \mathscr{P}_{ab}$ such that on the set $C_a([a,b];R)$ the inequality*

$$|\ell(v)(t) + \ell_1(v)(t)| \leqslant \ell_0(|v|)(t) \quad \text{for } t \in [a,b] \tag{7}$$

holds. Let, moreover,

$$\ell_0 \in \mathscr{S}_{ab}(a), \quad -\tfrac{1}{2}\ell_1 \in \mathscr{S}_{ab}(a). \tag{8}$$

Then problem (3), (4) has a unique solution.

Remark 2.3. Theorem 2.2 is nonimprovable in the sense that condition (7) cannot be replaced by the condition

$$|\ell(v)(t) + \ell_1(v)(t)| \leqslant (1+\varepsilon)\ell_0(|v|)(t) \quad \text{for } t \in [a,b], \tag{9}$$

and assumption (8) can be replaced neither by the assumption

$$\ell_0 \in \mathscr{S}_{ab}(a), \quad -\frac{1}{2+\varepsilon}\ell_1 \in \mathscr{S}_{ab}(a),$$

nor by the assumption

$$(1-\varepsilon)\ell_0 \in \mathscr{S}_{ab}(a), \quad -\tfrac{1}{2}\ell_1 \in \mathscr{S}_{ab}(a),$$

no matter how small $\varepsilon > 0$ would be (see Examples 4.1 and 4.2).

Remark 2.4. In [13] (see also [5]) effective nonimprovable sufficient conditions are established for an operator $\ell \in \tilde{\mathscr{L}}_{ab}$ to belong to the set $\mathscr{S}_{ab}(a)$. Therefore from Theorem 2.2 it immediately follows.

Corollary 2.1. *Let $\mu(t) \leqslant t$ for $t \in [a,b]$, the functions p, τ satisfy one of the following conditions*:

(a)

$$\int_a^t p(s) \int_a^{\tau(s)} p(\xi)\,d\xi\,ds \leqslant \alpha \int_a^t p(s)\,ds \quad \text{for } t \in [a,b],$$

where $\alpha \in]0,1[$;

(b)

$$\int_a^b p(s)\sigma(s) \int_s^{\tau(s)} p(\xi)\,d\xi \exp\left[\int_s^b p(\eta)\,d\eta\right] ds < 1,$$

where $\sigma(t) = \frac{1}{2}(1 + \operatorname{sgn}(\tau(t) - t))$ *for* $t \in [a,b]$;

(c) $\int_a^{\tau^*} p(s)\,ds \neq 0$ *and*

$$\operatorname{ess\,sup}\left\{\int_t^{\tau(t)} p(s)\,ds \colon t \in [a,b]\right\} < \lambda^*,$$

where

$$\lambda^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp(x\int_a^{\tau^*} p(s)\,\mathrm{d}s) - 1}\right) : x > 0\right\},$$

$$\tau^* = \operatorname{ess\ sup}\{\tau(t) \colon t \in [a,b]\},$$

and let the functions g, μ *satisfy one of the following conditions*:

(d)

$$\int_a^b g(s)\,\mathrm{d}s \leqslant 2;$$

(e)

$$\int_a^b g(s)\left(\int_{\mu(s)}^s g(\xi)\exp\left[\frac{1}{2}\int_{\mu(\xi)}^s g(\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi\right)\mathrm{d}s \leqslant 4;$$

(f) $g \not\equiv 0$ *and*

$$\operatorname{ess\ sup}\left\{\int_{\mu(t)}^t g(s)\,\mathrm{d}s \colon t \in [a,b]\right\} < 2\eta^*,$$

where

$$\eta^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp((x/2)\int_a^b g(s)\,\mathrm{d}s) - 1}\right) : x > 0\right\}.$$

Then problem (5), (4) *has a unique solution.*

Remark 2.5. The condition $\mu(t) \leqslant t$ for $t \in [a,b]$ is also necessary condition for an operator $\ell(v)(t) \overset{\mathrm{def}}{=} -g(t)v(\mu(t))$ to belong to the set $\mathscr{S}_{ab}(a)$ (see [7]).

Proof of Theorem 2.2. According to Theorem 2.1 it is sufficient to show that problem (3_0), (4_0) has only the trivial solution.

Let u be a solution of (3_0), (4_0). Then in view of (3_0), u satisfies

$$u'(t) = -\frac{1}{2}\,\ell_1(u)(t) + \left[\ell(u)(t) + \frac{1}{2}\,\ell_1(u)(t)\right], \qquad u(a) = 0. \tag{10}$$

By virtue of the assumption $-\frac{1}{2}\ell_1 \in \mathscr{S}_{ab}(a)$ and Theorem 2.1, the problem

$$\alpha'(t) = -\tfrac{1}{2}\ell_1(\alpha)(t) + \ell_0(|u|)(t) + \tfrac{1}{2}\ell_1(|u|)(t), \qquad \alpha(a) = 0 \tag{11}$$

has a unique solution α. Moreover, since $\ell_0, \ell_1 \in \mathscr{P}_{ab}$,

$$\alpha(t) \geqslant 0 \quad \text{for } t \in [a,b]. \tag{12}$$

In view of (7) and the condition $\ell_1 \in \mathscr{P}_{ab}$, from (11) we have

$$\alpha'(t) \geqslant -\tfrac{1}{2}\ell_1(\alpha)(t) + \ell(u)(t) + \tfrac{1}{2}\ell_1(u)(t) \quad \text{for } t \in [a,b],$$

$$(-\alpha(t))' \leqslant -\tfrac{1}{2}\ell_1(-\alpha)(t) + \ell(u)(t) + \tfrac{1}{2}\ell_1(u)(t) \quad \text{for } t \in [a,b].$$

The last two inequalities and (10), on account of the assumption $-\frac{1}{2}\ell_1 \in \mathcal{S}_{ab}(a)$ and Remark 2.2, yield

$$|u(t)| \leqslant \alpha(t) \quad \text{for } t \in [a,b]. \tag{13}$$

On the other hand, due to (13) and the conditions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, (11) results in

$$\alpha'(t) \leqslant \ell_0(\alpha)(t) \quad \text{for } t \in [a,b].$$

Since $\ell_0 \in \mathcal{S}_{ab}(a)$, the last inequality together with $\alpha(a) = 0$ yields $\alpha(t) \leqslant 0$ for $t \in [a,b]$, which, in view of (12), implies $\alpha \equiv 0$. Consequently, from (13) it follows that $u \equiv 0$. □

3. Nonlinear problem

Throughout this section we assume that $q \in K([a,b] \times R_+; R_+)$ is nondecreasing in the second argument, and satisfies

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)\,\mathrm{d}s = 0. \tag{14}$$

3.1. Main results

Theorem 3.1. *Let $c \in R_+$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$,*

$$h(v)\,\mathrm{sgn}\,v(a) \leqslant c \quad \textit{for } v \in C([a,b];R), \tag{15}$$

and on the set $B_c([a,b];R)$ the inequality

$$[F(v)(t) + \ell_1(v)(t)]\,\mathrm{sgn}\,v(t) \leqslant \ell_0(|v|)(t) + q(t, \|v\|_C) \quad \textit{for } t \in [a,b] \tag{16}$$

be fulfilled. Let, moreover,

$$\ell_0 \in \mathcal{S}_{ab}(a), \quad -\ell_1 \in \mathcal{S}_{ab}(a). \tag{17}$$

Then problem (1), (2) has at least one solution.

Remark 3.1. Theorem 3.1 is nonimprovable in the sense that condition (17) can be replaced neither by the condition

$$\ell_0 \in \mathcal{S}_{ab}(a), \quad -(1-\varepsilon)\ell_1 \in \mathcal{S}_{ab}(a)$$

nor by the condition

$$(1-\varepsilon)\ell_0 \in \mathcal{S}_{ab}(a), \quad -\ell_1 \in \mathcal{S}_{ab}(a),$$

no matter how small $\varepsilon > 0$ would be (see on Remark 3.1).

Theorem 3.2. *Let $\ell_1, \ell_0 \in \mathcal{P}_{ab}$,*

$$[h(v) - h(w)]\,\mathrm{sgn}(v(a) - w(a)) \leqslant 0 \quad \textit{for } v, w \in C([a,b];R) \tag{18}$$

and on the set $B_c([a,b];R)$, where $c = |h(0)|$, the inequality

$$[F(v)(t) - F(w)(t) + \ell_1(v-w)(t)]\,\mathrm{sgn}(v(t) - w(t)) \leqslant \ell_0(|v-w|)(t)$$

be fulfilled. Let, moreover, condition (17) *be satisfied. Then problem* (1), (2) *has a unique solution.*

Remark 3.2. Theorem 3.2 is nonimprovable in a certain sense (see on Remark 3.2).

Corollary 3.1. *Let* $c \in R_+$, *condition* (15) *be fulfilled, and*

$$f(t,x,y)\operatorname{sgn} x \leqslant q(t,|x|) \quad \text{for } t \in [a,b], \quad x,y \in R.$$

Let, moreover, $\mu(t) \leqslant t$ *for* $t \in [a,b]$, *the functions* p, τ *satisfy one of conditions* (a)–(c) *in Corollary* 2.1, *and the functions* g, μ *satisfy one of the following conditions*:

(d)

$$\int_a^b g(s)\,\mathrm{d}s \leqslant 1;$$

(e)

$$\int_a^b g(s)\left(\int_{\mu(s)}^s g(\xi)\exp\left[\int_{\mu(\xi)}^s g(\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi\right)\mathrm{d}s \leqslant 1;$$

(f) $g \not\equiv 0$ *and*

$$\operatorname{ess\,sup}\left\{\int_{\mu(t)}^t g(s)\,\mathrm{d}s\colon\ t \in [a,b]\right\} < \eta^*,$$

where

$$\eta^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp(x\int_a^b g(s)\,\mathrm{d}s) - 1}\right) : x > 0\right\}.$$

Then problem (6), (2) *has at least one solution.*

Corollary 3.2. *Let condition* (18) *be fulfilled, and*

$$[f(t,x_1,y_1) - f(t,x_2,y_2)]\operatorname{sgn}(x_1 - x_2) \leqslant 0 \quad \text{for } t \in [a,b], \ x_1,y_1,x_2,y_2 \in R.$$

Let, moreover, $\mu(t) \leqslant t$ *for* $t \in [a,b]$, *the functions* p, τ *satisfy one of conditions* (a)–(c) *in Corollary* 2.1, *and the functions* g, μ *satisfy one of conditions* (d)–(f) *in Corollary* 3.1. *Then problem* (6), (2) *has a unique solution.*

3.2. Auxiliary propositions and the proof of the main results

First, we formulate a result from [23, Theorem 1] in a form suitable for us.

Lemma 3.1. *Let there exist* $\ell_1 \in \tilde{\mathscr{L}}_{ab}$ *and a positive number* ρ *such that the problem*

$$u'(t) + \ell_1(u)(t) = 0, \quad u(a) = 0 \tag{19}$$

has only the trivial solution and for every $\delta \in\,]0,1[$ *and for an arbitrary function* $u \in \tilde{C}([a,b];R)$ *satisfying*

$$u'(t) + \ell_1(u)(t) = \delta[F(u)(t) + \ell_1(u)(t)], \quad u(a) = \delta h(u), \tag{20}$$

the estimate

$$\|u\|_C \leqslant \rho \tag{21}$$

holds. Then problem (1), (2) *has at least one solution.*

Definition 3.1. We say that the pair of operators (ℓ_0, ℓ_1) belongs to the set $\mathscr{A}$ if $\ell_0 \in \mathscr{P}_{ab}$, $\ell_1 \in \tilde{\mathscr{L}}_{ab}$, and there exists a positive number r such that for any $q^* \in L([a,b]; R_+)$ and $c \in R_+$, every function $u \in \tilde{C}([a,b];R)$ satisfying the inequalities $|u(a)| \leqslant c$ and

$$[u'(t) + \ell_1(u)(t)] \operatorname{sgn} u(t) \leqslant \ell_0(|u|)(t) + q^*(t) \quad \text{for } t \in [a,b] \tag{22}$$

admits the estimate

$$\|u\|_C \leqslant r(c + \|q^*\|_L). \tag{23}$$

Lemma 3.2. *Let* $(\ell_0, \ell_1) \in \mathscr{A}$, *there exist* $c \in R_+$ *such that*

$$h(v) \operatorname{sgn} v(a) \leqslant c \quad \textit{for } v \in C([a,b];R) \tag{24}$$

and on the set $B_c([a,b];R)$ *the inequality*

$$[F(v)(t) + \ell_1(v)(t)] \operatorname{sgn} v(t) \leqslant \ell_0(|v|)(t) + q(t, \|v\|_C) \quad \textit{for } t \in [a,b] \tag{25}$$

be fulfilled. Then problem (1), (2) *has at least one solution.*

Proof. First note that due to the condition $(\ell_0, \ell_1) \in \mathscr{A}$, the homogeneous problem (19) has only the trivial solution.

Let r be the number appearing in Definition 3.1. According to (14) there exists $\rho > 2rc$ such that

$$\frac{1}{x} \int_a^b q(s,x)\, ds < \frac{1}{2r} \quad \text{for } x > \rho.$$

Now assume that a function $u \in \tilde{C}([a,b];R)$ satisfies (20) for some $\delta \in]0,1[$. Then according to (24), u satisfies the inequality $|u(a)| \leqslant c$, i.e., $u \in B_c([a,b];R)$. By (25) the inequality (22) is fulfilled for $q^*(t) = q(t, \|u\|_C)$. Hence, by condition $(\ell_0, \ell_1) \in \mathscr{A}$ and the definition of the number ρ, we get estimate (21).

Since ρ depends neither on u nor on δ, from Lemma 3.1 it follows that problem (1), (2) has at least one solution. □

Lemma 3.3. *Let* $(\ell_0, \ell_1) \in \mathscr{A}$,

$$[h(v) - h(w)] \operatorname{sgn}(v(a) - w(a)) \leqslant 0 \quad \textit{for } v, w \in C([a,b];R), \tag{26}$$

and on the set $B_c([a,b];R)$, *where* $c = |h(0)|$, *the inequality*

$$\begin{aligned}&[F(v)(t) - F(w)(t) + \ell_1(v-w)(t)] \operatorname{sgn}(v(t) - w(t))\\ &\quad \leqslant \ell_0(|v-w|)(t) \quad \textit{for } t \in [a,b]\end{aligned} \tag{27}$$

be fulfilled. Then problem (1), (2) *has a unique solution.*

Proof. From (26) it follows that condition (24) is fulfilled with $c = |h(0)|$. By (27) we get that on the set $B_c([a,b];R)$ inequality (25) holds, where $q \equiv |F(0)|$. Consequently, all the assumptions of Lemma 3.2 are fulfilled which guarantees that problem (1), (2) has at least one solution. It remains to show that problem (1), (2) has at most one solution.

Let u_1, u_2 be solutions of problem (1), (2). Put $u(t) = u_1(t) - u_2(t)$ for $t \in [a,b]$. By (26) and (27) it is clear that

$$[u'(t) + \ell_1(u)(t)] \operatorname{sgn} u(t) \leqslant \ell_0(|u|)(t) \quad \text{for } t \in [a,b], \quad u(a) = 0.$$

Thus, the condition $(\ell_0, \ell_1) \in \mathscr{A}$ implies $u \equiv 0$, and consequently, $u_1 \equiv u_2$. □

Lemma 3.4. *Let $\ell_0 \in \tilde{\mathscr{L}}_{ab}$ and the homogeneous problem*

$$v'(t) = \ell_0(v)(t), \quad v(a) = 0$$

have only the trivial solution. Then there exists a positive number r_0 such that for any $q^* \in L([a,b];R)$ and $c \in R$, every solution v of the problem

$$v'(t) = \ell_0(v)(t) + q^*(t), \quad v(a) = c \tag{28}$$

admits the estimate

$$\|v\|_C \leqslant r_0(|c| + \|q^*\|_L). \tag{29}$$

Proof. Denote by

$$R \times L([a,b];R) = \{(c, q^*) \colon c \in R, q^* \in L([a,b];R)\}$$

the Banach space with the norm

$$\|(c,q^*)\|_{R\times L} = |c| + \|q^*\|_L$$

and denote by Ω the operator mapping every $(c,q^*) \in R \times L([a,b];R)$ to the solution v of problem (28). According to Theorem 1.4 in [20], $\Omega : R \times L([a,b];R) \to C([a,b];R)$ is a linear bounded operator. Denote by r_0 the norm of Ω. Then clearly for any $(c,q^*) \in R \times L([a,b];R)$, the inequality

$$\|\Omega(c,q^*)\|_C \leqslant r_0(|c| + \|q^*\|_L)$$

holds. Consequently, an arbitrary solution v of problem (28) admits estimate (29). □

Lemma 3.5. *Let $\ell_0, \ell_1 \in \mathscr{P}_{ab}$, $\ell_0 \in \mathscr{S}_{ab}(a)$, and $-\ell_1 \in \mathscr{S}_{ab}(a)$. Then $(\ell_0, \ell_1) \in \mathscr{A}$.*

Proof. Let $q^* \in L([a,b];R_+)$, $c \in R_+$, and $u \in \tilde{C}([a,b];R)$ satisfy inequalities $|u(a)| \leqslant c$ and (22). We show that (23) holds, where $r = r_0$ is the number appearing in Lemma 3.4.

It is clear that

$$u'(t) = -\ell_1(u)(t) + \tilde{q}(t), \tag{30}$$

where

$$\tilde{q}(t) = u'(t) + \ell_1(u)(t) \quad \text{for } t \in [a,b].$$

According to (22), we evidently have

$$\tilde{q}(t)\,\mathrm{sgn}\,u(t) \leqslant \ell_0(|u|)(t) + q^*(t) \quad \text{for } t \in [a,b]. \tag{31}$$

Furthermore, from (30), in view of the assumption $\ell_1 \in \mathscr{P}_{ab}$ and inequality (31), it follows that

$$\begin{aligned}[u(t)]_+' &\leqslant \ell_1([u]_-)(t) + \ell_0(|u|)(t) + q^*(t)\\ &= -\ell_1([u]_+)(t) + \ell_1(|u|)(t) + \ell_0(|u|)(t) + q^*(t) \quad \text{for } t \in [a,b]\end{aligned} \tag{32}$$

and

$$\begin{aligned}[u(t)]_-' &\leqslant \ell_1([u]_+)(t) + \ell_0(|u|)(t) + q^*(t)\\ &= -\ell_1([u]_-)(t) + \ell_1(|u|)(t) + \ell_0(|u|)(t) + q^*(t) \quad \text{for } t \in [a,b].\end{aligned} \tag{33}$$

Since $-\ell_1 \in \mathscr{S}_{ab}(a)$, according to Theorem 2.1, the problem

$$\alpha'(t) = -\ell_1(\alpha)(t) + \ell_1(|u|)(t) + \ell_0(|u|)(t) + q^*(t), \qquad \alpha(a) = c \tag{34}$$

has a unique solution α. Moreover, from (32)–(34), on account of conditions $-\ell_1 \in \mathscr{S}_{ab}(a)$ and $|u(a)| \leqslant c$, we get

$$[u(t)]_+ \leqslant \alpha(t), \qquad [u(t)]_- \leqslant \alpha(t) \quad \text{for } t \in [a,b]$$

and consequently,

$$|u(t)| \leqslant \alpha(t) \quad \text{for } t \in [a,b]. \tag{35}$$

By (35) and the condition $\ell_0, \ell_1 \in \mathscr{P}_{ab}$, (34) results in

$$\alpha'(t) \leqslant \ell_0(\alpha)(t) + q^*(t) \quad \text{for } t \in [a,b].$$

Since $\ell_0 \in \mathscr{S}_{ab}(a)$ and $\alpha(a) = c$, the latter inequality yields

$$\alpha(t) \leqslant v(t) \quad \text{for } t \in [a,b], \tag{36}$$

where v is a solution of problem (28). Now from (35) and (36), according to Lemma 3.4, we have estimate (23). □

Theorem 3.1 follows from Lemmas 3.2 and 3.5, Theorem 3.2 follows from Lemmas 3.3 and 3.5.

4. On Remarks 2.3, 3.1 and 3.2

On Remark 2.3. In Examples 4.1 and 4.2, there are constructed operators $\ell \in \tilde{\mathscr{L}}_{ab}$ such that homogeneous problem (3_0), (4_0) has a nontrivial solution. Then, according to Theorem 2.1, problem (3), (4) has either no solution or has infinitely many solutions.

Example 4.1. Let the operators $\ell, \ell_0 \in \tilde{\mathscr{L}}_{ab}$ be defined by

$$\ell(v)(t) \overset{\text{def}}{=} (1+\varepsilon)p(t)v(b), \qquad \ell_0(v)(t) \overset{\text{def}}{=} p(t)v(b), \tag{37}$$

where $\varepsilon > 0$, $p \in L([a,b]; R_+)$ and

$$\int_a^b p(s)\,\mathrm{d}s = \frac{1}{1+\varepsilon}. \tag{38}$$

According to Corollary 1.1(b) in [13] we have $\ell_0 \in \mathscr{S}_{ab}(a)$. Obviously, all the assumptions of Theorem 2.2 are fulfilled where $\ell_1 \equiv 0$, except of condition (7), instead of which condition (9) is satisfied.

On the other hand, problem (3_0), (4_0) has the nontrivial solution

$$u(t) = (1+\varepsilon)\int_a^t p(s)\,\mathrm{d}s \quad \text{for } t \in [a,b].$$

Example 4.1 shows that inequality (7) cannot be replaced by inequality (9), no matter how small $\varepsilon > 0$ would be. This example also shows that condition $\ell_0 \in \mathscr{S}_{ab}(a)$ cannot be replaced by condition $(1-\varepsilon)\ell_0 \in \mathscr{S}_{ab}(a)$, no matter how small $\varepsilon > 0$ would be.

Furthermore, this example shows that in condition (a) in Corollary 2.1, the assumption $\alpha \in]0,1[$ cannot be replaced by the assumption $\alpha \in]0,1]$, and in condition (b) in Corollary 2.1, the strict inequality cannot be replaced by the nonstrict one.

Example 4.2. Let $\varepsilon > 0$, $x_0 \in [0,1[$, and $\ell \in \tilde{\mathscr{L}}_{02}$ be an operator defined by

$$\ell(v)(t) \overset{\text{def}}{=} p(t)v(\tau(t)), \tag{39}$$

where

$$p(t) = \begin{cases} -\dfrac{1}{1+\varepsilon} & \text{for } t \in [0, 1-x_0[, \\ 1 & \text{for } t \in [1-x_0, 1[, \\ -(2+\varepsilon) & \text{for } t \in [1,2], \end{cases} \qquad \tau(t) = \begin{cases} 2 & \text{for } t \in [0, 1-x_0[, \\ 1 & \text{for } t \in [1-x_0, 2]. \end{cases}$$

Let, moreover,

$$\ell_0(v)(t) \overset{\text{def}}{=} p_0(t)v(\tau_0(t)), \qquad \ell_1(v)(t) \overset{\text{def}}{=} p_1(t)v(\tau_1(t)), \tag{40}$$

where

$$p_0(t) = \begin{cases} \dfrac{1}{1+\varepsilon} & \text{for } t \in [0, 1-x_0[, \\ 1 & \text{for } t \in [1-x_0, 1[, \\ 0 & \text{for } t \in [1,2], \end{cases} \qquad \tau_0(t) = \begin{cases} 2 & \text{for } t \in [0, 1-x_0[, \\ 1 & \text{for } t \in [1-x_0, 2], \end{cases}$$

$$p_1(t) = \begin{cases} 0 & \text{for } t \in [0,1[, \\ 2+\varepsilon & \text{for } t \in [1,2], \end{cases} \qquad \tau_1(t) = \begin{cases} 0 & \text{for } t \in [0,1[, \\ 1 & \text{for } t \in [1,2]. \end{cases}$$

It is clear that $\ell_0, \ell_1 \in \mathscr{P}_{02}$, ℓ_1 is a 0-Volterra operator, and condition (7) is fulfilled. Moreover,

$$\int_0^2 \ell_0(1)(s)\,\mathrm{d}s = \int_0^1 p_0(s)\,\mathrm{d}s = \frac{1+\varepsilon x_0}{1+\varepsilon} < 1$$

and

$$\frac{1}{2+\varepsilon}\int_0^2 \ell_1(1)(s)\,\mathrm{d}s = \frac{1}{2+\varepsilon}\int_1^2 p_1(s)\,\mathrm{d}s = 1.$$

Consequently, according to Corollary 1.1(b) and Theorem 1.3 in [13],

$$\ell_0 \in \mathscr{S}_{02}(0), \qquad -\frac{1}{2+\varepsilon}\ell_1 \in \mathscr{S}_{02}(0).$$

On the other hand, the function

$$u(t) = \begin{cases} t & \text{for } t \in [0,1[,\\ -(2+\varepsilon)(t-1)+1 & \text{for } t \in [1,2] \end{cases}$$

is a nontrivial solution of problem (3_0), (4_0).

Example 4.2 shows that the assumption $-\frac{1}{2}\ell_1 \in \mathscr{S}_{ab}(a)$ in Theorem 2.2 cannot be replaced by $-[1/(2+\varepsilon)]\ell_1 \in \mathscr{S}_{ab}(a)$, no matter how small $\varepsilon > 0$ would be.

This example also shows that condition (d) in Corollary 2.1 cannot be replaced by the condition

$$\int_a^b g(s)\,\mathrm{d}s \leqslant 2+\varepsilon$$

and condition (e) in Corollary 2.1 cannot be replaced by condition

$$\int_a^b g(s)\left(\int_{\mu(s)}^s g(\xi)\exp\left[\frac{1}{2}\int_{\mu(\xi)}^s g(\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi\right)\mathrm{d}s \leqslant 4+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

On Remark 3.1. Let $\ell \in \tilde{\mathscr{L}}_{ab}$ be defined by (37), where $\varepsilon > 0$, and $p \in L([a,b];R_+)$ satisfies (38). According to Example 4.1, problem (3_0), (4_0) has a nontrivial solution. By Remark 2.1 there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that problem (1), (2), where $F(v)(t) \overset{\text{def}}{=} \ell(v)(t) + q_0(t)$ for $t \in [a,b]$, $h(v) \equiv c_0$, has no solution, while conditions (15) and (16) are fulfilled, where $c = |c_0|$, $q \equiv |q_0|$, $\ell_0 \equiv \ell$, $\ell_1 \equiv 0$. On the other hand, according to Corollary 1.1(b) in [13] we have $(1-\varepsilon)\ell_0 \in \mathscr{S}_{ab}(a)$. Thus Example 4.1 shows that the condition $\ell_0 \in \mathscr{S}_{ab}(a)$ in Theorem 3.1 cannot be replaced by condition $(1-\varepsilon)\ell_0 \in \mathscr{S}_{ab}(a)$, no matter how small $\varepsilon > 0$ would be.

This example also shows that in condition (a) in Corollary 3.1, the assumption $\alpha \in]0,1[$ cannot be replaced by the assumption $\alpha \in]0,1]$, and in condition (b) in Corollary 3.1, the strict inequality cannot be replaced by the nonstrict one.

Example 4.3. *Let* $\varepsilon > 0$, $x_0 \in [0,1[$, $\ell \in \tilde{\mathscr{L}}_{03}$ *be an operator defined by* (39), *where*

$$p(t) = \begin{cases} -\dfrac{2}{2+\varepsilon} & \text{for } t \in [0, 1-x_0[, \\ 1 & \text{for } t \in [1-x_0, 1[, \\ 0 & \text{for } t \in [1,2[, \\ -(1+\varepsilon) & \text{for } t \in [2,3], \end{cases} \qquad \tau(t) = \begin{cases} 3 & \text{for } t \in [0, 1-x_0[, \\ 1 & \text{for } t \in [1-x_0, 3] \end{cases}$$

and ℓ_0, ℓ_1 be defined by (40), where

$$p_0(t) = \begin{cases} \dfrac{2}{2+\varepsilon} & \text{for } t \in [0, 1-x_0[, \\ 1 & \text{for } t \in [1-x_0, 1[, \\ 0 & \text{for } t \in [1,3], \end{cases} \qquad \tau_0(t) = \begin{cases} 3 & \text{for } t \in [0, 1-x_0[, \\ 1 & \text{for } t \in [1-x_0, 3]. \end{cases}$$

$$p_1(t) = \begin{cases} 0 & \text{for } t \in [0,2[, \\ 1+\varepsilon & \text{for } t \in [2,3], \end{cases} \qquad \tau_1(t) = \begin{cases} 0 & \text{for } t \in [0,2[, \\ 1 & \text{for } t \in [2,3]. \end{cases}$$

Put

$$z(t) = \begin{cases} 0 & \text{for } t \in [0,1[\cup [2,3], \\ \dfrac{-(1-\varepsilon/2)}{1-(1-\varepsilon/2)(t-1)} & \text{for } t \in [1,2[. \end{cases}$$

It is clear that $-z \in L([0,3]; R_+)$, $\ell_0, \ell_1 \in \mathscr{P}_{03}$, ℓ_1 is a 0-Volterra operator, and condition (7) is fulfilled. Moreover,

$$\int_0^3 \ell_0(1)(s)\,\mathrm{d}s = \int_0^1 p_0(s)\,\mathrm{d}s = \frac{2+\varepsilon x_0}{2+\varepsilon} < 1$$

and

$$(1-\varepsilon)\int_0^3 \ell_1(1)(s)\,\mathrm{d}s = (1-\varepsilon)\int_2^3 p_1(s)\,\mathrm{d}s = 1-\varepsilon^2 < 1.$$

Consequently, according to Corollary 1.1(b) and Theorem 1.3 in [13],

$$\ell_0 \in \mathscr{S}_{03}(0), \qquad -(1-\varepsilon)\ell_1 \in \mathscr{S}_{03}(0).$$

On the other hand, the function

$$u(t) = \begin{cases} t & \text{for } t \in [0,1[, \\ -(1-\frac{\varepsilon}{2})(t-1)+1 & \text{for } t \in [1,2[, \\ -(1+\varepsilon)(t-2)+\frac{\varepsilon}{2} & \text{for } t \in [2,3] \end{cases}$$

is a nontrivial solution of the problem

$$u'(t) = p(t)u(\tau(t)) + z(t)u(t), \qquad u(a) = 0.$$

Therefore, according to Remark 2.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that problem (1), (2) with $F(v)(t) \stackrel{\text{def}}{=} p(t)v(\tau(t)) + z(t)v(t) + q_0(t)$ for $t \in [a,b]$, $h(v) \equiv c_0$ has no solution, while conditions (15) and (16) are fulfilled, where $c = |c_0|$, $q \equiv |q_0|$.

Example 4.3 shows that the assumption $-\ell_1 \in \mathscr{S}_{ab}(a)$ in Theorem 3.1 cannot be replaced by $-(1-\varepsilon)\ell_1 \in \mathscr{S}_{ab}(a)$, no matter how small $\varepsilon > 0$ would be.

This example also shows that condition (d) in Corollary 3.1 cannot be replaced by the condition

$$\int_a^b g(s)\,\mathrm{d}s \leqslant 1 + \varepsilon$$

and condition (e) in Corollary 3.1 cannot be replaced by the condition

$$\int_a^b g(s) \left(\int_{\mu(s)}^s g(\xi) \exp\left[\int_{\mu(\xi)}^s g(\eta)\,\mathrm{d}\eta \right] \mathrm{d}\xi \right) \mathrm{d}s \leqslant 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

On Remark 3.2. Examples 4.1 and 4.3 also show that the assumptions imposed on the operators ℓ_0 and ℓ_1 in Theorem 3.2, resp. on the functions p, g, τ, and μ in Corollary 3.2, cannot be weakened.

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