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Periodic Problem Involving Quasilinear Differential Operator and Weak Singularity

(Dedicated to Ivan Kiguradze for his seventieth birthday anniversary)

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Abstract

We study the singular periodic boundary value problem of the form

$$\big(|u'|^{p-2}\,u'\big)' = f(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T),$$

where $1 < p < \infty$ and $f \in Car([0,T] \times (0,\infty))$ can have a repulsive space singularity at $x = 0$. Contrary to previous results by Mawhin and Jebelean, Liu Bing and Rachůnková and Tvrdý, we need not assume any strong force conditions. Our main existence results rely on a new antimaximum principle for periodic quasilinear periodic problem, which has an independent meaning.

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1. Introduction

This paper deals with singular periodic problems of the form

$$(\phi_p(u'))' = f(t,u), \tag{1.1}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.2}$$

where

$$0 < T < \infty, \quad 1 < p < \infty, \quad \phi_p(y) = |y|^{p-2}\,y \quad \text{for } \; y \in \mathbb{R} \tag{1.3}$$

and f satisfies the Carathéodory conditions on $[0,T] \times (0,\infty)$, i.e. f has the following properties: (i) for each $x \in (0,\infty)$ the function $f(.,x)$ is measurable on $[0,T]$; (ii) for almost every $t \in [0,T]$ the function $f(t,.)$ is continuous on $(0,\infty)$; (iii) for each compact set $K \subset (0,\infty)$ the function $m_{\rm K}(t) = \sup_{x\in K}\,|f(t,x)|$ is Lebesgue integrable on $[0,T]$.

Second order nonlinear differential equations or systems with singularities appear naturally in the description of particles submitted to Newtonian type forces or to forces caused by compressed gases, see e.g. [13], [16] or [17]. The mathematical interest in periodic singular problems increased considerably when the paper [23] by Lazer and Solimini appeared in 1987. Motivated by the model equation $u'' = a\,u^{-\alpha} + e(t)$ with $\alpha > 0$, $a \neq 0$ and e integrable on $[0,T]$, they investigated the existence of positive solutions to the Duffing equation $u'' = g(u) + e(t)$ using topological arguments and the lower and upper functions method. The restoring force g was allowed to have an attractive singularity or a strong repulsive singularity at origin. The results by Lazer and Solimini have been generalized or extended e.g. by Habets and Sanchez [19], Mawhin [28], del Pino, Manásevich and Montero [11], Omari and Ye [30], Zhang [43] and [45], Ge and Mawhin [18], Rachůnková and Tvrdý [33] or Rachůnková, Tvrdý and Vrkoč [38]. All of these papers, when dealing with the repulsive singularity, supposed that the strong force condition is satisfied. For the case of the weak singularity, first results were delivered by Rachůnková, Tvrdý and Vrkoč in [37]. Further results were delivered later also by Bonheure and De Coster [2] and Torres [40].

Regular periodic problems with ϕ– or p–Laplacian on the left hand side were considered by several authors, see e.g. del Pino, Manásevich and Murúa [12] or Yan [42]. General existence principles for the regular vector problem, based on the homotopy to the averaged nonlinearity, were presented by Manásevich and Mawhin in [26] (see also Mawhin [29]).

In the well-ordered case, the lower/upper functions method was extended to periodic problems with a ϕ–Laplacian operator on the left hand side by Cabada and Pouso in [7], Jiang and Wang in [22] and Staněk in [39]. The general existence principle valid also when lower/upper functions are non-ordered was given by Rachůnková and Tvrdý in [35] and, for the case when impulses are admitted, also in [34].

The singular periodic problem for the Liénard type equation

$$\left(|u'|^{p-2}\,u'\right)' + h(u)\,u' = g(u) + e(t) \tag{1.4}$$

with g having either an attractive singularity or a strong repulsive singularity at $x = 0$ was treated by Liu [25], Jebelean and Mawhin [20] and [21] and Rachůnková and Tvrdý [36]. Let us recall that a function g is said to have an *attractive* singularity at $x = 0$ if

$$\liminf_{x\to 0+}\; g(x) = -\infty.$$

Alternatively, we say that g has *a repulsive singularity* at the origin if

$$\limsup_{x\to 0+} g(x) = +\infty \tag{1.5}$$

and g has a *strong repulsive singularity* at the origin if

$$\lim_{x\to 0+} \int_x^1 g(s)\,\mathrm{d}s = +\infty. \tag{1.6}$$

For a more detailed survey of recent developments we refer to [32, Section 5].

The main goal of this paper is a new existence result, Theorem 4.4, for problem (1.1), (1.2). As in [37, Theorem 2.5] (see also [32, Theorem 5.26]), where the classical case $p = 2$ was treated, we need not assume that f satisfies any strong force condition. Our main tools are the lower and upper functions method and a generalization of a classical antimaximum principle to the quasilinear periodic problem

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = e(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{1.7}$$

established below in Theorem 3.2.

Our main result applies, in particular, to the Duffing type model problem

$$(\phi_p(u'))' + \left(\tfrac{\pi_p}{T}\right)^p u^{p-1} = a\,u^{-\alpha} + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{1.8}$$

where $1<p\leq 2,\ a>0,\ \alpha>0,\ e\in L_1[0,T]$, and $\lambda = \left(\frac{\pi_p}{T}\right)^p$ is the first eigenvalue of a homogeneous Dirichlet problem related to (1.7). In particular, we get that problem (1.8) has a positive solution if $\inf \operatorname{ess}\{e(t)\colon t\in[0,T]\} > 0$. It is worth mentioning that for $\alpha\in(0,1)$ the function $g(x) = a\,x^{-\alpha}$ does not satisfy the strong force condition (1.6).

2 Preliminaries

As usual, for an arbitrary subinterval I of $\mathbb{R}$ we denote by $C(I)$ the set of functions $x : I \to \mathbb{R}$ which are continuous on I; $C^1[0,T]$ stands for the set of functions $x \in C[0,T]$ with the first derivative continuous on $[0,T]$. Further, $L_1[0,T]$ is the set of functions $x : [0,T] \to \mathbb{R}$ which are measurable and Lebesgue integrable on $[0,T]$. $AC[0,T]$ is the set of functions absolutely continuous on $[0,T]$ and $AC_{loc}[0,T]$ is the set of functions absolutely continuous on each compact interval $I \subset [0,T]$. If $f : [0,T] \times (0,\infty) \to \mathbb{R}$ satisfies the Carathéodory conditions on $[0,T] \times (0,\infty)$ we write

$$f \in Car([0,T] \times (0,\infty)). \tag{2.1}$$

For $x \in L_1[0,T]$ we put

$$\|x\|_\infty = \sup_{t\in[0,T]} \operatorname{ess} |x(t)| \quad \text{and} \quad \overline{x} = \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s.$$

Definition 2.1 A function $u : [0,T] \to \mathbb{R}$ is a *solution* to problem (1.1), (1.2) if $\phi_p(u') \in AC[0,T]$, $u > 0$ on $[0,T]$, $(\phi_p(u'(t)))' = f(t,u(t))$ for a.e. $t \in [0,T]$, $u(0) = u(T)$ and $u'(0) = u'(T)$.

Notice that the requirement $\phi(u') \in AC[0,T]$ implies that $u \in C^1[0,T]$.

The singular problem (1.1), (1.2) will also be investigated through regular auxiliary problems of the form

$$(\phi_p(u'))' = \widetilde{f}(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{2.2}$$

or

$$(\phi_p(u'))' = \widetilde{f}(t,u), \quad u(a) = u(b) = 0, \tag{2.3}$$

where $\widetilde{f} \in Car([0,T] \times \mathbb{R})$ and $a,\, b \in \mathbb{R}$, $a < b$. As usual, by a *solution of problem* (2.2) we understand a function u such that $\phi_p(u') \in AC[0,T]$, (1.2) is true and $(\phi_p(u'(t)))' = \widetilde{f}(t,u(t))$ for a.e. $t \in [0,T]$. Analogously, u is a solution to (2.3) if $\phi_p(u') \in AC[a,b]$, $u(a) = u(b) = 0$ and $(\phi_p(u'(t)))' = \widetilde{f}(t,u(t))$ for a.e. $t \in [a,b]$.

The lower and upper functions method combined with the topological degree argument is an important tool for proofs of solvability of boundary value problems. For our purposes, the definitions of lower and upper functions associated with problems (2.2) or (2.3) given below are suitable. First, let us introduce the following notation.

Notation 2.2 For a given interval $[a,b] \subset \mathbb{R}$, we denote by $W[a,b]$ the set of functions $u \in C[a,b]$ for which there is an at most finite set $D_u \subset (a,b)$ such that $\phi(u') \in AC_{loc}([0,T] \setminus D_u)$ and, moreover, for all $t \in [a,b)$ and $s \in (a,b]$, the limits

$$u'(t+) := \lim_{\tau\to t+} u'(\tau) \quad \text{and} \quad u'(s-) := \lim_{\tau\to s-} u'(\tau)$$

are defined and finite and

$$u'(t+) > u'(t-) \quad \text{for all } t \in D_u. \tag{2.4}$$

Definition 2.3 We say that a function $\sigma \in C[0,T]$ is a *lower function* of problem (2.2) if $\sigma \in W[0,T]$ and

$$(\phi(\sigma'(t)))' \geq \widetilde{f}(t,\sigma(t)) \text{ for a.e. } t \in [0,T], \quad \sigma(0) = \sigma(T),\ \sigma'(0) \geq \sigma'(T). \tag{2.5}$$

If $\sigma \in C[0,T]$ is such that $-u \in W[0,T]$ and relations (2.5) hold with the reversed inequalities, we say that σ is *an upper function* of (2.2).

Analogously, $\sigma \in C[a,b]$ is a *lower function* of (2.3) if $u \in W[a,b]$ and

$$(\phi(\sigma'(t)))' \geq \widetilde{f}(t,\sigma(t)) \text{ for a.e. } t \in [a,b], \quad \sigma(a) \leq 0,\ \sigma(b) \leq 0, \tag{2.6}$$

hold. If $-u \in W[a,b]$ and the inequalities in (2.6) are reversed, then σ is called an *upper function* of (2.2) or of (2.3), respectively.

The next two assertions based on the lower and upper functions method will be useful for our purposes.

Proposition 2.4 ([32, Lemma 5.9]) *Assume* (1.3) *and* $\widetilde{f} \in Car([0,T] \times \mathbb{R})$. *Furthermore, let* σ_1 *and* σ_2 *be a lower and an upper function of* (2.2) *and let there be* $m \in L_1[0,T]$ *such that*

$$\widetilde{f}(t,x) > m(t) \ (\textit{or } \widetilde{f}(t,x) < m(t)) \quad \textit{for a.e. } t \in [0,T] \quad \textit{and all} \quad x \in \mathbb{R}.$$

Then problem (2.2) *has a solution* u *such that*

$$\min\{\sigma_1(\tau_u), \sigma_2(\tau_u)\} \leq u(\tau_u) \leq \max\{\sigma_1(\tau_u), \sigma_2(\tau_u)\} \quad \textit{for some } \tau_u \in [0,T].$$

Proposition 2.5 ([6, Theorem 3.5]) *Assume* (1.3), $\widetilde{f} \in Car([0,T] \times \mathbb{R})$ *and let* $a, b \in \mathbb{R}$, $a < b$ *be given. Furthermore, let* σ_1 *and* σ_2 *be a lower and an upper function of* (2.3) *such that* $\sigma_1 \leq \sigma_2$ *on* $[a,b]$. *Then problem* (2.3) *has a solution* u *such that* $\sigma_1 \leq u \leq \sigma_2$ *on* $[a,b]$.

3 Sign properties of quasilinear periodic problems

First, let us recall some basic known facts concerning initial value problems of the form

$$(\phi_p(u'))' + \lambda\, \phi_p(u) = 0, \tag{3.1}$$

$$u(t_0) = 0, \quad u'(t_0) = d, \tag{3.2}$$

where $1 < p < \infty$, $t_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $d \in \mathbb{R}$. As in [9] (see also e.g. [1], [10], [14], [15], [44], [46], [27]), let us put

$$\pi_p = 2\,(p-1)^{1/p} \int_0^1 (1-s^p)^{-1/p}\, \mathrm{d}s.$$

Clearly, $\pi_2 = \pi$. Furthermore, it is known that

$$\pi_p = 2\,(p-1)^{\frac{1}{p}}\, \frac{(\pi/p)}{\sin(\pi/p)}\,. \tag{3.3}$$

(See [15, Sec. 1.1.2], but take into account that our definition differs slightly from that used in [15], where $\pi_p = 2\int_0^1 (1-s^p)^{-1/p}\,\mathrm{d}s$.) It is known (see [15, Theorem 1.1.1]) that for each $t_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $d \in \mathbb{R}$ problem (3.1), (3.2) has a unique solution u on $\mathbb{R}$ which can be, by [9, sec. 3]), expressed as

$$u(t) = d\,\lambda^{-1/p}\,\sin_p(\lambda^{1/p}\,(t-t_0)) \quad \text{for } t \in \mathbb{R},$$

where the function $\sin_p : \mathbb{R} \to [-(p-1)^{1/p}, (p-1)^{1/p}]$ is defined as follows.

Let $w\colon [0, \pi_p/2] \to [0, (p-1)^{1/p}]$ be the inverse function to

$$x \in [0, (p-1)^{1/p}] \to \int_0^x \frac{\mathrm{d}s}{(1-\frac{s^p}{p-1})^{1/p}} \in [0, \pi_p/2].$$

Further, put $\widetilde{w}(t) = w(\pi_p - t)$ for $t \in [\pi_p/2, \pi_p]$ and then $\widetilde{w}(t) = -\widetilde{w}(-t)$ for $t \in [-\pi_p, 0]$. Finally, define $\sin_p : \mathbb{R} \to \mathbb{R}$ as the $2\,\pi_p$–periodic extension of $\widetilde{w}$ to the whole $\mathbb{R}$. In particular, if $d = 0$, then $u \equiv 0$ on $\mathbb{R}$. Obviously, we have

$$\begin{aligned} &\sin_p(t) = 0 && \text{if and only if } t = n\,\pi_p,\ n \in \mathbb{N}\cup\{0\},\\ &\sin_p(t) = (p-1)^{1/p} && \text{if and only if } t = (2\,n+1)\,\frac{\pi_p}{2},\ n \in \mathbb{N}\cup\{0\}, \end{aligned}$$

and

$$\sin_p(t) > 0 \qquad \text{if and only if } t \in (2\,n\,\pi_p, (2\,n+1)\,\pi_p),\ n \in \mathbb{N}\cup\{0\}.$$

As a corollary, we immediately obtain that for given $a,\ b \in \mathbb{R},\ a < b$, the corresponding Dirichlet problem

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = 0, \qquad u(a) = u(b) = 0 \tag{3.4}$$

possesses a nontrivial solution, i.e. λ is an eigenvalue for (3.4) if and only if

$$\lambda \in \left\{ \left(\tfrac{n\,\pi_p}{b-a}\right)^p : n \in \mathbb{N}\cup\{0\} \right\}. \tag{3.5}$$

In particular, $\lambda = \left(\frac{\pi_p}{T}\right)^p$ is the least eigenvalue for (3.4) with $b - a = T$, wherefrom the following assertion follows.

Proposition 3.1 *Let $1 < p < \infty$, $a, b \in \mathbb{R}$, $a < b$, and let $\lambda = \left(\frac{\pi_p}{T}\right)^p$, where π_p is given by* (3.3). *Then problem* (3.4) *has a nontrivial solution if and only if $b - a = n\,T$ for some positive integer n.*

It is easy to check that the function

$$G(t,s) = \frac{T}{2\,\pi}\,\sin\left(\tfrac{\pi}{T}\,|t-s|\right), \quad t, s \in [0,T],$$

is the Green function for $v'' + \left(\frac{\pi}{T}\right)^2 v = 0$, $v(0) = v(T)$, $v'(0) = v'(T)$, and $G(t,s)$ is nonnegative on $[0,T]\times[0,T]$. More generally, for the linear periodic problem the following *antimaximum* principle is valid:

Let $\mu \in L_1[0,T]$ be such that $0 \le \mu(t) \le \left(\frac{\pi}{T}\right)^2$ for a.e. $t \in [0,T]$ and $\overline{\mu} > 0$ and let $v \in AC^1[0,T]$ satisfy the periodic conditions (1.2) *and*

$$v''(t) + \mu(t)\, v(t) \ge 0 \quad \textit{for a.e. } t \in [0,T].$$

Then v is nonnegative on $[0,T]$.

Next, we will show that for the quasilinear periodic problem (1.7) an analogous assertion holds although, in general, no tools like the Green function are available.

Theorem 3.2 *Let $1 < p < \infty$ and let $\mu \in L_1[0,T]$ be such that*

$$\overline{\mu} > 0 \quad \textit{and} \quad 0 \le \mu(t) \le \left(\tfrac{\pi_p}{T}\right)^p \quad \textit{for a.e. } t \in [0,T], \tag{3.6}$$

where π_p is given by (3.3)*. Then $v \ge 0$ on $[0,T]$ holds for each $v \in W[0,T]$ (see* Notation 2.2*) such that*

$$(\phi_p(v'(t)))' + \mu(t)\, \phi_p(v(t)) \ge 0 \qquad \textit{for a.e. } t \in [0,T], \tag{3.7}$$

$$v(0) = v(T), \quad v'(0) \ge v'(T). \tag{3.8}$$

Proof. Let $v \in W[0,T]$ and $D = \{t_1, t_2, \ldots, t_m\} \subset (0,T)$ be such that $\phi_p(v') \in AC([0,T] \setminus D)$ and (3.6) and (3.7) hold. Put $t_0 = 0$ and $t_{m+1} = T$. Without any loss of generality we may assume that v does not vanish on $[0,T]$.

Step 1. First, we show that

$$\max\{v(t) \colon t \in [0,T]\} > 0. \tag{3.9}$$

Assuming, on the contrary, that $v \le 0$ on $[0,T]$, we get by (3.7)

$$(\phi_p(v'(t)))' \ge -\mu(t)\, \phi_p(v(t)) \ge 0 \qquad \text{for a.e. } t \in [0,T].$$

This, together with condition (2.4), means that v' is nondecreasing on $[0,T]$. Therefore, v may satisfy boundary conditions (3.8) if and only if $v = v(0) \le 0$ on $[0,T]$, i.e. the previous relation reduces to

$$-\mu(t)\, (-v(0))^{p-1} \ge 0 \qquad \text{for a.e. } t \in [0,T].$$

Since $\mu \ge 0$ a.e. on $[0,T]$ and $\overline{\mu} > 0$, this is possible if and only if $v(0) = 0$, i.e. $v = 0$ on $[0,T]$, which contradicts our assumption that v does not vanish on $[0,T]$.

Step 2. We show that $\min\{v(t) \colon t \in [0,T]\} \ge 0$. Assume on the contrary that

$$\min\{v(t) \colon t \in [0,T]\} < 0. \tag{3.10}$$

Let us extend v and μ to $T-$periodic functions on $\mathbb{R}$ and denote these extensions by the same symbols. Then μ is locally integrable on $\mathbb{R}$ and $v \in W(I)$ for each compact interval $I \subset \mathbb{R}$. Furthermore, inequality (3.7) holds for a.e. $t \in [0,T]$.

Let $t^* \in [0,T]$ be such that

$$v(t^*) = \max\{v(t) \colon t \in [0,T]\}.$$

By (3.8) we may assume that $t^* \in [0,T)$. Furthermore, by *Step* 1, $v(t^*) > 0$ and there are $a \in [-T, t^*)$ and $b \in (t^*, 2T]$ such that $v(a) = v(b) = 0$, $v > 0$ on (a,b). Due to (3.10), we have

$$0 < b - a < T. \tag{3.11}$$

Furthermore, using (3.6) and (3.7), we get

$$(\phi_p(v'(t)))' + \left(\tfrac{\pi_p}{T}\right)^p \phi_p(v(t)) \geq (\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) \geq 0 \text{ for a.e. } t \in [a,b]. \tag{3.12}$$

Now, put

$$a_0 = a - \frac{T - b + a}{2}, \quad b_0 = a_0 + T$$

and

$$\sigma_2(t) = d\left(\tfrac{T}{\pi_p}\right) \sin_p\left(\left(\tfrac{\pi_p}{T}\right)(t - a_0)\right) \quad \text{for } t \in \mathbb{R}$$

with $d > 0$ large enough so that $\sigma_2(t) > v(t) \geq 0$ on $[a,b]$. We have

$$(\phi_p(\sigma_2'(t)))' + \left(\tfrac{\pi_p}{T}\right)^p \phi_p(\sigma_2(t)) = 0 \text{ for a.e. } t \in [a,b]. \tag{3.13}$$

Thus, σ_2 is an upper function for (3.4). Moreover, since $v \in W[a,b]$, $v(a) = v(b)$ and (3.12) holds for a.e. $t \in [a,b]$, the function $\sigma_1 = v$ is a lower function for (3.4). Hence, by Proposition 2.5, where we put $\widetilde{f}(t,x) = -\left(\tfrac{\pi_p}{T}\right)^p \phi_p(x)$ for t, $x \in \mathbb{R}$, there exists a nontrivial solution u to (3.4). By (3.11), this contradicts Proposition 3.1. ∎

Example 3.3 The previous comparison result is optimal in the sense that for any $\lambda > \left(\tfrac{\pi_p}{T}\right)^p$ we can find $v \in W[0,T]$ which changes sign on $[0,T]$ and satisfies (3.7) (with $\mu(t) \equiv \lambda$) and (3.8).

To see this, denote $\mu = \left(\tfrac{\pi_p}{T}\right)^p$ and consider two cases:

$$\text{(i)} \quad \lambda \in (\mu, 2^p\mu) \qquad \text{and} \qquad \text{(ii)} \quad \lambda \geq 2^p\mu.$$

Case (i). Let $\lambda \in (\mu, 2^p\mu)$. Then we have $\lambda^{1/p} - \mu^{1/p} < \mu^{1/p}$. Let us define

$$t_0 = \frac{T\,\left(\lambda^{1/p} - \mu^{1/p}\right)}{2\,\lambda^{1/p}} \quad \text{and} \quad \bar{\lambda} = \lambda\left(\frac{\mu^{1/p}}{\lambda^{1/p} - \mu^{1/p}}\right)^p.$$

Then $0 < t_0 < \frac{T}{2}$, $\mu < \lambda < \bar{\lambda}$, $\lambda^{1/p}\,(T - 2\,t_0) = \pi_p$ and $\bar{\lambda}^{1/p}\,t_0 = \frac{\pi_p}{2}$. Let us consider the function

$$v(t) = \begin{cases} \bar{\lambda}^{-1/p} \sin_p\left(\bar{\lambda}^{1/p}\,(t - t_0)\right) & \text{if } \ t \in [0, t_0), \\ \lambda^{-1/p} \sin_p\left(\lambda^{1/p}\,(t - t_0)\right) & \text{if } \ t \in [t_0, T - t_0], \\ \bar{\lambda}^{-1/p} \sin_p\left(\bar{\lambda}^{1/p}\,(T - t - t_0)\right) & \text{if } \ t \in (T - t_0, T]. \end{cases}$$

We can see from the definition of v that $v \in W[0,T]$ (with $v \in C^1([0,T] \setminus \{t_0\})$), $v(t_0) = v(T - t_0) = 0$ and

$$v < 0 \quad \text{on} \quad [0, t_0) \cup (T - t_0, T] \quad \text{and} \quad v > 0 \quad \text{on} \quad (t_0, T - t_0),$$

i.e. v changes its sign on $[0,T]$. Having in mind the definitions of $\bar\lambda$ and v, we can verify that (3.7) is true for $\mu(t) \equiv \lambda$. Moreover,

$$v(0) = v(T) = -\left(\frac{p-1}{\bar\lambda}\right)^{1/p} < 0, \;\; v'(0) = v'(T) = 0 \text{ and } \;v'(t_0-) = v'(t_0+) = 1.$$

In particular, (3.8) holds, i.e. v has the desired properties.

Case (ii). Let $\lambda \geq 2^p \mu$. Put $t_1 = \left(\frac{T\left(\lambda^{1/p} - 2\mu^{1/p}\right)}{2\,\lambda^{1/p}}\right)$. Then $0 \leq t_1 < t_0 < \frac{T}{2}$ and we can define the function v by

$$v(t) = \begin{cases} -((p-1)/\lambda)^{1/p} & \text{if} \quad t \in [0, t_1), \\ \lambda^{-1/p} \sin_p\left(\lambda^{1/p}\,(t - t_0)\right) & \text{if} \quad t \in [t_1, T - t_1], \\ -((p-1)/\lambda)^{1/p} & \text{if} \quad t \in (T - t_1, T]. \end{cases}$$

Similarly to *Case* (i), $v \in W[0,T]$. Furthermore,

$$v > 0 \quad \text{on} \quad [0, t_0) \cup (T - t_0, T] \quad \text{and} \quad v < 0 \quad \text{on} \quad (t_0, T - t_0).$$

Therefore v changes its sign on $[0,T]$ and, similarly to *Case* (i), we can verify that v satisfies relations (3.7) (with $\mu(t) \equiv \lambda$) and (3.8).

4 Main results

First, let us recall the following a priori estimate. Its proof is an easy modification of the proof of [35, Lemma 2.4] (see also [32, Lemma 5.8]), where strict inequalities occur in place of non-strict ones.

Lemma 4.1 *Let $1 < p < \infty$ and $\psi \in L_1[0,T]$. Then*

$$\|v'\|_\infty \leq \phi_p^{-1}(\|\psi\|_1) \tag{4.1}$$

holds for each $v \in C^1[0,T]$ fulfilling $\phi_p(v') \in AC[0,T]$, $v(0) = v(T)$, $v'(0) = v'(T)$ and $(\phi_p(v'(t)))' \geq \psi(t)$ (or $(\phi_p(v'(t)))' \leq \psi(t)$) for a.e. $t \in [0,T]$.

Next, we prove an existence principle which relies on the comparison of the given problem (1.1), (1.2) with the related quasilinear problem fulfilling the antimaximum principle.

Theorem 4.2 *Assume* (1.3) *and* (2.1) *and* $2 \leq p < \infty$. *Furthermore, let $r > 0$, $A \geq r$, $\mu, \beta \in L_1[0,T]$ be such that $\mu(t) \geq 0$ a.e. on $[0,T]$, $\overline{\mu} > 0$,*

$$\overline{\beta} \leq 0 \quad \text{and} \quad f(t,x) \leq \beta(t) \qquad \text{for a.e. } t \in [0,T] \text{ and all } x \in [A,B] \tag{4.2}$$

and

$$f(t,x) + \mu(t)\,\phi_p(x-r) \geq 0 \qquad \textit{for a.e. } t \in [0,T] \quad \textit{and all} \quad x \in [r,B], \tag{4.3}$$

where

$$B - A \geq \frac{T}{2}\,\phi_p^{-1}(\|m\|_1),$$
$$m(t) = \max\big\{\sup\{f(t,x) \colon x \in [r,A]\}, \beta(t), 0\big\} \quad \textit{for a.e. } t \in [0,T] \tag{4.4}$$

and

$$\begin{cases} v \geq 0 \quad \textit{on} \quad [0,T] \ \textit{holds for each} \ \ v \in C^1[0,T] \ \textit{such that} \\ \quad \phi_p(v') \in AC[0,T], \\ \quad (\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) \geq 0 \ \ \textit{for a.e. } t \in [0,T], \\ \quad v(0) = v(T), \quad v'(0) = v'(T). \end{cases} \tag{4.5}$$

Then problem (1.1), (1.2) *has a solution* u *such that*

$$r \leq u \leq B \quad \textit{on} \quad [0,T] \quad \textit{and} \quad \|u'\|_\infty < \phi_p^{-1}(\|m\|_1). \tag{4.6}$$

Proof.

Part I. First, assume that $\overline{\beta} < 0$.

Step 1. Put

$$\widetilde{f}(t,x) = \begin{cases} f(t,r) - \mu(t)\,\phi_p(x-r) & \text{if } \ x \leq r, \\ f(t,x) & \text{if } \ x \in [r,B], \\ f(t,B) & \text{if } \ x \geq B \end{cases} \tag{4.7}$$

and consider problem (2.2). We have $\widetilde{f} \in Car([0,T]\times\mathbb{R})$. Furthermore, by (4.2)–(4.7), the inequalities

$$\widetilde{f}(t,x) \leq \beta(t) \quad \text{if } \ x \in [A,\infty) \tag{4.8}$$

and

$$\widetilde{f}(t,x) + \mu(t)\,\phi_p(x-r) \geq 0 \quad \text{for all } \ x \in \mathbb{R} \tag{4.9}$$

are valid for a.e. $t \in [0,T]$. In particular, in view of (4.7) we have

$$\widetilde{f}(t,x) \geq h(t) := -\mu(t)\,\phi_p(B-r) \quad \text{for a.e. } \ t \in [0,T] \text{ and all } \ x \in \mathbb{R}, \tag{4.10}$$

with $h \in L_1[0,T]$.

By (4.9), $\sigma_2 \equiv r$ is an upper function of (2.2). Further, if $b = \beta - \overline{\beta}$, then $b \in L_1[0,T]$ and $\overline{b} = 0$ and it is easy to see that there is a uniquely defined $\sigma_0 \in C^1[0,T]$ such that $\phi_p(\sigma_0') \in AC[0,T]$,

$$(\phi_p(\sigma_0'(t)))' = b(t) \quad \text{for a.e. } \ t \in [0,T] \quad \text{and} \quad \sigma_0(0) = \sigma_0(T) = 0.$$

Now, let us choose $c^* > 0$ such that $c^* + \sigma_0 \ge A$ on $[0,T]$ and define $\sigma_1 = c^* + \sigma_0$. By (4.8) we have

$$\sigma_1(0) = \sigma_1(T) = c^*,$$

$$(\phi_p(\sigma_1'(t)))' = b(t) = \beta(t) - \overline{\beta} > \beta(t) \ge \widetilde{f}(t,\sigma_1(t)) \ \text{ for a.e. } t \in [0,T]$$

and

$$\phi_p(\sigma_0'(T)) - \phi_p(\sigma_0'(0)) = T\,\overline{b} = 0.$$

Consequently, σ_1 is a lower function of (2.2). Therefore, by (4.10) and Proposition 2.4, the regular problem (2.2) has a solution u such that $u(t_u) \ge r$ for some $t_u \in [0,T]$.

Step 2. We show that

$$u(t) \ge r \quad \text{for } \ t \in [0,T]. \tag{4.11}$$

To this aim, set $v = u - r$. By virtue of (4.9), we have

$$(\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) = \widetilde{f}(t,u(t)) + \mu(t)\,\phi_p(u(t)-r) \ge 0$$

for a.e. $t \in [0,T]$. By (4.5) it follows that $v(t) \ge 0$ on $[0,T]$, i.e. (4.11) is true.

Step 3. We show that

$$u(t) < B \quad \text{for } \ t \in [0,T]. \tag{4.12}$$

Indeed, by the definition of m and by (4.7) and (4.8) we have

$$\widetilde{f}(t,x) \le m(t) \quad \text{for a.e. } t \in [0,T] \ \text{ and all } \ x \ge r.$$

Hence, we can use Lemma 4.1 to get the estimate

$$\|u'\|_\infty \le \phi_p^{-1}(\|m\|_1). \tag{4.13}$$

If $u \ge A$ were valid on $[0,T]$, then taking into account the periodicity of u' and (4.8), we would get

$$0 = \int_0^T \widetilde{f}(t,u(t))\,\mathrm{d}t \le \int_0^T \beta(t)\,\mathrm{d}t = T\,\overline{\beta} < 0,$$

a contradiction. Hence,

$$\min\{u(s) \colon s \in [0,T]\} < A.$$

Now, assume that

$$u^* := \max\{u(s) \colon s \in [0,T]\} > A$$

and extend u to be $T-$periodic on $\mathbb{R}$. There are s_1, s_2 and $s^* \in \mathbb{R}$ such that

$$s_1 < s^* < s_2, \quad s_2 - s_1 < T, \quad u(s_1) = u(s_2) = A \quad \text{and} \quad u(s^*) = u^* > A.$$

In particular, due to (4.13),

$$2\,(u(s^*) - A) = \int_{s_1}^{s^*} u'(s)\,\mathrm{d}s + \int_{s_2}^{s^*} u'(s)\,\mathrm{d}s \le T\,\phi_p^{-1}(\|m\|_1),$$

wherefrom the estimate

$$u(t) - A \le \frac{T}{2}\,\phi_p^{-1}(\|m\|_1) \le B - A \ \text{ on } \ [0,T]$$

follows. Thus, (4.12) is true.

Step 4. Estimates (4.11) and (4.12) mean that $r \le u \le B$ holds on $[0,T]$. In view of (4.7), we conclude that u is a solution to (1.1), (1.2).

Part II. Now, let $\overline{\beta} = 0$. Put $n_0 = \max\{\frac{1}{r}, \frac{1}{B-A}, 3\}$. For an arbitrary $n \in \mathbb{N}$, define

$$\widetilde{f}_n(t,x) = \begin{cases} f(t,r) & \text{if } x \le r, \\ f(t,x) & \text{if } x \in [r,A], \\ f(t,x) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{x-A}{x-A+1}\right) & \text{if } x \in (A,B], \\ f(t,B) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{B-A}{B-A+1}\right) & \text{if } x \ge B. \end{cases} \tag{4.14}$$

Taking into account (4.2), we get

$$\begin{aligned} \widetilde{f}_n(t,x) &= f(t,x) - \mu(t)\,\phi_p\left(\tfrac{1}{n}\,\tfrac{x-A}{x-A+1}\right) \le \beta(t) - \mu(t)\,\phi_p\left(\tfrac{1}{n}\,\tfrac{x-A}{x-A+1}\right) \\ &\le \beta(t) - \mu(t)\,\phi_p(\tfrac{1}{2n^2}) \qquad \text{if } \ x \in [A + \tfrac{1}{n}, B] \end{aligned}$$

and

$$\widetilde{f}_n(t,x) = f(t,B) - \mu(t)\,\phi_p\left(\tfrac{1}{n}\,\tfrac{B-A}{B-A+1}\right) \le \beta(t) - \mu(t)\,\phi_p(\tfrac{1}{2n^2}) \quad \text{if } \ x \ge B$$

for a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$ such that $n \ge n_0$. Thus,

$$\begin{cases} \widetilde{f}_n(t,x) \le \beta_n(t) := \beta(t) - \mu(t)\,\phi_p(\frac{1}{2n^2}) \\ \qquad \text{for } \ x \ge A + \frac{1}{n}\,, \ \text{ for a.e. } \ t \in [0,T] \quad \text{and all } \ n \ge n_0. \end{cases} \tag{4.15}$$

Clearly,

$$\overline{\beta_n} < 0 \quad \text{and} \quad \beta_n(t) \le \beta(t) \ \text{ for a.e. } t \in [0,T]. \tag{4.16}$$

Furthermore, by (4.3) and (4.14), we have

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \tfrac{1}{n})\right) \ge f(t,r) \ge 0 \quad \text{if } \ x \in [r - \tfrac{1}{n}, r],$$

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \tfrac{1}{n})\right) \ge f(t,x) + \mu(t)\,\phi_p\,(x - r) \ge 0 \quad \text{if } \ x \in [r, A],$$

and, taking into account that $\xi^{p-1}+\eta^{p-1}\le(\xi+\eta)^{p-1}$ holds for all $\xi,\ \eta\ge 0$ and each $p\ge 2$,

$$\begin{aligned}
&\widetilde{f}_n(t,x)+\mu(t)\,\phi_p\left(x-(r-\tfrac{1}{n})\right)\\
&\quad=f(t,x)-\mu(t)\,\phi_p\left(\tfrac{1}{n}\,\tfrac{x-A}{x-A+1}\right)+\mu(t)\,\phi_p\left(x-r+\tfrac{1}{n}\right)\\
&\qquad\ge f(t,x)+\mu(t)\,\phi_p\left(x-r\right)\ge 0\quad\text{if }\ x\in[A,B]
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{f}_n(t,x)+\mu(t)\,\phi_p\left(x-(r-\tfrac{1}{n})\right)\\
&\quad=f(t,B)-\mu(t)\,\phi_p\left(\tfrac{1}{n}\,\tfrac{B-A}{B-A+1}\right)+\mu(t)\,\phi_p\left(x-r+\tfrac{1}{n}\right)\\
&\qquad\ge f(t,B)+\mu(t)\,\phi_p\left(B-r\right)\ge 0\quad\text{if }\ x\ge B.
\end{aligned}$$

To summarize,

$$\widetilde{f}_n(t,x)+\mu(t)\,\phi_p\left(x-(r-\tfrac{1}{n})\right)\ge 0\quad\text{for all }\ x\ge r-\tfrac{1}{n}. \tag{4.17}$$

For a.e. $t\in[0,T]$ and all $n\in\mathbb{N}$, put

$$\widetilde{m}_n(t):=\max\left\{\,\sup\{\widetilde{f}_n(t,x)\colon x\in[r-\tfrac{1}{n},A+\tfrac{1}{n}]\},b_n(t),0\right\}.$$

In view of (4.4), (4.14) and (4.16), we have

$$0\le\widetilde{m}_n(t)\le m(t)\qquad\text{for a.e. }t\in[0,T]\quad\text{and}\quad n\ge n_0.$$

This together with (4.15)-(4.17) means that, for each $n\in\mathbb{N}$ large enough, *Part* I of this proof ensures the existence of a solution u_n to the auxiliary problem

$$(\phi_p(u_n'))'=\widetilde{f}_n(t,u_n),\quad u_n(0)=u_n(T),\quad u_n'(0)=u_n'(T)$$

which satisfies the estimates

$$r-\tfrac{1}{n}\le u_n(t)\le B+\tfrac{1}{n}\quad\text{on }\ [0,T]\quad\text{and}\quad\|u_n'\|_\infty\le\phi_p^{-1}(\|m\|_1).$$

Now, notice that

$$|\widetilde{f}_n(t,x)-\widetilde{f}(t,x)|\le\mu(t)\,\phi_p\left(\tfrac{1}{n}\right)\qquad\text{for a.e. }\ t\in[0,T],\ \text{all }\ x\in\mathbb{R}\text{ and all }\ n\in\mathbb{N},$$

where

$$\widetilde{f}(t,x)=\begin{cases}f(t,r)&\text{if }\ x\le r,\\ f(t,x)&\text{if }\ x\in[r,B],\\ f(t,B)&\text{if }\ x\ge B.\end{cases}$$

Thus, in a standard way (using the Arzelà-Ascoli and the Lebesgue Dominated Convergence Theorem) we can show that the sequence $\{u_n\}_{n=1}^\infty$ contains a subsequence which converges in $C^1[0,T]$ to a solution u of the problem

$$(\phi_p(u'))'=\widetilde{f}(t,u),\quad u(0)=u(T),\quad u'(0)=u'(T)$$

which satisfies necessarily the estimate (4.6), i.e. solves also (1.1), (1.2). ∎

The next supplementary assertion concerning the case $1 < p < 2$ follows immediately from *Part* I of the previous proof.

Theorem 4.3 *Let all assumptions of Theorem* 4.2 *be satisfied, with the exceptions that* $1 < p < 2$ *is allowed and* $\overline{\beta} < 0$ *is required in* (4.2). *Then problem* (1.1), (1.2) *has a solution* u *such that* (4.6) *is true.*

Theorems 3.2, 4.2 and 4.3 yield the following new existence criterion.

Theorem 4.4 *Assume* (1.3) *and* (2.1). *Furthermore, let* $1<p<\infty$, *and let* $r>0$, $A \geq r$, $B > A$ *and* $\beta \in L_1[0,T]$ *be such that* (4.2) (*with* $\overline{\beta} < 0$ *if* $1 < p < 2$) *and* (4.3) *hold, where*

$$B - A \geq \tfrac{T}{2}\,\phi_p^{-1}(\|m\|_1)$$

and

$$m(t) > \max\big\{\sup\{f(t,x)\colon x \in [r,A]\}, \beta(t), 0\big\} \quad \textit{for a.e. } t \in [0,T].$$

Then problem (1.1), (1.2) *has a solution* u *such that* (4.6) *is true.*

In particular, for the Duffing type equation $(\phi_p(u'))' = g(u) + e(t)$ we have

Corollary 4.5 *Let* $1 < p < \infty$. *Suppose that* $f(t,x) = g(x) + e(t)$ *for* $x \in (0,\infty)$ *and a.e.* $t \in [0,T]$, *where* $g \in C(0,\infty)$, $e \in L_1[0,T]$ *and*

$$\overline{e} + \limsup_{x\to\infty} g(x) < 0, \tag{4.18}$$

and there is $r > 0$ *such that*

$$e(t) + g(x) + \left(\tfrac{\pi_p}{T}\right)^p (x-r)^{p-1} \geq 0 \qquad \textit{for a.e. } t \in [0,T] \textit{ and all } x \geq r. \tag{4.19}$$

Then problem (1.1), (1.2) *has a solution* u *such that* $u \geq r$ *on* $[0,T]$.

Proof. Due to (4.18), we can find $A \geq r$ such that

$$g(x) + \overline{e} < \frac{1}{2}\left(\overline{e} + \limsup_{x\to\infty} g(x)\right) < 0 \quad \text{for } x \in [A,\infty).$$

Consequently,

$$f(t,x) = g(x) + e(t) = (g(x) + \overline{e}) + (e(t) - \overline{e}) < \frac{1}{2}\left(\overline{e} + \limsup_{x\to\infty} g(x)\right) + e(t) - \overline{e}$$

for a.e.$t \in [0,T]$ and all $x \in [A,\infty)$. Therefore, (4.2) holds with

$$\beta(t) := e(t) + \tfrac{1}{2}\left(\limsup_{x\to\infty} g(x) - \overline{e}\right),$$

$\overline{\beta} < 0$ and $B > A$ arbitrarily large. Finally, by virtue of (4.19), f satisfies (4.3) with $B > r$ arbitrarily large. Now, the assertion follows by Theorem 4.4. ∎

Remark 4.6 Notice that the assertion of Corollary 4.5 remains valid also when the assumption (4.18) is replaced by a slightly weaker assumption that there is an $A > r$ such that $g(x) + \overline{e} \le 0$ for $x \ge A$.

Finally, let us consider the model problem

$$(\phi_p(u'))' + k\,u^{p-1} = \frac{a}{u^\alpha} + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{4.20}$$

Corollary 4.7 *Let* $1 < p < \infty$, $e \in L_1[0,T]$, $a > 0$, $\alpha > 0$, $k \ge 0$. *Furthermore, let*

$$e_* := \inf\ \mathrm{ess}\{e(t) \colon t \in [0,T]\} > -\infty, \quad \mu = \left(\tfrac{\pi_p}{T}\right)^p$$

and let one of the following cases hold:

$$\begin{cases} k = 0,\ 1 < p < \infty,\ \overline{e} < 0 & \textit{and} \quad e_* + a\left(\frac{\alpha+p-1}{p-1}\right)\left(\frac{(p-1)\,\mu}{\alpha\, a}\right)^{\frac{\alpha}{\alpha+p-1}} > 0, \\ 0 < k < \mu,\ 1 < p < \infty & \textit{and} \quad e_* + a\left(\frac{\alpha+p-1}{p-1}\right)\left(\frac{(p-1)\,(\mu-k)}{\alpha\, a}\right)^{\frac{\alpha}{\alpha+p-1}} > 0, \\ k = \mu,\ 1 < p \le 2 & \textit{and} \quad e_* > 0. \end{cases} \tag{4.21}$$

Then problem (4.20) *has a positive solution.*

Proof. Denote

$$g(x) := -k\,x^{p-1} + \frac{a}{x^\alpha} + e(t) \ \text{ for } a.e.\ t \in [0,T] \text{ and all } x > 0.$$

To apply Corollary 4.5 we need to find conditions under which assumptions (4.18) and (4.19) are satisfied. It is easy to see that if $k > 0$, then condition (4.18) is satisfied for all $e \in L_1[0,T]$, while in the case $k = 0$ this condition holds whenever $\overline{e} < 0$. Denote

$$h_r(x) := \frac{a}{x^\alpha} + \mu\,(x-r)^{p-1} - k\,x^{p-1} \ \text{ for } r > 0 \text{ and } x \ge r \ \text{ or } r = 0 \text{ and } x > 0.$$

We can see that to verify condition (4.19) it suffices to show that

$$\exists\, r > 0 \quad \text{such that} \quad e_* + h_r(x) \ge 0 \quad \text{for all } x \ge r. \tag{4.22}$$

Since $\lim_{x\to\infty} h_r(x) = -\infty$ if $k > \mu$, $p > 1$ and $r \ge 0$ and also if $k = \mu$, $p > 2$ and $r > 0$, condition (4.22) cannot be satisfied in these cases. It remains to consider the following two possibilities:

(i) $1 < p \le 2$, $\quad 0 \le k \le \mu$,
(ii) $2 < p < \infty$, $\quad 0 \le k < \mu$.

Case (i). If $1 < p \le 2$, then the inequality $(x-r)^{p-1} \ge x^{p-1} - r^{p-1}$ holds for all $r \ge 0$ and $x \ge r$. Therefore, $h_r(x) \ge h_0(x) - \mu\, r^{p-1}$ for all $x \ge r$, i.e.

$$\varkappa(r) \ge \varkappa(0) - \mu\, r^{p-1} \quad \text{for all } k \in [0,\mu],$$

where $\varkappa(r)$ stands for

$$\varkappa(r) := \inf\{h_r(x) \colon x \in (r,\infty)\} \quad \text{for } r \geq 0.$$

It follows that condition (4.22) is satisfied provided

$$e_* + \varkappa(0) > 0 \text{ and } r = \left(\frac{e_* + \varkappa(0)}{\mu}\right)^{\frac{1}{p-1}}.$$

Notice that

$$\begin{cases} \varkappa(0) = a \left(\dfrac{\alpha+p-1}{p-1}\right)\left(\dfrac{(p-1)\ (\mu-k)}{\alpha\, a}\right)^{\frac{\alpha}{\alpha+p-1}} & \text{if } k \in [0,\mu), \\ \varkappa(0) = 0 & \text{if } k = \mu. \end{cases} \tag{4.23}$$

In particular, if $k=\mu$, problem (4.20) possesses a positive solution whenever $e_* > 0$.

Case (ii). Let $p>2$. First, assume that $k=0$. Then for each $r \geq 0$ there is exactly one $\widetilde{x}_r \in (r,\infty)$ such that $\varkappa(r) = h_r(\widetilde{x}_r)$. We can check that

$$\lim_{r\to 0+} \widetilde{x}_r = \widetilde{x}_0 = \left(\frac{\alpha\, a}{(p-1)\,\mu}\right)^{\frac{1}{\alpha+p-1}} \quad \text{and} \quad \lim_{r\to 0+} \varkappa(r) = \lim_{r\to 0+} h_r(\widetilde{x}_r) = h_0(\widetilde{x}_0) = \varkappa(0),$$

where $\varkappa(0)$ is given by (4.23). In particular, if $\varkappa(0) + e_* > 0$, then there is $r>0$ such that $\varkappa(r) + e_* > 0$, which means that (4.22) and hence also (4.19) are satisfied.

Now, assume that $0<k<\mu$ and let $e_* \geq 0$. Denote

$$\widetilde{g}_r(x) = \mu\,(x-r)^{p-1} - k\,x^{p-1} \quad \text{for } r>0 \quad \text{and} \quad x \geq r.$$

We have $\widetilde{g}\,'_r(x)=0$ if and only if $x=\widetilde{x}_r$, where $\widetilde{x}_r := r\left(1-\left(\frac{k}{\mu}\right)^{\frac{1}{p-2}}\right)^{-1}$ and

$$\widetilde{g}_r(\widetilde{x}_r) = \mu\, r^{p-1}\left(1-\left(\tfrac{k}{\mu}\right)^{\frac{1}{p-2}}\right)^{1-p}\left(\left(\tfrac{k}{\mu}\right)^{\frac{p-1}{p-2}} - k\right) = \inf\{\widetilde{g}_r(x) \colon x \in (0,\infty)\}.$$

Furthermore, $\widetilde{g}_r$ is strictly increasing on $[\widetilde{x}_r,\infty)$ and $\widetilde{g}_r(x) \geq 0$ for all $x \geq \xi_r$, where

$$\xi_r := r\left(1-\left(\frac{k}{\mu}\right)^{\frac{1}{p-1}}\right)^{-1} \in (\widetilde{x}_r,\infty).$$

Thus, $e_* + h_r(x) = e_* + \frac{a}{x^\alpha} + \widetilde{g}_r(x) \geq 0$ for $x \geq \xi_r$. On the other hand, for $x \in [r,\xi_r]$ we have $e_* + h_r(x) \geq e_* + \frac{a}{\xi_r{}^\alpha} + \widetilde{g}_r(\widetilde{x}_r)$. Consequently, condition (4.22) is satisfied whenever

$$e_* + \frac{a}{\xi_r{}^\alpha} + \widetilde{g}_r(\widetilde{x}_r) \geq 0 \quad \text{for some } r>0. \tag{4.24}$$

Now, since $\lim_{r\to 0+} \widetilde{x}_r = \lim_{r\to 0+} \xi_r = 0$, we have

$$\lim_{r\to 0+} \widetilde{g}_r(\widetilde{x}_r) = 0 \quad \text{and} \quad \lim_{r\to 0+} \frac{a}{\xi_r{}^\alpha} = \infty.$$

Thus, we can see that (4.24) and hence also (4.22) hold for some $r>0$ small enough.

Finally, let us consider the case that $p>2,\ 0<k<\mu$ and $e_*=-\eta_*<0$. We want again to show that (4.22) is true. To this aim let us rewrite the inequality $e_*+h_r(x)\geq 0$ as

$$\widetilde{h}_r(x):=-\frac{\eta_*}{x^{p-1}}+\frac{a}{x^{\alpha+p-1}}+\mu\left(1-\tfrac{r}{x}\right)^{p-1}\geq k.$$

First, notice that $\lim_{x\to\infty}\widetilde{h}_r(x)=\mu>k$ and there is $\delta>0$ such that

$$\widetilde{h}_r(r)=\frac{a-\eta_*\,r^{\alpha}}{r^{\alpha+p-1}}\geq k \quad \text{for all } r\in(0,\delta).$$

Indeed, it suffices to choose $\delta>0$ in such a way that both inequalities

$$\delta^{\alpha+p-1}\leq\frac{a}{2k} \qquad \text{and} \qquad \eta_*\,\delta^{\alpha}\leq\frac{a}{2}$$

hold. Furthermore, $\widetilde{h}\,'_r(x)=0$ if and only if

$$\rho_r(x)=\sigma(x) \tag{4.25}$$

holds, where

$$\rho_r(x):=r\,\mu\,(p-1)\left(1-\tfrac{r}{x}\right)^{p-2} \quad \text{and} \quad \sigma(x):=\frac{a\,(\alpha+p-1)}{x^{\alpha+p-2}}-\frac{\eta_*\,(p-1)}{x^{p-2}}.$$

We can see that, for each $r>0$, the function ρ_r is strictly increasing on $[r,\infty)$, while $\rho_r(r)=0$ and $\lim_{x\to\infty}\rho_r(x)=r\mu\,(p-1)>0$. On the other hand, $\sigma'(x)=0$ if and only if $x=\xi_*$, where

$$\xi_*=\left(\frac{a\,(\alpha+p-1)\,(\alpha+p-2)}{\eta_*\,(p-1)\,(p-2)}\right)^{\frac{1}{\alpha}}.$$

Furthermore,σ is strictly decreasing on$(0,\xi_*]$, strictly increasing on $[\xi_*,\infty)$, $\sigma(\xi_*)<0$, $\lim_{x\to\infty}\sigma(x)=0$ and $\sigma(x)=0$ if and only if $x=\xi$, where

$$\xi=\left(\frac{a\,(\alpha+p-1)}{\eta_*\,(p-1)}\right)^{\frac{1}{\alpha}}.$$

As a result, for each $r\in(0,\xi)$ there is exactly one point $\widetilde{x}_r\in(r,\xi)$ such that (4.25) holds for $x=\widetilde{x}_r$. Consequently, $\widetilde{h}_r(\widetilde{x}_r)=\inf\{\widetilde{h}_r(x)\colon x\in[r,\infty)\}$. Now, we can show that

$$\lim_{r\to 0+}\widetilde{h}_r(\widetilde{x}_r)=\widetilde{h}_0(\xi)=\mu-\left(\frac{\alpha\,a}{p-1}\right)\left(\frac{\eta_*\,(p-1)}{a\,(\alpha+p-1)}\right)^{\frac{\alpha+p-1}{\alpha}}.$$

Therefore, if

$$\mu-k>\left(\frac{\alpha\,a}{p-1}\right)\left(\frac{\eta_*\,(p-1)}{a\,(\alpha+p-1)}\right)^{\frac{\alpha+p-1}{\alpha}}, \tag{4.26}$$

then there exists $r\in(0,\min\{\delta,\xi\})$ such that $\widetilde{h}_r(x)\geq k$ holds for each $x\geq r$. In other words, provided (4.26) is true, condition (4.22) is satisfied. Now, it is a question of routine to verify that condition (4.26) is equivalent to condition $\varkappa(0)+e_*>0$ with $\varkappa(0)$ given by (4.23). Thus, making use of Corollary 4.5, we can summarize that problem (4.20) has a positive solution whenever one of the cases from (4.21) occurs. ■

Remark 4.8 Notice that $\lim_{x\to\infty} h_r(x) = -\infty$ if $k > \mu$, $p > 1$ and $r \geq 0$ and also if $k = \mu$, $p > 2$ and $r > 0$. Hence condition (4.19) cannot be satisfied and our method fails in these cases. Furthermore, the results obtained for the classical case $p = 2$ by Rachůnková, Tvrdý and Vrkoč [37] are contained in those presented in this section. So, the following natural questions arise:

Open problems

(i) What can be said about the existence or nonexistence of positive solutions to problem (4.20) in the cases $k > \mu$, $p > 1$ and $k = \mu$, $p > 2$?

(ii) Is it possible to weaken the condition $\inf \operatorname{ess}\{e(t) : t \in [0,T]\} > 0$ for the existence of a positive solution of (4.20) in the resonance case $k = \left(\frac{\pi_p}{T}\right)^p$ in a way similar to that used in the classical case $p = 2$ by Bonheure and De Coster [2] or by Bonheure, Fonda and Smets [3] ?

(iii) Is it possible, in a way similar to that used in the classical case $p = 2$ by Torres (see [40, Section 2]), to describe the sign properties of solutions to the quasilinear problem

$$(\phi_p(u'))' + \mu(t)\,\phi_p(u) = e(t), \quad u(0) = u(T),\ u'(0) = u'(T)$$

with $\mu \in L^q[0,T]$, $q > 1$, and $e \in L^1[0,T]$, $e \geq 0$ a.e. on $[0,T]$, in more details then those provided here by Theorem 3.2 ?

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