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NONPOSITIVE SOLUTIONS OF ONE FUNCTIONAL DIFFERENTIAL INEQUALITY

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We establish efficient conditions guaranteeing that every solution of the problem

$$u'(t) \geq \ell(u)(t), \quad u(a) \geq h(u),$$

where $\ell\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ and $h\colon C([a,b];\mathbb{R}) \to \mathbb{R}$ are linear bounded operators, is nonpositive. The results obtained are very useful for the investigation of the question of solvability and unique solvability of nonlocal boundary-value problems for first-order functional differential equations in both linear and nonlinear cases.

1. Introduction and Notation

On an interval $[a,b]$, we consider the functional differential inequality

$$u'(t) \geq \ell(u)(t), \tag{1.1}$$

where $\ell\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a linear bounded operator. A solution of inequality (1.1) is understood as an absolutely continuous function $u\colon [a,b] \to \mathbb{R}$ satisfying inequality (1.1) almost everywhere on the interval $[a,b]$.

Theorems on differential inequalities play a very important role in the theory of differential equations. For example, the well-known Gronwall inequality is also a corollary of a certain theorem on differential inequalities. Various types of differential inequalities are studied in the literature (see, e.g., [1, 3–6, 8, 9, 11, 13, 15–17]). In the present paper, we establish efficient sufficient conditions guaranteeing that every solution of inequality (1.1) that satisfies the condition

$$u(a) \geq h(u) \tag{1.2}$$

with linear bounded functional $h\colon C([a,b];\mathbb{R}) \to \mathbb{R}$ is nonpositive on the interval $[a,b]$. The statements obtained here can be used in the investigation of the question of solvability and unique solvability of nonlocal boundary-value problems for functional differential equations in both linear and nonlinear cases.

In order to simplify the formulation of the main results we introduce the following definition:

Definition 1.1. *Let $h \in F_{ab}$. An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set $\widetilde{V}_{ab}^{-}(h)$ $\big($resp., $\widetilde{V}_{ab}^{+}(h)\big)$ if every solution of problem (1.1), (1.2) is nonpositive (resp., nonnegative).*

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As indicated above, the aim of the paper is to find conditions guaranteeing the inclusions $\ell \in \widetilde{V}_{ab}^{-}(h)$ and $\ell \in \widetilde{V}_{ab}^{+}(h)$. In the case where the functional h is given by the formula

$$h(v) \stackrel{\mathrm{df}}{=} \lambda v(b) \quad \text{for } v \in C([a,b];\mathbb{R})$$

with $\lambda \geq 0$, the sets $\widetilde{V}_{ab}^{+}(h)$ and $\widetilde{V}_{ab}^{-}(h)$ are described in detail (see [7, 8]). In [14], the case where $h \in PF_{ab}$ is considered. However, the general case of h has not been studied yet.

We assume throughout the paper that the functional $h \in F_{ab}$ is defined by the formula

$$h(v) \stackrel{\mathrm{df}}{=} \lambda v(b) + h_0(v) - h_1(v) \quad \text{for } v \in C([a,b];\mathbb{R}), \tag{1.3}$$

where $\lambda > 0$ and $h_0, h_1 \in PF_{ab}$. There is no loss of generality in assuming this because any linear bounded functional can be represented in this form.

The following notation is used in what follows:

(1) $\mathbb{N}$ is the set of all natural numbers, $\mathbb{R}$ is the set of all real numbers, and $\mathbb{R}_+ = [0, +\infty[$. If $x \in \mathbb{R}$, then we set

$$[x]_+ = \frac{|x| + x}{2}, \quad [x]_- = \frac{|x| - x}{2}.$$

(2) $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $v\colon [a,b] \to \mathbb{R}$ endowed with the norm

$$\|v\|_C = \max\{|v(t)| \colon t \in [a,b]\}.$$

(3) $\widetilde{C}([a,b];D)$, where $D \subseteq \mathbb{R}$, is the set of absolutely continuous functions $v\colon [a,b] \to D$.

(4) $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p\colon [a,b] \to \mathbb{R}$ endowed with the norm

$$\|p\|_L = \int_a^b |p(s)|ds.$$

(5) $L([a,b];D) = \{p \in L([a,b];\mathbb{R}) \colon p\colon [a,b] \to D\}$, where $D \subseteq \mathbb{R}$.

(6) $C([a,b];D) = \{v \in C([a,b];\mathbb{R}) \colon v\colon [a,b] \to D\}$, where $D \subseteq \mathbb{R}$.

(7) $\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$, and P_{ab} is the set of operators $\ell \in \mathcal{L}_{ab}$ mapping the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.

(8) F_{ab} is the set of linear bounded functionals $h\colon C([a,b];\mathbb{R}) \to \mathbb{R}$, and PF_{ab} is the set of functionals $h \in F_{ab}$ mapping the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$.

Definition 1.2. *Let $t_0 \in [a,b]$. We say that $\ell \in \mathcal{L}_{ab}$ is a t_0-Volterra operator if, for arbitrary $a_1 \in [a,t_0]$, $b_1 \in [t_0,b]$, $a_1 \neq b_1$, and $v \in C([a,b];\mathbb{R})$ with the property*

$$v(t) = 0 \quad \textit{for } t \in [a_1,b_1],$$

the following relation is true:

$$\ell(v)(t) = 0 \quad \textit{for a.e. } t \in [a_1, b_1].$$

2. Preliminary Remarks

Recall that we suppose that $\ell \in \mathcal{L}_{ab}$ and $h \in F_{ab}$. The following two assumptions are natural:

(A) If $h(1) = 1$, then the operator ℓ is supposed to be nontrivial in the sense that the condition $\ell(1) \not\equiv 0$ is satisfied.

(B) $\widetilde{h} \not\equiv 0$, where the functional $\widetilde{h}$ is defined by the formula

$$\widetilde{h}(v) = h(v) - v(a) \quad \text{for } v \in C([a,b];\mathbb{R}).$$

Remark 2.1. It follows from Definition 1.1 that if $\ell \in \widetilde{V}^{-}_{ab}(h)$ $\big($resp., $\ell \in \widetilde{V}^{+}_{ab}(h)\big)$, then the homogeneous problem

$$u'(t) = \ell(u)(t), \quad u(a) = h(u) \tag{2.1}$$

has only the trivial solution. Therefore, the inclusion $\ell \in \widetilde{V}^{-}_{ab}(h)$ $\big($resp., $\ell \in \widetilde{V}^{+}_{ab}(h)\big)$ guarantees the unique solvability of the problem

$$u'(t) = \ell(u)(t) + q(t), \quad u(a) = h(u) + c \tag{2.2}$$

for every $q \in L([a,b];\mathbb{R})$ and $c \in \mathbb{R}$. This fact follows from the Fredholm property of problem (2.2) (see, e.g., [2, 10]; in the case where the operator ℓ is strongly bounded, see also [1, 12, 18]). Moreover, under the condition $\ell \in \widetilde{V}^{-}_{ab}(h)$ $\big($resp., $\ell \in \widetilde{V}^{+}_{ab}(h)\big)$, the unique solution of problem (2.2) is nonpositive (resp., nonnegative) whenever $q \in L([a,b];\mathbb{R}_+)$ and $c \in \mathbb{R}_+$.

Remark 2.2. It is easy to verify that the condition $(-P_{ab}) \cap \widetilde{V}^{-}_{ab}(h) \neq \varnothing$ yields

$$h(1) > 1. \tag{2.3}$$

Indeed, if $\ell \in (-P_{ab}) \cap \widetilde{V}^{-}_{ab}(h)$ and $h(1) \leq 1$, then the function $u \equiv 1$ is a positive solution of problem (1.1), (1.2), which contradicts the inclusion $\ell \in \widetilde{V}^{-}_{ab}(h)$.

On the other hand, if, together with (2.3), the inequality $h_0(1) \leq 1$ holds, then the zero operator belongs to the set $\widetilde{V}^{-}_{ab}(h)$. Indeed, let $u \in \widetilde{C}([a,b];\mathbb{R})$ satisfy (1.2) and let

$$u'(t) \geq 0 \quad \text{for a.e. } t \in [a,b].$$

Then it is clear that

$$u(a) \leq u(t) \leq u(b) \quad \text{for } t \in [a,b]. \tag{2.4}$$

By virtue of condition (2.4) and the assumption $h_0, h_1 \in PF_{ab}$, it follows from (1.2) that

$$u(a) \geq \lambda u(b) + h_0(u) - h_1(u) \geq u(a)h_0(1) + (\lambda - h_1(1))\, u(b).$$

Taking condition (2.4) and the assumption $h_0(1) \le 1$ into account, we get

$$(\lambda - h_1(1))\, u(b) \le (1 - h_0(1))\, u(a) \le (1 - h_0(1))\, u(b),$$

and, thus,

$$(h(1) - 1)\, u(b) \le 0.$$

The last inequality and (2.3) result in $u(b) \le 0$. Hence, condition (2.4) guarantees that $u(t) \le 0$ for $t \in [a, b]$, and, thus, $0 \in \widetilde{V}^-_{ab}(h)$.

We have shown that condition (2.3) is necessary for the validity of the relation $(-P_{ab}) \cap \widetilde{V}^-_{ab}(h) \ne \varnothing$, and the conditions (2.3) and $h_0(1) \le 1$ are sufficient for the inclusion $0 \in \widetilde{V}^-_{ab}(h)$ to hold.

Definition 2.1. *An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set $S_{ab}(a)$ $\big($resp., $S_{ab}(b)\big)$ if every solution u of inequality (1.1) that satisfies the condition $u(a) \ge 0$ $\big($resp., $u(b) \le 0\big)$ is nonnegative (resp., nonpositive).*

Remark 2.3. The sets $S_{ab}(a)$ and $S_{ab}(b)$ were investigated in [6].

3. Auxiliary Statements

In this section, auxiliary statements are given. More precisely, properties of the sets U^-_{ab} and $\widetilde{U}^+_{ab}(h)$ are studied that are very useful in the investigation of the validity of the desired inclusion $\ell \in \widetilde{V}^-_{ab}(h)$.

3.1. Formulation of Results. We first formulate all results; the proofs are given in the next subsection.

Definition 3.1. *Let $h \in F_{ab}$. An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set U^-_{ab}, if problem (1.1), (1.2) does not have nontrivial nonnegative solutions.*

Remark 3.1. It follows immediately from Definitions 1.1 and 3.1 that $\widetilde{V}^-_{ab}(h) \subseteq U^-_{ab}(h)$.

Since the set $U^-_{ab}(h)$ is wider than $\widetilde{V}^-_{ab}(h)$, conditions for the inclusion $\ell \in U^-_{ab}$ can be obtained relatively easy. In Theorem 3.1 (Theorem 3.2), the case $\ell \in P_{ab}$ $(-\ell \in P_{ab})$ is considered, whereas Theorems 3.3 and 3.4 concern the case where $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in P_{ab}$.

Theorem 3.1. *Let $\ell \in P_{ab}$ and*

$$h_1(1) < \lambda. \tag{3.1}$$

Let, moreover, there exist a function $\gamma \in \widetilde{C}([a, b]; \mathbb{R}_+)$ such that

$$\gamma'(t) \le \ell(\gamma)(t) \quad \text{for a.e. } t \in [a, b], \tag{3.2}$$

$$\gamma(a) < h(\gamma). \tag{3.3}$$

Then $\ell \in U^-_{ab}(h)$.

Remark 3.2. If $\ell \in P_{ab}$, $h(1) \geq 1$, and $h_1(1) < \lambda$, then the operator ℓ belongs to the set $U_{ab}^{-}(h)$ without any additional assumptions. Indeed, since the operator ℓ is supposed to be nontrivial in the case where $h(1) = 1$, the function

$$\gamma(t) = 1 + \int_a^t \ell(1)(s)ds \quad \text{for } t \in [a,b]$$

satisfies conditions (3.2) and (3.3).

Theorem 3.2. *Let* $-\ell \in P_{ab}$ *and*

$$h(1) > 1, \quad h_0(1) \leq 1. \tag{3.4}$$

Then $\ell \in U_{ab}^{-}(h)$ *if and only if there exists a function* $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *satisfying conditions (3.2) and (3.3).*

Theorem 3.3. *Let* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in P_{ab}$, *and*

$$h(1) \leq 1, \quad h_1(1) < \lambda. \tag{3.5}$$

If, moreover,

$$\int_a^b \ell_1(1)(s)\,ds < (\lambda - h_1(1)) \min\left\{1, \frac{1}{\lambda}\right\} \tag{3.6}$$

and

$$\int_a^b \ell_0(1)(s)\,ds > \frac{(1-h_0(1)) \min\left\{1, \frac{1}{\lambda}\right\}}{(\lambda - h_1(1)) \min\left\{1, \frac{1}{\lambda}\right\} - \int_a^b \ell_1(1)(s)\,ds} - 1, \tag{3.7}$$

then $\ell \in U_{ab}^{-}(h)$.

Theorem 3.4. *Let* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in P_{ab}$, *and*

$$h(1) > 1, \quad h_1(1) < \lambda. \tag{3.8}$$

Let, moreover, inequality (3.6) hold and

$$\int_a^b \ell_0(1)(s)\,ds > \omega\left(\int_a^b \ell_1(1)(s)\,ds\right), \tag{3.9}$$

where

$$\omega(y) = \begin{cases} \dfrac{(y + h_1(1))\left(1 - \frac{1}{\lambda}h_1(1)\right)}{1 - \frac{1}{\lambda}h_1(1) - y} - (h_0(1) + \lambda - 1) \\ \qquad \text{if} \quad \lambda \geq 1, \quad y < \dfrac{(h(1) - 1)\left(1 - \frac{1}{\lambda}h_1(1)\right)}{\lambda - 1 + h_0(1)}, \\ \dfrac{\left(y + \frac{1}{\lambda}h_1(1)\right)\left(1 - \frac{1}{\lambda}h_1(1)\right)}{1 - \frac{1}{\lambda}h_1(1) - y} - \left(\dfrac{1}{\lambda}h_0(1) + \dfrac{\lambda - 1}{\lambda}\right) \\ \qquad \text{if} \quad \lambda \geq 1, \quad y \geq \dfrac{(h(1) - 1)\left(1 - \frac{1}{\lambda}h_1(1)\right)}{\lambda - 1 + h_0(1)}, \\ \dfrac{\left(y + \frac{1-\lambda}{\lambda} + \frac{1}{\lambda}h_1(1)\right)(\lambda - h_1(1))}{\lambda - h_1(1) - y} - \dfrac{1}{\lambda}h_0(1) \\ \qquad \text{if} \quad \lambda < 1, \quad y < \dfrac{(h(1) - 1)(\lambda - h_1(1))}{h_0(1)}, \\ \dfrac{(y + 1 - \lambda + h_1(1))(\lambda - h_1(1))}{\lambda - h_1(1) - y} - h_0(1) \\ \qquad \text{if} \quad \lambda < 1, \quad y \geq \dfrac{(h(1) - 1)(\lambda - h_1(1))}{h_0(1)}. \end{cases} \tag{3.10}$$

Then $\ell \in U_{ab}^-(h)$.

We now introduce the following definition:

Definition 3.2. *Let* $h \in F_{ab}$. *An operator* $\ell \in \mathcal{L}_{ab}$ *is said to belong to the set* $\widetilde{U}_{ab}^+(h)$ *if there is no nonpositive solution* u *of inequality (1.1) that satisfies the condition*

$$u(a) > h(u). \tag{3.11}$$

Remark 3.3. It is clear that $\widetilde{U}_{ab}^+(0) = \mathcal{L}_{ab}$ and $\widetilde{V}_{ab}^+(h) \subseteq \widetilde{U}_{ab}^+(h)$.

Theorem 3.5. *Let* $\ell \in P_{ab}$ *and* $h \in PF_{ab}$ *be such that the inequality* $h(1) \leq 1$ *is true. If there exists a function* $\gamma \in \widetilde{C}([a, b];]0, +\infty[)$ *satisfying the conditions*

$$\gamma'(t) \geq \ell(\gamma)(t) \quad \text{for a.e. } t \in [a, b], \tag{3.12}$$

$$\gamma(a) \geq h(\gamma), \tag{3.13}$$

then $\ell \in \widetilde{U}_{ab}^+(h)$.

3.2. Proofs. We first recall a result established in [6].

Lemma 3.1 ([6], Theorem 1.1)**.** *Let* $\ell \in P_{ab}$. *Then* $\ell \in \mathcal{S}_{ab}(a)$ *if and only if there exists a function* $\gamma \in \widetilde{C}([a, b];]0, +\infty[)$ *satisfying condition (3.12).*

Proof of Theorem 3.1. Let u be a nonnegative solution of problem (1.1), (1.2). We show that $u \equiv 0$. Since $\ell \in P_{ab}$ and u is a nonnegative function, it follows from (1.1) that

$$0 \le u(a) \le u(t) \le u(b) \quad \text{for } t \in [a,b]. \tag{3.14}$$

Assume that $u(b) > 0$. Then condition (1.2), in view of relations (3.1) and (3.14) and the assumption that $h_0, h_1 \in PF_{ab}$, results in

$$u(a) \ge \lambda u(b) + h_0(u) - h_1(u) \ge (\lambda - h_1(1))\, u(b) > 0.$$

Consequently, relation (3.14) yields

$$u(t) > 0 \quad \text{for } t \in [a,b]. \tag{3.15}$$

We set

$$v(t) = ru(t) - \gamma(t) \quad \text{for } t \in [a,b],$$

where

$$r = \max\left\{\frac{\gamma(t)}{u(t)} : t \in [a,b]\right\}.$$

According to relations (3.3) and (3.15) and the assumption that $\gamma \in \widetilde{C}([a,b];\mathbb{R}_+)$, we get

$$r > 0. \tag{3.16}$$

It is obvious that

$$v(t) \ge 0 \quad \text{for } t \in [a,b] \tag{3.17}$$

and there exists $t_0 \in [a,b]$ such that

$$v(t_0) = 0. \tag{3.18}$$

Taking relations (1.1), (3.2), (3.16), and (3.17) and the assumption that $\ell \in P_{ab}$ into account, we now obtain

$$v'(t) \ge \ell(v)(t) \ge 0 \quad \text{for a.e. } t \in [a,b]. \tag{3.19}$$

Therefore, relation (3.19), with regard for (3.17) and (3.18), yields

$$0 = v(a) \le v(t) \le v(b) \quad \text{for } t \in [a,b]. \tag{3.20}$$

However, using relations (1.2), (3.1), (3.3), (3.16), and (3.20) and the assumption that $h_0, h_1 \in PF_{ab}$, we get

$$0 = v(a) > \lambda v(b) + h_0(v) - h_1(v) \ge (\lambda - h_1(1))\, v(b) \ge 0,$$

a contradiction.

The contradiction obtained proves that $u(b) \le 0$. However, relation (3.14) then implies that $u \equiv 0$, and, thus, $\ell \in U_{ab}^{-}(h)$.

The theorem is proved.

Proof of Theorem 3.2. First, assume that there exists a function $\gamma \in \widetilde{C}([a,b];]0+\infty[)$ satisfying relations (3.2) and (3.3). Let u be a nonnegative solution of problem (1.1), (1.2). We show that $u \equiv 0$. Assume that, on the contrary, there exists $t^* \in [a,b]$ such that

$$u(t^*) > 0. \tag{3.21}$$

We set

$$v(t) = r\gamma(t) - u(t) \quad \text{for } t \in [a,b],$$

where

$$r = \max\left\{\frac{u(t)}{\gamma(t)} : t \in [a,b]\right\}.$$

According to (3.21), inequality (3.16) holds. It is clear that condition (3.17) is satisfied and there exists $t_0 \in [a,b]$ such that (3.18) is true. Taking relations (1.1), (3.2), (3.16), and (3.17) and the assumption that $-\ell \in P_{ab}$ into account, we obtain

$$v'(t) \le \ell(v)(t) \le 0 \quad \text{for a.e. } t \in [a,b]. \tag{3.22}$$

Therefore, with regard for (3.17) and (3.18), relation (3.22) yields

$$0 = v(b) \le v(t) \le v(a) \quad \text{for } t \in [a,b]. \tag{3.23}$$

However, using relations (1.2), (3.3), (3.4), (3.16), and (3.23) and the assumption that $h_0, h_1 \in PF_{ab}$, we get

$$0 = \lambda v(b) = r\lambda\gamma(b) - \lambda u(b) > v(a) - h_0(v) + h_1(v) \ge v(a)\,(1 - h_0(1)) \ge 0,$$

a contradiction. The contradiction obtained proves that $u \equiv 0$, and, thus, $\ell \in U_{ab}^{-}(h)$.

Now assume that $\ell \in U_{ab}^{-}(h)$. We first show that the homogeneous problem (2.1) has only the trivial solution. Let u be a solution of problem (2.1). Using Remark 2.2, we get $0 \in \widetilde{V}_{ab}^{-}(h)$. Therefore, according to Remark 2.1, the problem

$$\alpha'(t) = \ell([u]_-)(t), \tag{3.24}$$

$$\alpha(a) = h(\alpha) \tag{3.25}$$

has a unique solution α, and the following relation is true:

$$\alpha(t) \ge 0 \quad \text{for } t \in [a,b]. \tag{3.26}$$

Using relations (1.1), (1.2), (3.24), and (3.25) and the assumption that $-\ell \in P_{ab}$, we get

$$v'(t) = \ell([u]_+)(t) \le 0 \quad \text{for a.e. } t \in [a,b], \quad v(a) = h(v),$$

where

$$v(t) = u(t) + \alpha(t) \quad \text{for } t \in [a,b]. \tag{3.27}$$

Consequently, using the inclusion $0 \in \widetilde{V}_{ab}^{-}(h)$, we obtain $v(t) \ge 0$ for $t \in [a,b]$, and, thus,

$$-u(t) \le \alpha(t) \quad \text{for } t \in [a,b]. \tag{3.28}$$

With regard for relation (3.26), inequality (3.28) implies that

$$[u(t)]_- \le \alpha(t) \quad \text{for } t \in [a,b].$$

Therefore, in view of the assumption that $-\ell \in P_{ab}$, Eq. (3.24) yields

$$\alpha'(t) \ge \ell(\alpha)(t) \quad \text{for a.e. } t \in [a,b]. \tag{3.29}$$

Consequently, α is a nonnegative function satisfying conditions (3.25) and (3.29). Hence, the assumption that $\ell \in U_{ab}^{-}(h)$ implies that $\alpha \equiv 0$, and, thus, relation (3.28) yields

$$u(t) \ge 0 \quad \text{for } t \in [a,b]. \tag{3.30}$$

Since $-u$ is also solution of the homogeneous problem (2.1), according to the statements proved above we have $-u(t) \ge 0$ for $t \in [a,b]$. Consequently, $u \equiv 0$, i.e., the homogeneous problem (2.1) has only the trivial solution. By virtue of the Fredholm property of problem (2.2) (see, e.g., [2, 10]), the problem

$$\gamma'(t) = \ell(\gamma)(t), \quad \gamma(a) = h(\gamma) + 1 - h(1) \tag{3.31}$$

has a unique solution γ. Setting

$$\bar{\gamma}(t) = \gamma(t) - 1 \quad \text{for } t \in [a,b]$$

and using (3.31), we get

$$\bar{\gamma}'(t) \le \ell(\bar{\gamma})(t) \quad \text{for a.e. } t \in [a,b], \quad \bar{\gamma}(a) = h(\bar{\gamma}).$$

As above, one can now show that $\bar{\gamma}(t) \ge 0$ for $t \in [a,b]$. Therefore, in view of the assumption that $h(1) > 1$, it follows from (3.31) that γ is a positive function satisfying inequalities (3.2) and (3.3).

The theorem is proved.

Proof of Theorem 3.3. Let u be a nonnegative solution of problem (1.1), (1.2). We show that $u \equiv 0$. Assume that, on the contrary, $u \not\equiv 0$. We set

$$x_0 = \int_a^b \ell_0(1)(s)\,ds, \quad y_0 = \int_a^b \ell_1(1)(s)\,ds, \tag{3.32}$$

$$M = \max\{u(t) \colon t \in [a,b]\}, \quad m = \min\{u(t) \colon t \in [a,b]\} \tag{3.33}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \quad u(t_m) = m. \tag{3.34}$$

Obviously,

$$M > 0, \quad m \geq 0, \tag{3.35}$$

and either

$$t_m < t_M \tag{3.36}$$

or

$$t_m \geq t_M. \tag{3.37}$$

First, assume that (3.36) holds. The integration of (1.1) from a to t_m and from t_M to $b,$ in view of relations (3.33)–(3.35) and the assumption that $\ell_0, \ell_1 \in P_{ab},$ yields

$$u(a) - m \leq \int_a^{t_m} \ell_1(u)(s)\,ds - \int_a^{t_m} \ell_0(u)(s)\,ds \leq M \int_a^{t_m} \ell_1(1)(s)\,ds, \tag{3.38}$$

$$M - u(b) \leq \int_{t_M}^{b} \ell_1(u)(s)\,ds - \int_{t_M}^{b} \ell_0(u)(s)\,ds \leq M \int_{t_M}^{b} \ell_1(1)(s)\,ds. \tag{3.39}$$

Moreover, with regard for relation (3.33) and the assumption that $h_0, h_1 \in PF_{ab},$ condition (1.2) implies that

$$u(a) - \lambda u(b) \geq h_0(u) - h_1(u) \geq m h_0(1) - M h_1(1). \tag{3.40}$$

It follows from (3.38)–(3.40) that

$$M\,(\lambda - h_1(1)) - m\,(1 - h_0(1)) \leq M \left(\int_a^{t_m} \ell_1(1)(s)\,ds + \lambda \int_{t_M}^{b} \ell_1(1)(s)\,ds \right),$$

i.e.,

$$M \left((\lambda - h_1(1)) \min\left\{1, \frac{1}{\lambda}\right\} - y_0 \right) \leq m\,(1 - h_0(1)) \min\left\{1, \frac{1}{\lambda}\right\}. \tag{3.41}$$

Now assume that (3.37) holds. The integration of (1.1) from t_M to $t_m,$ in view of relations (3.33)–(3.35) and the assumption that $\ell_0, \ell_1 \in P_{ab},$ results in

$$M - m \leq \int_{t_M}^{t_m} \ell_1(u)(s)\,ds - \int_{t_M}^{t_m} \ell_0(u)(s)\,ds \leq M \int_{t_M}^{t_m} \ell_1(1)(s)\,ds. \tag{3.42}$$

It is not difficult to verify that, by virtue of (3.5) and (3.42), inequality (3.41) is true.

We have proved that inequality (3.41) is satisfied in both cases (3.36) and (3.37). On the other hand, the integration of (1.1) from a to b, in view of relations (3.32) and (3.33) and the assumption that $\ell_0, \ell_1 \in P_{ab}$, yields

$$u(a) - u(b) \leq \int_a^b \ell_1(u)(s)\, ds - \int_a^b \ell_0(u)(s)\, ds \leq My_0 - mx_0,$$

i.e.,

$$mx_0 \leq My_0 + u(b) - u(a). \tag{3.43}$$

Moreover, condition (1.2) implies that

$$u(b) - u(a) \leq u(b)\,(1-\lambda) - h_0(u) + h_1(u), \tag{3.44}$$

$$u(b) - u(a) \leq u(a)\left(\frac{1}{\lambda} - 1\right) - \frac{1}{\lambda}\, h_0(u) + \frac{1}{\lambda}\, h_1(u). \tag{3.45}$$

First, assume that $\lambda \leq 1$. Inequalities (3.43) and (4.44), together with relation (3.33) and the assumption that $h_0, h_1 \in PF_{ab}$, result in

$$mx_0 \leq My_0 + M\,(1-\lambda) - mh_0(1) + Mh_1(1). \tag{3.46}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get, from (3.41) and (3.46), the relation $m > 0$ and the inequality

$$(\lambda - h_1(1) - y_0)\,(x_0 + h_0(1)) \leq (y_0 + 1 - \lambda + h_1(1))\,(1 - h_0(1))\,,$$

which, in view of (3.6) and (3.32), contradicts (3.7).

Now assume that $\lambda > 1$. Inequalities (3.43) and (3.45), together with relation (3.33) and the assumption that $h_0, h_1 \in PF_{ab}$, yield

$$mx_0 \leq My_0 - m\,\frac{\lambda - 1}{\lambda} - \frac{1}{\lambda}\, mh_0(1) + \frac{1}{\lambda}\, Mh_1(1). \tag{3.47}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get, from (3.41) and (3.47), the relation $m > 0$ and the inequality

$$\left(1 - \frac{1}{\lambda}\, h_1(1) - y_0\right)\left(x_0 + \frac{\lambda-1}{\lambda} + \frac{1}{\lambda}\, h_0(1)\right) \leq \left(y_0 + \frac{1}{\lambda}\, h_1(1)\right)\frac{1 - h_0(1)}{\lambda},$$

which, in view of (3.6) and (3.32), contradicts (3.7).

The contradictions obtained prove the relation $u \equiv 0$, and, thus, $\ell \in U^-_{ab}(h)$.

The theorem is proved.

Proof of Theorem 3.4. Let u be a nonnegative solution of problem (1.1), (1.2). We show that $u \equiv 0$. Assume that, on the contrary, $u \not\equiv 0$. We define the numbers x_0, y_0 and M, m by formulas (3.32) and (3.33), respectively, and choose $t_M, t_m \in [a,b]$ such that relations (3.34) hold. Obviously, condition (3.35) is satisfied, and either relation (3.36) or relation (3.37) is true.

First, assume that relation (3.36) is true. By analogy with the proof of Theorem 3.3, one can prove inequality (3.41). Consequently, in view of (2.3) and (3.35), we get

$$M\left((\lambda - h_1(1))\min\left\{1, \frac{1}{\lambda}\right\} - y_0\right) \leq m\,(\lambda - h_1(1))\min\left\{1, \frac{1}{\lambda}\right\}. \tag{3.48}$$

Now assume that relation (3.37) is true. By analogy with the proof of Theorem 3.3, it can be shown that relation (3.42) is satisfied. Consequently, it is not difficult to verify that, by virtue of (3.1) and (3.42), inequality (3.48) is true.

We have proved that inequality (3.48) is satisfied in both cases (3.36) and (3.37). On the other hand, by analogy with the proof of Theorem 3.3, one can obtain inequalities (3.43)–(3.45).

First, assume that

$$\lambda \geq 1, \quad y_0 < \frac{(h(1)-1)\left(1-\frac{1}{\lambda}h_1(1)\right)}{\lambda - 1 + h_0(1)}.$$

Relations (3.43) and (3.44), together with formulas (3.33) and the assumption that $h_0, h_1 \in PF_{ab}$, result in

$$mx_0 \leq My_0 - m(\lambda - 1) - mh_0(1) + Mh_1(1). \tag{3.49}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get, from (3.48) and (3.49), the relation $m > 0$ and the inequality

$$\left(1 - \frac{1}{\lambda}h_1(1) - y_0\right)(x_0 + \lambda - 1 + h_0(1)) \leq (y_0 + h_1(1))\left(1 - \frac{1}{\lambda}h_1(1)\right),$$

which, in view of (3.6) and (3.32), contradicts (3.9) with ω given by (3.10).

Now assume that

$$\lambda \geq 1, \quad y_0 \geq \frac{(h(1)-1)\left(1-\frac{1}{\lambda}h_1(1)\right)}{\lambda - 1 + h_0(1)}.$$

Inequalities (3.43) and (3.45), together with formulas (3.33) and the assumption that $h_0, h_1 \in PF_{ab}$, result in

$$mx_0 \leq My_0 - m\,\frac{\lambda - 1}{\lambda} - \frac{1}{\lambda}\,mh_0(1) + \frac{1}{\lambda}\,Mh_1(1). \tag{3.50}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get, from (3.48) and (3.50), the relation $m > 0$ and the inequality

$$\left(1 - \frac{1}{\lambda}\,h_1(1) - y_0\right)\left(x_0 + \frac{\lambda - 1}{\lambda} + \frac{1}{\lambda}\,h_0(1)\right) \leq \left(y_0 + \frac{1}{\lambda}\,h_1(1)\right)\frac{\lambda - h_1(1)}{\lambda},$$

which, in view of (3.6) and (3.32), contradicts (3.9) with ω given by (3.10).

Now assume that

$$\lambda < 1, \quad y_0 < \frac{(h(1)-1)\,(\lambda - h_1(1))}{h_0(1)}.$$

Inequalities (3.43) and (3.45), together with formulas (3.33) and the assumption that $h_0, h_1 \in PF_{ab}$, result in

$$mx_0 \le My_0 + M\,\frac{1-\lambda}{\lambda} - \frac{1}{\lambda}\,mh_0(1) + \frac{1}{\lambda}\,Mh_1(1). \tag{3.51}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get, from (3.48) and (3.51), the relation $m > 0$ and the inequality

$$(\lambda - h_1(1) - y_0)\left(x_0 + \frac{1}{\lambda}\,h_0(1)\right) \le \left(y_0 + \frac{1-\lambda}{\lambda} + \frac{1}{\lambda}\,h_1(1)\right)(\lambda - h_1(1)),$$

which, in view of (3.6) and (3.32), contradicts (3.9) with ω given by (3.10).

Finally, assume that

$$\lambda < 1, \quad y_0 \ge \frac{(h(1)-1)\,(\lambda - h_1(1))}{h_0(1)}.$$

Inequalities (3.43) and (3.44), together with formulas (3.33) and the assumption that $h_0, h_1 \in PF_{ab}$, result in

$$mx_0 \le My_0 + M\,(1-\lambda) - mh_0(1) + Mh_1(1). \tag{3.52}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get, from (3.48) and (3.52), the relation $m > 0$ and the inequality

$$(\lambda - h_1(1) - y_0)\,(x_0 + h_0(1)) \le (y_0 + 1 - \lambda + h_1(1))\,(\lambda - h_1(1)),$$

which, in view of (3.6) and (3.32), contradicts (3.9) with ω given by (3.10).

The contradictions obtained prove the relation $u \equiv 0$, and, thus, $\ell \in U_{ab}^{-}(h)$.

The theorem is proved.

Proof of Theorem 3.5. By virtue of inequality (3.12) and the assumption that $\ell \in P_{ab}$, Lemma 3.1 guarantees that $\ell \in \mathcal{S}_{ab}(a)$.

Let u be a nonpositive solution of problem (1.1), (3.11). It is not difficult to verify that

$$u(a) < 0. \tag{3.53}$$

Indeed, if $u(a) = 0$, then inequality (1.1), in view of the inclusion $\ell \in \mathcal{S}_{ab}(a)$, yields $u(t) \ge 0$ for $t \in [a,b]$. Hence we get $u \equiv 0$, which contradicts relation (3.11).

We set

$$w(t) = \gamma(a)u(t) - u(a)\gamma(t) \quad \text{for } t \in [a,b].$$

Using relations (1.1), (3.12), and (3.53) and the assumption that $\gamma(a) > 0$, we immediately obtain

$$w'(t) \ge \ell(w)(t) \quad \text{for a.e. } t \in [a,b], \tag{3.54}$$

$$w(a) = 0. \tag{3.55}$$

Therefore, the inclusion $\ell \in \mathcal{S}_{ab}(a)$ implies that

$$w(t) \geq 0 \quad \text{for } t \in [a,b]. \tag{3.56}$$

On the other hand, it follows from relations (3.11), (3.13), (3.53), and (3.56) and the assumptions $\gamma(a) > 0$ and $h \in PF_{ab}$ that

$$w(a) > h(w) \geq 0,$$

which contradicts relation (3.55).

The contradiction obtained proves that there is no nonpositive solution of problem (1.1), (3.11), and, thus, $\ell \in \widetilde{U}^{+}_{ab}(h)$.

The theorem is proved.

4. Main Results

In this section, we give main results of the paper, which are efficient conditions under which the operator ℓ belongs to the set $\widetilde{V}^{-}_{ab}(h)$. The results are formulated in Secs. 4.1–4.3, and their proofs are presented in Sec. 4.5.

We first give a rather theoretical statement.

Proposition 4.1. *Let $h \in F_{ab}$. Then $\ell \in \widetilde{V}^{-}_{ab}(h)$ if and only if $\ell \in U^{-}_{ab}(h)$ and there exists $\bar{\ell} \in P_{ab}$ such that $\ell + \bar{\ell} \in \widetilde{V}^{-}_{ab}(h)$.*

We now present a general result.

Theorem 4.1. *Let $\ell \in S_{ab}(b) \cap \widetilde{U}^{+}_{ab}(h_0)$. Then $\ell \in \widetilde{V}^{-}_{ab}(h)$ if and only if there exists a function $\gamma \in \widetilde{C}([a,b];\mathbb{R}_+)$ satisfying conditions (3.2) and (3.3).*

4.1. Case $\ell \in P_{ab}$. The following statements can be proved in the case where $\ell \in P_{ab}$:

Theorem 4.2. *Let $\ell \in P_{ab} \cap \widetilde{U}^{+}_{ab}(h_0)$ be a b-Volterra operator and let condition (3.4) be satisfied. Then $\ell \in \widetilde{V}^{-}_{ab}(h)$ if and only if $\ell \in S_{ab}(b)$.*

Corollary 4.1. *Let $\ell \in P_{ab}$ be a b-Volterra operator and let condition (3.4) be satisfied. If, moreover, there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ such that condition (3.12) is satisfied and*

$$\gamma(a) \geq h_0(\gamma), \tag{4.1}$$

then $\ell \in \widetilde{V}^{-}_{ab}(h)$.

Corollary 4.2. *Let $\ell \in P_{ab}$ be a b-Volterra operator and let*

$$h(1) > 1, \quad h_0(1) < 1. \tag{4.2}$$

Assume that

$$h_0(\varphi_1) > 0 \tag{4.3}$$

and there exist $m, k \in \mathbb{N}$ *such that* $m > k$ *and*

$$\varrho_m(t) \le \varrho_k(t) \quad \text{for } t \in [a,b], \tag{4.4}$$

where $\varrho_1 \equiv 1$ *and*

$$\varrho_{i+1}(t) \stackrel{\text{df}}{=} \frac{h_0(\varphi_i)}{1-h_0(1)} + \varphi_i(t) \quad \text{for } t \in [a,b], \quad i \in \mathbb{N}, \tag{4.5}$$

$$\varphi_i(t) \stackrel{\text{df}}{=} \int_a^t \ell(\varrho_i)(s)ds \quad \text{for } t \in [a,b], \quad i \in \mathbb{N}. \tag{4.6}$$

Then $\ell \in \widetilde{V}^-_{ab}(h)$.

Remark 4.1. It follows from Corollary 4.2 (for $k = 1$ and $m = 2$) that if $\ell \in P_{ab}$ is a b-Volterra operator, condition (4.2) is satisfied, and relation (4.3) holds with φ_1 given by (4.6), then $\ell \in \widetilde{V}^-_{ab}(h)$, provided that

$$\int_a^b \ell(1)(s)ds \le 1 - h_0(1).$$

Corollary 4.3. *Let* $\ell \in P_{ab}$ *be a* b*-Volterra operator and let condition (4.2) be satisfied. Then the operator* ℓ *belongs to the set* $\widetilde{V}^-_{ab}(h)$, *provided that* $\ell \in \widetilde{V}^+_{ab}(h_0)$.

Remark 4.2. Recall that efficient conditions guaranteeing the validity of the inclusion $\ell \in \widetilde{V}^+_{ab}(h_0)$ are stated in [14].

4.2. Case $-\ell \in P_{ab}$***.*** The following statements can be proved in the case where $-\ell \in P_{ab}$.

Theorem 4.3. *Let* $-\ell \in P_{ab}$ *and let condition (3.4) be satisfied. Then* $\ell \in \widetilde{V}^-_{ab}(h)$ *if and only if* $\ell \in U^-_{ab}(h)$.

Corollary 4.4. *Let* $-\ell \in P_{ab}$ *and let condition (3.4) be satisfied. Assume that at least one of the following conditions is satisfied:*

(a) there exist $m, k \in \mathbb{N}$ *and a constant* $\delta \in [0,1[$ *such that* $m > k$ *and*

$$\varrho_m(t) \le \delta\varrho_k(t) \quad \text{for } t \in [a,b], \tag{4.7}$$

where $\varrho_1 \equiv 1$, $\varrho_{i+1} \equiv \vartheta(\varrho_i)$ *for* $i \in \mathbb{N}$, *and*

$$\vartheta(v)(t) \stackrel{\text{df}}{=} \frac{\widetilde{h}(v)}{h(1)-1} - \frac{z(v)(a)}{h(1)-1} - z(v)(t) \quad \text{for } t \in [a,b], \quad v \in C([a,b];\mathbb{R}), \tag{4.8}$$

$$\widetilde{h}(v) \stackrel{\text{df}}{=} h(z(v)), \quad z(v)(t) \stackrel{\text{df}}{=} \int_t^b \ell(v)(s)\,ds \quad \text{for } t \in [a,b], \quad v \in C([a,b];\mathbb{R}); \tag{4.9}$$

(b) there exists $\bar{\ell} \in P_{ab}$ *such that*

$$h(z_0) > z_0(a), \tag{4.10}$$

$$z_0(a)\,(1 - h(z_1)) + h(z_0)z_1(a) < h(z_0), \tag{4.11}$$

and the inequality

$$\ell(1)(t)\vartheta(v)(t) - \ell(\vartheta(v))(t) \le \bar{\ell}(v)(t) \quad \text{for a.e. } t \in [a,b] \tag{4.12}$$

holds on the set $\{v \in C([a,b];\mathbb{R}_+) \colon v(a) = h(v)\}$, *where the operator* ϑ *is defined by (4.8) and (4.9),*

$$z_0(t) = \exp\left(\int_t^b |\ell(1)(s)|ds\right) \quad \text{for } t \in [a,b], \tag{4.13}$$

and

$$z_1(t) = \int_t^b \bar{\ell}(1)(s)\exp\left(\int_t^s |\ell(1)(\xi)|\,d\xi\right)ds \quad \text{for } t \in [a,b]. \tag{4.14}$$

Then $\ell \in \widetilde{V}_{ab}^{-}(h)$.

Remark 4.3. Let $-\ell \in P_{ab}$ and let condition (3.4) be satisfied. Then it follows from Corollary 4.4(a) (for $k = 1$ and $m = 2$) that $\ell \in \widetilde{V}_{ab}^{-}(h)$, provided that

$$\int_a^b |\ell(1)(s)|\,ds < 1 - \frac{1 + h_1(1)}{\lambda + h_0(1)}.$$

Moreover, it follows from Corollary 4.4(b) (with $\bar{\ell} \equiv 0$) that $\ell \in \widetilde{V}_{ab}^{-}(h)$, provided that ℓ is a b-Volterra operator and condition (4.10) is satisfied, i.e.,

$$z_0(a) < h(z_0),$$

where the function z_0 is given by (4.13).

4.3. Case $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in P_{ab}$. The following statements can be proved in the case where the operator is regular, i.e., admits the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in P_{ab}$.

Theorem 4.4. *Let* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in P_{ab}$, *let*

$$h_1(1) < \lambda, \quad h_0(1) \le 1, \tag{4.15}$$

and let

$$\int_a^b \ell_0(1)(s)ds \leq (1-h_0(1)) \min\left\{1, \frac{1}{\lambda}\right\}. \tag{4.16}$$

Then $\ell \in \widetilde{V}_{ab}^-(h)$ *if and only if* $\ell \in U_{ab}^-(h)$.

Theorem 4.5. *Let* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in P_{ab}$, *let* $h \in F_{ab}$, *and let condition (2.3) be satisfied. If*

$$\ell_0 \in \widetilde{V}_{ab}^-(h), \quad -\ell_1 \in \widetilde{V}_{ab}^-(h), \tag{4.17}$$

then $\ell \in \widetilde{V}_{ab}^-(h)$.

4.4. Further Remarks. We introduce an operator $\varphi\colon C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ by setting

$$\varphi(w)(t) \overset{\text{df}}{=} w(a+b-t) \quad \text{for } t \in [a,b], \quad w \in C([a,b];\mathbb{R}).$$

Let

$$\widehat{\ell}(w)(t) \overset{\text{df}}{=} -\ell\left(\varphi(w)\right)(a+b-t) \quad \text{for a.e. } t \in [a,b] \text{ and all } w \in C([a,b];\mathbb{R}),$$

$$\widehat{h}(w) \overset{\text{df}}{=} \frac{1}{\lambda}\, v(b) - \frac{1}{\lambda}\, h_0\left(\varphi(w)\right) + \frac{1}{\lambda}\, h_1\left(\varphi(w)\right) \quad \text{for } w \in C([a,b];\mathbb{R}).$$

It is clear that if u is a solution of problem (1.1), (1.2), then the function $v \overset{\text{df}}{=} -\varphi(u)$ is a solution of the problem

$$v'(t) \geq \widehat{\ell}(v)(t), \quad v(a) \geq \widehat{h}(v), \tag{4.18}$$

and, vice versa, if v is a solution of problem (4.18), then the function $u \overset{\text{df}}{=} -\varphi(v)$ is a solution of problem (1.1), (1.2).

Consequently, the following relation is true:

$$\ell \in \widetilde{V}_{ab}^+(h) \Leftrightarrow \widehat{\ell} \in \widetilde{V}_{ab}^-\left(\widehat{h}\right).$$

Therefore, efficient conditions guaranteeing the validity of the inclusion $\ell \in \widetilde{V}_{ab}^+(h)$ can be immediately derived from the results stated in Secs. 4.1–4.3. For example, Corollary 4.1 of Sec. 4.1 immediately yields the following statement:

Corollary 4.5. *Let* $-\ell \in P_{ab}$ *be an* a*-Volterra operator and let*

$$h(1) < 1, \quad h_1(1) \leq \lambda.$$

If, moreover, there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *such that condition (3.2) is satisfied and*

$$\lambda\gamma(b) \geq h_1(\gamma),$$

then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

4.5. Proofs. To prove the statements formulated in Secs. 4 and 4.5 we need the following lemmas:

Lemma 4.1. *Let* $h \in F_{ab}$ *and* $\ell \in U_{ab}^{-}(h)$. *Then* $\ell + \bar{\ell} \in U_{ab}^{-}(h)$ *for every* $\bar{\ell} \in P_{ab}$.

Proof. The required statement follows immediately from Definition 3.1.

Lemma 4.2 ([6], Theorem 1.6)**.** *Suppose that* $\ell \in P_{ab}$ *is a* b*-Volterra operator and there exists a function* $\gamma \in \widetilde{C}([a,b];\mathbb{R}_+)$ *satisfying condition (3.12) and such that*

$$\gamma(t) > 0 \quad \textit{for } t \in]a,b].$$

Then $\ell \in \mathcal{S}_{ab}(b)$.

Proof of Proposition 4.1. First, assume that $\ell \in \widetilde{V}_{ab}^{-}(h)$. Then, according to Remark 3.1, we have $\ell \in U_{ab}^{-}(h)$. Moreover, it is clear that the inclusion $\ell + \bar{\ell} \in \widetilde{V}_{ab}^{-}(h)$ is true with $\bar{\ell} \equiv 0$.

Now assume that $\ell \in U_{ab}^{-}(h)$ and there exists an operator $\bar{\ell} \in P_{ab}$ such that $\ell + \bar{\ell} \in \widetilde{V}_{ab}^{-}(h)$. Let u be a solution of problem (1.1), (1.2). We show that the function u is nonpositive.

According to the assumption that $\ell + \bar{\ell} \in \widetilde{V}_{ab}^{-}(h)$ and Remark 2.1, the problem

$$\alpha'(t) = \left(\ell + \bar{\ell}\right)(\alpha)(t) - \bar{\ell}([u]_+)(t), \tag{4.19}$$

$$\alpha(a) = h(\alpha) \tag{4.20}$$

has a unique solution α, and the following relation is true:

$$\alpha(t) \geq 0 \quad \text{for } t \in [a,b]. \tag{4.21}$$

Relations (1.1), (1.2), (4.19), and (4.20) and the assumption that $\bar{\ell} \in P_{ab}$ yield

$$v'(t) \geq (\ell + \bar{\ell})(v)(t) \quad \text{for a.e. } t \in [a,b], \quad v(a) \geq h(v),$$

where

$$v(t) = u(t) - \alpha(t) \quad \text{for } t \in [a,b]. \tag{4.22}$$

Consequently, using the inclusion $\ell + \bar{\ell} \in \widetilde{V}_{ab}^{-}(h)$, we obtain $v(t) \leq 0$ for $t \in [a,b]$, and, thus,

$$u(t) \leq \alpha(t) \quad \text{for } t \in [a,b]. \tag{4.23}$$

With regard for relation (4.21), inequality (4.23) implies that

$$[u(t)]_+ \leq \alpha(t) \quad \text{for } t \in [a,b].$$

Therefore, in view of the assumption that $\bar{\ell} \in P_{ab}$, Eq. (4.19) yields

$$\alpha'(t) \geq (\ell + \bar{\ell})(\alpha)(t) - \bar{\ell}(\alpha)(t) = \ell(\alpha)(t) \quad \text{for a.e. } t \in [a, b]. \tag{4.24}$$

Consequently, α is a nonnegative function satisfying conditions (4.20) and (4.24). Hence, the assumption that $\ell \in U^-_{ab}(h)$ implies that $\alpha \equiv 0$, and, thus, relation (4.23) yields

$$u(t) \leq 0 \quad \text{for } t \in [a, b]. \tag{4.25}$$

Therefore, the inclusion $\ell \in \widetilde{V}^-_{ab}(h)$ is true.

The proposition is proved.

Proof of Theorem 4.1. First, assume that $\ell \in \widetilde{V}^-_{ab}(h)$. According to Remark 2.1, the problem

$$\gamma'(t) = \ell(\gamma)(t), \quad \gamma(a) = h(\gamma) - 1 \tag{4.26}$$

has a unique solution γ, and, moreover, the following relation is true:

$$\gamma(t) \geq 0 \quad \text{for } t \in [a, b]. \tag{4.27}$$

The function γ obviously satisfies conditions (3.2) and (3.3).

Now assume that there exists a function $\gamma \in \widetilde{C}([a, b]; \mathbb{R}_+)$ satisfying conditions (3.2) and (3.3). We show that $\ell \in \widetilde{V}^-_{ab}(h)$. Let u be a solution of problem (1.1), (1.2). It is clear that either

$$u(b) > 0 \tag{4.28}$$

or

$$u(b) \leq 0. \tag{4.29}$$

Assume that condition (4.28) is satisfied. We set

$$w(t) = \gamma(b)u(t) - u(b)\gamma(t) \quad \text{for } t \in [a, b].$$

Using (1.1), (3.2), and (4.28), we get

$$w'(t) \geq \ell(w)(t) \quad \text{for a.e. } t \in [a, b], \tag{4.30}$$

$$w(b) = 0. \tag{4.31}$$

Therefore, the assumption that $\ell \in S_{ab}(b)$ yields

$$w(t) \leq 0 \quad \text{for } t \in [a, b]. \tag{4.32}$$

On the other hand, relations (1.2), (3.3), (4.28), (4.31), and (4.32), and the assumption that $h_1 \in PF_{ab}$ imply that

$$w(a) > \lambda w(b) + h_0(w) - h_1(w) \geq h_0(w).$$

Consequently, the function w is a nonpositive solution of the problem

$$w'(t) \geq \ell(w)(t), \quad w(a) > h_0(w),$$

which contradicts the assumption that $\ell \in \widetilde{U}^+_{ab}(h_0)$.

The contradiction obtained proves that u satisfies condition (4.29). In view of (1.1) and (4.29), the assumption that $\ell \in \mathcal{S}_{ab}(b)$ now yields relation (4.25), and, thus, $\ell \in \widetilde{V}^-_{ab}(h)$.

The theorem is proved.

Proof of Theorem 4.2. First, assume that $\ell \in \mathcal{S}_{ab}(b)$. It is clear that, in view of relation (2.3) and the assumption that $\ell \in P_{ab}$, the function $\gamma \equiv 1$ satisfies conditions (3.2) and (3.3). Hence, by virtue of Theorem 4.1, we get $\ell \in \widetilde{V}^-_{ab}(h)$.

Now let $\ell \in \widetilde{V}^-_{ab}(h)$. Assume that, on the contrary, $\ell \notin \mathcal{S}_{ab}(b)$. Then there exists a solution u of inequality (1.1) satisfying the relations $u(b) = c$ and

$$u(t_0) > 0, \tag{4.33}$$

where $c \leq 0$ and $t_0 \in]a, b[$. According to the assumption that $\ell \in \widetilde{V}^-_{ab}(h)$ and Remark 2.1, the problem

$$u_0'(t) = \ell(u_0)(t), \tag{4.34}$$

$$u_0(a) = h(u_0) - 1 \tag{4.35}$$

has a unique solution u_0, and, moreover,

$$u_0(t) \geq 0 \quad \text{for } t \in [a, b]. \tag{4.36}$$

It is not difficult to verify that

$$u_0(b) > 0. \tag{4.37}$$

Indeed, assume that inequality (4.37) does not hold. Then, in view of (4.36), we find $u_0(b) = 0$. Hence, by virtue of relation (4.36) and the assumption that $h_1 \in PF_{ab}$, condition (4.35) yields

$$u_0(a) = \lambda u_0(b) + h_0(u_0) - h_1(u_0) - 1 < h_0(u_0),$$

which, together with (4.34) and (4.36), contradicts the assumption that $\ell \in \widetilde{U}^+_{ab}(h_0)$. The contradiction obtained proves relation (4.37).

Since $\ell \notin \mathcal{S}_{ab}(b)$, in view of relations (4.34), (4.36), and (4.37) and the assumption that $\ell \in P_{ab}$ it follows from Lemma 4.2 that there exists $a_0 \in]a, b[$ such that

$$u_0(t) = 0 \quad \text{for } t \in [a, a_0], \tag{4.38}$$

$$u_0(t) > 0 \quad \text{for } t \in]a_0, b]. \tag{4.39}$$

Let $\widetilde{\ell}$ denote the restriction of the operator ℓ to the space $C([a_0, b]; \mathbb{R})$. By virtue of conditions (4.34) and (4.39), we get

$$u_0'(t) = \widetilde{\ell}(u_0)(t) \quad \text{for a.e. } t \in [a_0, b], \quad u_0(t) > 0 \quad \text{for } t \in]a_0, b],$$

and, thus, Lemma 4.2 guarantees the validity of the inclusion $\widetilde{\ell} \in \mathcal{S}_{a_0 b}(b)$. It follows from inequality (1.1) and condition (4.34) that

$$w'(t) \geq \widetilde{\ell}(w)(t) \quad \text{for a.e. } t \in [a_0, b], \quad w(b) = 0, \tag{4.40}$$

where

$$w(t) = u(t) - \frac{c}{u_0(b)}\, u_0(t) \quad \text{for } t \in [a_0, b].$$

Since $\widetilde{\ell} \in \mathcal{S}_{a_0 b}(b)$, relations (4.40) result in $w(t) \leq 0$ for $t \in [a_0, b]$, i.e.,

$$u(t) \leq \frac{c}{u_0(b)}\, u_0(t) \quad \text{for } t \in [a_0, b].$$

Using the latter inequality and relations (4.33) and (4.39), we get

$$a < t_0 < a_0. \tag{4.41}$$

We now set

$$v(t) = u(t) + (u(a) - h(u))\, u_0(t) \quad \text{for } t \in [a, b]. \tag{4.42}$$

It is clear that

$$v'(t) \geq \ell(v)(t) \quad \text{for a.e. } t \in [a, b], \quad v(a) = h(v).$$

Consequently, by virtue of the assumption that $\ell \in \widetilde{V}_{ab}^{-}(h)$, the inequality $v(t) \leq 0$ holds for $t \in [a, b]$. Finally, in view of (4.38) and (4.41), relation (4.42) yields

$$0 \geq v(t_0) = u(t_0) + (u(a) - h(u))\, u_0(t_0) = u(t_0),$$

which contradicts inequality (4.33).

The contradiction obtained proves the inclusion $\ell \in \mathcal{S}_{ab}(b)$.

The theorem is proved.

Proof of Corollary 4.1. According to Lemma 4.2, inequality (3.12) yields $\ell \in \mathcal{S}_{ab}(b)$. On the other hand, by virtue of conditions (3.12), (3.4), and (4.1), using Theorem 3.5 we get $\ell \in \widetilde{U}_{ab}^{+}(h_0)$. Consequently, the assertion of the corollary follows from Theorem 4.2.

Proof of Corollary 4.2. We set

$$\gamma(t) = \sum_{j=k+1}^{m} \varrho_j(t) \quad \text{for } t \in [a,b].$$

In view of condition (4.3), where the function φ_1 is given by (4.6), we get $\gamma \in \widetilde{C}([a,b];\,]0,+\infty[)$. On the other hand, by virtue of relations (4.4)–(4.6) and the assumption that $\ell \in P_{ab}$, it is clear that the function γ satisfies conditions (3.12) and (4.1). Consequently, the conditions of Corollary 4.1 are satisfied.

Proof of Corollary 4.3. According to the assumption that $\ell \in P_{ab} \cap \widetilde{V}_{ab}^{+}(h_0)$, Theorem 2.1 in [14] guarantees that there exists a function $\gamma \in \widetilde{C}([a,b];\,]0,+\infty[)$ satisfying conditions (3.12) and such that

$$\gamma(a) > h_0(\gamma).$$

Consequently, the conditions of Corollary 4.1 are satisfied.

Proof of Theorem 4.3. The validity of the theorem follows immediately from Proposition 4.1 (with $\bar{\ell} \equiv -\ell$) and Remark 2.2.

Proof of Corollary 4.4. (a) It is not difficult to verify that the function γ defined by the formula

$$\gamma(t) = \sum_{j=1}^{m} \varrho_j(t) - \delta \sum_{j=1}^{k} \varrho_j(t) \quad \text{for } t \in [a,b]$$

is positive and satisfies conditions (3.2) and (3.3). Consequently, the assertion of the corollary follows from Theorems 3.2 and 4.3.

(b) According to relations (4.10) and (4.11), there exists $\varepsilon > 0$ such that

$$\gamma_0\,(\varepsilon - h(z_1))\,z_0(a) + \gamma_0 h(z_0) z_1(a) \le 1, \tag{4.43}$$

where $\gamma_0 = (h(z_0) - z_0(a))^{-1}$. We set

$$\begin{aligned}\gamma(t) = \gamma_0 \Bigg[&(\varepsilon - h(z_1)) \exp\left(\int_t^b |\ell(1)(s)|\,ds \right) \\ &+ \exp\left(\int_a^b |\ell(1)(s)|\,ds \right) \int_a^t \bar{\ell}(1)(s) \exp\left(\int_t^s |\ell(1)(\xi)|\,d\xi \right) ds \\ &+ h(z_0) \int_t^b \bar{\ell}(1)(s) \exp\left(\int_t^s |\ell(1)(\xi)|\,d\xi \right) ds \Bigg] \quad \text{for } t \in [a,b],\end{aligned}$$

where the functions z_0 and z_1 are defined by (4.13) and (4.14), respectively. It is not difficult to verify that γ is a solution of the problem

$$\gamma'(t) = \ell(1)(t)\gamma(t) - \bar{\ell}(1)(t), \tag{4.44}$$

$$\gamma(a) = h(\gamma) - \varepsilon. \tag{4.45}$$

In view of inequalities (3.4) and (4.10) and the assumption that $h_0, h_1 \in PF_{ab}$, it follows from the definition of the function γ that $\gamma(b) > 0$, and, thus, the relation $\gamma(t) > 0$ holds for $t \in [a,b]$. Since $-\ell, \bar{\ell} \in P_{ab}$, equality (4.44) implies that $\gamma(t) \le \gamma(a)$ for $t \in [a,b]$. With regard for inequality (4.43), conditions (4.44) and (4.45) yield

$$\gamma'(t) \le \ell(1)(t)\gamma(t) - \bar{\ell}(\gamma)(t) \quad \text{for a.e. } t \in [a,b], \quad \gamma(a) < h(\gamma).$$

Consequently, Theorem 4.3 guarantees the validity of the inclusion

$$\widetilde{\ell} \in \widetilde{V}_{ab}^{-}(h), \tag{4.46}$$

where

$$\widetilde{\ell}(v)(t) \stackrel{\text{df}}{=} \ell(1)(t)v(t) - \bar{\ell}(v)(t) \quad \text{for a.e. } t \in [a,b] \text{ and all } v \in C([a,b];\mathbb{R}).$$

Since $-\ell \in P_{ab}$, in order to prove the inclusion $\ell \in \widetilde{V}_{ab}^{-}(h)$ it is sufficient to show that $\ell \in U_{ab}^{-}(h)$ (see Theorem 4.3). Hence, let u be a nonnegative solution of problem (1.1), (1.2). We show that $u \equiv 0$. We set

$$w(t) = \vartheta(v)(t) \quad \text{for } t \in [a,b], \tag{4.47}$$

where the operator ϑ is defined by (4.8) and (4.6), and

$$v(t) = u(t) + \frac{u(a) - h(u)}{h(1) - 1} \quad \text{for } t \in [a,b].$$

Obviously,

$$v(t) \ge u(t) \quad \text{for } t \in [a,b]$$

and

$$v'(t) \ge \ell(v)(t) \quad \text{for a.e. } t \in [a,b], \quad v(a) = h(v), \tag{4.48}$$

$$w'(t) = \ell(v)(t) \quad \text{for a.e. } t \in [a,b], \quad w(a) = h(w). \tag{4.49}$$

It follows from (4.48) and (4.49) that

$$y'(t) \ge 0 \quad \text{for a.e. } t \in [a,b], \quad y(a) = h(y),$$

where $y(t) = v(t) - w(t)$ for $t \in [a, b]$. By virtue of Remark 2.2, we have $0 \in \widetilde{V}_{ab}^{-}(h)$. Consequently, $y(t) \leq 0$ for $t \in [a, b]$, i.e.,

$$0 \leq u(t) \leq v(t) \leq w(t) \quad \text{for } t \in [a, b]. \tag{4.50}$$

On the other hand, using relations (4.12) and (4.47)–(4.50) and the assumption that $-\ell, \bar{\ell} \in P_{ab}$, we get

$$\begin{aligned} w'(t) &= \ell(v)(t) \geq \ell(1)(t)w(t) + \ell(w)(t) - \ell(1)(t)w(t) \\ &= \ell(1)(t)w(t) + \ell(\vartheta(v))(t) - \ell(1)(t)\vartheta(v)(t) \geq \ell(1)(t)w(t) - \bar{\ell}(v)(t) \\ &\geq \ell(1)(t)w(t) - \bar{\ell}(w)(t) = \widetilde{\ell}(w)(t) \quad \text{for a.e. } t \in [a, b]. \end{aligned}$$

Taking (4.46) and (4.49) into account, we find $w(t) \leq 0$ for $t \in [a, b]$. Hence, relation (4.50) implies that $u \equiv 0$, and, thus, $\ell \in \widetilde{V}_{ab}^{-}(h)$.

The corollary is proved.

Proof of Theorem 4.4. Assume that $\ell \in U_{ab}^{-}(h)$. Since $\ell_1 \in P_{ab}$, Lemma 4.1 guarantees that $\ell_0 = \ell + \ell_1 \in U_{ab}^{-}(h)$.

We show that $\ell_0 \in \widetilde{V}_{ab}^{-}(h)$. Assume, on the contrary, that there exists a solution u of the inequality

$$u'(t) \geq \ell_0(u)(t) \tag{4.51}$$

that satisfies condition (1.2) and is not nonpositive on the interval $[a, b]$. Then, in view of the inclusion $\ell_0 \in U_{ab}^{-}(h)$ proved above, it is clear that u takes both positive and negative values, i.e.,

$$M > 0, \quad m > 0, \tag{4.52}$$

where

$$M = \max\{u(t) : t \in [a, b]\}, \quad m = -\min\{u(t) : t \in [a, b]\}. \tag{4.53}$$

We now choose $t_M, t_m \in [a, b]$ such that

$$u(t_M) = M, \quad u(t_m) = -m. \tag{4.54}$$

It is obvious that either

$$t_M < t_m \tag{4.55}$$

or

$$t_M > t_m. \tag{4.56}$$

If inequality (4.55) holds, then the integration of (4.51) from t_M to t_m, in view of relations (4.52) and (4.53) and the assumption that $\ell_0 \in P_{ab}$, results in

$$M + m \le -\int_{t_M}^{t_m} \ell_0(u)(s)\,ds \le m\int_a^b \ell_0(1)(s)\,ds.$$

Hence, by virtue of (4.16) and the second inequality in (4.52), we get $M \le 0$, which contradicts the first inequality in (4.52).

If inequality (4.56) holds, then the integration of (4.51) from a to t_m and from t_M to b, in view of relations (4.52) and (4.53) and the assumption that $\ell_0 \in P_{ab}$, yields

$$u(a) + m \le -\int_a^{t_m} \ell_0(u)(s)\,ds \le m\int_a^{t_m} \ell_0(1)(s)\,ds, \tag{4.57}$$

$$M - u(b) \le -\int_{t_M}^{b} \ell_0(u)(s)\,ds \le m\int_{t_M}^{b} \ell_0(1)(s)\,ds. \tag{4.58}$$

On the other hand, in view of relation (4.53) and the assumption that $h_0, h_1 \in PF_{ab}$, condition (1.2) implies that

$$u(a) - \lambda u(b) \ge h_0(u) - h_1(u) \ge -mh_0(1) - Mh_1(1). \tag{4.59}$$

It now follows from (4.57)–(4.59) that

$$M\,(\lambda - h_1(1)) + m\,(1 - h_0(1)) \le m\left(\int_a^{t_m} \ell_0(1)(s)\,ds + \lambda\int_{t_M}^{b} \ell_0(1)(s)\,ds\right),$$

i.e.,

$$M\,(\lambda - h_1(1))\min\left\{1, \frac{1}{\lambda}\right\} + m\,(1 - h_0(1))\min\left\{1, \frac{1}{\lambda}\right\} \le m\int_a^b \ell_0(1)(s)\,ds.$$

Hence, by virtue of (3.1), (4.16), and the second inequality in (4.52), we find $M \le 0$, which contradicts the first inequality in (4.52).

The contradictions obtained prove the inclusion $\ell_0 \in \widetilde{V}_{ab}^-(h)$.

We now set $\bar{\ell} \equiv \ell_1$. Since $\ell \in U_{ab}^-(h)$ and $\ell + \bar{\ell} = \ell_0 \in \widetilde{V}_{ab}^-(h)$, Proposition 4.1 yields $\ell \in \widetilde{V}_{ab}^-(h)$.

The converse implication follows immediately from Remark 3.1.

The theorem is proved.

Proof of Theorem 4.5. It is easy to verify that $\ell \in U_{ab}^-(h)$. Indeed, the assumption that $-\ell_1 \in \widetilde{V}_{ab}^-(h)$ yields $-\ell_1 \in U_{ab}^-(h)$ (see Remark 3.1), and, thus, in view of Lemma 4.1, we get $\ell = -\ell_1 + \ell_0 \in U_{ab}^-(h)$.

We now set $\bar{\ell} \equiv \ell_1$. Then it is clear that $\bar{\ell} \in P_{ab}$ and $\ell + \bar{\ell} = \ell_0 \in \widetilde{V}_{ab}^{-}(h)$. Consequently, Proposition 4.1 yields $\ell \in \widetilde{V}_{ab}^{-}(h)$.

The theorem is proved.

5. Differential Inequalities with Argument Deviations

In this section, we give some corollaries of the main results for operators with argument deviations. More precisely, efficient criteria are proved below for the validity of the inclusion $\ell \in \widetilde{V}_{ab}^{-}(h)$ in the case where the operator ℓ is given by one of the following formulas:

$$\ell(v)(t) \overset{\text{df}}{=} p(t)v(\tau(t)) \quad \text{for a.e. } t \in [a,b] \text{ and all } v \in C([a,b];\mathbb{R}), \tag{5.1}$$

$$\ell(v)(t) \overset{\text{df}}{=} -g(t)v(\mu(t)) \quad \text{for a.e. } t \in [a,b] \text{ and all } v \in C([a,b];\mathbb{R}), \tag{5.2}$$

$$\ell(v)(t) \overset{\text{df}}{=} p(t)v(\tau(t)) - g(t)v(\mu(t)) \quad \text{for a.e. } t \in [a,b] \text{ and all } v \in C([a,b];\mathbb{R}). \tag{5.3}$$

Here, we suppose that $p, g \in L([a,b];\mathbb{R}_+)$ and $\tau, \mu\colon [a,b] \to [a,b]$ are measurable functions.

Throughout this section, the following notation is used:

$$\mu_* = \operatorname{ess\,inf}\{\mu(t)\colon t \in [a,b]\}, \quad \tau^* = \operatorname{ess\,sup}\{\tau(t)\colon t \in [a,b]\}, \tag{5.4}$$

$$\alpha(t) = \exp\left(\int_t^b g(s)\,ds\right), \quad \beta(t) = \exp\left(\int_a^t p(s)\,ds\right) \quad \text{for } t \in [a,b]. \tag{5.5}$$

We first formulate all results; their proofs are given later, in Sec. 5.1 below.

Theorem 5.1. *Let condition (3.1) be satisfied and let*

$$h(1) < 1. \tag{5.6}$$

Assume that

$$0 < \int_a^b p(s)\,ds \le (1 - h_0(1)) \min\left\{1, \frac{1}{\lambda}\right\} \tag{5.7}$$

and

$$\operatorname{ess\,inf}\left\{\int_t^{\tau(t)} p(s)\,ds\colon t \in [a,b]\right\} > \eta_*, \tag{5.8}$$

where

$$\eta_* = \inf\left\{\frac{1}{x}\ln\frac{x\beta^x(\tau^*)}{\beta^x(\tau^*) + (h(\beta^x) - 1)\,(1 - h(1))^{-1}}\colon\ x > 0,\ h(\beta^x) > 1\right\}. \tag{5.9}$$

Then the operator ℓ *given by (5.1) belongs to the set* $\widetilde{V}_{ab}^-(h)$.

Corollary 5.1. *Let inequalities (3.1) and (5.6) be true. Assume that condition (5.7) is satisfied and*

$$\operatorname{ess\,inf}\left\{\int_t^{\tau(t)} p(s)\,ds\colon\ t \in [a,b]\right\} > \xi_*, \tag{5.10}$$

where

$$\xi_* = \inf\left\{\frac{\|p\|_L}{y}\ln\frac{ye^y\,(1 - h(1))}{\|p\|_L\,(e^y - 1)\,(1 - h_0(1))}\colon\ y > \ln\frac{1 - h_0(1)}{\lambda - h_1(1)}\right\}. \tag{5.11}$$

Then the operator ℓ *defined by (5.1) belongs to the set* $\widetilde{V}_{ab}^-(h)$.

Theorem 5.2. *Let conditions (3.1) and (5.6) be satisfied. Assume that* $\tau(t) \geq t$ *for a.e.* $t \in [a,b]$,

$$\int_a^b p(s)ds > \ln\frac{1 - h_0(1)}{\lambda - h_1(1)}, \tag{5.12}$$

and at least one of the following conditions is satisfied:

(a) $h_0(z_0) > 0$ *and*

$$\max\left\{\frac{h_0(z_1) + (1 - h_0(1))\,z_1(t)}{h_0(z_0) + (1 - h_0(1))\,z_0(t)}\colon\ t \in [a,b]\right\} < 1 - \frac{h_0(z_0)}{1 - h_0(1)}, \tag{5.13}$$

where

$$z_0(t) = \int_a^t p(s)\,ds \quad \textit{for } t \in [a,b], \tag{5.14}$$

$$z_1(t) = \int_a^t p(s)\left(\int_a^{\tau(s)} p(\xi)d\xi\right)ds \quad \textit{for } t \in [a,b]; \tag{5.15}$$

(b) the following relations are true:

$$h_0(\beta) < 1, \tag{5.16}$$

and

$$\frac{h_0(\gamma_0)}{1-h_0(\beta)}\,\beta(b)+\gamma_0(b)<1, \tag{5.17}$$

where

$$\gamma_0(t)=\int_a^t p(s)\left(\int_s^{\tau(s)} p(\xi)d\xi\right)\exp\left(\int_s^t p(\eta)d\eta\right)ds \quad \text{for } t\in[a,b]; \tag{5.18}$$

(c) $h_0(1)\neq 0$ *and*

$$\operatorname{ess\,sup}\left\{\int_t^{\tau(t)} p(s)ds\colon\ t\in[a,b]\right\}<\kappa^*, \tag{5.19}$$

where

$$\kappa^*=\sup\left\{\frac{\|p\|_L}{x}\ln\frac{xe^x(1-h_0(1))}{\|p\|_L(e^x-1)}\colon\ 0<x<\ln\frac{1}{h_0(1)}\right\}. \tag{5.20}$$

Then the operator ℓ *given by (5.1) belongs to the set* $\widetilde{V}_{ab}^-(h)$.

Theorem 5.3. *Let conditions (2.3) and (4.2) be satisfied. Assume that* $\tau(t)\geq t$ *for almost every* $t\in[a,b]$ *and at least one of the following conditions is satisfied:*

(a) inequality (5.13) holds, where the functions z_0 *and* z_1 *are defined by (5.14) and (5.15), respectively;*

(b) inequalities (5.16) and (5.17) hold, where the function γ_0 *is given by (5.18);*

(c) $h_0(1)\neq 0$ *and condition (5.19) is satisfied, where the number* κ^* *is defined by (5.20).*

Then the operator ℓ *given by (5.1) belongs to the set* $\widetilde{V}_{ab}^-(h)$.

Remark 5.1. If $h_0(z_0)>0$, where z_0 is defined by (5.14), then the strict inequality (5.13) in Theorem 5.3(a) can be weakened. More precisely, the following assertion is true:

Theorem 5.4. *Let conditions (2.3) and (4.2) be satisfied. Assume that* $\tau(t)\geq t$ *for almost every* $t\in[a,b]$,

$$h_0(z_0)>0,$$

and

$$\max\left\{\frac{h_0(z_1)+(1-h_0(1))\,z_1(t)}{h_0(z_0)+(1-h_0(1))\,z_0(t)}\colon\ t\in[a,b]\right\}\leq 1-\frac{h_0(z_0)}{1-h_0(1)}, \tag{5.21}$$

where the functions z_0 *and* z_1 *are defined by (5.14) and (5.15), respectively. Then the operator* ℓ *given by (5.1) belongs to the set* $\widetilde{V}_{ab}^-(h)$.

Theorem 5.5. *Let conditions (2.3) and (3.4) be satisfied. Assume that*

$$\operatorname{ess\,sup}\left\{ \int_{\mu(t)}^{t} g(s)\,ds\colon\ t \in [a,b] \right\} < \omega^*, \tag{5.22}$$

where

$$\omega^* = \sup\left\{ \frac{1}{x} \ln \frac{x\alpha^x(\mu_*)}{\alpha^x(\mu_*) - f(x)}\colon\ x > 0,\ \widehat{h}(\alpha^x) > \alpha^x(a) \right\},$$

$$f(x) \overset{\text{df}}{=} \frac{\widehat{h}(\alpha^x) - \alpha^x(a)}{h(1) - 1} \quad \text{for } x > 0, \tag{5.23}$$

$$\widehat{h}(v) \overset{\text{df}}{=} \min\{h(1), h(v)\} \quad \text{for } v \in C([a,b];\mathbb{R}).$$

Then the operator ℓ *given by (5.2) belongs to the set* $\widetilde{V}_{ab}^{-}(h)$.

Corollary 5.2. *Let conditions (2.3) and (3.4) be satisfied. Assume that* $g \not\equiv 0$ *and*

$$\operatorname{ess\,sup}\left\{ \int_{\mu(t)}^{t} g(s)ds\colon\ t \in [a,b] \right\} < \xi^*,$$

where

$$\xi^* = \sup\left\{ \frac{\|g\|_L}{y} \ln \frac{ye^y\,(h(1)-1)}{\|g\|_L(e^y - 1)\,(\lambda + h_0(1))}\colon\ 0 < y < \ln \frac{\lambda + h_0(1)}{1 + h_1(1)} \right\}. \tag{5.24}$$

Then the operator ℓ *defined by (5.2) belongs to the set* $\widetilde{V}_{ab}^{-}(h)$.

Theorem 5.6. *Let conditions (2.3) and (4.2) be satisfied. Assume that* $g \not\equiv 0$ *and*

$$\max\left\{ \frac{z_1(a) - h(z_1) + (h(1)-1)\,z_1(t)}{z_0(a) - h(z_0) + (h(1)-1)\,z_0(t)}\colon\ t \in [a,b] \right\} < 1 - \frac{z_0(a) - h(z_0)}{h(1) - 1}, \tag{5.25}$$

where

$$z_0(t) = \int_t^b g(s)\,ds \quad \text{for } t \in [a,b],$$

$$z_1(t) = \int_t^b g(s) \left(\int_{\mu(s)}^{b} g(\xi)\,d\xi \right) ds \quad \text{for } t \in [a,b].$$

Then the operator ℓ *given by (5.2) belongs to the set* $\widetilde{V}_{ab}^{-}(h)$.

Theorem 5.7. *Let conditions (2.3) and (3.4) be satisfied. Assume that inequalities (4.10) and (4.11) are true, where*

$$z_0(t) = \exp\left(\int_t^b g(s)\,ds\right) \quad \textit{for } t \in [a,b],$$

$$z_1(t) = \int_t^b g(s)\sigma(s)\left(\int_{\mu(s)}^s g(\xi)\,d\xi\right)\exp\left(\int_t^s g(\eta)\,d\eta\right)ds \quad \textit{for } t \in [a,b], \tag{5.26}$$

and

$$\sigma(t) = \frac{1}{2}\,(1 + \operatorname{sgn}\,(t - \mu(t))) \quad \textit{for a.e. } t \in [a,b]. \tag{5.27}$$

Then the operator ℓ *given by (5.2) belongs to the set* $\widetilde{V}_{ab}^{-}(h)$.

Theorem 5.8. *Let conditions (3.1) and (3.5) be satisfied. If*

$$\int_a^b g(s)\,ds < (\lambda - h_1(1))\min\left\{1, \frac{1}{\lambda}\right\} \tag{5.28}$$

and

$$\frac{(1 - h_0(1))\min\left\{1, \frac{1}{\lambda}\right\}}{(\lambda - h_1(1))\min\left\{1, \frac{1}{\lambda}\right\} - \int_a^b g(s)\,ds} - 1 < \int_a^b p(s)\,ds \le (1 - h_0(1))\min\left\{1, \frac{1}{\lambda}\right\},$$

then the operator ℓ *given by (5.3) belongs to the set* $\widetilde{V}_{ab}^{-}(h)$.

Theorem 5.9. *Let conditions (2.3) and (3.4) be satisfied. Assume that inequality (5.28) is true and*

$$\omega\left(\int_a^b g(s)\,ds\right) < \int_a^b p(s)\,ds \le (1 - h_0(1))\min\left\{1, \frac{1}{\lambda}\right\},$$

where the function ω *is defined by (3.10). Then the operator* ℓ *given by (5.3) belongs to the set* $\widetilde{V}_{ab}^{-}(h)$.

Corollary 5.3. *Let conditions (3.1) and (3.4) be satisfied. Assume that either*

$$h(1) \le 1, \quad \frac{1 - h_0(1)}{\lambda - h_1(1)} - 1 < \int_a^b p(s)\,ds \le (1 - h_0(1))\min\left\{1, \frac{1}{\lambda}\right\}$$

or

$$h(1) > 1, \quad \int_a^b p(s)\,ds \le (1 - h_0(1)) \min\left\{1, \frac{1}{\lambda}\right\}.$$

Then the operator ℓ *given by (5.1) belongs to the set* $\widetilde{V}_{ab}^-(h)$.

Theorem 5.10. *Let conditions (2.3) and (3.4) be satisfied. Assume that the functions* p *and* τ *satisfy condition (5.7) or the assumptions of Theorem 5.3 or 5.4, whereas the functions* g *and* μ *satisfy the assumptions of Theorem 5.5, 5.6, or 5.7. Then the operator* ℓ *given by (5.3) belongs to the set* $\widetilde{V}_{ab}^-(h)$.

5.1. Proofs. We give the following lemmas prior to the proof of the statements formulated above:

Lemma 5.1. *Let the functional* h *be defined by formula (1.3), where* $\lambda > 0$ *and* $h_0, h_1 \in PF_{ab}$ *are such that conditions (3.1) and (5.6) are satisfied. Also assume that the operator* ℓ *is defined by (5.1),* $p \not\equiv 0$, *and condition (5.8) is satisfied, where the number* η_* *is defined by (5.9). Then there exists a function* $\gamma \in \widetilde{C}([a,b];\,]0,+\infty[)$ *that satisfies inequalities (3.2) and (3.3).*

Proof. According to (5.8) with η^* given by (5.9), there exist $x_0 > 0$ and $\varepsilon > 0$ such that

$$h(\beta^{x_0}) \ge 1 + \varepsilon \tag{5.29}$$

and the following relation is true:

$$\int_t^{\tau(t)} p(s)\,ds \ge \frac{1}{x_0} \ln \frac{x_0 \beta^{x_0}(\tau^*)}{\beta^{x_0}(\tau^*) + (h(\beta^{x_0}) - 1 - \varepsilon)\,(1 - h(1))^{-1}} \quad \text{for a.e. } t \in [a,b]. \tag{5.30}$$

We set

$$\delta = \frac{h(\beta^{x_0}) - 1 - \varepsilon}{1 - h(1)}. \tag{5.31}$$

By virtue of conditions (5.6) and (5.29), we get $\delta \ge 0$. Hence, relation (5.30) yields

$$e^{x_0 \int_t^{\tau(t)} p(s)\,ds} \ge \frac{x_0 \beta^{x_0}(\tau^*)}{\beta^{x_0}(\tau^*) + \delta} \ge \frac{x_0 \beta^{x_0}(\tau(t))}{\beta^{x_0}(\tau(t)) + \delta} \quad \text{for a.e. } t \in [a,b].$$

Consequently,

$$x_0 e^{x_0 \int_a^t p(s)\,ds} \le e^{x_0 \int_a^{\tau(t)} p(s)\,ds} + \delta \quad \text{for a.e. } t \in [a,b]. \tag{5.32}$$

We now set

$$\gamma(t) = e^{x_0 \int_a^t p(s)\,ds} + \delta \quad \text{for } t \in [a,b].$$

It is clear that $\gamma(t) > 0$ for $t \in [a,b]$, and, using condition (5.32), we get

$$\ell(\gamma)(t) = p(t)\left(e^{x_0 \int_a^{\tau(t)} p(s)\,ds} + \delta\right) \geq x_0 p(t) e^{x_0 \int_a^{t} p(s)\,ds} = \gamma'(t) \quad \text{for a.e. } t \in [a,b],$$

i.e., inequality (3.2) holds. On the other hand, in view of equality (5.31) and the assumption that $h(1) < 1$, inequality (3.3) is satisfied.

The lemma is proved.

Lemma 5.2. *Let the operator* ℓ *be defined by (5.2), let* $h \in F_{ab}$ *satisfy condition (2.3), and let inequality (5.22) be true, where the number* ω^* *is defined by (5.23). Then there exists a function* $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *that satisfies inequalities (3.2) and (3.3).*

Proof. According to (5.22) with ω^* given by (5.23), there exist $x_0 > 0$ and $\varepsilon > 0$ such that

$$\widehat{h}(\alpha^{x_0}) \geq \alpha^{x_0}(a) + \varepsilon \tag{5.33}$$

and the inequality

$$\int_{\mu(t)}^{t} g(s)ds \leq \frac{1}{x_0} \ln \frac{x_0 \alpha^{x_0}(\mu_*)}{\alpha^{x_0}(\mu_*) - \left(\widehat{h}(\alpha^{x_0}) - \alpha^{x_0}(a) - \varepsilon\right)(h(1)-1)^{-1}} \tag{5.34}$$

holds for almost every $t \in [a,b]$. We set

$$\delta = \frac{\widehat{h}(\alpha^{x_0}) - \alpha^{x_0}(a) - \varepsilon}{h(1) - 1}. \tag{5.35}$$

By virtue of conditions (2.3), (5.23), and (5.33), we get $\delta \in [0,1[$. Hence, relation (5.34) yields

$$e^{x_0 \int_{\mu(t)}^{t} g(s)\,ds} \leq \frac{x_0 \alpha^{x_0}(\mu_*)}{\alpha^{x_0}(\mu_*) - \delta} \leq \frac{x_0 \alpha^{x_0}(\mu(t))}{\alpha^{x_0}(\mu(t)) - \delta} \quad \text{for a.e. } t \in [a,b].$$

Consequently,

$$e^{x_0 \int_{\mu(t)}^{b} g(s)\,ds} - \delta \leq x_0 e^{x_0 \int_t^b g(s)\,ds} \quad \text{for a.e. } t \in [a,b]. \tag{5.36}$$

We now set

$$\gamma(t) = e^{x_0 \int_t^b g(s)\,ds} - \delta \quad \text{for } t \in [a,b].$$

It is clear that $\gamma(t) > 0$ for $t \in [a,b]$, and, using condition (5.36), we get

$$\ell(\gamma)(t) = -g(t)\left(e^{x_0 \int_{\mu(t)}^{b} g(s)\,ds} - \delta\right) \geq -x_0 g(t) e^{x_0 \int_t^b g(s)\,ds} = \gamma'(t) \quad \text{for a.e. } t \in [a,b],$$

i.e., inequality (3.2) holds. On the other hand, in view of relations (5.23) and (5.35) and the assumption that $h(1) > 1$, inequality (3.3) is satisfied.

The lemma is proved.

We are now in a position to prove Theorems 5.1–5.10.

Proof of Theorem 5.1. Let the operator ℓ be defined by (5.1). It is clear that $\ell \in P_{ab}$, and condition (5.7) implies the validity of relation (4.16) with $\ell_0 \equiv \ell$. According to Lemma 5.1, there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying conditions (3.2) and (3.3), which guarantees the validity of the inclusion $\ell \in U_{ab}^{-}(h)$ (see Theorem 3.1). Consequently, in view of Theorem 4.4 (with $\ell_0 \equiv \ell$ and $\ell_1 \equiv 0$), we get $\ell \in \widetilde{V}_{ab}^{-}(h)$.

The theorem is proved.

Proof of Corollary 5.1. It is not difficult to verify that

$$\frac{x\beta^x(\tau^*)}{\beta^x(\tau^*) + (h(\beta^x) - 1)\,(1 - h(1))^{-1}} \leq \frac{x\beta^x(b)}{\beta^x(b) + \big((\lambda - h_1(1))\, e^{x\|p\|_L} + h_0(1) - 1\big)\,(1 - h(1))^{-1}}$$

$$= \frac{xe^{x\|p\|_L}\,(1 - h(1))}{\big(e^{x\|p\|_L} - 1\big)\,(1 - h_0(1))}$$

for every $x > 0$ such that

$$(\lambda - h_1(1))\, e^{x\|p\|_L} > 1 - h_0(1).$$

Therefore, $\eta_* \leq \xi_*$, where η_* and ξ_* are defined by (5.9) and (5.11), respectively. Consequently, the assertion of the corollary follows immediately from Theorem 5.1.

Proof of Theorem 5.2. Let the operator ℓ be defined by (5.1). It is clear that $\ell \in P_{ab}$ and ℓ is a b-Volterra operator. According to Theorems 4.1 and 4.2, and Corollary 4.2 in [14], we conclude that each of conditions (a)–(c) guarantees the validity of the inclusion $\ell \in \widetilde{V}_{ab}^{+}(h_0)$. Moreover, by virtue of Theorem 2.1 in [14], there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ that satisfies inequality (3.12). Therefore, Lemma 4.2 guarantees that $\ell \in S_{ab}(b)$. Furthermore, the inclusion $\ell \in \widetilde{V}_{ab}^{+}(h_0)$ proved above yields $\ell \in \widetilde{U}_{ab}^{+}(h_0)$ (see Remark 3.3).

On the other hand, since we suppose that $\tau(t) \geq t$ for almost every $t \in [a,b]$, condition (5.12) implies the validity of (5.10), where ξ_* is defined by (5.11). Therefore, by analogy with the proof of Corollary 5.1, it can be shown that relation (5.8) is satisfied with η_* given by (5.9), and, thus, according to Lemma 5.1, there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ that satisfies conditions (3.2) and (3.3).

Consequently, by virtue of Theorem 4.1, we get $\ell \in \widetilde{V}_{ab}^{-}(h)$.

The theorem is proved.

Proof of Theorem 5.3. Let the operator ℓ be defined by (5.1). It is clear that $\ell \in P_{ab}$ and ℓ is a b-Volterra operator. According to Theorems 4.1 and 4.2, and Corollary 4.2 in [14], we conclude that each of conditions (a)–(c) guarantees the validity of the inclusion $\ell \in \widetilde{V}_{ab}^{+}(h_0)$. Therefore, the assumptions of Corollary 4.3 are satisfied.

The theorem is proved.

Proof of Theorem 5.4. Let the operator ℓ be defined by (5.1). It is clear that $\ell \in P_{ab}$ and ℓ is a b-Volterra operator. Using condition (5.21), one can easily verify that

$$\varrho_3(t) \leq \varrho_2(t) \quad \text{for } t \in [a,b],$$

where the functions ϱ_2 and ϱ_3 are defined by (4.5) and (4.6). Consequently, the assumptions of Corollary 4.2 are satisfied with $k = 2$ and $m = 3$.

The theorem is proved.

Proof of Theorem 5.5. The assertion of the theorem follows immediately from Lemma 5.2 and Theorem 4.3.

Proof of Corollary 5.2. It is not difficult to verify that

$$\frac{x\alpha^x(\mu_*)}{\alpha^x(\mu_*) - \left(\widehat{h}(\alpha^x) - \alpha^x(a)\right)(h(1)-1)^{-1}} \geq \frac{x\alpha^x(a)}{\alpha^x(a) - (\lambda + h_0(1) - (1 + h_1(1))\,\alpha^x(a))\,(h(1)-1)^{-1}}$$

$$= \frac{xe^{x\|g\|_L}\,(h(1)-1)}{\left(e^{x\|g\|_L} - 1\right)(\lambda + h_0(1))}$$

for every $x > 0$ such that

$$\lambda + h_0(1) > (1 + h_1(1))\,e^{x\|g\|_L}.$$

Therefore, $\xi^* \leq \omega^*$, where ω^* and ξ^* are defined by (5.23) and (5.24), respectively. Consequently, the validity of the corollary follows immediately from Theorem 5.5.

Proof of Theorem 5.6. Let the operator ℓ be defined by (5.2). It is clear that $-\ell \in P_{ab}$. According to condition (5.25), there exists $\delta \in [0, 1[$ such that the inequality

$$\frac{z_1(a) - h(z_1)}{h(1)-1} + z_1(t) \leq \left(\delta - \frac{z_0(a) - h(z_0)}{h(1)-1}\right)\left(\frac{z_0(a) - h(z_0)}{h(1)-1} + z_0(t)\right)$$

holds for $t \in [a, b]$. However, this means that

$$\varrho_3(t) \leq \delta\varrho_2(t) \quad \text{for } t \in [a, b],$$

where the functions ϱ_2 and ϱ_3 are defined in Corollary 4.4(a). Consequently, the assumptions of Corollary 4.4(a) are satisfied with $k = 2$ and $m = 3$.

The theorem is proved.

Proof of Theorem 5.7. Let the operators ℓ and $\bar{\ell}$ be defined by formula (5.2) and the relation

$$\bar{\ell}(v)(t) \overset{\text{df}}{=} g(t)\sigma(t)\left(\int_{\mu(t)}^{t} g(s)v(\mu(s))ds\right) \quad \text{for a.e. } t \in [a, b] \text{ and all } v \in C([a, b]; \mathbb{R}),$$

respectively, where the function σ is given by (5.27). It is clear that $-\ell \in P_{ab}$, $\bar{\ell} \in P_{ab}$, and

$$\ell(1)(t)\vartheta(v)(t) - \ell(\vartheta(v))(t) = g(t)\int_{\mu(t)}^{t} g(s)v(\mu(s))ds \leq \bar{\ell}(v)(t) \quad \text{for a.e. } t \in [a, b] \text{ and all } v \in C([a, b]; \mathbb{R}_+),$$

where the operator ϑ is defined by (4.8) and (4.9), and, thus, condition (4.12) is satisfied on the set $C([a,b];\mathbb{R}_+)$. Therefore, the assumptions of Corollary 4.4(b) are satisfied.

The theorem is proved.

Proof of Theorem 5.8. Let the operator ℓ be defined by (5.3), let

$$\ell_0(v)(t) \stackrel{\text{df}}{=} p(t)v(\tau(t)) \quad \text{for a.e. } t \in [a,b] \text{ and all } v \in C([a,b];\mathbb{R}), \tag{5.37}$$

and let

$$\ell_1(v)(t) \stackrel{\text{df}}{=} g(t)v(\mu(t)) \quad \text{for a.e. } t \in [a,b] \text{ and all } v \in C([a,b];\mathbb{R}). \tag{5.38}$$

It is clear that $\ell_0, \ell_1 \in P_{ab}$ and $\ell = \ell_0 - \ell_1$. Therefore, the validity of the theorem follows from Theorems 3.3 and 4.4.

Proof of Theorem 5.9. Let the operators ℓ, ℓ_0, and ℓ_1 be defined by (5.3), (5.37), and (5.38), respectively. It is clear that $\ell_0, \ell_1 \in P_{ab}$ and $\ell = \ell_0 - \ell_1$. Therefore, the assertion of the theorem follows from Theorems 3.4 and 4.4.

Proof of Corollary 5.3. The validity of the corollary follows immediately from Theorems 5.8 and 5.9 with $g \equiv 0$.

Proof of Theorem 5.10. Let the operators ℓ, ℓ_0, and ℓ_1 be defined by (5.3), (5.37), and (5.38), respectively. It is clear that $\ell_0, \ell_1 \in P_{ab}$ and $\ell = \ell_0 - \ell_1$. Therefore, the assertion of the theorem follows immediately from Theorem 4.5, Theorems 5.3–5.7, and Corollary 5.3.

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