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On oscillation and nonoscillation of two-dimensional linear differential systems

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Dedicated to the memory of Professor Levan Magnaradze on the occasion of his 100th birthday anniversary

Abstract. New oscillation and nonoscillation criteria are established for the two-dimensional system $u' = q(t)v$, $v' = -p(t)u$, where $p, q\colon [0, +\infty[\to \mathbb{R}$ are locally integrable functions, $q(t) \geq 0$ for a.e. $t \geq 0$, and $\int_0^{+\infty} q(s)\mathrm{d}s < +\infty$.

Keywords. Two-dimensional system of linear ODE, oscillation, nonoscillation.

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1 Formulation of the main results

Consider the system

$$\begin{aligned} u' &= q(t)v, \\ v' &= -p(t)u, \end{aligned} \tag{1.1}$$

where $p, q\colon [0, +\infty[\to \mathbb{R}$ are locally Lebesgue integrable functions. Under a *solution* of system (1.1) we understand a vector-function $(u, v)\colon [0, +\infty[\to \mathbb{R}^2$ with locally absolutely continuous components satisfying equalities (1.1) almost everywhere in $[0, +\infty[$. A solution (u, v) of system (1.1) is said to be *nontrivial* if $u \not\equiv 0$ in any neighborhood of $+\infty$. A nontrivial solution (u, v) of system (1.1) is called *oscillatory* if the function u has a sequence of zeros tending to infinity.

Definition 1.1. System (1.1) is said to be *oscillatory* if every nontrivial solution of this system is oscillatory, and *nonoscillatory* otherwise.

It is well known (see, e.g., [9] or [6, Lemma 2]) that if

$$q(t) \geq 0 \quad \text{for a.e. } t \geq 0 \tag{1.2}$$

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and system (1.1) has at least one nontrivial oscillatory solution, then every solution of (1.1) is oscillatory. Moreover, it is clear that system (1.1) is nonoscillatory provided that $q(t) = 0$ for a.e. $t \geq t_0$. Therefore we will assume throughout the paper that relation (1.2) holds and $q \not\equiv 0$ in any neighborhood of $+\infty$.

In the case where $\int_0^{+\infty} q(s)\mathrm{d}s = +\infty$, conditions for the oscillation and nonoscillation of system (1.1) are contained, e.g., in [1–13] (see also the references therein). In the present paper, we consider the case where

$$\int_0^{+\infty} q(s)\mathrm{d}s < +\infty.$$

Let

$$f(t) := \int_t^{+\infty} q(s)\mathrm{d}s \quad \text{for } t \geq 0.$$

For any $\lambda > 1$, we put

$$c(t;\lambda) := (\lambda - 1) f^{\lambda-1}(t) \int_0^t \frac{q(s)}{f^{\lambda}(s)} \Big(\int_0^s f^{\lambda}(\xi) p(\xi)\mathrm{d}\xi \Big)\mathrm{d}s \quad \text{for } t \geq 0.$$

The following theorem is an analogue of the well-known Hartman–Wintner theorem (see, e.g., [4, Theorem 7.3]).

Theorem 1.1. *Let there exist $\lambda > 1$ such that either*

$$\lim_{t\to+\infty} c(t;\lambda) = +\infty$$

or

$$-\infty < \liminf_{t\to+\infty} c(t;\lambda) < \limsup_{t\to+\infty} c(t;\lambda).$$

Then system (1.1) *is oscillatory.*

If we take this theorem into account, then it is obvious that, for given $\lambda > 1$, the following two cases remain uncovered: the first case where

$$\text{there exists a finite limit } \lim_{t\to+\infty} c(t;\lambda), \tag{1.3}$$

and the other case where

$$\liminf_{t\to+\infty} c(t;\lambda) = -\infty.$$

Below, we establish new oscillation and nonoscillation criteria assuming that (1.3) holds for some $\lambda > 1$. Having such λ, we denote

$$Q(t;\lambda) := \frac{1}{f^{\lambda-1}(t)} \Big(c_0(\lambda) - \int_0^t f^{\lambda}(s) p(s)\mathrm{d}s \Big) \quad \text{for } t \geq 0,$$

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where

$$c_0(\lambda) = \lim_{t\to+\infty} c(t;\lambda). \tag{1.4}$$

Moreover, for any $\mu < 1$, we put

$$H(t;\mu) := f^{1-\mu}(t)\int_0^t f^{\mu}(s)p(s)\mathrm{d}s \quad \text{for } t \geq 0.$$

Finally, let

$$Q_*(\lambda) = \liminf_{t\to+\infty} Q(t;\lambda), \qquad Q^*(\lambda) = \limsup_{t\to+\infty} Q(t;\lambda),$$
$$H_*(\mu) = \liminf_{t\to+\infty} H(t;\mu), \qquad H^*(\mu) = \limsup_{t\to+\infty} H(t;\mu).$$

Now we are in a position to formulate our main results, their proofs are given later in Section 3.

Theorem 1.2. *Let there exist $\lambda > 1$ such that condition* (1.3) *holds and*

$$\limsup_{t\to+\infty} \frac{-1}{f^{\lambda-1}(t)\ln f(t)}\big(c_0(\lambda) - c(t;\lambda)\big) > \frac{1}{4}, \tag{1.5}$$

where the number $c_0(\lambda)$ is defined by formula (1.4)*. Then system* (1.1) *is oscillatory.*

Corollary 1.1. *Let there exist $\lambda > 1$ such that condition* (1.3) *holds, $Q_*(\lambda) > -\infty$, and*

$$\limsup_{t\to+\infty} \frac{-1}{\ln f(t)} \int_0^t f(s)p(s)\mathrm{d}s > \frac{1}{4}. \tag{1.6}$$

Then system (1.1) *is oscillatory.*

Corollary 1.2. *Let there exist $\lambda > 1$ and $\mu < 1$ such that condition* (1.3) *holds and*

$$\liminf_{t\to+\infty}\big(Q(t;\lambda) + H(t;\mu)\big) > \frac{1}{4(\lambda-1)} + \frac{1}{4(1-\mu)}. \tag{1.7}$$

Then system (1.1) *is oscillatory.*

Corollary 1.3. *Let there exist $\lambda > 1$ such that condition* (1.3) *holds and either*

$$Q_*(\lambda) > \frac{1}{4(\lambda-1)} \tag{1.8}$$

or

$$H_*(\mu) > \frac{1}{4(1-\mu)} \tag{1.9}$$

for some $\mu < 1$. Then system (1.1) *is oscillatory.*

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Remark 1.1. It might seem that if assumption (1.9) is satisfied, then assumption (1.3) is redundant. However, it follows from Lemma 2.12 that, under assumption (1.9), the function $c(\cdot\,;\lambda)$ possesses a limit for every $\lambda > 1$ and

$$\lim_{t\to+\infty} c(t;\lambda) > -\infty.$$

If this limit is equal to $+\infty$ then system (1.1) is oscillatory according to Theorem 1.1. Therefore assumption (1.3) is necessary in a certain sense, also in the case where inequality (1.9) is supposed to be satisfied.

The next theorem deals with the upper limit of the sum on the left-hand side of inequality (1.7) and thus it complements Corollary 1.2 in a certain sense.

Theorem 1.3. *Let there exist* $\lambda > 1$ *and* $\mu < 1$ *such that condition* (1.3) *holds and*

$$\limsup_{t\to+\infty}\big(Q(t;\lambda) + H(t;\mu)\big) > \frac{\lambda^2}{4(\lambda-1)} + \frac{\mu^2}{4(1-\mu)}. \tag{1.10}$$

Then system (1.1) *is oscillatory.*

In view of Corollary 1.3, it is natural to restrict our further consideration to the case where

$$Q_*(\lambda) \le \frac{1}{4(\lambda-1)} \quad \text{and} \quad H_*(\mu) \le \frac{1}{4(1-\mu)}.$$

Theorem 1.4. *Let there exist* $\lambda > 1$ *and* $\mu < 1$ *such that condition* (1.3) *holds and either*

$$\frac{\lambda(2-\lambda)}{4(\lambda-1)} \le Q_*(\lambda) \le \frac{1}{4(\lambda-1)}, \tag{1.11}$$

$$H^*(\mu) > \frac{\mu^2}{4(1-\mu)} + \frac{1+\sqrt{1-4(\lambda-1)Q_*(\lambda)}}{2} \tag{1.12}$$

or

$$\frac{\mu(2-\mu)}{4(1-\mu)} \le H_*(\mu) \le \frac{1}{4(1-\mu)}, \tag{1.13}$$

$$Q^*(\lambda) > \frac{\lambda^2}{4(\lambda-1)} - \frac{1-\sqrt{1-4(1-\mu)H_*(\mu)}}{2}. \tag{1.14}$$

Then system (1.1) *is oscillatory.*

If both conditions (1.11) and (1.13) are satisfied, then the oscillation criteria (1.12) and (1.14) can be slightly refined as shown in the following two statements.

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Theorem 1.5. *Let there exist* $\lambda > 1$ *and* $\mu < 1$ *such that conditions* (1.3), (1.11), *and* (1.13) *are fulfilled and*

$$\limsup_{t\to+\infty}\big(Q(t;\lambda)+H(t;\mu)\big) > Q_*(\lambda)+H_*(\mu) + \frac{1}{2}\Big(\sqrt{1-4(\lambda-1)Q_*(\lambda)}+\sqrt{1-4(1-\mu)H_*(\mu)}\Big). \quad (1.15)$$

Then system (1.1) *is oscillatory.*

Corollary 1.4. *Let there exist* $\lambda > 1$ *and* $\mu < 1$ *such that conditions* (1.3), (1.11), *and* (1.13) *hold. Then either of the inequalities*

$$Q^*(\lambda) > Q_*(\lambda)+\frac{1}{2}\Big(\sqrt{1-4(\lambda-1)Q_*(\lambda)}+\sqrt{1-4(1-\mu)H_*(\mu)}\Big) \quad (1.16)$$

and

$$H^*(\mu) > H_*(\mu)+\frac{1}{2}\Big(\sqrt{1-4(\lambda-1)Q_*(\lambda)}+\sqrt{1-4(1-\mu)H_*(\mu)}\Big) \quad (1.17)$$

guarantees that system (1.1) *is oscillatory.*

At the end of this section, we present two nonoscillation results.

Theorem 1.6. *Let there exist* $\lambda > 1$ *such that condition* (1.3) *holds and either*

$$-\frac{(2\lambda-3)(2\lambda-1)}{4(\lambda-1)} < Q_*(\lambda), \quad Q^*(\lambda) < \frac{1}{4(\lambda-1)} \quad (1.18)$$

or

$$-\frac{(3-2\mu)(1-2\mu)}{4(1-\mu)} < H_*(\mu), \quad H^*(\mu) < \frac{1}{4(1-\mu)} \quad (1.19)$$

for some $\mu < 1$. *Then system* (1.1) *is nonoscillatory.*

Remark 1.2. It might seem that if assumption (1.19) is satisfied, then assumption (1.3) is redundant. However, it follows from Lemmas 2.11 and 2.12 that, under assumption (1.19), the function $c(\cdot\,;\lambda)$ possesses a finite limit for every $\lambda > 1$. Therefore assumption (1.3) is necessary also in the case where inequalities (1.19) are supposed to be satisfied.

Theorem 1.7. *Let there exist* $\lambda > 1$ *such that condition* (1.3) *holds and either*

$$-\infty < Q_*(\lambda) \le -\frac{(2\lambda-3)(2\lambda-1)}{4(\lambda-1)}\,, \quad (1.20)$$

$$Q^*(\lambda) < Q_*(\lambda)+1-\lambda+\sqrt{1-4(\lambda-1)Q_*(\lambda)} \quad (1.21)$$

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or

$$-\infty < H_*(\mu) \le -\frac{(3-2\mu)(1-2\mu)}{4(1-\mu)}\,, \tag{1.22}$$

$$H^*(\mu) < H_*(\mu) + \mu - 1 + \sqrt{1 - 4(1-\mu)H_*(\mu)} \tag{1.23}$$

for some $\mu < 1$. *Then system* (1.1) *is nonoscillatory.*

Remark 1.3. It might seem that if assumptions (1.22), (1.23) are satisfied, then assumption (1.3) is redundant. However, it follows from Lemmas 2.11 and 2.12 that, under assumption (1.22), (1.23), the function $c(\cdot\,;\lambda)$ possesses a finite limit for every $\lambda > 1$. Therefore, assumption (1.3) in the previous theorem is necessary also in the case where inequalities (1.22), (1.23) are supposed to be satisfied.

2 Auxiliary lemmas

Lemma 2.1. *Let* $\lambda > 1$ *be such that*

$$\liminf_{t\to+\infty} c(t;\lambda) > -\infty \tag{2.1}$$

and (u, v) *be a solution of system* (1.1) *satisfying the relation*

$$u(t) \neq 0 \quad \text{for } t \ge a \tag{2.2}$$

with some $a \ge 0$. *Then*

$$\int_a^{+\infty} q(s) f^{\lambda-2}(s)\Big[f(s)\rho(s) + \frac{\lambda}{2}\Big]^2 \mathrm{d}s < +\infty, \tag{2.3}$$

where

$$\rho(t) = \frac{v(t)}{u(t)} \quad \text{for } t \ge a. \tag{2.4}$$

Proof. In view of (1.1), relation (2.4) yields

$$\rho'(t) = -p(t) - q(t)\rho^2(t) \quad \text{for a.e. } t \ge a. \tag{2.5}$$

Multiplying both sides of equality (2.5) by the expression $f^\lambda(t)$ and integrating them from a to t, one gets

$$\begin{aligned} f^\lambda(t)\rho(t) = f^\lambda(a)\rho(a) - \int_a^t f^\lambda(s)p(s)\mathrm{d}s + \frac{\lambda^2}{4}\int_a^t q(s) f^{\lambda-2}(s)\mathrm{d}s \\ - \int_a^t q(s) f^{\lambda-2}(s)\Big[f(s)\rho(s) + \frac{\lambda}{2}\Big]^2 \mathrm{d}s \quad \text{for } t \ge a, \end{aligned}$$

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whence it follows that

$$f^{\lambda-1}(t)\Big[f(t)\rho(t)+\frac{\lambda}{2}\Big]=-\int_a^t q(s)f^{\lambda-2}(s)\Big[f(s)\rho(s)+\frac{\lambda}{2}\Big]^2\mathrm{d}s \\ -\int_a^t f^{\lambda}(s)p(s)\mathrm{d}s+\frac{\lambda(\lambda-2)}{4(\lambda-1)}\,f^{\lambda-1}(t)+\delta \quad (2.6)$$

for $t\geq a$, where

$$\delta=f^{\lambda}(a)\rho(a)+\frac{\lambda^2}{4(\lambda-1)}\,f^{\lambda-1}(a).$$

Now we multiply both sides of equality (2.6) by the expression $q(t)f^{-\lambda}(t)$ and integrate them from a to t and thus we get

$$\int_a^t \frac{q(s)}{f(s)}\Big[f(s)\rho(s)+\frac{\lambda}{2}\Big]\mathrm{d}s \\ =-\int_a^t \frac{q(s)}{f^{\lambda}(s)}\Big(\int_a^s f^{\lambda}(\xi)p(\xi)\mathrm{d}\xi\Big)\mathrm{d}s \\ -\int_a^t \frac{q(s)}{f^{\lambda}(s)}\Big(\int_a^s q(\xi)f^{\lambda-2}(\xi)\Big[f(\xi)\rho(\xi)+\frac{\lambda}{2}\Big]^2\mathrm{d}\xi\Big)\mathrm{d}s \\ +\frac{\lambda(\lambda-2)}{4(\lambda-1)}\,\ln\frac{f(a)}{f(t)}+\frac{\delta}{\lambda-1}\Big(\frac{1}{f^{\lambda-1}(t)}-\frac{1}{f^{\lambda-1}(a)}\Big)\quad \text{for } t\geq a. \quad (2.7)$$

Suppose on the contrary that inequality (2.3) does not hold, i.e.,

$$\int_a^{+\infty} q(s)f^{\lambda-2}(s)\Big[f(s)\rho(s)+\frac{\lambda}{2}\Big]^2\mathrm{d}s=+\infty. \quad (2.8)$$

Obviously, for any $t\geq\tau\geq a$, we have

$$f^{\lambda-1}(t)\int_a^t \frac{q(s)}{f^{\lambda}(s)}\Big(\int_a^s q(\xi)f^{\lambda-2}(\xi)\Big[f(\xi)\rho(\xi)+\frac{\lambda}{2}\Big]^2\mathrm{d}\xi\Big)\mathrm{d}s \\ \geq f^{\lambda-1}(t)\int_\tau^t \frac{q(s)}{f^{\lambda}(s)}\Big(\int_a^s q(\xi)f^{\lambda-2}(\xi)\Big[f(\xi)\rho(\xi)+\frac{\lambda}{2}\Big]^2\mathrm{d}\xi\Big)\mathrm{d}s \\ \geq f^{\lambda-1}(t)\int_\tau^t \frac{q(s)}{f^{\lambda}(s)}\mathrm{d}s\int_a^\tau q(\xi)f^{\lambda-2}(\xi)\Big[f(\xi)\rho(\xi)+\frac{\lambda}{2}\Big]^2\mathrm{d}\xi \\ =\frac{1}{\lambda-1}\Big(1-\frac{f^{\lambda-1}(t)}{f^{\lambda-1}(\tau)}\Big)\int_a^\tau q(s)f^{\lambda-2}(s)\Big[f(s)\rho(s)+\frac{\lambda}{2}\Big]^2\mathrm{d}s.$$

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Consequently, on account of equality (2.8) and the relation

$$\lim_{t\to+\infty} f(t) = 0, \tag{2.9}$$

we get

$$\lim_{t\to+\infty} f^{\lambda-1}(t) \int_a^t \frac{q(s)}{f^{\lambda}(s)} \Big(\int_a^s q(\xi) f^{\lambda-2}(\xi) \Big[f(\xi)\rho(\xi) + \frac{\lambda}{2} \Big]^2 \mathrm{d}\xi \Big) \mathrm{d}s = +\infty. \tag{2.10}$$

Moreover, by virtue of equality (2.9), it is clear that

$$\lim_{t\to+\infty} f^{\lambda-1}(t) \ln \frac{f(a)}{f(t)} = 0. \tag{2.11}$$

Therefore, by virtue of relations (2.1) and (2.9)–(2.11), equality (2.7) implies that there exists $a_1 > a$ such that

$$\begin{aligned} &\int_a^t \frac{q(s)}{f(s)} \Big[f(s)\rho(s) + \frac{\lambda}{2} \Big] \mathrm{d}s \\ &\quad \le -\frac{1}{2} \int_a^t \frac{q(s)}{f^{\lambda}(s)} \Big(\int_a^s q(\xi) f^{\lambda-2}(\xi) \Big[f(\xi)\rho(\xi) + \frac{\lambda}{2} \Big]^2 \mathrm{d}\xi \Big) \mathrm{d}s \end{aligned} \tag{2.12}$$

for $t \ge a_1$. By using Hölder's inequality we get

$$\begin{aligned} &\Big(\int_a^t \frac{q(s)}{f(s)} \Big[f(s)\rho(s) + \frac{\lambda}{2} \Big] \mathrm{d}s \Big)^2 \\ &\quad \le \int_a^t \frac{q(s)}{f^{\lambda}(s)} \, \mathrm{d}s \int_a^t q(s) f^{\lambda-2}(s) \Big[f(s)\rho(s) + \frac{\lambda}{2} \Big]^2 \mathrm{d}s \\ &\quad \le \frac{1}{(\lambda-1) f^{\lambda-1}(t)} \int_a^t q(s) f^{\lambda-2}(s) \Big[f(s)\rho(s) + \frac{\lambda}{2} \Big]^2 \mathrm{d}s \end{aligned} \tag{2.13}$$

for $t \ge a$ and thus inequality (2.12) yields

$$\begin{aligned} &\frac{\lambda-1}{4} \Big[\int_a^t \frac{q(s)}{f^{\lambda}(s)} \Big(\int_a^s q(\xi) f^{\lambda-2}(\xi) \Big[f(\xi)\rho(\xi) + \frac{\lambda}{2} \Big]^2 \mathrm{d}\xi \Big) \mathrm{d}s \Big]^2 \\ &\quad \le \frac{1}{f^{\lambda-1}(t)} \int_a^t q(s) f^{\lambda-2}(s) \Big[f(s)\rho(s) + \frac{\lambda}{2} \Big]^2 \mathrm{d}s \quad \text{for } t \ge a_1. \end{aligned}$$

Therefore we have

$$v'(t) \ge \frac{\lambda-1}{4} \frac{q(t)}{f(t)} v^2(t) \quad \text{for a.e. } t \ge a_1, \tag{2.14}$$

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where

$$v(t) = \int_a^t \frac{q(s)}{f^\lambda(s)} \Big(\int_a^s q(\xi) f^{\lambda-2}(\xi) \Big[f(\xi)\rho(\xi) + \frac{\lambda}{2} \Big]^2 \mathrm{d}\xi \Big) \mathrm{d}s \quad \text{for } t \geq a_1.$$

Now relation (2.14) leads to

$$\frac{1}{v(a_1)} \geq \frac{1}{v(a_1)} - \frac{1}{v(t)} \geq \frac{\lambda-1}{4} \int_{a_1}^t \frac{q(s)}{f(s)}\, \mathrm{d}s = \frac{\lambda-1}{4} \ln \frac{f(a_1)}{f(t)} \quad \text{for } t \geq a_1,$$

which, in view of equality (2.9), yields the contradiction $1 \geq +\infty$. □

Lemma 2.2. *Let* (u, v) *be a solution of system* (1.1) *satisfying relation* (2.2) *with some* $a \geq 0$. *Moreover, let* $\lambda > 1$ *be such that inequality* (2.3) *holds, where the function* ρ *is defined by formula* (2.4). *Then there exists a finite limit*

$$\lim_{t \to +\infty} c(t; \lambda).$$

Proof. Analogously to the proof of Lemma 2.1 we obtain relation (2.6), whence, in view of assumption (2.3), we get

$$\begin{aligned} f^{\lambda-1}(t)\Big[f(t)\rho(t) + \frac{\lambda}{2} \Big] &= \delta(a) - \int_0^t f^\lambda(s) p(s) \mathrm{d}s + \frac{\lambda(\lambda-2)}{4(\lambda-1)}\, f^{\lambda-1}(t) \\ &\quad + \int_t^{+\infty} q(s) f^{\lambda-2}(s) \Big[f(s)\rho(s) + \frac{\lambda}{2} \Big]^2 \mathrm{d}s \quad \text{for } t \geq a, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} \delta(a) &= f^\lambda(a)\rho(a) + \frac{\lambda^2}{4(\lambda-1)}\, f^{\lambda-1}(a) \\ &\quad + \int_0^a f^\lambda(s) p(s) \mathrm{d}s - \int_a^{+\infty} q(s) f^{\lambda-2}(s) \Big[f(s)\rho(s) + \frac{\lambda}{2} \Big]^2 \mathrm{d}s. \end{aligned}$$

Multiplying both sides of equality (2.15) by the expression $q(t) f^{-\lambda}(t)$ and integrating them from a to t, one gets

$$\begin{aligned} \int_a^t \frac{q(s)}{f(s)} \Big[f(s)\rho(s) + \frac{\lambda}{2} \Big] \mathrm{d}s &= \frac{1}{(\lambda-1) f^{\lambda-1}(t)} \big(\delta(a) - c(t; \lambda) \big) + \frac{\lambda(\lambda-2)}{4(\lambda-1)} \ln \frac{f(a)}{f(t)} \\ &\quad + I(t) - \frac{1}{(\lambda-1) f^{\lambda-1}(a)} \big(\delta(a) - c(a; \lambda) \big) \quad \text{for } t \geq a, \end{aligned} \tag{2.16}$$

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where, for $t \geq a$,

$$I(t) := \int_a^t \frac{q(s)}{f^{\lambda}(s)} \left(\int_s^{+\infty} q(\xi) f^{\lambda-2}(\xi) \left[f(\xi)\rho(\xi) + \frac{\lambda}{2} \right]^2 \mathrm{d}\xi \right) \mathrm{d}s. \tag{2.17}$$

Obviously, we have

$$0 \leq I(t) \leq I(\tau) + \int_\tau^t \frac{q(s)}{f^{\lambda}(s)} \, \mathrm{d}s \int_\tau^{+\infty} q(s) f^{\lambda-2}(s) \left[f(s)\rho(s) + \frac{\lambda}{2} \right]^2 \mathrm{d}s$$

for $t \geq \tau \geq a$. The latter inequalities imply, on account of (2.9), that

$$\begin{aligned} 0 &\leq \limsup_{t\to+\infty} f^{\lambda-1}(t) I(t) \\ &\leq \frac{1}{\lambda - 1} \int_\tau^{+\infty} q(s) f^{\lambda-2}(s) \left[f(s)\rho(s) + \frac{\lambda}{2} \right]^2 \mathrm{d}s \quad \text{for } \tau \geq a \end{aligned}$$

and thus, in view of assumption (2.3), we obtain

$$\lim_{t\to+\infty} f^{\lambda-1}(t) I(t) = 0. \tag{2.18}$$

On the other hand, by using Hölder's inequality, we can check that inequality (2.13) holds for $t \geq a$, whence we get

$$\begin{aligned} &\left(\int_a^t \frac{q(s)}{f(s)} \left[f(s)\rho(s) + \frac{\lambda}{2} \right] \mathrm{d}s \right)^2 \\ &\quad \leq \frac{1}{(\lambda - 1) f^{\lambda-1}(t)} \int_a^{+\infty} q(s) f^{\lambda-2}(s) \left[f(s)\rho(s) + \frac{\lambda}{2} \right]^2 \mathrm{d}s \quad \text{for } t \geq a. \end{aligned}$$

Consequently, we have

$$\lim_{t\to+\infty} f^{\lambda-1}(t) \int_a^t \frac{q(s)}{f(s)} \left[f(s)\rho(s) + \frac{\lambda}{2} \right] \mathrm{d}s = 0, \tag{2.19}$$

because the function f satisfies relation (2.9). It is also obvious that equality (2.11) holds. Therefore, according to conditions (2.9), (2.11), (2.18), and (2.19), it follows from expression (2.16) that

$$\lim_{t\to+\infty} c(t; \lambda) = \delta(a).$$ □

The following statement follows from Lemmas 2.1 and 2.2 and their proofs.

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Lemma 2.3. *Let $\lambda > 1$ be such that condition* (2.1) *holds and* (u, v) *be a solution of system* (1.1) *satisfying relation* (2.2) *with some $a \geq 0$. Then inequality* (2.3) *is satisfied, where the function ρ is defined by formula* (2.4)*, and*

$$\begin{aligned} f^{\lambda-1}(t)\Big[f(t)\rho(t)+\frac{\lambda}{2}\Big] &= c_0(\lambda)-\int_0^t f^{\lambda}(s)p(s)\mathrm{d}s+\frac{\lambda(\lambda-2)}{4(\lambda-1)}\,f^{\lambda-1}(t) \\ &\quad+\int_t^{+\infty} q(s)f^{\lambda-2}(s)\Big[f(s)\rho(s)+\frac{\lambda}{2}\Big]^2\mathrm{d}s \quad \text{for } t \geq a, \end{aligned} \tag{2.20}$$

where $c_0(\lambda)$ is given by equality (1.4).

Lemma 2.4. *Let system* (1.1) *be nonoscillatory and $\lambda > 1$ be such that conditions* (1.3) *and* (1.11) *hold. Then every nontrivial solution* (u, v) *of system* (1.1) *admits the estimate*

$$\liminf_{t\to+\infty}\Big[f(t)\frac{v(t)}{u(t)}+\frac{\lambda}{2}\Big] \geq \frac{\lambda-1-\sqrt{1-4(\lambda-1)Q_*(\lambda)}}{2}. \tag{2.21}$$

Proof. Let (u, v) be a nontrivial solution of system (1.1). Since (1.1) is nonoscillatory, there exists $a \geq 0$ such that (2.2) holds. According to Lemma 2.3, relations (2.3) and (2.20) are fulfilled, where the function ρ is defined by formula (2.4) and $c_0(\lambda)$ is given by equality (1.4). It follows from (2.20) that

$$\begin{aligned} f(t)\rho(t)+\frac{\lambda}{2} &= Q(t;\lambda)+\frac{\lambda(\lambda-2)}{4(\lambda-1)} \\ &\quad+\frac{1}{f^{\lambda-1}(t)}\int_t^{+\infty} q(s)f^{\lambda-2}(s)\Big[f(s)\rho(s)+\frac{\lambda}{2}\Big]^2\mathrm{d}s \quad \text{for } t \geq a. \end{aligned} \tag{2.22}$$

Now we put

$$m = \liminf_{t\to+\infty}\Big[f(t)\rho(t)+\frac{\lambda}{2}\Big]. \tag{2.23}$$

If $m = +\infty$, then the desired inequality (2.21) holds trivially. Now assume that $m < +\infty$. Then it follows from relation (2.22) that

$$m \geq Q_*(\lambda)+\frac{\lambda(\lambda-2)}{4(\lambda-1)}. \tag{2.24}$$

If $Q_*(\lambda) = \frac{\lambda(2-\lambda)}{4(\lambda-1)}$, then the desired inequality (2.21) is fulfilled because $m \geq 0$. Hence, we suppose in what follows that $Q_*(\lambda) > \frac{\lambda(2-\lambda)}{4(\lambda-1)}$ and thus $m > 0$ (see relation (2.24)). Let $\varepsilon \in\,]0, m]$ be arbitrary and choose $t_\varepsilon \geq a$ such that

$$f(t)\rho(t)+\frac{\lambda}{2} \geq m-\varepsilon, \quad Q(t;\lambda) \geq Q_*(\lambda)-\varepsilon \quad \text{for } t \geq t_\varepsilon.$$

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Then from equality (2.22) we get

$$f(t)\rho(t) + \frac{\lambda}{2} \geq Q_*(\lambda) - \varepsilon + \frac{\lambda(\lambda-2)}{4(\lambda-1)} + \frac{(m-\varepsilon)^2}{\lambda-1} \quad \text{for } t \geq t_\varepsilon,$$

which implies that

$$m \geq Q_*(\lambda) - \varepsilon + \frac{\lambda(\lambda-2)}{4(\lambda-1)} + \frac{(m-\varepsilon)^2}{\lambda-1}.$$

Since $\varepsilon \in]0, m]$ is arbitrary, the latter inequality leads to

$$m^2 - (\lambda-1)m + (\lambda-1)Q_*(\lambda) + \frac{\lambda(\lambda-2)}{4} \leq 0.$$

Consequently, we have

$$m \geq \frac{1}{2}\Big(\lambda - 1 - \sqrt{1 - 4(\lambda-1)Q_*(\lambda)}\Big)$$

which, in view of notation (2.23), proves the desired estimate (2.21). □

Lemma 2.5. *Let system* (1.1) *be nonoscillatory and* $\mu < 1$ *be such that condition* (1.13) *holds. Then every nontrivial solution* (u, v) *of system* (1.1) *admits the estimate*

$$\limsup_{t\to+\infty} f(t)\frac{v(t)}{u(t)} \leq \frac{-1 + \sqrt{1 - 4(1-\mu)H_*(\mu)}}{2}. \tag{2.25}$$

Proof. Let (u, v) be a nontrivial solution of system (1.1). Since (1.1) is nonoscillatory, there exists $a \geq 0$ such that (2.2) holds. Define the function ρ by formula (2.4). Then, in view of (1.1), relation (2.4) yields that (2.5) is satisfied. Multiplying both sides of equality (2.5) by the expression $f^{\mu}(t)$ and integrating them from τ to t, one gets

$$\begin{aligned} f^{\mu}(t)\rho(t) = \delta_1(\tau) - \int_0^t f^{\mu}(s)p(s)\mathrm{d}s \\ - \int_\tau^t q(s)f^{\mu-2}(s)\big[f(s)\rho(s)\big(\mu + f(s)\rho(s)\big)\big]\mathrm{d}s \quad \text{for } t \geq \tau \geq a, \end{aligned}$$

where

$$\delta_1(\tau) = f^{\mu}(\tau)\rho(\tau) + \int_0^\tau f^{\mu}(s)p(s)\mathrm{d}s \quad \text{for } \tau \geq a, \tag{2.26}$$

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whence we get, for $t \geq \tau \geq a$,

$$f(t)\rho(t) = \delta_1(\tau) f^{1-\mu}(t) - H(t;\mu) - f^{1-\mu}(t)\int_\tau^t q(s) f^{\mu-2}(s)\big[f(s)\rho(s)\big(\mu + f(s)\rho(s)\big)\big]\mathrm{d}s. \quad (2.27)$$

Now we put

$$M = \limsup_{t\to+\infty} f(t)\rho(t). \quad (2.28)$$

If $M = -\infty$, then the desired inequality (2.25) holds trivially. Now assume that $M > -\infty$. According to the inequality $-x(\mu + x) \leq \frac{\mu^2}{4}$ for $x \in \mathbb{R}$, it follows from relation (2.27) that

$$f(t)\rho(t) \leq \delta_1(\tau) f^{1-\mu}(t) - H(t;\mu) + \frac{\mu^2}{4(1-\mu)} \quad \text{for } t \geq \tau \geq a.$$

Hence, in view of equality (2.9), we get

$$M \leq -H_*(\mu) + \frac{\mu^2}{4(1-\mu)}. \quad (2.29)$$

If $H_*(\mu) = \frac{\mu(2-\mu)}{4(1-\mu)}$, then the desired inequality (2.25) is fulfilled because $M \leq -\frac{\mu}{2}$. Hence, we suppose in the sequel that $H_*(\mu) > \frac{\mu(2-\mu)}{4(1-\mu)}$ and thus $M < -\frac{\mu}{2}$ (see relation (2.29)). Let $\varepsilon \in\,]0, -M - \frac{\mu}{2}]$ be arbitrary and choose $t_\varepsilon \geq a$ such that

$$f(t)\rho(t) \leq M + \varepsilon, \quad H(t;\mu) \geq H_*(\mu) - \varepsilon \quad \text{for } t \geq t_\varepsilon. \quad (2.30)$$

Since $M + \varepsilon \leq -\frac{\mu}{2}$, it is easy to check that

$$f(s)\rho(s)\big(\mu + f(s)\rho(s)\big) \geq (M+\varepsilon)(\mu + M + \varepsilon) \quad \text{for } s \geq t_\varepsilon. \quad (2.31)$$

Therefore, by using relations (2.30) and (2.31), from equality (2.27) with $\tau = t_\varepsilon$ we get

$$f(t)\rho(t) \leq \delta_1(t_\varepsilon) f^{1-\mu}(t) - H_*(\mu) + \varepsilon - \frac{(M+\varepsilon)(\mu+M+\varepsilon)}{1-\mu}\Big(1 - \frac{f^{1-\mu}(t)}{f^{1-\mu}(t_\varepsilon)}\Big) \quad \text{for } t \geq t_\varepsilon,$$

which yields that

$$M \leq -H_*(\mu) + \varepsilon - \frac{(M+\varepsilon)(\mu+M+\varepsilon)}{1-\mu}$$

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because the function f satisfies relation (2.9). Since $\varepsilon \in \,]0, -M - \frac{\mu}{2}]$ is arbitrary, the latter inequality leads to

$$M^2 + M + (1-\mu)H_*(\mu) \leq 0.$$

Consequently, we have

$$M \leq \frac{1}{2}\Big(-1 + \sqrt{1 - 4(1-\mu)H_*(\mu)}\Big)$$

which, in view of notation (2.28), proves the desired estimate (2.25). □

Lemma 2.6. *Let* $\lambda > 1$ *be such that* (1.3) *holds and* $\alpha \in \mathbb{R}$. *Then system* (1.1) *is nonoscillatory if and only if the system*

$$\begin{aligned} x' &= q(t)y, \\ y' &= g_1(t)x + g_2(t)y \end{aligned} \tag{2.32}$$

is nonoscillatory[1]*, where*

$$\begin{aligned} g_1(t) &= -\frac{q(t)}{f^2(t)}\Big(Q^2(t;\lambda) + (2\alpha+\lambda)Q(t;\lambda) + \alpha(\alpha+1)\Big), \\ g_2(t) &= -\frac{2q(t)}{f(t)}\Big(Q(t;\lambda) + \alpha\Big) \end{aligned} \tag{2.33}$$

for a.e. $t \geq 0$.

Proof. Suppose that system (1.1) is nonoscillatory and (u, v) is a solution of this system satisfying relation (2.2) with some $a \geq 0$. Put

$$x(t) = \exp\Big(\int_a^t q(s)\rho(s)\mathrm{d}s\Big) \quad \text{for } t \geq a,$$

$$y(t) = \rho(t)\exp\Big(\int_a^t q(s)\rho(s)\mathrm{d}s\Big) \quad \text{for } t \geq a,$$

where

$$\rho(t) = \frac{v(t)}{u(t)} - \frac{Q(t;\lambda)+\alpha}{f(t)} \quad \text{for } t \geq a.$$

It is not difficult to verify that (x, y) is a solution of system (2.32) on the interval $[a, +\infty[$. Since $x(t) \neq 0$ for $t \geq a$, system (2.32) is nonoscillatory as well.

[1] Solutions of (2.32) and the notion of nonoscillation are understood in the same sense as for system (1.1).

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Suppose now that system (2.32) is nonoscillatory and (x, y) is a solution of this system satisfying the relation

$$x(t) \neq 0 \quad \text{for } t \geq a$$

with some $a \geq 0$. Put

$$u(t) = \exp\Big(\int_a^t q(s)\sigma(s)\mathrm{d}s\Big) \quad \text{for } t \geq a,$$
$$v(t) = \sigma(t)\exp\Big(\int_a^t q(s)\sigma(s)\mathrm{d}s\Big) \quad \text{for } t \geq a,$$

where

$$\sigma(t) = \frac{x(t)}{y(t)} + \frac{Q(t;\lambda)+\alpha}{f(t)} \quad \text{for } t \geq a.$$

As above, it is easy to check that (u, v) is a solution of system (1.1) on the interval $[a, +\infty[$. Since $u(t) \neq 0$ for $t \geq a$, system (1.1) is also nonoscillatory. □

Lemma 2.7. *Let* $\mu < 1$ *and* $\beta \in \mathbb{R}$. *System* (1.1) *is nonoscillatory if and only if system* (2.32) *is nonoscillatory, where*

$$g_1(t) = -\frac{q(t)}{f^2(t)}\big(H^2(t;\mu) - (2\beta+\mu)H(t;\mu) + \beta(\beta+1)\big),$$
$$g_2(t) = \frac{2q(t)}{f(t)}\big(H(t;\mu) - \beta\big)$$

for a.e. $t \geq 0$.

Proof. The proof is analogous to that of Lemma 2.6 and thus it is omitted. □

Lemma 2.8. *Let* (x, y) *be a nontrivial oscillatory solution of system* (2.32) *with locally integrable functions* $g_1, g_2\colon [0, +\infty[\to \mathbb{R}$. *Then, for any* $a \geq 0$, *there exist* $t_2 > t_1 > a$ *such that*

$$x(t) \neq 0 \quad \textit{for } t \in]t_1, t_2[, \quad x(t_1) = 0, \quad x(t_2) = 0. \tag{2.34}$$

Proof. Let $a \geq 0$ be arbitrary. Since (x, y) is a nontrivial oscillatory solution of system (2.32), there exists $t_0 > a$ such that $x(t_0) = 0$. Put

$$A = \big\{t \geq t_0 : x(s) = 0 \text{ for } s \in [t_0, t]\big\}.$$

Clearly, $t_0 \in A$ and thus $A \neq \varnothing$. Moreover, the set A is bounded from above, because the solution (x, y) is nontrivial. Let $t_1 = \sup A$. Obviously, $t_1 > a$ and

$$x(t_1) = 0. \tag{2.35}$$

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Moreover, we have $y(t_1) \neq 0$ because otherwise, in view of (2.35), the solution (x, y) would be trivial. Therefore there exists $\tau_1 > t_1$ such that

$$y(t) \neq 0 \quad \text{for } t \in [t_1, \tau_1]. \tag{2.36}$$

By virtue of (1.2) and (2.36), the first equality in (2.32) yields the monotonicity of the function x on $[t_1, \tau_1]$, which, together with (2.35), guarantees that

$$x(t) \neq 0 \quad \text{for } t \in]t_1, \tau_1] \tag{2.37}$$

(under the opposite assumption we would get a contradiction with the equality $t_1 = \sup A$). Since (x, y) is an oscillatory solution of system (2.32), there exists $b > \tau_1$ such that $x(b) = 0$. Let

$$B = \{t > \tau_1 : x(t) = 0\}.$$

Clearly, $b \in B$ and thus $B \neq \varnothing$. Moreover, the set B is bounded from below because $x(\tau_1) \neq 0$. Let $t_2 = \inf B$. Obviously, $t_2 > \tau_1$,

$$x(t_2) = 0 \tag{2.38}$$

and

$$x(t) \neq 0 \quad \text{for } t \in]\tau_1, t_2[. \tag{2.39}$$

Therefore, relations (2.35) and (2.37)–(2.39) yield the validity of the desired properties (2.34). By the arbitrariness of $a \geq 0$, we complete the proof. □

Lemma 2.9. *Let* $\lambda > 1$ *be such that condition* (1.3) *holds,* $\alpha \in \mathbb{R}$*, and*

$$Q^2(t;\lambda) + (2\alpha + \lambda)Q(t;\lambda) + \alpha(\alpha + 1) \leq 0 \quad \textit{for } t \geq a \tag{2.40}$$

with some $a \geq 0$*. Then system* (1.1) *is nonoscillatory.*

Proof. According to Lemma 2.6, it suffices to show that system (2.32) is nonoscillatory, where the functions g_1 and g_2 are defined by (2.33). Suppose on the contrary that (x, y) is a nontrivial oscillatory solution of system (2.32). Then, by virtue of Lemma 2.8, there exist $t_1 > a$ and $t_2 > t_1$ such that relations (2.34) are fulfilled. We can assume without loss of generality that

$$x(t) > 0 \quad \text{for } t \in]t_1, t_2[. \tag{2.41}$$

Obviously, $y(t_1) \neq 0$ because otherwise, in view of (2.34), the solution (x, y) would be trivial.

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If $y(t_1) < 0$, then there exists $t_3 \in]t_1, t_2[$ such that

$$y(t) < 0 \quad \text{for } t \in [t_1, t_3]. \tag{2.42}$$

Taking relations (1.2), (2.34), and (2.42) into account, it follows from the first equality in (2.32) that

$$x(t) = \int_{t_1}^{t} q(s)y(s)\mathrm{d}s \le 0 \quad \text{for } t \in [t_1, t_3],$$

which contradicts inequality (2.41). The contradiction obtained proves that

$$y(t_1) > 0. \tag{2.43}$$

The second equality in (2.32) yields

$$\Big(y(t)\exp\Big(-\int_a^t g_2(s)\mathrm{d}s\Big)\Big)' = g_1(t)\exp\Big(-\int_a^t g_2(s)\mathrm{d}s\Big)x(t) \quad \text{for a.e. } t \ge a,$$

whence by using (1.2), (2.40), (2.41) and (2.43), we get

$$y(t) > 0 \quad \text{for } t \in [t_1, t_2]. \tag{2.44}$$

However, in view of relations (1.2) and (2.44), it follows from the first equality in (2.32) that

$$x'(t) \ge 0 \quad \text{for a.e. } t \in [t_1, t_2],$$

which is in contradiction with properties (2.34). □

Analogously, one can prove

Lemma 2.10. *Let $\mu < 1$ and $\beta \in \mathbb{R}$ be such that*

$$H^2(t;\mu) - (2\beta + \mu)H(t;\mu) + \beta(\beta + 1) \le 0 \quad \textit{for } t \ge a \tag{2.45}$$

with some $a \ge 0$. Then system (1.1) is nonoscillatory.

Lemma 2.11. *Let there exist a number $\mu < 1$ such that*

$$H^*(\mu) < +\infty. \tag{2.46}$$

Then, for every $\lambda > 1$, the function $c(\cdot;\lambda)$ possesses a limit as $t \to +\infty$ and

$$\lim_{t \to +\infty} c(t;\lambda) < +\infty. \tag{2.47}$$

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Proof. According to assumption (2.46), there are numbers $M \in \mathbb{R}$ and $t_0 \geq 0$ such that

$$f(t) < 1, \quad H(t;\mu) \leq M \quad \text{for } t \geq t_0. \tag{2.48}$$

We first note that

$$\int_{t_0}^{+\infty} q(s) f^{\lambda-2}(s) \ln \frac{1}{f(s)}\, \mathrm{d}s < +\infty \tag{2.49}$$

which can be checked by direct calculation. It is clear that

$$\int_0^t f(s)p(s)\mathrm{d}s = H(t;\mu) + (1-\mu)\int_0^t \frac{q(s)}{f(s)}\, H(s;\mu)\mathrm{d}s \quad \text{for } t \geq 0$$

and thus, using relations (2.9) and (2.48), we get

$$\limsup_{t\to+\infty} \frac{1}{\ln \frac{1}{f(t)}} \int_0^t f(s)p(s)\mathrm{d}s \leq M(1-\mu).$$

Consequently, there exists $t_1 \geq t_0$ such that

$$\frac{1}{\ln \frac{1}{f(t)}} \int_0^t f(s)p(s)\mathrm{d}s \leq M_0 \quad \text{for } t \geq t_1, \tag{2.50}$$

where $M_0 = M(1-\mu) + 1$.

Now let $\lambda > 1$ be arbitrary. It is easy to verify that

$$c'(t;\lambda) = (\lambda-1)q(t)f^{\lambda-2}(t)\int_0^t f(s)p(s)\mathrm{d}s \quad \text{for a.e. } t \geq 0.$$

The integration of the latter equality from τ to t leads to

$$c(t;\lambda) = c(\tau;\lambda) + (\lambda-1)\int_\tau^t q(s)f^{\lambda-2}(s)\Big(\int_0^s f(\xi)p(\xi)\mathrm{d}\xi\Big)\mathrm{d}s \quad \text{for } t \geq \tau \geq 0.$$

Therefore, using relation (2.50), we get

$$c(t;\lambda) \leq c(\tau;\lambda) + M_0(\lambda-1)\int_\tau^t q(s)f^{\lambda-2}(s)\ln\frac{1}{f(s)}\mathrm{d}s \quad \text{for } t \geq \tau \geq t_1$$

which, in view of inequality (2.49), leads to

$$\limsup_{t\to+\infty} c(t;\lambda) \leq c(\tau;\lambda) + M_0(\lambda-1)\int_\tau^{+\infty} q(s)f^{\lambda-2}(s)\ln\frac{1}{f(s)}\mathrm{d}s \quad \text{for } \tau \geq t_1$$

whence we get $\limsup_{t\to+\infty} c(t;\lambda) < +\infty$ and

$$\limsup_{t\to+\infty} c(t;\lambda) \leq \liminf_{\tau\to+\infty} c(\tau;\lambda),$$

i.e., relation (2.47) holds. □

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Analogously, one can prove

Lemma 2.12. *Let there exist a number* $\mu < 1$ *such that*

$$H_*(\mu) > -\infty.$$

Then, for every $\lambda > 1$, *the function* $c(\cdot\,;\lambda)$ *possesses a limit as* $t \to +\infty$ *and*

$$\lim_{t\to+\infty} c(t;\lambda) > -\infty.$$

3 Proofs of the main results

Proof of Theorem 1.1. Assume on the contrary that system (1.1) is nonoscillatory and (u, v) is a solution of this system satisfying (2.2) with some $a \geq 0$. Then, by virtue of Lemma 2.1, inequality (2.3) holds, where the function ρ is defined by (2.4). Therefore it follows from Lemma 2.2 that there exists a finite limit $\lim_{t\to+\infty} c(t;\lambda)$, which contradicts the assumptions of the theorem. □

Proof of Theorem 1.2. Assume on the contrary that system (1.1) is nonoscillatory and (u, v) is a solution of this system satisfying (2.2) with some $a \geq 0$. According to Lemma 2.3, relations (2.3) and (2.20) are satisfied, where the function ρ is defined by (2.4).

Multiplying both sides of equality (2.20) by the expression $q(t) f^{-\lambda}(t)$ and integrating them from a to t, one gets

$$\begin{aligned}
&\int_a^t \frac{q(s)}{f(s)}\Big[f(s)\rho(s) + \frac{\lambda}{2}\Big]\mathrm{d}s \\
&\qquad = \frac{1}{(\lambda-1) f^{\lambda-1}(t)}\big(c_0(\lambda) - c(t;\lambda)\big) \\
&\qquad\quad + \int_a^t \frac{q(s)}{f^{\lambda}(s)}\Big(\int_s^{+\infty} q(\xi) f^{\lambda-2}(\xi)\Big[f(\xi)\rho(\xi) + \frac{\lambda}{2}\Big]^2 \mathrm{d}\xi\Big)\mathrm{d}s \\
&\qquad\quad + \frac{\lambda(\lambda-2)}{4(\lambda-1)} \ln\frac{f(a)}{f(t)} - \frac{1}{(\lambda-1) f^{\lambda-1}(a)}\big(c_0(\lambda) - c(a;\lambda)\big) \quad \text{for } t \geq a,
\end{aligned}$$

whence

$$\begin{aligned}
\frac{1}{f^{\lambda-1}(t)}\big(c_0(\lambda) - c(t;\lambda)\big) &= \delta_0 + \frac{\lambda(\lambda-2)}{4} \ln\frac{f(t)}{f(a)} \\
&\quad + \int_a^t \frac{q(s)}{f(s)}\Big[f(s)\rho(s) + \frac{\lambda}{2}\Big]\Big(\lambda - 1 - \Big[f(s)\rho(s) + \frac{\lambda}{2}\Big]\Big)\mathrm{d}s \\
&\quad - \frac{1}{f^{\lambda-1}(t)}\int_t^{+\infty} q(s) f^{\lambda-2}(s)\Big[f(s)\rho(s) + \frac{\lambda}{2}\Big]^2 \mathrm{d}s \quad \text{for } t \geq a,
\end{aligned}$$

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where

$$\delta_0 = \frac{1}{f^{\lambda-1}(a)}\Big(c_0(\lambda) - c(a;\lambda) + \int_a^{+\infty} q(s) f^{\lambda-2}(s)\Big[f(s)\rho(s) + \frac{\lambda}{2}\Big]^2 \mathrm{d}s\Big).$$

By using the inequality $4x(\lambda - 1 - x) \leq (\lambda - 1)^2$ for $x \in \mathbb{R}$, the latter relation gives

$$\frac{1}{f^{\lambda-1}(t)}\big(c_0(\lambda) - c(t;\lambda)\big) \leq \delta_0 - \frac{1}{4}\ln\frac{f(t)}{f(a)}.$$

Consequently, in view of equality (2.9), we get

$$\limsup_{t\to+\infty} \frac{-1}{f^{\lambda-1}(t)\ln f(t)}\big(c_0(\lambda) - c(t;\lambda)\big) \leq \frac{1}{4},$$

which is in contradiction with assumption (1.5). □

Proof of Corollary 1.1. It is not difficult to verify that

$$c(t;\lambda) = \int_0^t f^{\lambda}(s)p(s)\mathrm{d}s - f^{\lambda-1}(t)\int_0^t f(s)p(s)\mathrm{d}s \quad \text{for } t \geq 0$$

and thus we have

$$\frac{-1}{f^{\lambda-1}(t)\ln f(t)}\big(c_0(\lambda) - c(t;\lambda)\big)$$
$$= -\frac{Q(t;\lambda)}{\ln f(t)} + \frac{-1}{\ln f(t)}\int_0^t f(s)p(s)\mathrm{d}s \quad \text{for } t \geq 0.$$

The latter equality, on account of relations (1.6), (2.9), and the assumption $Q_*(\lambda) > -\infty$, yields the validity of inequality (1.5). Consequently, the assertion of the corollary follows from Theorem 1.2. □

Proof of Corollary 1.2. It is not difficult to verify that the equalities

$$Q(t;\lambda) + H(t;\mu)$$
$$= (\lambda - \mu) f^{1-\mu}(t)\int_0^t \frac{q(s)}{f^{2-\mu}(s)} Q(s;\lambda)\mathrm{d}s + \frac{c_0(\lambda)}{f^{\lambda-\mu}(0)} f^{1-\mu}(t) \quad (3.1)$$

and

$$\frac{-1}{f^{\lambda-1}(t)\ln f(t)}\big(c_0(\lambda) - c(t;\lambda)\big)$$
$$= \frac{c_0(\lambda)}{f^{\lambda-1}(0)}\frac{1}{\ln\frac{1}{f(t)}} + \frac{\lambda-1}{\ln\frac{1}{f(t)}}\int_0^t \frac{q(s)}{f(s)} Q(s;\lambda)\mathrm{d}s \quad (3.2)$$

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hold for $t \geq 0$. Therefore, in view of assumption (1.7) and relation (2.9), it follows from equality (3.1) that

$$\liminf_{t\to+\infty} f^{1-\mu}(t)\int_0^t \frac{q(s)}{f^{2-\mu}(s)}\, Q(s;\lambda)\mathrm{d}s > \frac{1}{4(\lambda-1)(1-\mu)}. \tag{3.3}$$

On the other hand, for $t \geq 0$ we have

$$\int_0^t \frac{q(s)}{f(s)}\, Q(s;\lambda)\mathrm{d}s = f^{1-\mu}(t)\int_0^t \frac{q(s)}{f^{2-\mu}(s)}\, Q(s;\lambda)\mathrm{d}s$$
$$+ (1-\mu)\int_0^t \frac{q(s)}{f(s)}\Big[f^{1-\mu}(s)\int_0^s \frac{q(\xi)}{f^{2-\mu}(\xi)}\, Q(\xi;\lambda)\mathrm{d}\xi\Big]\mathrm{d}s$$

and thus, using relations (2.9) and (3.3), we get

$$\liminf_{t\to+\infty} \frac{1}{\ln\frac{1}{f(t)}}\int_0^t \frac{q(s)}{f(s)}\, Q(s;\lambda)\mathrm{d}s > \frac{1}{4(\lambda-1)}. \tag{3.4}$$

Now, by virtue of (2.9) and (3.4), it follows from equality (3.2) that inequality (1.5) is satisfied. Consequently, by Theorem 1.2 system (1.1) is oscillatory. □

Proof of Corollary 1.3. Assume that $\lambda > 1$ is such that (1.3) holds.

First, suppose that (1.8) is fulfilled. It is not difficult to verify that, for any $\mu < 1$, equality (3.1) holds for $t \geq 0$, which, in view of relation (2.9), yields the validity of inequality (1.7). Therefore, by virtue of Corollary 1.2, system (1.1) is oscillatory.

Now, assume that $\mu < 1$ is such that inequality (1.9) holds. It is clear that

$$\int_0^t f(s)p(s)\mathrm{d}s = H(t;\mu) + (1-\mu)\int_0^t \frac{q(s)}{f(s)}\, H(s;\mu)\mathrm{d}s \quad \text{for } t \geq 0$$

and thus, using relations (1.9) and (2.9), we get

$$\liminf_{t\to+\infty} \frac{1}{\ln\frac{1}{f(t)}}\int_0^t f(s)p(s)\mathrm{d}s > \frac{1}{4}. \tag{3.5}$$

On the other hand, it is easy to verify that

$$c'(t;\lambda) = (\lambda-1)q(t)f^{\lambda-2}(t)\int_0^t f(s)p(s)\mathrm{d}s \quad \text{for a.e. } t \geq 0.$$

The integration of the latter equality from t to τ leads to

$$c(\tau;\lambda)-c(t;\lambda) = (\lambda-1)\int_t^\tau q(s)f^{\lambda-2}(s)\Big(\int_0^s f(\xi)p(\xi)\mathrm{d}\xi\Big)\mathrm{d}s \quad \text{for } \tau \geq t \geq 0.$$

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Taking now into account notation (1.4), we get

$$c_0(\lambda)-c(t;\lambda) = (\lambda-1)\int_t^{+\infty} q(s)f^{\lambda-2}(s)\ln\frac{1}{f(s)}\Big(\frac{1}{\ln\frac{1}{f(s)}}\int_0^s f(\xi)p(\xi)\mathrm{d}\xi\Big)\mathrm{d}s$$

for $t \geq 0$, whence, using relations (2.9) and (3.5), inequality (1.5) follows. Consequently, the assertion of the corollary follows from Theorem 1.2. □

Proof of Theorem 1.3. Assume on the contrary that system (1.1) is nonoscillatory and (u, v) is a solution of this system satisfying (2.2) with some $a \geq 0$. Define the function ρ by (2.4). Then, using (1.1), we obtain equality (2.5). Multiplying both sides of (2.5) by the expression $f^{\mu}(t)$ and integrating them from τ to t, one gets equality (2.27), where $\delta_1(\tau)$ is defined by formula (2.26). On the other hand, Lemma 2.3 yields the validity of relations (2.3) and (2.20).

Now, it follows from equalities (2.20) and (2.27) that

$$\begin{aligned} Q(t;\lambda) + H(t;\mu) &= \frac{\lambda^2}{4(\lambda-1)} + \delta_1(\tau)f^{1-\mu}(t) \\ &- f^{1-\mu}(t)\int_\tau^t q(s)f^{\mu-2}(s)\big[f(s)\rho(s)\big(\mu + f(s)\rho(s)\big)\big]\mathrm{d}s \\ &- \frac{1}{f^{\lambda-1}(t)}\int_t^{+\infty} q(s)f^{\lambda-2}(s)\Big[f(s)\rho(s)+\frac{\lambda}{2}\Big]^2\mathrm{d}s \quad \text{for } t \geq \tau \geq a. \end{aligned} \tag{3.6}$$

Putting $\tau = a$ and using the inequality $-x(\mu + x) \leq \frac{\mu^2}{4}$ for $x \in \mathbb{R}$, we obtain from equality (3.6) the inequality

$$Q(t;\lambda) + H(t;\mu) \leq \frac{\lambda^2}{4(\lambda-1)} + \frac{\mu^2}{4(1-\mu)} + \delta_1(a)f^{1-\mu}(t) \quad \text{for } t \geq a.$$

Therefore, by virtue of relation (2.9), we get

$$\limsup_{t\to+\infty}\big(Q(t;\lambda) + H(t;\mu)\big) \leq \frac{\lambda^2}{4(\lambda-1)} + \frac{\mu^2}{4(1-\mu)},$$

which contradicts assumption (1.10). □

Proof of Theorem 1.4. Assume on the contrary that system (1.1) is nonoscillatory and (u, v) is a solution of this system satisfying (2.2) with some $a \geq 0$. Define the function ρ by formula (2.4). Then, using (1.1), we obtain equality (2.5). Multiplying both sides of (2.5) by the expression $f^{\mu}(t)$ and integrating them from τ to t, one gets equality (2.27), where $\delta_1(\tau)$ is defined by formula (2.26). On the other hand, Lemma 2.3 yields the validity of relations (2.3) and (2.20).

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Suppose that assumptions (1.11) and (1.12) (respectively, assumptions (1.13) and (1.14)) are fulfilled and $\varepsilon > 0$ is arbitrary. Then, according to Lemma 2.4 (respectively, Lemma 2.5), there exists $t_\varepsilon \geq a$ such that

$$\begin{aligned} f(t)\rho(t) &\geq m - \varepsilon \quad \text{for } t \geq t_\varepsilon, \\ \text{resp.,} \quad f(t)\rho(t) &\leq M + \varepsilon \quad \text{for } t \geq t_\varepsilon, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} m &= -\frac{1}{2}\Big(1 + \sqrt{1 - 4(\lambda - 1)Q_*(\lambda)}\Big), \\ \text{resp.,} \quad M &= \frac{1}{2}\Big(-1 + \sqrt{1 - 4(1 - \mu)H_*(\mu)}\Big). \end{aligned} \tag{3.8}$$

By using relation (3.7) and the inequality $-x(\mu + x) \leq \frac{\mu^2}{4}$ for $x \in \mathbb{R}$, it follows from equality (2.27) with $\tau = a$ (respectively, from equality (2.20)) that

$$H(t;\mu) \leq \delta_1(a) f^{1-\mu}(t) - m + \varepsilon + \frac{\mu^2}{4(1-\mu)}\left(1 - \frac{f^{1-\mu}(t)}{f^{1-\mu}(a)}\right) \quad \text{for } t \geq t_\varepsilon,$$

$$\text{resp.,} \quad Q(t;\lambda) \leq M + \varepsilon + \frac{\lambda^2}{4(\lambda - 1)} \quad \text{for } t \geq t_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and equality (2.9) holds, the latter inequality yields

$$H^*(\mu) \leq -m + \frac{\mu^2}{4(1-\mu)}, \quad \text{resp.,} \quad Q^*(\lambda) \leq M + \frac{\lambda^2}{4(\lambda - 1)}$$

which, together with notation (3.8), contradicts assumption (1.12) (respectively, (1.14)). □

Proof of Theorem 1.5. Assume on the contrary that system (1.1) is nonoscillatory and (u, v) is a solution of this system satisfying (2.2) with some $a \geq 0$. Define the function ρ by formula (2.4). Then, using (1.1), we get equality (2.5). Multiplying both sides of (2.5) by the expression $f^{\mu}(t)$ and integrating them from τ to t, one gets equality (2.27), where $\delta_1(\tau)$ is defined by formula (2.26). On the other hand, Lemma 2.3 yields the validity of relations (2.3) and (2.20). Therefore, it follows from equalities (2.20) and (2.27) that relation (3.6) is satisfied.

Put

$$m = \liminf_{t \to +\infty}\left[f(t)\rho(t) + \frac{\lambda}{2}\right], \quad M = \limsup_{t \to +\infty} f(t)\rho(t). \tag{3.9}$$

Then Lemmas 2.4 and 2.5, in view of assumptions (1.11) and (1.13), lead to the inequalities

$$\begin{aligned} m &\geq \frac{1}{2}\Big(\lambda - 1 - \sqrt{1 - 4(\lambda - 1)Q_*(\lambda)}\Big) \geq 0, \\ M &\leq \frac{1}{2}\Big(-1 + \sqrt{1 - 4(1 - \mu)H_*(\mu)}\Big) \leq -\frac{\mu}{2}. \end{aligned} \tag{3.10}$$

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Suppose that $M < -\frac{\mu}{2}$ and $m > 0$. Let $0 < \varepsilon \leq \min\{-M - \frac{\mu}{2}, m\}$ be arbitrary and choose $t_\varepsilon \geq a$ such that

$$f(t)\rho(t) + \frac{\lambda}{2} \geq m - \varepsilon, \quad f(t)\rho(t) \leq M + \varepsilon \quad \text{for } t \geq t_\varepsilon. \tag{3.11}$$

Since we have $M + \varepsilon \leq -\frac{\mu}{2}$, it is easy to check that

$$f(s)\rho(s)\big(\mu + f(s)\rho(s)\big) \geq (M + \varepsilon)(\mu + M + \varepsilon) \quad \text{for } s \geq t_\varepsilon.$$

Consequently, in view of relation (2.9), from equality (3.6) with $\tau = t_\varepsilon$ we get

$$\begin{aligned} Q(t;\lambda) + H(t;\mu) \leq \frac{\lambda^2}{4(\lambda-1)} &- \frac{(M+\varepsilon)(\mu+M+\varepsilon)}{1-\mu} \\ &- \frac{(m-\varepsilon)^2}{\lambda-1} + \delta_2(t_\varepsilon) f^{1-\mu}(t) \quad \text{for } t \geq t_\varepsilon, \end{aligned} \tag{3.12}$$

where

$$\delta_2(t_\varepsilon) = \delta_1(t_\varepsilon) + \frac{(M+\varepsilon)(\mu+M+\varepsilon)}{(1-\mu) f^{1-\mu}(t_\varepsilon)}.$$

Since $\varepsilon > 0$ is arbitrary, by virtue of relation (2.9), it follows from inequality (3.12) that

$$\limsup_{t\to+\infty}\big(Q(t;\lambda) + H(t;\mu)\big) \leq \frac{\lambda^2}{4(\lambda-1)} - \frac{M(\mu+M)}{1-\mu} - \frac{m^2}{\lambda-1}. \tag{3.13}$$

If $M = -\frac{\mu}{2}$ (respectively, $m = 0$), then we use the fact that

$$\begin{aligned} -f^{1-\mu}(t)\int_\tau^t q(s) f^{\mu-2}(s)\big[f(s)\rho(s)\big(\mu + f(s)\rho(s)\big)\big]\mathrm{d}s \\ \leq \frac{\mu^2}{4(1-\mu)} = -\frac{M(\mu+M)}{1-\mu} \quad \text{for } t \geq \tau \geq a, \end{aligned}$$

resp.,

$$\begin{aligned} -\frac{1}{f^{\lambda-1}(t)}\int_t^{+\infty} q(s) f^{\lambda-2}(s)\Big[f(s)\rho(s) + \frac{\lambda}{2}\Big]^2\mathrm{d}s \\ \leq 0 = -\frac{m^2}{\lambda-1} \quad \text{for } t \geq 0 \end{aligned}$$

and we also arrive at inequality (3.13).

Consequently, inequalities (3.13) and (3.10) lead to a contradiction with assumption (1.15). □

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Proof of Corollary 1.4. It is easy to show that

$$\limsup_{t\to+\infty}\big(Q(t;\lambda)+H(t;\mu)\big)\ge Q^*(\lambda)+H_*(\mu)$$

and

$$\limsup_{t\to+\infty}\big(Q(t;\lambda)+H(t;\mu)\big)\ge H^*(\mu)+Q_*(\lambda).$$

Consequently, in both cases (1.16) and (1.17), inequality (1.15) is satisfied, and thus the assertion of the corollary follows immediately from Theorem 1.5. □

Proof of Theorem 1.6. Assume that inequalities (1.18) (respectively, (1.19)) hold. Then there exists $a\ge 0$ such that

$$-\frac{(2\lambda-3)(2\lambda-1)}{4(\lambda-1)}\le Q(t;\lambda)\le\frac{1}{4(\lambda-1)}\quad\text{for } t\ge a,$$

$$\text{resp.,}\quad -\frac{(3-2\mu)(1-2\mu)}{4(1-\mu)}\le H(t;\mu)\le\frac{1}{4(1-\mu)}\quad\text{for } t\ge a.$$

The latter inequalities yield

$$Q^2(t;\lambda)+\frac{2\lambda^2-4\lambda+1}{2(\lambda-1)}\,Q(t;\lambda)-\frac{(2\lambda-1)(2\lambda-3)}{16(\lambda-1)^2}\le 0\quad\text{for } t\ge a,$$

resp.,

$$H^2(t;\mu)+\frac{2\mu^2-4\mu+1}{2(1-\mu)}\,H(t;\mu)-\frac{(1-2\mu)(3-2\mu)}{16(1-\mu)^2}\le 0\quad\text{for } t\ge a$$

and thus relation (2.40) with $\alpha=-\frac{2\lambda-1}{4(\lambda-1)}$ (respectively, relation (2.45) with $\beta=-\frac{1-2\mu}{4(1-\mu)}$) is fulfilled. Consequently, according to Lemma 2.9 (respectively, Lemma 2.8), system (1.1) is nonoscillatory. □

Proof of Theorem 1.7. Assume that inequalities (1.20) and (1.21) (respectively, inequalities (1.22) and (1.23)) are fulfilled. It follows from assumption (1.21) (respectively, (1.23)) that

$$Q^*(\lambda)<\frac{1}{4(\lambda-1)},\quad\text{resp.,}\quad H^*(\mu)<\frac{1}{4(1-\mu)}.$$

Choose $\varepsilon>0$ such that

$$Q^*(\lambda)+\varepsilon\le\frac{1}{4(\lambda-1)},\quad\text{resp.,}\quad H^*(\mu)+\varepsilon\le\frac{1}{4(1-\mu)}\tag{3.14}$$

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and

$$Q^*(\lambda)+\varepsilon \le Q_*(\lambda)-\varepsilon+1-\lambda+\sqrt{1-4(\lambda-1)(Q_*(\lambda)-\varepsilon)},$$
$$\text{resp.,}\quad H^*(\mu)+\varepsilon \le H_*(\mu)-\varepsilon+\mu-1+\sqrt{1-4(1-\mu)(H_*(\mu)-\varepsilon)}. \tag{3.15}$$

Let

$$\alpha=-\frac{1}{2}-Q^*(\lambda)-\varepsilon+\frac{1}{2}\sqrt{1-4(\lambda-1)(Q^*(\lambda)+\varepsilon)},$$
$$\text{resp.,}\quad \beta=-\frac{1}{2}+H^*(\mu)+\varepsilon-\frac{1}{2}\sqrt{1-4(1-\mu)(H^*(\mu)+\varepsilon)}.$$

By using inequality (3.14) it is easy to verify that

$$\alpha \ge -\frac{\lambda^2}{4(\lambda-1)},\quad \text{resp.,}\quad \beta \le \frac{\mu^2}{4(1-\mu)}. \tag{3.16}$$

Thus it follows from the definition of the number α (respectively, β) that

$$Q^*(\lambda)+\varepsilon=-\alpha-\frac{\lambda}{2}+\sqrt{\lambda^2/4+\alpha(\lambda-1)},$$
$$\text{resp.,}\quad H^*(\mu)+\varepsilon=\beta+\frac{\mu}{2}+\sqrt{\mu^2/4-\beta(1-\mu)}. \tag{3.17}$$

Observe that inequality (3.15) leads to

$$\alpha^2+\big(1+2(Q_*(\lambda)-\varepsilon)\big)\alpha+(Q_*(\lambda)-\varepsilon)(Q_*(\lambda)-\varepsilon+\lambda)\le 0,$$
$$\text{resp.,}\quad \beta^2+\big(1-2(H_*(\mu)-\varepsilon)\big)\beta+(H_*(\mu)-\varepsilon)(H_*(\mu)-\varepsilon-\mu)\le 0,$$

whence, in view of relation (3.16), we get

$$Q_*(\lambda)-\varepsilon \ge -\alpha-\frac{\lambda}{2}-\sqrt{\lambda^2/4+\alpha(\lambda-1)},$$
$$\text{resp.,}\quad H_*(\mu)-\varepsilon \ge \beta+\frac{\mu}{2}-\sqrt{\mu^2/4-\beta(1-\mu)}. \tag{3.18}$$

Finally, there exists $a \ge 0$ such that

$$Q_*(\lambda)-\varepsilon \le Q(t;\lambda) \le Q^*(\lambda)+\varepsilon \quad \text{for } t \ge a,$$
$$\text{resp.,}\quad H_*(\mu)-\varepsilon \le H(t;\mu) \le H^*(\mu)+\varepsilon \quad \text{for } t \ge a$$

which, together with relations (3.17) and (3.18), guarantees that the inequalities

$$-\alpha-\frac{\lambda}{2}-\sqrt{\lambda^2/4+\alpha(\lambda-1)} \le Q(t;\lambda) \le -\alpha-\frac{\lambda}{2}+\sqrt{\lambda^2/4+\alpha(\lambda-1)}$$

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resp.,

$$\beta + \frac{\mu}{2} - \sqrt{\mu^2/4 - \beta(1-\mu)} \le H(t;\mu) \le \beta + \frac{\mu}{2} + \sqrt{\mu^2/4 - \beta(1-\mu)}$$

hold for $t \ge a$. However, this means that inequality (2.40) (respectively, inequality (2.45)) is satisfied and thus the assertion of the theorem follows from Lemma 2.9 (respectively, Lemma 2.10). □

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