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ON A BOUNDARY-VALUE PROBLEM OF ANTIPERIODIC TYPE FOR FIRST-ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NON-VOLTERRA TYPE

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We establish unimprovable (in a certain sense) sufficient conditions for the solvability and unique solvability of the boundary-value problem

$$u'(t) = F(u)(t), \quad u(a) + \lambda u(b) = h(u),$$

where $F\colon C([a,b];R) \to L([a,b];R)$ is a continuous operator satisfying the Caratheodory conditions, $h\colon C([a,b];R) \to R$ is a continuous functional, and $\lambda \in R_+$.

Introduction

The following notation is used throughout the paper:

R is the set of all real numbers and $R_+ = [0, +\infty[$.

$C([a,b];R)$ is the Banach space of continuous functions $u\colon [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \le t \le b\}$.

$C([a,b];R_+) = \{u \in C([a,b];R)\colon u(t) \ge 0 \text{ for } t \in [a,b]\}$.

$\widetilde{C}([a,b];R)$ is the set of absolutely continuous functions $u\colon [a,b] \to R$.

$B^i_{\lambda c}([a,b];R) = \{u \in C([a,b];R)\colon (u(a) + \lambda u(b)) \operatorname{sgn}((2-i)u(a) + (i-1)u(b)) \le c\}$, where $c \in R$, $i = 1, 2$.

$L([a,b];R)$ is the Banach space of Lebesgue-integrable functions $p\colon [a,b] \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

$L([a,b];R_+) = \{p \in L([a,b];R)\colon p(t) \ge 0 \text{ for almost all } t \in [a,b]\}$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau\colon [a,b] \to [a,b]$.

$\widetilde{\mathcal{L}}_{ab}$ is the set of linear operators $\ell\colon C([a,b];R) \to L([a,b];R)$ for which there is a function $\eta \in L([a,b];R_+)$ such that

$$|\ell(v)(t)| \le \eta(t)\|v\|_C \quad \text{for } t \in [a,b], \quad v \in C([a,b];R).$$

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \widetilde{\mathcal{L}}_{ab}$ transforming the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$.

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K_{ab} is the set of continuous operators $F\colon C([a,b];R) \to L([a,b];R)$ satisfying the Caratheodory conditions, i.e., for every $r > 0,$ there exists $q_r \in L([a,b];R_+)$ such that

$$|F(v)(t)| \le q_r(t) \quad \text{for } t \in [a,b], \ \ \|v\|_C \le r.$$

$K([a,b] \times A; B),$ where $A \subseteq R^2$ and $B \subseteq R,$ is the set of functions $f\colon [a,b] \times A \to B$ satisfying the Caratheodory conditions, i.e., $f(\cdot, x)\colon [a,b] \to B$ is a measurable function for all $x \in A,$ $f(t,\cdot)\colon A \to B$ is a continuous function for almost all $t \in [a,b],$ and, for every $r > 0,$ there exists $q_r \in L([a,b];R_+)$ such that

$$|f(t,x)| \le q_r(t) \quad \text{for } t \in [a,b], \ x \in A, \ \|x\| \le r.$$

$$[x]_+ = \frac{1}{2}(|x| + x), \quad [x]_- = \frac{1}{2}(|x| - x).$$

A solution of the equation

$$u'(t) = F(u)(t), \tag{0.1}$$

where $F \in K_{ab},$ is understood as a function $u \in \widetilde{C}([a,b];R)$ satisfying Eq. (0.1) almost everywhere in $[a,b].$

Consider the problem of the existence and uniqueness of a solution of Eq. (0.1) satisfying the boundary condition

$$u(a) + \lambda u(b) = h(u), \tag{0.2}$$

where $\lambda \in R_+$ and $h\colon C([a,b];R) \to R$ is a continuous functional.

General boundary-value problems for functional differential equations were studied very extensively. Numerous interesting general results were obtained (see, e.g., [1–27] and references therein), but only a few efficient criteria for the solvability of special boundary-value problems for functional differential equations are known even in the linear case. In the present paper, we try to fill the existing gap to a certain extent. More precisely, in Sec. 1, we give unimprovable efficient sufficient conditions for the solvability and unique solvability of problem (0.1), (0.2). Sections 2, 3, and 4 are devoted to auxiliary propositions, the proofs of the main results, and examples verifying their optimality, respectively.

All results are concretized for a differential equation with deviating arguments of the form

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + f(t, u(t), u(\nu(t))), \tag{0.3}$$

where $p, g \in L([a,b];R_+),$ $\tau, \mu, \nu \in \mathcal{M}_{ab},$ and $f \in K([a,b] \times R^2; R).$

A special case of the boundary-value problem considered is a Cauchy problem (for $\lambda = 0$ and $h \equiv$ Const). In this case, the theorems presented below coincide with the results obtained in [5]. Boundary-value problems of periodic type (i.e., for $\lambda < 0$) for linear and nonlinear equations were studied in [14] and [15], respectively.

The following result is known from the general theory of linear boundary-value problems for functional differential equations (see, e.g., [3, 19, 27]):

Theorem 0.1. *Let* $\ell \in \widetilde{\mathcal{L}}_{ab}$*. Then the problem*

$$u'(t) = \ell(u)(t) + q_0(t), \quad u(a) + \lambda u(b) = c_0, \tag{0.4}$$

where $q_0 \in L([a,b];R)$ *and* $c_0 \in R,$ *is uniquely solvable if and only if the corresponding homogeneous problem*

$$u'(t) = \ell(u)(t), \tag{0.1$_0$}$$

$$u(a) + \lambda u(b) = 0 \tag{0.2$_0$}$$

has only the trivial solution.

Remark 0.1. It follows from the Riesz–Schauder theory that if $\ell \in \widetilde{\mathcal{L}}_{ab}$ and problem (0.1$_0$), (0.2$_0$) has a nontrivial solution, then there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that problem (0.4) does not have solutions.

1. Main Results

Throughout the paper, we assume that $q \in K([a,b] \times R_+; R_+)$ is nondecreasing in the second argument and such that

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)ds = 0. \tag{1.1}$$

Theorem 1.1. *Let* $\lambda \in \,]0,1],$ $c \in R_+,$

$$h(v) \operatorname{sgn} v(a) \le c \quad \textit{for} \;\; v \in C([a,b];R), \tag{1.2}$$

and let there exist

$$\ell_0, \ell_1 \in \mathcal{P}_{ab} \tag{1.3}$$

such that the following inequality holds on the set $B^1_{\lambda c}([a,b];R):$

$$[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)] \operatorname{sgn} v(t) \le q(t, \|v\|_C) \quad \textit{for} \;\; t \in [a,b]. \tag{1.4}$$

If, moreover,

$$\|\ell_0(1)\|_L < 1, \qquad \|\ell_1(1)\|_L < \alpha(\lambda), \tag{1.5}$$

where

$$\alpha(\lambda) = \begin{cases} -\lambda + 2\sqrt{1 - \|\ell_0(1)\|_L} & \textit{for} \quad \|\ell_0(1)\|_L < 1 - \lambda^2, \\ \dfrac{1}{\lambda}\,(1 - \|\ell_0(1)\|_L) & \textit{for} \quad \|\ell_0(1)\|_L \ge 1 - \lambda^2, \end{cases} \tag{1.6}$$

then problem (0.1), (0.2) has at least one solution.

Remark 1.2. Theorem 1.1 is unimprovable in a certain sense. More precisely, the second inequality in (1.5) cannot be replaced by

$$\|\ell_1(1)\|_L < (1+\varepsilon)\alpha(\lambda),$$

no matter how small $\varepsilon > 0$ may be (see Examples 4.1–4.3).

Theorem 1.2. *Let* $\lambda \in \,]0,1]$, $c \in R_+$,

$$h(v)\,\mathrm{sgn}\,v(b) \le c \quad \textit{for} \;\; v \in C([a,b];R), \tag{1.7}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that the following inequality holds on the set* $B^2_{\lambda c}([a,b];R)$:

$$[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)]\,\mathrm{sgn}\,v(t) \ge -q(t, \|v\|_C) \quad \textit{for} \;\; t \in [a,b]. \tag{1.8}$$

If, moreover,

$$\|\ell_0(1)\|_L + \lambda\|\ell_1(1)\|_L < \lambda, \tag{1.9}$$

then problem (0.1), (0.2) has at least one solution.

Remark 1.3. Theorem 1.2 is unimprovable in a certain sense. More precisely, inequality (1.9) cannot be replaced by

$$\|\ell_0(1)\|_L + \lambda\|\ell_1(1)\|_L < \lambda + \varepsilon,$$

no matter how small $\varepsilon > 0$ may be (see Examples 4.4 and 4.5).

Remark 1.4. Let $\lambda \in [1,+\infty[$. We define an operator $\psi\colon L([a,b];R) \to L([a,b];R)$ by the formula

$$\psi(w)(t) \overset{\text{df}}{=} w(a+b-t) \quad \text{for} \;\; t \in [a,b].$$

Let φ be a restriction of ψ to the space $C([a,b];R)$. We set $\vartheta = \dfrac{1}{\lambda}$ and

$$\widehat{F}(w)(t) \overset{\text{df}}{=} -\psi(F(\varphi(w)))(t), \qquad \widehat{h}(w) \overset{\text{df}}{=} \vartheta h(\varphi(w)).$$

It is clear that if u is a solution of problem (0.1), (0.2), then the function $v \overset{\text{df}}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{F}(v)(t), \qquad v(a) + \vartheta v(b) = \widehat{h}(v), \tag{1.10}$$

and, vice versa, if v is a solution of problem (1.10), then the function $u \overset{\text{df}}{=} \varphi(v)$ is a solution of problem (0.1), (0.2).

Therefore, the following theorems follow immediately from Theorems 1.1 and 1.2:

Theorem 1.3. *Suppose that* $\lambda \in [1, +\infty[$, $c \in R_+$, *condition (1.7) is satisfied, and there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that inequality (1.8) holds on the set* $B^2_{\lambda c}([a,b];R)$. *If, moreover,*

$$\|\ell_1(1)\|_L < 1, \qquad \|\ell_0(1)\|_L < \beta(\lambda), \tag{1.11}$$

where

$$\beta(\lambda) = \begin{cases} -\dfrac{1}{\lambda} + 2\sqrt{1 - \|\ell_1(1)\|_L} & \text{for} \quad \|\ell_1(1)\|_L < 1 - \dfrac{1}{\lambda^2}, \\ \lambda\,(1 - \|\ell_1(1)\|_L) & \text{for} \quad \|\ell_1(1)\|_L \geq 1 - \dfrac{1}{\lambda^2}, \end{cases} \tag{1.12}$$

then problem (0.1), (0.2) has at least one solution.

Theorem 1.4. *Suppose that* $\lambda \in [1, +\infty[$, $c \in R_+$, *condition (1.2) is satisfied, and there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that inequality (1.4) holds on the set* $B^1_{\lambda c}([a,b];R)$. *If, moreover,*

$$\lambda\|\ell_1(1)\|_L + \|\ell_0(1)\|_L < 1, \tag{1.13}$$

then problem (0.1), (0.2) has at least one solution.

Remark 1.5. In view of Remarks 1.1–1.3, it is clear that Theorems 1.3 and 1.4 are also unimprovable.

Next, we establish theorems on the unique solvability of problem (0.1), (0.2).

Theorem 1.5. *Let* $\lambda \in\ ]0,1]$,

$$[h(v) - h(w)]\,\mathrm{sgn}(v(a) - w(a)) \leq 0 \quad \text{for} \ \ v, w \in C([a,b];R), \tag{1.14}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that the following inequality holds on the set* $B^1_{\lambda c}([a,b];R)$, $c = |h(0)|$:

$$[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)]\,\mathrm{sgn}(v(t) - w(t)) \leq 0. \tag{1.15}$$

Also assume that relation (1.5), where $\alpha(\lambda)$ *is defined by (1.6), is satisfied. Then problem (0.1), (0.2) is uniquely solvable.*

Theorem 1.6. *Let* $\lambda \in\ ]0,1]$,

$$[h(v) - h(w)]\,\mathrm{sgn}(v(b) - w(b)) \leq 0 \quad \text{for} \ \ v, w \in C([a,b];R), \tag{1.16}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that the following inequality holds on the set* $B^2_{\lambda c}([a,b];R)$, $c = |h(0)|$:

$$[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)]\,\mathrm{sgn}(v(t) - w(t)) \ \geq 0. \tag{1.17}$$

Also assume that relation (1.9) is satisfied. Then problem (0.1), (0.2) is uniquely solvable.

According to Remark 1.3, Theorems 1.5 and 1.6 yield the following results:

Theorem 1.7. *Suppose that* $\lambda \in [1,+\infty[$, *condition (1.16) is satisfied, and there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that inequality (1.17) holds on the set* $B^2_{\lambda c}([a,b];R)$, *where* $c = |h(0)|$. *Also assume that relation (1.11), where* $\beta(\lambda)$ *is defined by (1.12), is satisfied. Then problem (0.1), (0.2) is uniquely solvable.*

Theorem 1.8. *Suppose that* $\lambda \in [1,+\infty[$, *condition (1.14) is satisfied, and there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that inequality (1.15) holds on the set* $B^1_{\lambda c}([a,b];R)$, *where* $c = |h(0)|$. *Also assume that relation (1.13) is satisfied. Then problem (0.1), (0.2) is uniquely solvable.*

Remark 1.6. Theorems 1.5–1.8 are unimprovable in a certain sense (see Examples 4.1–4.5).

For an equation of the type (0.3), Theorems 1.1–1.8 yield the following assertions:

Corollary 1.1. *Suppose that* $\lambda \in \,]0,1]$, $c \in R_+$, *condition (1.2) is satisfied, and*

$$f(t,x,y)\operatorname{sgn} x \le q(t) \quad \textit{for} \;\; t \in [a,b], \;\; x,y \in R, \tag{1.18}$$

where $q \in L([a,b];R_+)$. *If, moreover,*

$$\int_a^b p(s)ds < 1, \qquad \int_a^b g(s)ds < \gamma(\lambda), \tag{1.19}$$

where

$$\gamma(\lambda) = \begin{cases} -\lambda + 2\sqrt{1 - \displaystyle\int_a^b p(s)ds} & \textit{for} \quad \displaystyle\int_a^b p(s)ds < 1 - \lambda^2, \\[2ex] \dfrac{1}{\lambda}\left(1 - \displaystyle\int_a^b p(s)ds\right) & \textit{for} \quad \displaystyle\int_a^b p(s)ds \ge 1 - \lambda^2, \end{cases} \tag{1.20}$$

then problem (0.3), (0.2) has at least one solution.

Corollary 1.2. *Suppose that* $\lambda \in \,]0,1]$, $c \in R_+$, *condition (1.7) is satisfied, and*

$$f(t,x,y)\operatorname{sgn} x \ge -q(t) \quad \textit{for} \;\; t \in [a,b], \;\; x,y \in R, \tag{1.21}$$

where $q \in L([a,b];R_+)$. *If, moreover,*

$$\lambda \int_a^b g(s)ds + \int_a^b p(s)ds < \lambda, \tag{1.22}$$

then problem (0.3), (0.2) has at least one solution.

Corollary 1.3. *Suppose that* $\lambda \in [1, +\infty[,\ c \in R_+,$ *conditions (1.7) and (1.21) are satisfied, and*

$$\int_a^b g(s)ds < 1, \qquad \int_a^b p(s)ds < \delta(\lambda), \tag{1.23}$$

where

$$\delta(\lambda) = \begin{cases} -\dfrac{1}{\lambda} + 2\sqrt{1 - \displaystyle\int_a^b g(s)ds} & \text{for} \quad \displaystyle\int_a^b g(s)ds < 1 - \dfrac{1}{\lambda^2}, \\ \lambda\left(1 - \displaystyle\int_a^b g(s)ds\right) & \text{for} \quad \displaystyle\int_a^b g(s)ds \ge 1 - \dfrac{1}{\lambda^2}. \end{cases} \tag{1.24}$$

Then problem (0.3), (0.2) has at least one solution.

Corollary 1.4. *Suppose that* $\lambda \in [1, +\infty[,\ c \in R_+,$ *conditions (1.2) and (1.18) are satisfied, and*

$$\lambda \int_a^b g(s)ds + \int_a^b p(s)ds < 1. \tag{1.25}$$

Then problem (0.3), (0.2) has at least one solution.

Corollary 1.5. *Suppose that* $\lambda \in\ ]0, 1],$ *condition (1.14) is satisfied, and*

$$[f(t, x_1, y_1) - f(t, x_2, y_2)]\,\mathrm{sgn}(x_1 - x_2) \le 0 \quad \text{for} \ \ t \in [a, b],\ \ x_1, x_2, y_1, y_2 \in R. \tag{1.26}$$

Also assume that relation (1.19), where $\gamma(\lambda)$ *is defined by (1.20), is satisfied. Then problem (0.3), (0.2) is uniquely solvable.*

Corollary 1.6. *Suppose that* $\lambda \in\ ]0, 1],$ *relations (1.16) and (1.22) are satisfied, and*

$$[f(t, x_1, y_1) - f(t, x_2, y_2)]\,\mathrm{sgn}(x_1 - x_2) \ge 0 \quad \text{for} \ \ t \in [a, b],\ \ x_1, x_2, y_1, y_2 \in R. \tag{1.27}$$

Then problem (0.3), (0.2) is uniquely solvable.

Corollary 1.7. *Suppose that* $\lambda \in [1, +\infty[$ *and conditions (1.16) and (1.27) are satisfied. Also assume that relation (1.23), where* $\delta(\lambda)$ *is defined by (1.24), is true. Then problem (0.3), (0.2) is uniquely solvable.*

Corollary 1.8. *Suppose that* $\lambda \in [1, +\infty[$ *and conditions (1.14), (1.25), and (1.26) are satisfied. Then problem (0.3), (0.2) is uniquely solvable.*

2. Auxiliary Propositions

First we formulate Theorem 1 from [22] in the form suitable for what follows.

Lemma 2.1. *Suppose that there exist a positive number ρ and an operator $\ell \in \widetilde{\mathcal{L}}_{ab}$ such that the homogeneous problem (0.1$_0$), (0.2$_0$) has only the trivial solution, and, for every $\delta \in \,]0,1[$ and an arbitrary function $u \in \widetilde{C}([a,b];R)$ such that*

$$u'(t) = \ell(u)(t) + \delta[F(u)(t) - \ell(u)(t)], \qquad u(a) + \lambda u(b) = \delta h(u), \tag{2.1}$$

the following estimate is true:

$$\|u\|_C \le \rho. \tag{2.2}$$

Then problem (0.1), (0.2) has at least one solution.

Definition 2.1. *We say that an operator $\ell \in \widetilde{\mathcal{L}}_{ab}$ belongs to the set $U_i(\lambda)$, $i \in \{1,2\}$, if there exists a positive number r such that, for any $q^* \in L([a,b];R_+)$ and $c \in R_+$, every function $u \in \widetilde{C}([a,b];R)$ satisfying the inequalities*

$$[u(a) + \lambda u(b)] \operatorname{sgn}\left((2-i)u(a) + (i-1)u(b)\right) \le c, \tag{2.3}$$

$$(-1)^{i+1}[u'(t) - \ell(u)(t)] \operatorname{sgn} u(t) \le q^*(t) \quad \textit{for } t \in [a,b], \tag{2.4}$$

admits the estimate

$$\|u\|_C \le r\left(c + \|q^*\|_L\right). \tag{2.5}$$

Lemma 2.2. *Let $i \in \{1,2\}$, $c \in R_+$,*

$$h(v) \operatorname{sgn}((2-i)v(a) + (i-1)v(b)) \le c \quad \textit{for} \quad v \in C([a,b];R), \tag{2.6}$$

and let there exist $\ell \in U_i(\lambda)$ such that, on the set $B^i_{\lambda c}([a,b];R)$, the following inequality is satisfied:

$$(-1)^{i+1}[F(v)(t) - \ell(v)(t)] \operatorname{sgn} v(t) \le q(t, \|v\|_C) \quad \textit{for } t \in [a,b]. \tag{2.7}$$

Then problem (0.1), (0.2) has at least one solution.

Proof. First note that, due to the condition $\ell \in U_i(\lambda)$, the homogeneous problem (0.1$_0$), (0.2$_0$) has only the trivial solution.

Let r be the number appearing in Definition 2.1. According to (1.1), there exists $\rho > 2rc$ such that

$$\frac{1}{x}\int_a^b q(s,x)ds < \frac{1}{2r} \quad \text{for } x > \rho.$$

Now assume that a function $u \in \widetilde{C}([a,b];R)$ satisfies (2.1) for some $\delta \in \,]0,1[$. Then, according to (2.6), u satisfies inequality (2.3), i.e., $u \in B^i_{\lambda c}([a,b];R)$. Taking (2.1) and (2.7) into account, we conclude that inequality (2.4) holds for $q^*(t) = q(t, \|u\|_C)$. Hence, by using the condition $\ell \in U_i(\lambda)$ and the definition of the number ρ, we get estimate (2.2).

Since ρ depends neither on u nor on δ, it follows from Lemma 2.1 that problem (0.1), (0.2) has at least one solution.

The lemma is proved.

Lemma 2.3. *Let* $i \in \{1,2\}$, *let*

$$[h(u_1) - h(u_2)] \operatorname{sgn}((2-i)(u_1(a) - u_2(a)) + (i-1)(u_1(b) - u_2(a))) \le 0 \tag{2.8}$$

$$\text{for} \quad u_1, u_2 \in C([a,b];R),$$

and let there exist $\ell \in U_i(\lambda)$ *such that, on the set* $B^i_{\lambda c}([a,b];R)$, *where* $c = |h(0)|$, *the following inequality is satisfied:*

$$(-1)^{i+1}[F(u_1)(t) - F(u_2)(t) - \ell(u_1 - u_2)(t)] \operatorname{sgn}(u_1(t) - u_2(t)) \le 0. \tag{2.9}$$

Then problem (0.1), (0.2) is uniquely solvable.

Proof. It follows from (2.8) that condition (2.6), where $c = |h(0)|$, is satisfied. By virtue of (2.9), inequality (2.7), where $q \equiv |F(0)|$, holds on the set $B^i_{\lambda c}([a,b];R)$. Consequently, all assumptions of Lemma 2.2 are satisfied, which guarantees that problem (0.1), (0.2) has at least one solution. It remains to show that problem (0.1), (0.2) has at most one solution.

Let u_1 and u_2 be arbitrary solutions of problem (0.1), (0.2). We set $u(t) = u_1(t) - u_2(t)$ for $t \in [a,b]$. Then, by virtue of (2.8) and (2.9), we get

$$[u(a) + \lambda u(b)] \operatorname{sgn}((2-i)u(a) + (i-1)u(b)) \le 0,$$

$$(-1)^{i+1}[u'(t) - \ell(u)(t)] \operatorname{sgn} u(t) \le 0 \quad \text{for } t \in [a,b].$$

This, together with the condition $\ell \in U_i(\lambda)$, yields $u \equiv 0$. Consequently, $u_1 \equiv u_2$.

The lemma is proved.

Lemma 2.4. *Suppose that* $\lambda \in \,]0,1]$ *and the operator* ℓ *admits the representation* $\ell = \ell_0 - \ell_1$, *where* ℓ_0 *and* ℓ_1 *satisfy conditions (1.3) and (1.5), in which* α *is defined by (1.6). Then* ℓ *belongs to the set* $U_1(\lambda)$.

Proof. Let $q^* \in L([a,b];R_+)$, $c \in R_+$, and $u \in \widetilde{C}([a,b];R)$ satisfy (2.3) and (2.4) for $i = 1$. We prove relation (2.5), where

$$r = \begin{cases} \dfrac{\|\ell_1(1)\|_L + 1 + \lambda}{1 - \|\ell_0(1)\|_L - \frac{1}{4}(\|\ell_1(1)\|_L + \lambda)^2} & \text{if } \|\ell_0(1)\|_L < 1 - \lambda^2, \\[2ex] \dfrac{\|\ell_1(1)\|_L + 1 + \lambda}{1 - \|\ell_0(1)\|_L - \lambda\|\ell_1(1)\|_L} & \text{if } \|\ell_0(1)\|_L \geq 1 - \lambda^2. \end{cases} \tag{2.10}$$

It is clear that

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \widetilde{q}(t), \tag{2.11}$$

where

$$\widetilde{q}(t) = u'(t) - \ell(u)(t) \quad \text{for } t \in [a,b]. \tag{2.12}$$

Obviously,

$$\widetilde{q}(t) \operatorname{sgn} u(t) \leq q^*(t) \quad \text{for } t \in [a,b] \tag{2.13}$$

and

$$[u(a) + \lambda u(b)] \operatorname{sgn} u(a) \leq c. \tag{2.14}$$

First assume that u does not change its sign. According to (2.14) and the assumption $\lambda \in \,]0,1]$, we obtain

$$|u(a)| \leq c. \tag{2.15}$$

We choose $t_0 \in [a,b]$ so that

$$|u(t_0)| = \|u\|_C. \tag{2.16}$$

By virtue of (1.3) and (2.13), relation (2.11) yields

$$|u(t)|' \leq \|u\|_C \, \ell_0(1)(t) + q^*(t) \quad \text{for } t \in [a,b]. \tag{2.17}$$

The integration of (2.17) from a to t_0 with regard for (1.3), (2.15), and (2.16) results in

$$\|u\|_C - c \leq \|u\|_C - |u(a)| \leq \|u\|_C \int_a^{t_0} \ell_0(1)(s)ds + \int_a^{t_0} q^*(s)ds \leq \|u\|_C \|\ell_0(1)\|_L + \|q^*\|_L.$$

Thus,

$$\|u\|_C \,(1-\|\ell_0(1)\|_L) \le c + \|q^*\|_L,$$

and, consequently, estimate (2.5) holds.

Now assume that u changes its sign. We set

$$M = \max\{u(t)\colon t\in[a,b]\}, \quad m = -\min\{u(t)\colon t\in[a,b]\} \tag{2.18}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \quad u(t_m) = -m. \tag{2.19}$$

It is obvious that $M>0$, $m>0$, and either

$$t_m < t_M \tag{2.20}$$

or

$$t_m > t_M. \tag{2.21}$$

First, assume that relation (2.20) is satisfied. It is clear that there exists $\alpha_2 \in\ ]t_m, t_M[$ such that

$$u(t)>0 \quad \text{for } \alpha_2 < t \le t_M, \quad u(\alpha_2)=0. \tag{2.22}$$

Let

$$\alpha_1 = \inf\{t\in[a,t_m]\colon u(s)<0 \ \text{ for } \ t\le s\le t_m\}.$$

It is obvious that

$$u(t)<0 \quad \text{for } \alpha_1 < t \le t_m \qquad \text{and} \qquad u(\alpha_1)=0 \quad \text{if } \alpha_1 > a. \tag{2.23}$$

It follows from relations (2.14) and (2.23) and the assumption $\lambda \in\ ]0,1]$ that

$$u(\alpha_1) \ge -\lambda[u(b)]_+ - c \ge -\lambda M - c. \tag{2.24}$$

The integration of (2.11) from α_1 to t_m and from α_2 to t_M with regard for (1.3), (2.13), (2.18), (2.19), (2.22), (2.23), and (2.24) yields

$$m - \lambda M - c \le m + u(\alpha_1) \le M\int_{\alpha_1}^{t_m} \ell_1(1)(s)ds + m\int_{\alpha_1}^{t_m} \ell_0(1)(s)ds + \int_{\alpha_1}^{t_m} q^*(s)ds,$$

$$M \le M\int_{\alpha_2}^{t_M} \ell_0(1)(s)ds + m\int_{\alpha_2}^{t_M} \ell_1(1)(s)ds + \int_{\alpha_2}^{t_M} q^*(s)ds.$$

Using the last two inequalities, we obtain

$$m(1-C_1) \le M(A_1+\lambda) + \|q^*\|_L + c, \qquad M(1-D_1) \le mB_1 + \|q^*\|_L, \tag{2.25}$$

where

$$A_1 = \int\limits_{\alpha_1}^{t_m} \ell_1(1)(s)ds, \qquad B_1 = \int\limits_{\alpha_2}^{t_M} \ell_1(1)(s)ds,$$

$$C_1 = \int\limits_{\alpha_1}^{t_m} \ell_0(1)(s)ds, \qquad D_1 = \int\limits_{\alpha_2}^{t_M} \ell_0(1)(s)ds.$$

Due to the first inequality in (1.5), we have $C_1 < 1$ and $D_1 < 1$. Consequently, relation (2.25) yields

$$\begin{aligned} 0 < m(1-C_1)(1-D_1) &\le (A_1+\lambda)(mB_1 + \|q^*\|_L) + \|q^*\|_L + c \\ &\le m(A_1+\lambda)B_1 + (\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1 + \lambda), \\ 0 < M(1-C_1)(1-D_1) &\le B_1(M(A_1+\lambda) + \|q^*\|_L + c) + \|q^*\|_L \\ &\le M(A_1+\lambda)B_1 + (\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1 + \lambda). \end{aligned} \tag{2.26}$$

Obviously,

$$(1-C_1)(1-D_1) \ge 1 - (C_1+D_1) \ge 1 - \|\ell_0(1)\|_L > 0. \tag{2.27}$$

If $\|\ell_0(1)\|_L \ge 1-\lambda^2$, then, according to (1.6) and the second inequality in (1.5), we obtain $\|\ell_1(1)\|_L < \lambda$. Hence, $B_1 < \lambda$ and

$$(A_1+\lambda)B_1 = A_1B_1 + \lambda B_1 \le \lambda(A_1+B_1) \le \lambda\|\ell_1(1)\|_L.$$

By virtue of the last inequality and (2.27), relation (2.26) yields

$$\begin{aligned} m &\le r_0(\|\ell_1(1)\|_L + 1 + \lambda)(c + \|q^*\|_L), \\ M &\le r_0(\|\ell_1(1)\|_L + 1 + \lambda)(c + \|q^*\|_L), \end{aligned} \tag{2.28}$$

where

$$r_0 = (1 - \|\ell_0(1)\|_L - \lambda\|\ell_1(1)\|_L)^{-1}. \tag{2.29}$$

Therefore, estimate (2.5) is true.

If $\|\ell_0(1)\|_L < 1-\lambda^2$, then, by virtue of the inequalities

$$4(A_1+\lambda)B_1 \le (A_1+B_1+\lambda)^2 \le (\|\ell_1(1)\|_L+\lambda)^2$$

and (2.27), relation (2.26) yields

$$m \le r_1(\|\ell_1(1)\|_L+1+\lambda)(c+\|q^*\|_L),$$
$$M \le r_1(\|\ell_1(1)\|_L+1+\lambda)(c+\|q^*\|_L), \tag{2.30}$$

where

$$r_1 = \left[1-\|\ell_0(1)\|_L-\frac{1}{4}(\|\ell_1(1)\|_L+\lambda)^2\right]^{-1}. \tag{2.31}$$

Therefore, estimate (2.5) is valid.

Now assume that (2.21) is satisfied. Obviously, there exists $\alpha_4 \in]t_m, t_M[$ such that

$$u(t)<0 \quad \text{for } \alpha_4<t\le t_m, \qquad u(\alpha_4)=0. \tag{2.32}$$

Let

$$\alpha_3 = \inf\{t\in[a,t_M]\colon u(s)>0 \ \text{for } t\le s\le t_M\}.$$

Obviously,

$$u(t)>0 \quad \text{for } \alpha_3<t\le t_M \qquad \text{and} \qquad u(\alpha_3)=0 \quad \text{if } \alpha_3>a. \tag{2.33}$$

It follows from relations (2.14) and (2.33) and the assumption $\lambda\in]0,1]$ that

$$u(\alpha_3) \le \lambda[u(b)]_- + c \le \lambda m + c. \tag{2.34}$$

The integration of (2.11) from α_3 to t_M and from α_4 to t_m with regard for (1.3), (2.13), (2.18), (2.19), (2.32), (2.33), and (2.34) results in

$$M-\lambda m-c \le M-u(\alpha_3) \le M\int_{\alpha_3}^{t_M}\ell_0(1)(s)ds + m\int_{\alpha_3}^{t_M}\ell_1(1)(s)ds + \int_{\alpha_3}^{t_M}q^*(s)ds,$$

$$m \le M\int_{\alpha_4}^{t_m}\ell_1(1)(s)ds + m\int_{\alpha_4}^{t_m}\ell_0(1)(s)ds + \int_{\alpha_4}^{t_m}q^*(s)ds.$$

Using the last two inequalities, we obtain

$$M(1-C_2) \le m(A_2+\lambda) + \|q^*\|_L + c, \qquad m(1-D_2) \le MB_2 + \|q^*\|_L, \tag{2.35}$$

where

$$A_2 = \int_{\alpha_3}^{t_M} \ell_1(1)(s)ds, \qquad B_2 = \int_{\alpha_4}^{t_m} \ell_1(1)(s)ds,$$

$$C_2 = \int_{\alpha_3}^{t_M} \ell_0(1)(s)ds, \qquad D_2 = \int_{\alpha_4}^{t_m} \ell_0(1)(s)ds.$$

Due to the first inequality in (1.5), we have $C_2 < 1$ and $D_2 < 1$. Consequently, relation (2.35) yields

$$\begin{aligned} 0 < M(1-C_2)(1-D_2) &\le (A_2+\lambda)(MB_2 + \|q^*\|_L) + \|q^*\|_L + c \\ &\le M(A_2+\lambda)B_2 + (\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1 + \lambda), \\ 0 < m(1-C_2)(1-D_2) &\le B_2(m(A_2+\lambda) + \|q^*\|_L + c) + \|q^*\|_L \\ &\le m(A_2+\lambda)B_2 + (\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1 + \lambda). \end{aligned} \tag{2.36}$$

Obviously,

$$(1-C_2)(1-D_2) \ge 1 - (C_2 + D_2) \ge 1 - \|\ell_0(1)\|_L > 0. \tag{2.37}$$

If $\|\ell_0(1)\|_L \ge 1-\lambda^2$, then, according to relation (1.6) and the second inequality in (1.5), we obtain $\|\ell_1(1)\|_L < \lambda$. Hence, $B_2 < \lambda$ and

$$(A_2+\lambda)B_2 = A_2B_2 + \lambda B_2 \le \lambda(A_2+B_2) \le \lambda\|\ell_1(1)\|_L.$$

By virtue of the last inequality and (2.37), relation (2.36) yields (2.28), where r_0 is defined by (2.29). Therefore, estimate (2.5) is valid.

If $\|\ell_0(1)\|_L < 1-\lambda^2$, then, by virtue of the inequalities

$$4(A_2+\lambda)B_2 \le (A_2+B_2+\lambda)^2 \le (\|\ell_1(1)\|_L + \lambda)^2$$

and (2.37), relation (2.36) yields (2.30), where r_1 is defined by (2.31). Therefore, estimate (2.5) holds.

The lemma is proved.

Lemma 2.5. *Suppose that $\lambda \in \,]0,1]$ and the operator ℓ admits the representation $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 satisfy conditions (1.3) and (1.9). Then ℓ belongs to the set $U_2(\lambda)$.*

Proof. Let $q^* \in L([a,b];R_+)$, $c \in R_+$ and $u \in \widetilde{C}([a,b];R)$ satisfy (2.3) and (2.4) for $i=2$. We prove relation (2.5), where

$$r = \frac{\lambda\|\ell_0(1)\|_L + 1 + \lambda}{\lambda - \lambda\|\ell_1(1)\|_L - \|\ell_0(1)\|_L}. \tag{2.38}$$

It is obvious that u satisfies (2.11), where $\widetilde{q}$ is defined by (2.12). Clearly,

$$-\widetilde{q}(t)\operatorname{sgn} u(t) \le q^*(t) \quad \text{for} \quad t \in [a,b], \tag{2.39}$$

and

$$[u(a) + \lambda u(b)]\operatorname{sgn} u(b) \le c. \tag{2.40}$$

First, assume that u does not change its sign. According to (2.40) and the assumption $\lambda \in \,]0,1]$, we obtain

$$|u(b)| \le \frac{c}{\lambda}. \tag{2.41}$$

We choose $t_0 \in [a,b]$ so that (2.16) holds. Due to (1.3) and (2.41), relation (2.11) yields

$$-|u(t)|' \le \|u\|_C\, \ell_1(1)(t) + q^*(t) \quad \text{for} \quad t \in [a,b]. \tag{2.42}$$

The integration of (2.42) from t_0 to b with regard for (1.3), (2.41), and (2.16) results in

$$\|u\|_C - \frac{c}{\lambda} \le \|u\|_C - |u(b)| \le \|u\|_C \int_{t_0}^{b} \ell_1(1)(s)ds + \int_{t_0}^{b} q^*(s)ds \le \|u\|_C\|\ell_1(1)\|_L + \|q^*\|_L.$$

Thus,

$$\|u\|_C\,(1 - \|\ell_1(1)\|_L) \le \frac{c + \|q^*\|_L}{\lambda},$$

and, consequently, estimate (2.5) holds.

Now assume that u changes its sign. We define numbers M and m by (2.18) and choose $t_M, t_m \in [a,b]$ so that (2.19) is satisfied. It is obvious that $M>0$, $m>0$, and either (2.20) or (2.21) is valid.

First, assume that relation (2.21) holds. It is clear that there exists $\alpha_1 \in \,]t_M, t_m[$ such that

$$u(t) > 0 \quad \text{for} \;\; t_M \le t < \alpha_1, \qquad u(\alpha_1) = 0. \tag{2.43}$$

Let

$$\alpha_2 = \sup\{t \in [t_m, b] \colon u(s) < 0 \;\text{ for }\; t_m \le s \le t\}.$$

Obviously,

$$u(t) < 0 \quad \text{for } t_m \le t < \alpha_2 \qquad \text{and} \qquad u(\alpha_2) = 0 \quad \text{if } \alpha_2 < b. \tag{2.44}$$

Using relations (2.40) and (2.44) and the assumption $\lambda \in \,]0,1]$, we obtain

$$u(\alpha_2) \ge -\frac{1}{\lambda}[u(a)]_+ - \frac{c}{\lambda} \ge -\frac{M}{\lambda} - \frac{c}{\lambda}. \tag{2.45}$$

The integration of (2.11) from t_M to α_1 and from t_m to α_2 with regard for (1.3), (2.18), (2.19), (2.39), (2.43), (2.44), and (2.45) gives

$$M \le M\int_{t_M}^{\alpha_1} \ell_1(1)(s)ds + m\int_{t_M}^{\alpha_1} \ell_0(1)(s)ds + \int_{t_M}^{\alpha_1} q^*(s)ds,$$

$$m - \frac{M}{\lambda} - \frac{c}{\lambda} \le m + u(\alpha_2) \le M\int_{t_m}^{\alpha_2} \ell_0(1)(s)ds + m\int_{t_m}^{\alpha_2} \ell_1(1)(s)ds + \int_{t_m}^{\alpha_2} q^*(s)ds.$$

The last two inequalities yield

$$M(1 - A_1) \le mC_1 + \|q^*\|_L, \qquad m(1 - B_1) \le M\left(D_1 + \frac{1}{\lambda}\right) + \|q^*\|_L + \frac{c}{\lambda}, \tag{2.46}$$

where

$$A_1 = \int_{t_M}^{\alpha_1} \ell_1(1)(s)ds, \qquad B_1 = \int_{t_m}^{\alpha_2} \ell_1(1)(s)ds,$$

$$C_1 = \int_{t_M}^{\alpha_1} \ell_0(1)(s)ds, \qquad D_1 = \int_{t_m}^{\alpha_2} \ell_0(1)(s)ds.$$

Due to (1.9), we have $A_1 < 1$ and $B_1 < 1$. Consequently, relation (2.46) yields

$$\begin{aligned} 0 < M(1-A_1)(1-B_1) &\le C_1\left(M\left(D_1+\frac{1}{\lambda}\right)+\|q^*\|_L+\frac{c}{\lambda}\right)+\|q^*\|_L \\ &\le MC_1\left(D_1+\frac{1}{\lambda}\right)+\left(\|\ell_0(1)\|_L+1+\frac{1}{\lambda}\right)(\|q^*\|_L+c)\,, \\ 0 < m(1-A_1)(1-B_1) &\le \left(D_1+\frac{1}{\lambda}\right)(mC_1+\|q^*\|_L)+\|q^*\|_L+\frac{c}{\lambda} \\ &\le mC_1\left(D_1+\frac{1}{\lambda}\right)+\left(\|\ell_0(1)\|_L+1+\frac{1}{\lambda}\right)(\|q^*\|_L+c)\,. \end{aligned} \tag{2.47}$$

Obviously,

$$(1-A_1)(1-B_1) \ge 1-(A_1+B_1) \ge 1-\|\ell_1(1)\|_L > 0. \tag{2.48}$$

According to relation (1.9) and the assumption $\lambda \in \,]0,1]$, we obtain $\|\ell_0(1)\|_L < \dfrac{1}{\lambda}$. Hence, $C_1 < \dfrac{1}{\lambda}$ and

$$C_1\left(D_1+\frac{1}{\lambda}\right) = C_1D_1+\frac{1}{\lambda}C_1 \le \frac{1}{\lambda}(C_1+D_1) \le \frac{1}{\lambda}\|\ell_0(1)\|_L.$$

By virtue of the last inequality, (2.48), and the assumption $\lambda \in \,]0,1]$, relation (2.47) yields

$$\begin{aligned} M &\le r_0\,(\lambda\|\ell_0(1)\|_L+1+\lambda)\,(c+\|q^*\|_L), \\ m &\le r_0\,(\lambda\|\ell_0(1)\|_L+1+\lambda)\,(c+\|q^*\|_L), \end{aligned} \tag{2.49}$$

where

$$r_0 = (\lambda-\lambda\|\ell_1(1)\|_L-\|\ell_0(1)\|_L)^{-1}. \tag{2.50}$$

Therefore, estimate (2.5) holds.

Now assume that relation (2.20) is valid. Obviously, there exists $\alpha_3 \in \,]t_m, t_M[$ such that

$$u(t) < 0 \quad \text{for } t_m \le t < \alpha_3, \qquad u(\alpha_3) = 0. \tag{2.51}$$

Let

$$\alpha_4 = \sup\{t \in [t_M, b] \colon u(s) > 0 \text{ for } t_M \le s \le t\}.$$

It is clear that

$$u(t) > 0 \quad \text{for } t_M \le t < \alpha_4 \qquad \text{and} \qquad u(\alpha_4) = 0 \quad \text{if } \alpha_4 < b. \tag{2.52}$$

It follows from relations (2.40) and (2.52) and the assumption $\lambda \in \,]0,1]$ that

$$u(\alpha_4) \leq \frac{1}{\lambda}[u(a)]_- + \frac{c}{\lambda} \leq \frac{m}{\lambda} + \frac{c}{\lambda}. \tag{2.53}$$

The integration of (2.11) from t_m to α_3 and from t_M to α_4 with regard for (1.3), (2.18), (2.19), (2.39), (2.51), (2.52), and (2.53) yields

$$m \leq M \int_{t_m}^{\alpha_3} \ell_0(1)(s)ds + m \int_{t_m}^{\alpha_3} \ell_1(1)(s)ds + \int_{t_m}^{\alpha_3} q^*(s)ds,$$

$$M - \frac{m}{\lambda} - \frac{c}{\lambda} \leq M - u(\alpha_4) \leq M \int_{t_M}^{\alpha_4} \ell_1(1)(s)ds + m \int_{t_M}^{\alpha_4} \ell_0(1)(s)ds + \int_{t_M}^{\alpha_4} q^*(s)ds.$$

Using the last two inequalities, we get

$$m(1 - A_2) \leq MC_2 + \|q^*\|_L, \qquad M(1 - B_2) \leq m\left(D_2 + \frac{1}{\lambda}\right) + \|q^*\|_L + \frac{c}{\lambda}, \tag{2.54}$$

where

$$A_2 = \int_{t_m}^{\alpha_3} \ell_1(1)(s)ds, \qquad B_2 = \int_{t_M}^{\alpha_4} \ell_1(1)(s)ds,$$

$$C_2 = \int_{t_m}^{\alpha_3} \ell_0(1)(s)ds, \qquad D_2 = \int_{t_M}^{\alpha_4} \ell_0(1)(s)ds.$$

Due to (1.9), we have $A_2 < 1$ and $B_2 < 1$. Consequently, relation (2.54) yields

$$\begin{aligned} 0 < m(1 - A_2)(1 - B_2) &\leq C_2\left(m\left(D_2 + \frac{1}{\lambda}\right) + \|q^*\|_L + \frac{c}{\lambda}\right) + \|q^*\|_L \\ &\leq mC_2\left(D_2 + \frac{1}{\lambda}\right) + \left(\|\ell_0(1)\|_L + 1 + \frac{1}{\lambda}\right)(\|q^*\|_L + c), \\ 0 < M(1 - A_2)(1 - B_2) &\leq \left(D_2 + \frac{1}{\lambda}\right)(MC_2 + \|q^*\|_L) + \|q^*\|_L + \frac{c}{\lambda} \\ &\leq MC_2\left(D_2 + \frac{1}{\lambda}\right) + \left(\|\ell_0(1)\|_L + 1 + \frac{1}{\lambda}\right)(\|q^*\|_L + c). \end{aligned} \tag{2.55}$$

Obviously,

$$(1 - A_2)(1 - B_2) \geq 1 - (A_2 + B_2) \geq 1 - \|\ell_1(1)\|_L > 0. \tag{2.56}$$

According to relation (1.9) and the assumption $\lambda \in \,]0,1],$ we obtain $\|\ell_0(1)\|_L < \dfrac{1}{\lambda}$. Hence, $C_2 < \dfrac{1}{\lambda}$ and

$$C_2\left(D_2 + \frac{1}{\lambda}\right) = C_2 D_2 + \frac{1}{\lambda}C_2 \leq \frac{1}{\lambda}(C_2 + D_2) \leq \frac{1}{\lambda}\|\ell_0(1)\|_L.$$

By virtue of the last inequality and (2.56), relation (2.55) yields (2.49), where r_0 is defined by (2.50). Therefore, estimate (2.5) is valid.

The lemma is proved.

3. Proofs of Main Results

Theorem 1.1 follows from Lemmas 2.2 and 2.4, Theorem 1.2 follows from Lemmas 2.2 and 2.5, Theorem 1.5 follows from Lemmas 2.3 and 2.4, and Theorem 1.6 follows from Lemmas 2.3 and 2.5.

Proof of Corollary 1.1. Conditions (1.18) and (1.19), where γ is defined by (1.20), obviously yield conditions (1.4) and (1.5), where α is defined by (1.6) and

$$\begin{gathered} F(v)(t) \overset{\text{df}}{=} p(t)v(\tau(t)) - g(t)v(\mu(t)) + f(t, v(t), v(\nu(t))), \\ \ell_0(v)(t) \overset{\text{df}}{=} p(t)v(\tau(t)), \qquad \ell_1(v)(t) \overset{\text{df}}{=} g(t)v(\mu(t)). \end{gathered} \tag{3.1}$$

Consequently, all assumptions of Theorem 1.1 are satisfied.

Proof of Corollary 1.5. Conditions (1.26) and (1.19), where γ is defined by (1.20), obviously yield conditions (1.15) and (1.5), where α is defined by (1.6) and $F,$ $\ell_0,$ and ℓ_1 are defined by (3.1). Consequently, all assumptions of Theorem 1.5 are satisfied.

Corollaries 1.2–1.4 and 1.6–1.8 can be proved by analogy.

4. On Remarks 1.1 and 1.2

On Remark 1.1. Let $\lambda \in \,]0,1]$ (for the case $\lambda = 0,$ see [5]). Denote by G the set of pairs $(x,y) \in R_+ \times R_+$ such that either

$$x < 1 - \lambda^2, \qquad y < 2\sqrt{1-x} - \lambda,$$

or

$$1 - \lambda^2 \leq x < 1, \qquad y < \frac{1-x}{\lambda}.$$

According to Theorem 1.1, if (1.2) is satisfied and there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that $(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in G$ and inequality (1.4) holds on the set $B^1_{\lambda c}([a,b];R)$, then problem (0.1), (0.2) is solvable.

Below, we give examples showing that, for any pair $(x_0, y_0) \not\in \overline{G}$, $x_0 \geq 0$, $y_0 \geq 0$, there exist functions $p_0 \in L([a,b];R)$, $-p_1 \in L([a,b];R_+)$, and $\tau \in \mathcal{M}_{ab}$ such that

$$\int_a^b [p_0(s)]_+ ds = x_0, \quad \int_a^b [p_0(s)]_- ds = y_0, \tag{4.1}$$

and the problem

$$u'(t) = p_0(t)u(\tau(t)) + p_1(t)u(t), \quad u(a) + \lambda u(b) = 0 \tag{4.2}$$

has a nontrivial solution. Then, by Remark 0.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that problem (0.1), (0.2), where

$$F(v)(t) \overset{\text{df}}{=} p_0(t)v(\tau(t)) + p_1(t)v(t) + q_0(t), \quad h(v) \overset{\text{df}}{=} c_0, \tag{4.3}$$

does not have solutions, while conditions (1.2) and (1.4) are satisfied with $\ell_0(v)(t) \overset{\text{df}}{=} [p_0(t)]_+ v(\tau(t))$, $\ell_1(v)(t) \overset{\text{df}}{=} [p_0(t)]_- v(\tau(t))$, $q \equiv |q_0|$, and $c = |c_0|$.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \not\in \overline{G}$, then (x_0, y_0) belongs to at least one of the following sets:

$$G_1 = \left\{(x,y) \in R_+ \times R_+ \colon 1 < x, \ \ 0 \leq y\right\},$$

$$G_2 = \left\{(x,y) \in R_+ \times R_+ \colon 1 - \lambda^2 \leq x \leq 1, \ \ \frac{1-x}{\lambda} < y\right\},$$

$$G_3 = \left\{(x,y) \in R_+ \times R_+ \colon 0 \leq x < 1 - \lambda^2, \ \ 2\sqrt{1-x} - \lambda < y\right\}.$$

Example 4.1. Let $(x_0, y_0) \in G_1$ and $\varepsilon > 0$ be such that $x_0 - \varepsilon \geq 1$ and $\lambda - \varepsilon > 0$. We set $a = 0$, $b = 4$, $t_0 = 3 + \dfrac{\varepsilon}{1+\varepsilon}$,

$$p_0(t) = \begin{cases} 0 & \text{for } \ t \in [0,1[, \\ -y_0 & \text{for } \ t \in [1,2[, \\ x_0 - 1 - \varepsilon & \text{for } \ t \in [2,3[, \\ 1 + \varepsilon & \text{for } \ t \in [3,4], \end{cases}$$

$$p_1(t) = \begin{cases} -\dfrac{\lambda - \varepsilon}{\lambda - (\lambda - \varepsilon)t} & \text{for } \ t \in [0,1[, \\ 0 & \text{for } \ t \in [1,4], \end{cases}$$

and

$$\tau(t) = \begin{cases} t_0 & \text{for } t \in [0,3[, \\ 4 & \text{for } t \in [3,4]. \end{cases}$$

Then relation (4.1) holds, and problem (4.2) has the nontrivial solution

$$u(t) = \begin{cases} -(\lambda-\varepsilon)t + \lambda & \text{for } t \in [0,1[, \\ \varepsilon & \text{for } t \in [1,3[, \\ -(1+\varepsilon)(t-3)+\varepsilon & \text{for } t \in [3,4]. \end{cases}$$

Example 4.2. Let $(x_0, y_0) \in G_2$ and $\varepsilon > 0$ be such that $\dfrac{1-x_0+\varepsilon}{\lambda} \leq y_0$ and $\lambda - \varepsilon > 0$. We set $a = 0$, $b = 4$, $t_0 = 2 + \dfrac{\varepsilon}{1-x_0+\varepsilon}$,

$$p_0(t) = \begin{cases} 0 & \text{for } t \in [0,1[, \\ -y_0 + \dfrac{1-x_0+\varepsilon}{\lambda} & \text{for } t \in [1,2[, \\ -\dfrac{1-x_0+\varepsilon}{\lambda} & \text{for } t \in [2,3[, \\ x_0 & \text{for } t \in [3,4], \end{cases}$$

$$p_1(t) = \begin{cases} -\dfrac{\lambda-\varepsilon}{\lambda-(\lambda-\varepsilon)t} & \text{for } t \in [0,1[, \\ 0 & \text{for } t \in [1,4], \end{cases}$$

and

$$\tau(t) = \begin{cases} t_0 & \text{for } t \in [0,2[, \\ 0 & \text{for } t \in [2,3[, \\ 4 & \text{for } t \in [3,4]. \end{cases}$$

Then relation (4.1) holds, and problem (4.2) has the nontrivial solution

$$u(t) = \begin{cases} -(\lambda-\varepsilon)t + \lambda & \text{for } t \in [0,1[, \\ \varepsilon & \text{for } t \in [1,2[, \\ -(1-x_0+\varepsilon)(t-2)+\varepsilon & \text{for } t \in [2,3[, \\ -x_0(t-3)-(1-x_0) & \text{for } t \in [3,4]. \end{cases}$$

Example 4.3. Let $(x_0, y_0) \in G_3$ and $\varepsilon > 0$ be such that $y_0 \geq 2\sqrt{1-x_0} - \lambda + \varepsilon$ and $\varepsilon < 1 - \sqrt{1-x_0}$. We set $a = 0$, $b = 5$,

$$p_0(t) = \begin{cases} -\sqrt{1-x_0} + \lambda & \text{for } t \in [0,1[, \\ 0 & \text{for } t \in [1, 3 - \sqrt{1-x_0} - \varepsilon[, \\ -1 & \text{for } t \in [3 - \sqrt{1-x_0} - \varepsilon, 3[, \\ -y_0 + 2\sqrt{1-x_0} - \lambda + \varepsilon & \text{for } t \in [3,4[, \\ x_0 & \text{for } t \in [4,5], \end{cases}$$

$$p_1(t) = \begin{cases} 0 & \text{for } t \in [0,1[\cup[3 - \sqrt{1-x_0} - \varepsilon, 5], \\ -\dfrac{1-x_0}{(1-x_0)(1-t) + \sqrt{1-x_0}} & \text{for } t \in [1,2[, \\ -\dfrac{\sqrt{1-x_0}}{\sqrt{1-x_0}(3-t) - (1-x_0)} & \text{for } t \in [2, 3 - \sqrt{1-x_0} - \varepsilon[, \end{cases}$$

and

$$\tau(t) = \begin{cases} 5 & \text{for } t \in [0,1[, \\ 1 & \text{for } t \in [1,3[, \\ 3 - \sqrt{1-x_0} & \text{for } t \in [3,4[, \\ 5 & \text{for } t \in [4,5]. \end{cases}$$

Then relation (4.1) holds, and problem (4.2) has the nontrivial solution

$$u(t) = \begin{cases} (\sqrt{1-x_0} - \lambda)t + \lambda & \text{for } t \in [0,1[, \\ (1-x_0)(1-t) + \sqrt{1-x_0} & \text{for } t \in [1,2[, \\ \sqrt{1-x_0}(3-t) - (1-x_0) & \text{for } t \in [2,3[, \\ -(1-x_0) & \text{for } t \in [3,4[, \\ x_0(5-t) - 1 & \text{for } t \in [4,5]. \end{cases}$$

On Remark 1.2. Let $\lambda \in \,]0,1]$. Denote by H the set of pairs $(x,y) \in R_+ \times R_+$ such that

$$x + \lambda y < \lambda.$$

By virtue of Theorem 1.2, if relation (1.7) is satisfied and there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and $q \in L([a,b]; R_+)$ such that $(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in H$ and inequality (1.8) holds on the set $B^2_{\lambda c}([a,b]; R)$, then problem (0.1), (0.2) is solvable.

Below, we give examples showing that, for any pair $(x_0, y_0) \notin \overline{H}$, $x_0 \geq 0$, $y_0 \geq 0$, there exist functions $p_0 \in L([a,b]; R)$, $p_1 \in L([a,b]; R_+)$, and $\tau \in \mathcal{M}_{ab}$ such that relation (4.1) is satisfied and problem (4.2) has a nontrivial solution. Then, by Remark 0.1, there exist $q_0 \in L([a,b]; R)$ and $c_0 \in R$ such that problem (0.1), (0.2), where F and h are defined by (4.3), does not have solutions, while conditions (1.7) and (1.8) are satisfied with $\ell_0(v)(t) \overset{\text{df}}{=} [p_0(t)]_+ v(\tau(t))$, $\ell_1(v)(t) \overset{\text{df}}{=} [p_0(t)]_- v(\tau(t))$, $q \equiv |q_0|$, and $c = |c_0|$.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin \overline{H}$, then (x_0, y_0) belongs to at least one of the following sets:

$$H_1 = \{(x,y) \in R_+ \times R_+ \colon \lambda < x, \;\; 0 \leq y\},$$

$$H_2 = \left\{(x,y) \in R_+ \times R_+ \colon 0 \leq x \leq \lambda, \;\; -\frac{x}{\lambda} + 1 < y\right\}.$$

Example 4.4. Let $(x_0, y_0) \in H_1$ and $\varepsilon > 0$ be such that $x_0 - \lambda \geq \varepsilon$ and $1 - \varepsilon > 0$. We set $a = 0$, $b = 4$, $t_0 = \dfrac{\lambda}{\lambda + \varepsilon}$,

$$p_0(t) = \begin{cases} \lambda + \varepsilon & \text{for } t \in [0,1[, \\ -y_0 & \text{for } t \in [1,2[, \\ x_0 - \lambda - \varepsilon & \text{for } t \in [2,3[, \\ 0 & \text{for } t \in [3,4], \end{cases}$$

$$p_1(t) = \begin{cases} 0 & \text{for } t \in [0,3[, \\ \dfrac{1-\varepsilon}{(1-\varepsilon)(t-4)+1} & \text{for } t \in [3,4], \end{cases}$$

and

$$\tau(t) = \begin{cases} 4 & \text{for } t \in [0,1[, \\ t_0 & \text{for } t \in [1,4]. \end{cases}$$

Then relation (4.1) holds, and problem (4.2) has the nontrivial solution

$$u(t) = \begin{cases} (\lambda + \varepsilon)t - \lambda & \text{for } t \in [0,1[, \\ \varepsilon & \text{for } t \in [1,3[, \\ (1-\varepsilon)(t-4)+1 & \text{for } t \in [3,4]. \end{cases}$$

Example 4.5. Let $(x_0, y_0) \in H_2$ and $\varepsilon > 0$ be such that $\dfrac{\lambda - x_0 + \varepsilon}{\lambda} \le y_0$ and $1 - \varepsilon > 0$. We set $a = 0$, $b = 4$, $t_0 = 2 - \dfrac{\varepsilon}{\lambda - x_0 + \varepsilon}$,

$$p_0(t) = \begin{cases} x_0 & \text{for } t \in [0,1[, \\ -\dfrac{\lambda - x_0 + \varepsilon}{\lambda} & \text{for } t \in [1,2[, \\ -y_0 + \dfrac{\lambda - x_0 + \varepsilon}{\lambda} & \text{for } t \in [2,3[, \\ 0 & \text{for } t \in [3,4], \end{cases}$$

$$p_1(t) = \begin{cases} 0 & \text{for } t \in [0,3[, \\ \dfrac{1-\varepsilon}{1-(1-\varepsilon)(4-t)} & \text{for } t \in [3,4], \end{cases}$$

and

$$\tau(t) = \begin{cases} 4 & \text{for } t \in [0,1[, \\ 0 & \text{for } t \in [1,2[, \\ t_0 & \text{for } t \in [2,4]. \end{cases}$$

Then relation (4.1) holds, and problem (4.2) has the nontrivial solution

$$u(t) = \begin{cases} -x_0 t + \lambda & \text{for } t \in [0,1[, \\ (\lambda - x_0 + \varepsilon)(2-t) - \varepsilon & \text{for } t \in [1,2[, \\ -\varepsilon & \text{for } t \in [2,3[, \\ (1-\varepsilon)(4-t) - 1 & \text{for } t \in [3,4]. \end{cases}$$

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