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On the Periodic Boundary Value Problem for First-Order Functional-Differential Equations

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1. STATEMENT OF THE PROBLEM AND BASIC NOTATION

In the present paper, we consider the boundary value problem

$$u'(t) = \ell(u)(t) + F(u)(t), \tag{1.1}$$

$$u(a) = u(b), \tag{1.2}$$

where $\ell : C([a, b]; R) \to L([a, b]; R)$ is a linear bounded operator and $F : C([a, b]; R) \to L([a, b]; R)$ is a continuous operator, not necessarily linear.

This problem, which is the subject of numerous studies, has long been attracting mathematicians' attention. Interesting results about its solvability can be found, e.g., in [1–11]. Nevertheless, problem (1.1), (1.2) has not been completely analyzed yet even for the linear case, in which Eq. (1.1) has the form

$$u'(t) = \ell(u)(t) + g(t). \tag{1.3}$$

In a sense, we fill the gap. More precisely, new effective criteria for the solvability and unique solvability of problems (1.3), (1.2) and (1.1), (1.2) are given in Sections 2 and 3. In Section 4, we construct examples justifying the optimality of these criteria. The results are further specialized for the equations

$$u'(t) = p(t)u(\tau(t)) + g(t), \tag{1.4}$$

$$u'(t) = p(t)u(\tau(t)) + f(t, u(\mu(t)), u(t)) \tag{1.5}$$

with deviating arguments.

We use the following notation:

R is the set of real numbers;

$R_+ = [0, +\infty[$;

$C([a, b]; R)$ is the space of continuous functions $u : [a, b] \to R$ with the norm

$$\|u\|_C = \max\{|u(t)| : \ a \le t \le b\};$$

$C([a, b]; R_+) = \{u \in C([a, b]; R) : \ u(t) \ge 0 \text{ for } t \in [a, b]\}$;

$C_0([a, b]; R) = \{u \in C([a, b]; R) : \ u(a) = u(b)\}$;

$\tilde{C}([a, b]; R)$ is the set of absolutely continuous functions $u : [a, b] \to R$;

$L([a, b]; R)$ is the space of Lebesgue integrable functions $p : [a, b] \to R$ with the norm

$$\|p\|_L = \int_a^b |p(s)| ds;$$

$L([a, b]; R_+) = \{p \in L([a, b]; R) : \ p(t) \ge 0 \text{ for } t \in]a, b[\,\}$;

$\mathscr{M}_{ab}$ is the set of measurable functions $\tau : [a, b] \to [a, b]$;

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$\mathscr{P}_{ab}$ is the set of linear positive operators $h : C([a,b];R) \to L([a,b];R)$, that is, linear operators that map $C([a,b];R_+)$ into $L([a,b];R_+)$;

$\mathscr{K}_{ab}$ is the set of continuous operators $F : C([a,b];R) \to L([a,b];R)$ such that

$$\sup\{|F(v)(\cdot)| : \|v\|_C \le r\} \in L([a,b];R_+)$$

for arbitrary $r > 0$;

$$[p]_+ = 2^{-1}(|p| + p); \qquad [p]_- = 2^{-1}(|p| - p);$$

$$J_\pm \equiv \int_a^b [p(s)]_\pm ds.$$

Throughout the following, we assume that $\ell : C([a,b];R) \to L([a,b];R)$ is a linear bounded operator, $F \in \mathscr{K}_{ab}$, $p, g \in L([a,b];R)$, $\tau, \mu \in \mathscr{M}_{ab}$, and $f : [a,b] \times R^2 \to R$ satisfies the local Carathéodory conditions. We deal with solutions of problem (1.1), (1.2) in the space $\tilde{C}([a,b];R)$.

2. THE LINEAR PROBLEM

Theorem 2.1. *Let*

$$\ell = \ell_0 - \ell_1, \quad \ell_0, \ell_1 \in \mathscr{P}_{ab}, \tag{2.1}$$

and let the inequalities

$$\|\ell_i(1)\|_L < 1, \tag{2.2}$$

$$\|\ell_i(1)\|_L / (1 - \|\ell_i(1)\|_L) < \|\ell_j(1)\|_L < 2 + 2(1 - \|\ell_i(1)\|_L)^{1/2} \tag{2.3}$$

be valid for some $i, j \in \{0, 1\}$, *where* $i \ne j$. *Then problem* (1.3), (1.2) *has a unique solution.*

Corollary 2.1. *Suppose that either* $J_+ < 1$ *and* $J_+/(1-J_+) < J_- < 2 + 2(1-J_+)^{1/2}$, *or* $J_- < 1$ *and* $J_-/(1-J_-) < J_+ < 2 + 2(1-J_-)^{1/2}$. *Then problem* (1.4), (1.2) *has a unique solution.*

Remark 2.1. Let H be the set of pairs $(x, y) \in R_+ \times R_+$ such that either $0 \le x < 1$ and $x/(1-x) < y < 2 + 2(1-x)^{1/2}$, or $0 \le y < 1$ and $y/(1-y) < x < 2 + 2(1-y)^{1/2}$. It follows from Theorem 2.1 that if $(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in H$, then problem (1.3), (1.2) is uniquely solvable. In Section 4, we consider examples showing that, for any pair $(x_0, y_0) \notin H$, there exist functions $p, g \in L([a,b];R)$ and $\tau \in \mathscr{M}_{ab}$ such that $y_0 = J_+$, $x_0 = J_-$, and problem (1.4), (1.2) has no solutions.

Theorem 2.2. *Let condition* (2.1) *be satisfied, and let the inequalities*

$$\|\ell_0(1)\|_L < 1, \qquad \|\ell_1(1)\|_L < 1, \tag{2.4}$$

$$\|\ell_i(1)\|_L / (1 - \|\ell_i(1)\|_L) < \|\ell_j(1)\|_L, \tag{2.5}$$

$$\sigma g(t) \ge 0 \quad for \quad t \in]a,b[, \quad g(t) \not\equiv 0, \tag{2.6}$$

be valid for some $\sigma \in \{-1, 1\}$ *and* $i, j \in \{0, 1\}$, *where* $i \ne j$. *Then problem* (1.3), (1.2) *has a unique solution* u *such that*

$$\sigma(-1)^i u(t) > 0 \quad for \quad t \in [a,b]. \tag{2.7}$$

Proof of Theorem 2.1. Let $i = 0$ and $j = 1$. The case in which $i = 1$ and $j = 0$ can be treated in a similar way.

By Theorem 1.1 in [8], it suffices to show that the homogeneous equation

$$u'(t) = \ell(u)(t) \tag{2.8}$$

does not have nontrivial solutions satisfying condition (1.2). Suppose the contrary: problem (2.8), (1.2) has a nontrivial solution u. We first suppose that u has a constant sign. Without loss of generality, we assume that $u(t) \geq 0$ for $t \in [a,b]$. We set

$$m = \min\{u(t): \ a \leq t \leq b\}, \qquad M = \max\{u(t): \ a \leq t \leq b\} \tag{2.9}$$

and take $t_1, t_2 \in [a,b]$ such that

$$u(t_1) = m, \qquad u(t_2) = M. \tag{2.10}$$

Obviously, $t_1 \neq t_2$, since otherwise we would have $\|\ell_0(1)\|_L = \|\ell_1(1)\|_L$ by virtue of (2.8) and (1.2), which contradicts the first inequality in (2.3). Consequently, either

$$t_1 < t_2, \tag{2.11}$$

or

$$t_2 < t_1. \tag{2.12}$$

If inequality (2.11) is valid, then, by integrating (2.8) from t_1 to t_2 and by taking account of (2.1), (2.9), and (2.10), we obtain

$$M - m = \int_{t_1}^{t_2} [\ell_0(u)(s) - \ell_1(u)(s)]\, ds \leq M \int_{t_1}^{t_2} \ell_0(1)(s) ds \leq M \|\ell_0(1)\|_L .$$

Now suppose that (2.12) holds. Then, by integrating (2.8) from a to t_2 and from t_1 to b and by taking account of (2.1), (2.9), and (2.10), we obtain

$$M - u(a) \leq M \int_a^{t_2} \ell_0(1)(s) ds, \quad u(b) - m \leq M \int_{t_1}^{b} \ell_0(1)(s) ds.$$

By summing the last two inequalities and by using (1.2), we obtain

$$M - m \leq M \|\ell_0(1)\|_L . \tag{2.13}$$

Consequently, in both cases, inequality (2.13) is valid.

On the other hand, by integrating (2.8) from a to b, we obtain

$$\int_a^b \ell_0(u)(s) ds = \int_a^b \ell_1(u)(s) ds.$$

This, together with (2.9), implies that

$$m \|\ell_1(1)\|_L \leq M \|\ell_0(1)\|_L . \tag{2.14}$$

By (2.13) and the first inequality in (2.3), we arrive at a contradiction:

$$M \leq M \left(\|\ell_0(1)\|_L + \|\ell_0(1)\|_L / \|\ell_1(1)\|_L\right) < M. \tag{2.15}$$

Now suppose that u changes its sign. We set

$$\bar{m} = -\min\{u(t): \ a \leq t \leq b\}, \qquad \bar{M} = \max\{u(t): \ a \leq t \leq b\} \tag{2.16}$$

and take $\alpha, \beta \in [a,b]$ such that

$$u(\alpha) = \bar{M}, \qquad u(\beta) = -\bar{m}. \tag{2.17}$$

Without loss of generality, we can assume that $\alpha < \beta$. By integrating (2.8) over the intervals $[\alpha, \beta]$, $[a, \alpha]$, and $[\beta, b]$ and by taking account of (2.1) and (2.17), we obtain

$$\bar{M} + \bar{m} \leq \bar{M} \int_{\alpha}^{\beta} \ell_1(1)(s)ds + \bar{m} \int_{\alpha}^{\beta} \ell_0(1)(s)ds, \tag{2.18}$$

$$\bar{M} - u(a) \leq \bar{M} \int_{a}^{\alpha} \ell_0(1)(s)ds + \bar{m} \int_{a}^{\alpha} \ell_1(1)(s)ds, \tag{2.19}$$

$$u(b) + \bar{m} \leq \bar{M} \int_{\beta}^{b} \ell_0(1)(s)ds + \bar{m} \int_{\beta}^{b} \ell_1(1)(s)ds. \tag{2.20}$$

If we add the last two inequalities, then we obtain

$$\bar{M} + \bar{m} \leq \bar{M} \int_{I} \ell_0(1)(s)ds + \bar{m} \int_{I} \ell_1(1)(s)ds, \tag{2.21}$$

where $I = [a, b] \setminus]\alpha, \beta[$. It follows from (2.18) and (2.21) that

$$\bar{M}(1 - C) \leq \bar{m}(A - 1), \qquad \bar{m}(1 - D) \leq \bar{M}(B - 1), \tag{2.22}$$

where $A = \int_I \ell_1(1)(s)ds$, $B = \int_\alpha^\beta \ell_1(1)(s)ds$, $C = \int_I \ell_0(1)(s)ds$, and $D = \int_\alpha^\beta \ell_0(1)(s)ds$. On the other hand, by (2.2), $C < 1$ and $D < 1$; therefore, it follows from (2.22) that $A > 1$, $B > 1$, and

$$(1 - C)(1 - D) \leq (A - 1)(B - 1). \tag{2.23}$$

Now, by taking account of the inequalities $(1 - C)(1 - D) \geq 1 - (C + D) = 1 - \|\ell_0(1)\|_L$ and $4(A - 1)(B - 1) \leq (A + B - 2)^2 = (\|\ell_1(1)\|_L - 2)^2$, from (2.23), we obtain $4\,(1 - \|\ell_0(1)\|_L) \leq (\|\ell_1(1)\|_L - 2)^2$, which contradicts condition (2.3). The proof of the theorem is complete.

Proof of Corollary 2.1. We set $\ell(u)(t) = p(t)u(\tau(t))$, $\ell_0(u)(t) = [p(t)]_+ u(\tau(t))$, and $\ell_1(u)(t) = [p(t)]_- u(\tau(t))$. Then Eq. (1.4) acquires the form (1.1), where ℓ satisfies condition (2.1). On the other hand, obviously, under the assumptions of Corollary 2.1, the operators ℓ_0 and ℓ_1 satisfy inequalities (2.2) and (2.3) for some $i, j \in \{0, 1\}$. The proof of the corollary is complete.

Proof of Theorem 2.2. Without loss of generality, we assume that $\sigma = -1$ and $i = 0$. By Theorem 2.1 and conditions (2.4) and (2.5), problem (1.3), (1.2) has a unique solution u.

We first show that $u(t) \neq 0$ for $a \leq t \leq b$. Suppose the contrary: u has at least one zero. We define the numbers $\bar{m}$ and $\bar{M}$ by relations (2.16) and choose $\alpha, \beta \in [a, b]$ so as to satisfy condition (2.17). By (2.6), $u(t) \not\equiv 0$. Therefore,

$$\bar{m} \geq 0, \qquad \bar{M} \geq 0, \qquad \bar{m} + \bar{M} > 0. \tag{2.24}$$

Suppose that $\beta < \alpha$. By integrating (1.3) from β to α and by taking account of (2.1), (2.6), (2.16), and (2.17), we obtain

$$\bar{M} + \bar{m} = \int_{\beta}^{\alpha} \ell_0(u)(s)ds - \int_{\beta}^{\alpha} \ell_1(u)(s)ds + \int_{\beta}^{\alpha} g(s)ds \leq \bar{M}\, \|\ell_0(1)\|_L + \bar{m}\, \|\ell_1(1)\|_L .$$

This, together with (2.4) and (2.24), leads to a contradiction: $\bar{M} + \bar{m} < \bar{M} + \bar{m}$. Now we suppose that $\alpha < \beta$. By integrating (1.3) from a to α and from β to b and by taking account of (2.1), (2.6), (2.16), and (2.17), we obtain inequalities (2.19) and (2.20); adding these, we see that inequality (2.21) is valid. From (2.4) and (2.24), we obtain the contradiction $\bar{M} + \bar{m} < \bar{M} + \bar{m}$. Consequently, $u(t) \neq 0$ for $t \in [a, b]$.

Now let us show that u satisfies condition (2.7). Suppose the contrary: $u(t) > 0$ for $t \in [a, b]$. We define the numbers m and M by (2.9) and choose $t_1, t_2 \in [a, b]$, $t_1 \neq t_2$, so as to satisfy (2.10). Arguing as in the proof of Theorem 2.1 and taking account of (2.6), we obtain (2.13). On the other hand, the integration of (1.3) from a to b with regard to (2.6) leads to the inequality

$$\int_a^b \ell_1(u)(s)ds < \int_a^b \ell_0(u)(s)ds.$$

This, together with (2.9), implies (2.14), which, with regard to (2.13) and (2.5), leads to the contradiction (2.15). This completes the proof of the theorem.

3. THE NONLINEAR PROBLEM

Theorem 3.1. *Let condition* (2.1) *be satisfied. Suppose that there exist* $i, j \in \{0, 1\}$ *and* q *belongs to* $L([a, b]; R_+)$ *such that* $i \neq j$,

$$\|\ell_i(1)\|_L < 1, \qquad \|\ell_i(1)\|_L / (1 - \|\ell_i(1)\|_L) < \|\ell_j(1)\|_L < 2(1 - \|\ell_i(1)\|_L)^{1/2}, \tag{3.1}$$

and the inequality

$$(-1)^i F(v)(t) \operatorname{sgn} v(t) \leq q(t) \tag{3.2}$$

is valid almost everywhere on $[a, b]$ *for each* $v \in C_0([a, b]; R)$. *Then problem* (1.1), (1.2) *has at least one solution.*

Corollary 3.1. *Suppose that there exists a function* $q \in L([a, b]; R_+)$ *such that either*

$$f(t, x, y) \operatorname{sgn} y \leq q(t) \quad \text{for} \quad t \in]a, b[, \quad x, y \in R,$$

$$J_+ < 1, \qquad J_+/(1 - J_+) < J_- < 2(1 - J_+)^{1/2},$$

or

$$f(t, x, y) \operatorname{sgn} y \geq -q(t) \quad \text{for} \quad t \in]a, b[, \quad x, y \in R,$$

$$J_- < 1, \qquad J_-/(1 - J_-) < J_+ < 2(1 - J_-)^{1/2}.$$

Then problem (1.5), (1.2) *has at least one solution.*

Theorem 3.2. *Let condition* (2.1) *be satisfied, and let inequalities* (3.1) *be valid for some* $i, j \in \{0, 1\}$, *where* $i \neq j$. *Moreover, suppose that*

$$(-1)^i [F(v)(t) - F(\bar{v})(t)] \operatorname{sgn}(v(t) - \bar{v}(t)) \leq 0 \tag{3.3}$$

almost everywhere on $[a, b]$ *for any* $v, \bar{v} \in C_0([a, b]; R)$. *Then problem* (1.1), (1.2) *has exactly one solution.*

Remark 3.1. Conditions (3.1) are optimal and cannot be weakened.

To prove Theorem 3.1, we need the following assertion.

Lemma 3.1. *Suppose that there exists a function* $\ell^* \in L([a, b]; R)$ *such that the inequality*

$$|\ell(u)(t)| \leq \ell^*(t)\|u\|_C \tag{3.4}$$

is valid for an arbitrary function $u \in C([a, b]; R)$ *almost everywhere on* $[a, b]$. *Moreover, suppose that there exists an* $r > 0$ *such that, for each* $\lambda \in [0, 1]$, *an arbitrary solution of the differential equation*

$$u'(t) = \ell(u)(t) + \lambda F(u)(t) \tag{3.5}$$

satisfying condition (1.2) *can be estimated as*

$$\|u\|_C \leq r. \tag{3.6}$$

Then problem (1.1), (1.2) *has at least one solution.*

This lemma is a special case of Corollary 2 in [7].

Now consider the differential inequality

$$(u'(t) - \ell(u)(t)) \operatorname{sgn} u(t) \le q(t), \tag{3.7}$$

where $q \in L([a,b]; R_+)$. A function $u \in \tilde{C}([a,b]; R)$ is referred to as a *solution* of problem (3.7), (1.2) if it satisfies condition (1.2) and satisfies the differential inequality (3.7) almost everywhere on $[a,b]$.

Lemma 3.2. *Let the operator ℓ satisfy the assumptions of Theorem* 3.1. *Then there exists a positive constant r_0 such that, for each function $q \in L([a,b]; R_+)$, an arbitrary solution u of problem* (3.7), (1.2) *admits the estimate* (3.6), *where $r = r_0 \|q\|_L$.*

Proof. Let us prove the lemma for $i = 0$ and $j = 1$. For $i = 1$ and $j = 0$, the proof can be performed in a similar way. We set

$$r_0 = \frac{1 + \|\ell_1(1)\|_L}{\|\ell_1(1)\|_L (1 - \|\ell_0(1)\|_L) - \|\ell_0(1)\|_L} + \frac{1 + \|\ell_1(1)\|_L}{1 - \|\ell_0(1)\|_L - (1/4)\|\ell_0(1)\|_L^2}, \quad r = r_0 \|q\|_L. \tag{3.8}$$

Let $q \in L([a,b]; R)$ be an arbitrarily given function, and let u be some solution of problem (3.7), (1.2). Without loss of generality, we can assume that $u(t) \not\equiv 0$. We first suppose that u has a constant sign. Let $m = \min\{|u(t)| : a \le t \le b\}$, $M = \max\{|u(t)| : a \le t \le b\}$, and $t_1, t_2 \in [a,b]$ be numbers such that $t_1 \ne t_2$, $|u(t_1)| = m$, and $|u(t_2)| = M$. Then, by (2.1) and (3.7), we have

$$|u(t)|' \le M\ell_0(1)(t) - m\ell_1(1)(t) + q(t). \tag{3.9}$$

Obviously, one of conditions (2.11) and (2.12) is satisfied. If condition (2.11) is valid, then, by integrating inequality (3.9) from t_1 to t_2, we obtain

$$M - m \le M \int_{t_1}^{t_2} \ell_0(1)(s)ds - m \int_{t_1}^{t_2} \ell_1(1)(s)ds + \int_{t_1}^{t_2} q(s)ds \le M\|\ell_0(1)\|_L + \|q\|_L.$$

If condition (2.12) holds, then the integration of inequality (3.9) from a to t_2 and from t_1 to b implies that

$$M - |u(a)| \le M \int_a^{t_2} \ell_0(1)(s)ds + \int_a^{t_2} q(s)ds, \qquad |u(b)| - m \le M \int_{t_1}^{b} \ell_0(1)(s)ds + \int_{t_1}^{b} q(s)ds.$$

By adding the last two inequalities, we obtain

$$M - m \le M\|\ell_0(1)\|_L + \|q\|_L. \tag{3.10}$$

Consequently, inequality (3.10) is valid in both cases considered above.

By integrating (3.9) from a to b, we obtain the inequality $m\|\ell_1(1)\|_L \le M\|\ell_0(1)\|_L + \|q\|_L$, which, together with (3.10), implies that $M(\|\ell_1(1)\|_L (1 - \|\ell_0(1)\|_L) - \|\ell_0\|_L) \le \|q\|_L (1 + \|\ell_1(1)\|_L)$. Now it follows from (3.8) that the estimate (3.6) is valid.

Now we suppose that u changes its sign. We set

$$u(t) = \begin{cases} u(t) & \text{if } \ a \le t \le b \\ u(t-b+a) & \text{if } \ b < t \le 2b-a, \end{cases}$$

$$\bar{\ell}_i(\bar{u})(t) = \begin{cases} \ell_i(u)(t) & \text{for } \ t \in [a,b] \\ \ell_i(u)(t-b+a) & \text{for } \ t \in \,]b, 2b-a[\,, \end{cases} \qquad i = 0, 1,$$

$$\bar{q}(t) = \begin{cases} q(t) & \text{for } \ t \in [a,b] \\ q(t-b+a) & \text{for } \ t \in \,]b, 2b-a]. \end{cases}$$

Obviously, $\bar{u}(a)=\bar{u}(b)=\bar{u}(2b-a)$, and the inequality

$$\left(\bar{u}'(t)-\bar{\ell}_0\left(\bar{u}\right)(t)+\bar{\ell}_1\left(\bar{u}\right)(t)\right)\operatorname{sgn}\bar{u}(t)\leq\bar{q}(t) \tag{3.11}$$

is valid almost everywhere on $[a,b]$.

Let $\bar{m}=-\min\left\{\bar{u}(t):\ a\leq t\leq 2b-a\right\}$ and $\bar{M}=\max\left\{\bar{u}(t):\ a\leq t\leq 2b-a\right\}$. Then there exist $\alpha_k,t_k\in[a,2b-a]$ $(k=1,2)$ such that $\alpha_k<t_k$ $(k=1,2)$, $[\alpha_1,t_1]\cap[\alpha_2,t_2]=\varnothing$,

$$(t_1-\alpha_1)+(t_2-\alpha_2)\leq b-a,$$

and

$$\begin{aligned}&\bar{u}(t)<0 \quad\text{for}\quad \alpha_1<t<t_1, && \bar{u}\left(t_1\right)=-\bar{m}, && \bar{u}\left(\alpha_1\right)=0,\\ &\bar{u}(t)>0 \quad\text{for}\quad \alpha_2<t<t_2, && \bar{u}\left(t_2\right)=\bar{M}, && \bar{u}\left(\alpha_2\right)=0.\end{aligned} \tag{3.12}$$

By integrating (3.11) from α_1 to t_1 and from α_2 to t_2 and by taking account of (3.12), we obtain

$$\begin{aligned}\bar{m}&\leq\bar{M}\int_{\alpha_1}^{t_1}\bar{\ell}_1(1)(s)ds+\bar{m}\int_{\alpha_1}^{t_1}\bar{\ell}_0(1)(s)ds+\int_{\alpha_1}^{t_1}\bar{q}(s)ds,\\ \bar{M}&\leq\bar{M}\int_{\alpha_2}^{t_2}\bar{\ell}_0(1)(s)ds+\bar{m}\int_{\alpha_2}^{t_2}\bar{\ell}_1(1)(s)ds+\int_{\alpha_2}^{t_2}\bar{q}(s)ds.\end{aligned} \tag{3.13}$$

Note that $\int_{\alpha_i}^{t_i}\bar{q}(s)ds\leq\|q\|_L$ $(i=1,2)$ and there exist nonempty sets $I_k\subset[a,b]$ $(k=1,2)$ such that $I_1\cap I_2=\varnothing$ and $\int_{\alpha_k}^{t_k}\bar{\ell}_n(1)(s)ds=\int_{I_k}\ell_n(1)(s)ds$ $(n=0,1;\ k=1,2)$. Therefore, it follows from (3.13) that $\bar{m}(1-C)\leq\bar{M}A+\|q\|_L$ and $\bar{M}(1-D)\leq\bar{m}B+\|q\|_L$, where $A=\int_{I_1}\ell_1(1)(s)ds$, $B=\int_{I_2}\ell_1(1)(s)ds$, $C=\int_{I_1}\ell_0(1)(s)ds$, and $D=\int_{I_2}\ell_0(1)(s)ds$. Consequently,

$$\begin{aligned}\bar{m}(1-C)(1-D)&\leq A\left(\bar{m}B+\|q\|_L\right)+\|q\|_L(1-D)\leq\bar{m}AB+\|q\|_L(A+1),\\ \bar{M}(1-C)(1-D)&\leq B\left(\bar{M}A+\|q\|_L\right)+\|q\|_L(1-C)\leq\bar{M}AB+\|q\|_L(B+1).\end{aligned}$$

However, since $4AB\leq(A+B)^2\leq\|\ell_1(1)\|_L^2$ and $(1-C)(1-D)\geq 1-\|\ell_0(1)\|_L$, it follows that

$$\bar{m}\leq\left(1+\|\ell_1(1)\|_L\right)\varrho\|q\|_L,\qquad \bar{M}\leq\left(1+\|\ell_1(1)\|_L\right)\varrho\|q\|_L,$$

where $\varrho=\left(1-\|\ell_0(1)\|_L-(1/4)\|\ell_1(1)\|_L^2\right)^{-1}$. Now, by (3.8), the validity of the estimate (3.6) becomes obvious, which completes the proof of the lemma.

Proof of Theorem 3.1. First, we note that, by (2.1), the operator ℓ satisfies condition (3.4), where $\ell^*(t)=\ell_0(1)(t)+\ell_1(1)(t)$.

Let r_0 be the positive constant occurring in Lemma 3.2, and let $r=r_0\|q\|_L$. By Lemma 3.1, to prove the theorem, it suffices to show that, for each $\lambda\in[0,1]$, an arbitrary solution of problem (3.5), (1.2) admits the estimate (3.6).

By condition (3.2), an arbitrary solution of problem (3.5), (1.2) is also a solution of problem (3.7), (1.2) provided that $\lambda\in[0,1]$. On the other hand, by Lemma 3.2, each solution of problem (3.7), (1.2) admits the estimate (3.6). The proof of the theorem is complete.

If $\ell(u)(t)=p(t)u(\tau(t))$ and $F(u)(t)=f(t,u(\mu(t)),u(t))$, then Theorem 3.1 implies Corollary 3.1.

Proof of Theorem 3.2. By (3.3), condition (3.2) with $q(t)=|F(0)(t)|$ is valid. Consequently, by Theorem 3.1, problem (1.1), (1.2) is solvable. It remains to show that it has at most one solution. Let u_1 and u_2 be arbitrary solutions of that problem, and let $u(t)=u_2(t)-u_1(t)$. Then, by condition (3.3), the function u is a solution of problem (3.7), (1.2) with $q(t)\equiv 0$. This, together with Lemma 3.2, implies that $u(t)\equiv 0$, i.e., $u_2(t)\equiv u_1(t)$. The proof of the theorem is complete.

4. ON REMARKS 2.1 AND 3.1

On Remark 2.1

Let $(x_0, y_0) \notin H$. Then, obviously, $(y_0, x_0) \notin H$; i.e., it suffices to consider the case in which $y_0 \geq x_0$. Note also that if, for some $p \in L([a,b];R)$ and $\tau \in \mathscr{M}_{ab}$, the problem

$$u'(t) = p(t)u(\tau(t)), \qquad u(a) = u(b) \tag{4.1}$$

has a nontrivial solution, then there exists a function $g \in L([a,b];R)$ such that problem (1.4), (1.2) has no solution. Accordingly, the functions p and τ in the examples below are constructed so as to ensure that problem (4.1) has a nontrivial solution and

$$J_+ = y_0, \qquad J_- = x_0. \tag{4.2}$$

Example 4.1. Let $x_0 \in [0,1]$ and $y_0 \geq 2 + 2\sqrt{1-x_0}$. We take $k \in [0,1]$ such that $4k/(k+1)^2 = x_0$ and set $a = 0$, $b = 5$, $c = y_0 - 4/(k+1)$, and

$$\tau(t) = \begin{cases} 1 & \text{for} \quad t \in [0,1] \cup [(3-k)/(k+1), 4-k] \cup [5-k, 5] \\ 3 & \text{for} \quad t \in]1, (3-k)/(k+1)[\\ 4-k & \text{for} \quad t \in]4-k, 5-k[\,, \end{cases}$$

$$p(t) = \begin{cases} 1/(k+1) & \text{for} \quad t \in]0,1[\cup [3, 4-k] \cup [5-k, 5] \\ 1/(1-k) & \text{for} \quad t \in]1, (3-k)/(k+1)[\\ -1/(k+1) & \text{for} \quad t \in](3-k)/(k+1), 3[\\ c & \text{for} \quad t \in]4-k, 5-k[\,. \end{cases}$$

One can readily see that $c \geq 0$, relation (4.2) is valid, and the function

$$u(t) = \begin{cases} t+k & \text{if} \quad 0 \leq t < 1 \\ 2+k-t & \text{if} \quad 1 \leq t < 3 \\ t+k-4 & \text{if} \quad 3 \leq t < 4-k \\ 0 & \text{if} \quad 4-k \leq t < 5-k \\ t+k-5 & \text{if} \quad 5-k \leq t \leq 5, \end{cases}$$

is a nontrivial solution of problem (4.1).

Example 4.2. Let $x_0 \geq 1$, $y_0 \geq 1$, $a = 0$, $b = 4$, and

$$\tau(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < 1 \\ 1 & \text{if} \quad 1 \leq t < 3 \\ 4 & \text{if} \quad 3 \leq t \leq 4, \end{cases} \qquad p(t) = \begin{cases} -1 & \text{if} \quad 0 < t < 1 \\ 1-x_0 & \text{if} \quad 1 < t < 2 \\ y_0 - 1 & \text{if} \quad 2 < t < 3 \\ 1 & \text{if} \quad 3 < t < 4. \end{cases}$$

Then relation (4.2) is valid, and the function $u(t) = \begin{cases} 1-t & \text{if} \quad 0 \leq t < 1 \\ 0 & \text{if} \quad 1 \leq t < 3, \\ t-3 & \text{if} \quad 3 \leq t \leq 4 \end{cases}$ is a solution of problem (4.1).

Example 4.3. Let $x_0 \in]0,1[\,$, $x_0 \leq y_0 \leq x_0/(1-x_0)$, $a = 0$, $b = 2$, $t_0 = 1/y_0 - (1-x_0)/x_0$, and

$$\tau(t) = \begin{cases} t_0 & \text{if} \quad 0 \leq t < 1 \\ 1 & \text{if} \quad 1 \leq t \leq 2, \end{cases} \qquad p(t) = \begin{cases} y_0 & \text{if} \quad 0 < t < 1 \\ -x_0 & \text{if} \quad 1 < t < 2. \end{cases}$$

Then $t_0 \in [0,1]$, relation (4.2) is valid, and the function

$$u(t) = \begin{cases} t + (1-x_0)/x_0 & \text{if} \quad 0 \leq t < 1 \\ 2 + (1-x_0)/x_0 - t & \text{if} \quad 1 \leq t \leq 2, \end{cases}$$

is a nontrivial solution of problem (4.1).

On Remark 3.1

Let $a = 0$, $b = 4$, $\varepsilon \in]0,1[$, and

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [0, 2-\varepsilon/2] \cup [3,4] \\ 1 & \text{for} \quad t \in]2-\varepsilon/2, 3[\,, \end{cases}$$

$$p(t) = \begin{cases} -1 & \text{for} \quad t \in]0,1[\cup]2-\varepsilon/2, 3[\cup]4-\varepsilon/2, 4[\\ 0 & \text{for} \quad t \in]1, 2-\varepsilon/2[\cup]3, 4-\varepsilon/2[\,, \end{cases}$$

$$h(t) = \begin{cases} 0 & \text{for} \quad t \in]0,1[\cup]2-\varepsilon/2, 3[\cup]4-\varepsilon/2, 4[\\ 1/(2-t) & \text{for} \quad t \in]1, 2-\varepsilon/2[\\ 1/(4-t) & \text{for} \quad t \in]3, 4-\varepsilon/2[\,. \end{cases}$$

Obviously, $J_- = 2+\varepsilon$, $J_+ = 0$, and the function $u(t) = \begin{cases} t & \text{if} \quad 0 \le t < 1 \\ 2-t & \text{if} \quad 1 \le t < 3 \\ t-4 & \text{if} \quad 3 \le t \le 4 \end{cases}$ is a nontrivial solution of the problem $u'(t) = p(t)u(\tau(t)) - h(t)u(t)$, $u(a) = u(b)$.

Consequently, there exists a function $g \in L(\,]a,b[; R)$ such that the problem

$$u'(t) = p(t)u(\tau(t)) - h(t)u(t) + g(t), \qquad u(a) = u(b),$$

has no solution. In other words, problem (1.1), (1.2) with $\ell(v)(t) \equiv -\ell_1(v)(t) \equiv p(t)v(\tau(t))$, $\ell_0(v)(t) \equiv 0$, and $F(v)(t) \equiv -h(t)v(t) + g(t)$ has no solution even though the operator F satisfies condition (3.2). Consequently, the second inequality in (3.1) cannot be replaced by the inequality $\|\ell_j(1)\|_L \le (2+\varepsilon)\left(1 - \|\ell_i(1)\|_L\right)^{1/2}$ however small $\varepsilon > 0$ is. As to the remaining inequalities in (3.1), Examples 4.2 and 4.3 imply that these inequalities cannot be replaced by nonstrict inequalities.

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REFERENCES

1. Azbelev, N.V., Maksimov, V.P., and Rakhmatullina, L.F., *Vvedenie v teoriyu funktsional'no-differentsial'nykh uravnenii* (Introduction to Theory of Functional-Differential Equations), Moscow, 1991.
2. Gelashvili, Sh. and Kiguradze, I., *Mem. Differential Equations Math. Phys.*, 1995, vol. 5, pp. 1–113.
3. Hale, J. K., *Arch. Rational Mech. Anal.*, 1964, vol. 15, pp. 289–304.
4. Kiguradze, I., *Mem. Differential Equations Math. Phys.*, 1997, vol. 10, pp. 134–137.
5. Kiguradze, I. and Půža, B., *Arch. Math.*, 1997, vol. 33, no. 3, pp. 197–212.
6. Kiguradze, I.T. and Půža, B., *Differents. Uravn.*, 1997, vol. 33, no. 2, pp. 185–194.
7. Kiguradze, I. and Půža, B., *Mem. Differential Equations Math. Phys.*, 1997, vol. 12, pp. 106–113.
8. Kiguradze, I. and Půža, B., *Czechoslovak Math. J.*, 1997, vol. 47, no. 2, pp. 341–373.
9. Kiguradze, I. and Půža, B., *Georgian Math. J.*, 1999, vol. 6, no. 1, pp. 47–66.
10. Mawhin, J., *J. Diff. Equat.*, 1971, vol. 10, pp. 240–261.
11. Půža, B., *Differents. Uravn.*, 1995, vol. 31, no. 11, pp. 1937–1938.
12. Schwabik, Š., Tvrdý, M., and Vejvoda, O., *Differential and Integral Equations: Boundary Value Problems and Adjoints*, Praha, 1979.