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Oscillatory properties of solutions to certain two-dimensional systems of non-linear ordinary differential equations

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ABSTRACT

For the system of non-linear equations

$$u' = g(t)|v|^{1/\alpha}\operatorname{sgn} v, \qquad v' = -p(t)|u|^{\alpha}\operatorname{sgn} u$$

with $\alpha > 0$ and locally integrable functions $g\colon [0,+\infty[\to [0,+\infty[$, $p\colon [0,+\infty[\to \mathbb{R}$, new oscillation criteria are given in both cases $\int_0^{+\infty} g(s)ds = +\infty$ and $\int_0^{+\infty} g(s)ds < +\infty$. As a consequence of the main results we derive, among others, a Hartman–Wintner type theorem for the system considered which, in the case $\int_0^{+\infty} g(s)ds < +\infty$, does not require the assumption $p \geq 0$ appearing in the existing literature.

1. Introduction

On the half-line $[0,+\infty[$, we consider the system

$$\begin{aligned} u' &= g(t)|v|^{1/\alpha}\operatorname{sgn} v,\\ v' &= -p(t)|u|^{\alpha}\operatorname{sgn} u, \end{aligned} \tag{1.1}$$

where $\alpha > 0$ and $p, g\colon [0,+\infty[\to \mathbb{R}$ are locally Lebesgue integrable functions.

A pair (u, v) is said to be *a solution to system* (1.1) *on the interval* $I \subseteq [0,+\infty[$ if the functions $u, v\colon I \to \mathbb{R}$ are absolutely continuous on every compact interval contained in I and satisfy equalities (1.1) almost everywhere in I. In the paper [14], Mirzov proved that all non-extendable solutions to system (1.1) are defined on the whole interval $[0,+\infty[$. Therefore, speaking about a solution to system (1.1), we assume without loss of generality that it is defined on the whole interval $[0,+\infty[$. Mirzov also proved (see, e.g., [15, Theorem 9.3]) that all non-zero solutions (u, v) to system (1.1) are proper, i.e., the inequality

$$\sup\left\{|u(\tau)| + |v(\tau)| : t \leq \tau < +\infty\right\} > 0$$

holds for every $t \geq 0$.

Definition 1.1. A solution (u, v) to system (1.1) is called *non-trivial* if $u \not\equiv 0$ on any neighborhood of $+\infty$. We say that a non-trivial solution (u, v) to system (1.1) is *oscillatory* if the function u has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

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It is well known (see [14, Theorem 1.1]) that a certain analog of Sturm's theorem holds for system (1.1) under the additional assumption

$$g(t) \geq 0 \quad \text{for a.e. } t \geq 0. \tag{1.2}$$

In particular, if inequality (1.2) holds and system (1.1) has an oscillatory solution, then any other non-trivial solution is also oscillatory. Moreover, under assumption (1.2), if (u, v) is an oscillatory solution to system (1.1), then together with u, the function v also oscillates. On the other hand, it is clear that if $g \equiv 0$ on some neighborhood of $+\infty$, then all non-trivial solutions to system (1.1) are non-oscillatory.

Therefore, we assume throughout the paper that inequality (1.2) holds and

$$\operatorname{meas}\left\{\tau \geq t : g(\tau) > 0\right\} > 0 \quad \text{for every } t \geq 0. \tag{1.3}$$

Definition 1.2. We say that system (1.1) is *oscillatory* if all its non-trivial solutions are oscillatory.

In the last two decades, many results have been obtained in oscillation theory of the equation

$$\left(r(t)|u'|^{q-1}\operatorname{sgn} u'\right)' + p(t)|u|^{q-1}\operatorname{sgn} u = 0 \tag{1.4}$$

which is referred as "half-linear equation" or alternatively "equation with the scalar q-Laplacian" (see survey given in [1]). Eq. (1.4) is usually considered under the assumptions $q > 1$, $r, p\colon [0, +\infty[\to \mathbb{R}$ are continuous and r is positive, and both cases

$$\int_0^{+\infty} r^{\frac{1}{1-q}}(s)\mathrm{d}s = +\infty \tag{1.5}$$

and

$$\int_0^{+\infty} r^{\frac{1}{1-q}}(s)\mathrm{d}s < +\infty \tag{1.6}$$

are studied in the existing literature. Solutions of (1.4) are understood in the classical sense (i.e., a solution u of (1.4) is a C^1 function such that $r|u'|^{q-1}\operatorname{sgn} u' \in C^1$ and satisfies (1.4) everywhere in an interval under consideration). Therefore, it is clear that if the function u is a solution to Eq. (1.4), then the vector function $(u, r|u'|^{q-1}\operatorname{sgn} u')$ is a solution to system (1.1) with

$$g(t) := r^{\frac{1}{1-q}}(t) \quad \text{for } t \geq 0,\ \alpha := q - 1.$$

Hence, Eq. (1.4) is a particular case of system (1.1) in which p, g are continuous and $g(t) > 0$ for all t. Observe that, even for this particular case of (1.1), there are certain gaps in the oscillation theory. For instance, an analog of the Hartman–Wintner theorem for Eq. (1.4) is well known in the case, where (1.5) holds, which allows one to derive further oscillation and non-oscillation criteria of Hille and Nehari type (see, e.g., [3,7,15,11,5,10,17] and survey given in [1]). As for the case, where (1.6) is satisfied, as far as we know, an analog of the Hartman–Wintner theorem and some Hille and Nehari type oscillations criteria are proved only under the additional assumption

$$p(t) \geq 0 \quad \text{for a.e. } t \geq 0 \tag{1.7}$$

(see, e.g., [7,6,16] and survey given in [1]). Moreover, so-called Kamenev type oscillation criteria are known for Eq. (1.4) and read as follows:

$$\limsup_{t \to +\infty} \Delta(t) = +\infty,$$

where the function Δ is expressed in terms of the coefficients r, p and the parameter q (see, e.g., [9,13,8]). But only a few results can be found in the existing literature for the contrary case, where

$$\limsup_{t \to +\infty} \Delta(t) < +\infty$$

(see, e,g., [13,8]).

The aim of this paper is to present a new Kamenev type oscillation criterion and its counterpart for the system (1.1) in both cases, where

$$\int_0^{+\infty} g(s)\mathrm{d}s = +\infty \tag{1.8$_1$}$$

and

$$\int_0^{+\infty} g(s)\mathrm{d}s < +\infty \tag{1.8$_2$}$$

(see Theorems 2.1 and 2.8). As consequences of the main results we also derive Hartman–Wintner type theorems for system (1.1), which essentially generalize previously known results.

2. Main results

In this section, we formulate all main results, their proofs are postponed till Section 4.

2.1. The case $\int_0^\infty g(s)\mathrm{d}s = +\infty$

We assume throughout this subsection that the coefficient g is non-integrable on $[0, +\infty[$, i.e., that g satisfies condition (1.8_1). Let

$$f_1(t) := \int_0^t g(s)\mathrm{d}s \quad \text{for } t \geq 0. \tag{2.1}$$

In view of assumptions (1.2), (1.3) and (1.8_1), we have

$$\lim_{t\to+\infty} f_1(t) = +\infty$$

and there exists a number $t_g \geq 0$ such that $f_1(t) > 0$ for $t > t_g$ and $f_1(t_g) = 0$. Since we are interested in behavior of solutions in the neighborhood of $+\infty$, we can assume without loss of generality that $t_g = 0$, i.e.,

$$f_1(t) > 0 \quad \text{for } t > 0.$$

For any $\kappa > \alpha$, $\beta > 0$ and $\lambda < \alpha$, we put

$$k_1(t;\kappa,\beta,\lambda) := \frac{1}{f_1^{\kappa\beta}(t)} \int_0^t \left[f_1^\beta(t) - f_1^\beta(s)\right]^\kappa f_1^\lambda(s)p(s)\mathrm{d}s \quad \text{for } t > 0 \tag{2.2}$$

and

$$c_1(t;\lambda) := \frac{\alpha-\lambda}{f_1^{\alpha-\lambda}(t)} \int_0^t \frac{g(s)}{f_1^{\lambda+1-\alpha}(s)} \left(\int_0^s f_1^\lambda(\xi)p(\xi)\mathrm{d}\xi\right)\mathrm{d}s \quad \text{for } t > 0. \tag{2.3}$$

Now we can formulate a main result of this subsection.

Theorem 2.1. *Let conditions (1.2), (1.3) and (1.8_1) hold, $\kappa > \alpha$, $\beta > 0$, $\lambda < \alpha$ and either*

$$\limsup_{t\to+\infty} k_1(t;\kappa,\beta,\lambda) = +\infty \tag{2.4}$$

or

$$\begin{cases} -\infty < \limsup\limits_{t\to+\infty} k_1(t;\kappa,\beta,\lambda) < +\infty, \\ \text{the function } c_1(\cdot;\lambda) \text{ does not possess a finite limit as } t \to +\infty. \end{cases} \tag{2.5}$$

Then system (1.1) is oscillatory.

Observe that condition (2.4) with $\beta = 1$, $\lambda = 0$ and $g \equiv 1$ reduces to the condition

$$\limsup_{t\to+\infty} \frac{1}{t^\kappa} \int_0^t (t-s)^\kappa p(s)\mathrm{d}s = +\infty \quad \text{for some } \kappa > \alpha \tag{2.6}$$

which is established in [9] and it is the half-linear extension of the classical Kamenev linear oscillation criterion (see [4]). Conditions (2.5) then give a possible counterpart of the oscillation criterion (2.6).

It is well known that system (1.1) is oscillatory provided that the function

$$M: t \mapsto \frac{1}{f_1(t)} \int_0^t g(s) \left(\int_0^s p(\xi)\mathrm{d}\xi\right)\mathrm{d}s \tag{2.7}$$

is bounded from below in some neighborhood of $+\infty$ and does not have a finite limit as $t \to +\infty$ (see, e.g., [15, Theorem 12.3]). The following corollary of Theorem 2.1 can be applied also in the case, where the lower limit of the function (2.7) is $-\infty$ (see Example 2.4).

For any $\lambda < \alpha$, we put

$$\begin{aligned} h_1(t;\lambda) := {} & \frac{\alpha_*!\,(\alpha-\lambda)^{\alpha_*}}{f_1^{\alpha_*(\alpha-\lambda)}(t)} \int_0^t \frac{g(s_{\alpha_*})}{f_1^{\lambda+1-\alpha}(s_{\alpha_*})} \left(\int_0^{s_{\alpha_*}} \frac{g(s_{\alpha_*-1})}{f_1^{\lambda+1-\alpha}(s_{\alpha_*-1})}\right. \\ & \times \left.\left(\int_0^{s_{\alpha_*-1}} \cdots \left(\int_0^{s_1} f_1^\lambda(\xi)p(\xi)\mathrm{d}\xi\right)\mathrm{d}s_1 \cdots\right)\mathrm{d}s_{\alpha_*-1}\right)\mathrm{d}s_{\alpha_*} \quad \text{for } t > 0, \end{aligned} \tag{2.8}$$

where $\alpha_* := \max\{2, \lfloor\alpha\rfloor + 1\}$ and $\lfloor\alpha\rfloor$ denotes the integer part of the number α.

Corollary 2.2. *Let conditions (1.2), (1.3) and (1.8_1) hold, $\lambda < \alpha$ and either*

$$\limsup_{t\to+\infty} h_1(t;\lambda) = +\infty$$

or

$$\begin{cases} -\infty < \limsup\limits_{t\to+\infty} h_1(t;\lambda) < +\infty, \\ \text{the function } c_1(\cdot;\lambda) \text{ does not possess a finite limit as } t \to +\infty. \end{cases}$$

Then system (1.1) is oscillatory.

Remark 2.3. Observe that if $\alpha < 2$, then $\alpha_* = 2$ in formula (2.8) and the function $h_1(\cdot;\lambda)$ can be expressed in the form

$$h_1(t;\lambda) := \frac{2(\alpha-\lambda)}{f_1^{2(\alpha-\lambda)}(t)} \int_0^t g(s) f_1^{2(\alpha-\lambda)-1}(s) c_1(s;\lambda) \mathrm{d}s \quad \text{for } t > 0.$$

Example 2.4. Consider system (1.1) in which $\alpha \in]0, 2[$, $g \equiv 1$ and

$$p(t) := (\alpha + 1 - t^2)\cos t - (\alpha+3)t \sin t \quad \text{for } t \geq 0.$$

It is clear that conditions (1.2), (1.3) and (1.8_1) hold. Moreover, it is easy to verify that

$$M(t) = t\cos t + (\alpha - 1)\sin t + \frac{\alpha - 1}{t}(\cos t - 1) \quad \text{for } t > 0,$$

where the function M is given by formula (2.7). Therefore, the relation

$$\liminf_{t\to+\infty} M(t) = -\infty$$

holds and thus, above-mentioned Mirzov's result cannot be applied in this case.

On the other hand, we have

$$c_1(t;0) = \alpha t \cos t, \qquad h_1(t;0) = 2\alpha^2 \sin t - \frac{4\alpha^3}{t^{2\alpha}} \int_0^t s^{2\alpha-1} \sin s \, \mathrm{d}s \quad \text{for } t > 0$$

which yields that

$$|h_1(t;0)| \leq 4\alpha^2 \quad \text{for } t > 0 \quad \text{and} \quad \liminf_{t\to+\infty} c_1(t;0) = -\infty.$$

Consequently, by virtue of Corollary 2.2 with $\lambda = 0$, system (1.1) is oscillatory.

Now we formulate a Hartman–Wintner type result which also follows from Theorem 2.1. For any $\lambda < \alpha$ and $\nu < 1$, we put

$$\widetilde{c}_1(t;\lambda,\nu) := \frac{1-\nu}{f_1^{1-\nu}(t)} \int_0^t \frac{g(s)}{f_1^{\nu}(s)} \left(\int_0^s f_1^{\lambda}(\xi) p(\xi) \mathrm{d}\xi \right) \mathrm{d}s \quad \text{for } t > 0.$$

Corollary 2.5. *Let conditions (1.2), (1.3) and (1.8_1) hold, $\lambda < \alpha$, $\nu < 1$ and either*

$$\lim_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu) = +\infty$$

or

$$-\infty < \liminf_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu) < \limsup_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu).$$

Then system (1.1) is oscillatory.

Observe that Corollary 2.5 with $\lambda = 0$ and $\nu = 0$ coincides with above-mentioned Mirzov's result, namely Theorem 12.3 from [15]. On the other hand, it is worth mentioning that Corollary 2.5 with $g \equiv 1$, $\lambda = 0$ and $\nu = 1 - \alpha$ is in compliance with Theorem 1.1 stated in [5].

Remark 2.6. Using integration by parts, it is easy to verify that for any $\lambda < \alpha$ and $\nu_1, \nu_2 < 1$, we have

$$\widetilde{c}_1(t;\lambda,\nu_2) = \frac{1-\nu_2}{1-\nu_1} \widetilde{c}_1(t;\lambda,\nu_1) + \frac{\nu_2 - \nu_1}{1-\nu_1} \frac{1-\nu_2}{f_1^{1-\nu_2}(t)} \int_0^t \frac{g(s)}{f_1^{\nu_2}(s)} \widetilde{c}_1(s;\lambda,\nu_1) \mathrm{d}s \quad \text{for } t > 0$$

whence we get the following assertions:

(i) There exists a finite limit $\lim_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu_2)$ if and only if there exists a finite limit $\lim_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu_1)$, in which case both limits are equal.

(ii) If $\nu_2 > \nu_1$ and $\liminf_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu_1) > -\infty$, then $\liminf_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu_2) > -\infty$.
(iii) If $\nu_2 > \nu_1$ and $\lim_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu_1) = \pm\infty$, then $\lim_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu_2) = \pm\infty$.

For half-linear equation (1.4), from Corollary 2.5 one gets the following generalization of Theorem 2.2.10 from [1].

Corollary 2.7. *Let $\lambda < q-1$ and relation (1.5) hold. Then each of following two conditions is sufficient for oscillation of (1.4):*

$$\lim_{t\to+\infty} \frac{1}{R_1(t)} \int_0^t r^{\frac{1}{1-q}}(s) \left(\int_0^s R_1^\lambda(\xi)p(\xi)\mathrm{d}\xi \right) \mathrm{d}s = +\infty,$$

$$\begin{aligned} -\infty &< \liminf_{t\to+\infty} \frac{1}{R_1(t)} \int_0^t r^{\frac{1}{1-q}}(s) \left(\int_0^s R_1^\lambda(\xi)p(\xi)\mathrm{d}\xi \right) \mathrm{d}s \\ &< \limsup_{t\to+\infty} \frac{1}{R_1(t)} \int_0^t r^{\frac{1}{1-q}}(s) \left(\int_0^s R_1^\lambda(\xi)p(\xi)\mathrm{d}\xi \right) \mathrm{d}s, \end{aligned}$$

where

$$R_1(t) := \int_0^t r^{\frac{1}{1-q}}(s)\mathrm{d}s \quad \textit{for } t \ge 0.$$

2.2. The case $\int_0^\infty g(s)\mathrm{d}s < +\infty$

Unlike Section 2.1, we assume throughout this subsection that the coefficient g is integrable on $[0,+\infty[$, i.e., that g satisfies condition (1.8_2). Let

$$f_2(t) := \int_t^{+\infty} g(s)\mathrm{d}s \quad \text{for } t \ge 0. \tag{2.9}$$

In view of assumptions (1.2), (1.3) and (1.8_2), we have

$$\lim_{t\to+\infty} f_2(t) = 0$$

and

$$f_2(t) > 0 \quad \text{for } t \ge 0.$$

For any $\kappa > \alpha$, $\beta < 0$ and $\lambda > \alpha$, we put

$$k_2(t;\kappa,\beta,\lambda) := f_2^{\kappa|\beta|}(t) \int_0^t \left[f_2^\beta(t) - f_2^\beta(s)\right]^\kappa f_2^\lambda(s)p(s)\mathrm{d}s \quad \text{for } t \ge 0 \tag{2.10}$$

and

$$c_2(t;\lambda) := (\lambda-\alpha)f_2^{\lambda-\alpha}(t) \int_0^t \frac{g(s)}{f_2^{\lambda+1-\alpha}(s)} \left(\int_0^s f_2^\lambda(\xi)p(\xi)\mathrm{d}\xi \right) \mathrm{d}s \quad \text{for } t \ge 0. \tag{2.11}$$

Analogously to the previous subsection, we can formulate the following main result and some of its consequences.

Theorem 2.8. *Let conditions (1.2), (1.3) and (1.8_2) hold, $\kappa > \alpha$, $\beta < 0$, $\lambda > \alpha$ and either*

$$\limsup_{t\to+\infty} k_2(t;\kappa,\beta,\lambda) = +\infty \tag{2.12}$$

or

$$\begin{cases} -\infty < \limsup_{t\to+\infty} k_2(t;\kappa,\beta,\lambda) < +\infty, \\ \textit{the function } c_2(\cdot;\lambda) \textit{ does not possess a finite limit as } t \to +\infty. \end{cases} \tag{2.13}$$

Then system (1.1) is oscillatory.

As for corollaries of Theorem 2.8, at first for any $\lambda > \alpha$, we put

$$\begin{aligned} h_2(t;\lambda) &:= \alpha_*!\,(\lambda-\alpha)^{\alpha_*} f_2^{\alpha_*(\lambda-\alpha)}(t) \int_0^t \frac{g(s_{\alpha_*})}{f_2^{\lambda+1-\alpha}(s_{\alpha_*})} \left(\int_0^{s_{\alpha_*}} \frac{g(s_{\alpha_*-1})}{f_2^{\lambda+1-\alpha}(s_{\alpha_*-1})} \right. \\ &\quad \times \left. \left(\int_0^{s_{\alpha_*-1}} \cdots \left(\int_0^{s_1} f_2^\lambda(\xi)p(\xi)\mathrm{d}\xi \right) \mathrm{d}s_1 \cdots \right) \mathrm{d}s_{\alpha_*-1} \right) \mathrm{d}s_{\alpha_*} \quad \text{for } t \ge 0, \end{aligned} \tag{2.14}$$

where $\alpha_* := \max\{2, \lfloor\alpha\rfloor + 1\}$ and $\lfloor\alpha\rfloor$ denotes the integer part of the number α.

Corollary 2.9. *Let conditions (1.2), (1.3) and (1.8_2) hold, $\lambda > \alpha$ and either*

$$\limsup_{t\to+\infty} h_2(t;\lambda) = +\infty$$

or

$$\begin{cases} -\infty < \limsup\limits_{t\to+\infty} h_2(t;\lambda) < +\infty, \\ \text{the function } c_2(\cdot;\lambda) \text{ does not possess a finite limit as } t \to +\infty. \end{cases}$$

Then system (1.1) is oscillatory.

Remark 2.10. Observe that if $\alpha < 2$, then $\alpha_* = 2$ in formula (2.14) and the function $h_2(\cdot;\lambda)$ can be expressed in the form

$$h_2(t;\lambda) := 2(\lambda-\alpha) f_2^{2(\lambda-\alpha)}(t) \int_0^t \frac{g(s)}{f_2^{2(\lambda-\alpha)+1}(s)} c_2(s;\lambda)\mathrm{d}s \quad \text{for } t \geq 0.$$

Now for any $\lambda > \alpha$ and $\nu > 1$, we put

$$\widetilde{c}_2(t;\lambda,\nu) := (\nu-1) f_2^{\nu-1}(t) \int_0^t \frac{g(s)}{f_2^{\nu}(s)} \left(\int_0^s f_2^{\lambda}(\xi) p(\xi)\mathrm{d}\xi \right) \mathrm{d}s \quad \text{for } t \geq 0.$$

Corollary 2.11. *Let conditions (1.2), (1.3) and (1.8_2) hold, $\lambda > \alpha$, $\nu > 1$ and either*

$$\lim_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu) = +\infty$$

or

$$-\infty < \liminf_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu) < \limsup_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu).$$

Then system (1.1) is oscillatory.

Observe that Corollary 2.11 with $\alpha = 1$ and $\nu = \lambda$ coincides with Theorem 1.1 stated in [12].

Remark 2.12. Using integration by parts, it is easy to verify that for any $\lambda > \alpha$ and $\nu_1, \nu_2 > 1$, we have

$$\widetilde{c}_2(t;\lambda,\nu_2) = \frac{\nu_2-1}{\nu_1-1}\widetilde{c}_2(t;\lambda,\nu_1) + \frac{\nu_1-\nu_2}{\nu_1-1}(\nu_2-1) f_2^{\nu_2-1}(t) \int_0^t \frac{g(s)}{f_2^{\nu_2}(s)} \widetilde{c}_2(s;\lambda,\nu_1)\mathrm{d}s \quad \text{for } t \geq 0$$

whence we get the following assertions:

(i) There exists a finite limit $\lim_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu_2)$ if and only if there exists a finite limit $\lim_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu_1)$, in which case both limits are equal.
(ii) If $\nu_2 < \nu_1$ and $\liminf_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu_1) > -\infty$, then $\liminf_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu_2) > -\infty$.
(iii) If $\nu_2 < \nu_1$ and $\lim_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu_1) = \pm\infty$, then $\lim_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu_2) = \pm\infty$.

As far as we know, a Hartman–Wintner type result for half-linear equation (1.4) in the case, where (1.6) is satisfied, is known only under the additional assumption (1.7) (see survey given in [1, Section 2.2]). We can eliminate this additional assumption and derive from Corollary 2.11 the following statement.

Corollary 2.13. *Let $\lambda > q-1$ and relation (1.6) hold. Then each of following two conditions is sufficient for oscillation of (1.4):*

$$\lim_{t\to+\infty} R_2(t) \int_0^t \frac{1}{r^{\frac{1}{q-1}}(s) R_2^2(s)} \left(\int_0^s R_2^{\lambda}(\xi) p(\xi)\mathrm{d}\xi \right) \mathrm{d}s = +\infty, \tag{2.15}$$

$$\begin{aligned} -\infty &< \liminf_{t\to+\infty} R_2(t) \int_0^t \frac{1}{r^{\frac{1}{q-1}}(s) R_2^2(s)} \left(\int_0^s R_2^{\lambda}(\xi) p(\xi)\mathrm{d}\xi \right) \mathrm{d}s \\ &< \limsup_{t\to+\infty} R_2(t) \int_0^t \frac{1}{r^{\frac{1}{q-1}}(s) R_2^2(s)} \left(\int_0^s R_2^{\lambda}(\xi) p(\xi)\mathrm{d}\xi \right) \mathrm{d}s, \end{aligned}$$

where

$$R_2(t) := \int_t^{+\infty} r^{\frac{1}{1-q}}(s)\mathrm{d}s \quad \text{for } t \geq 0.$$

Remark 2.14. Observe that if inequality (1.7) holds, then relation (2.15) is satisfied if and only if

$$\int_0^{+\infty} R_2^{\lambda}(s)p(s)\mathrm{d}s = +\infty.$$

Therefore, Theorem 2.2.11 stated in [1] follows now from Corollary 2.13 with $\lambda = q$.

3. Auxiliary lemmas

In this section, we present several lemmas on algebraic inequalities which we need to prove main results.

Lemma 3.1. *Let $\alpha > 0$ and $\omega \geq 0$. Then the inequality*

$$\omega|z| - \alpha|z|^{\frac{1+\alpha}{\alpha}} \leq \left(\frac{\omega}{1+\alpha}\right)^{1+\alpha} \tag{3.1}$$

is satisfied for all $z \in \mathbb{R}$.

Proof. It is sufficient to show that inequality (3.1) holds for every $z \geq 0$. We put

$$\ell(z) := \alpha z^{\frac{1+\alpha}{\alpha}} - \omega z + \left(\frac{\omega}{1+\alpha}\right)^{1+\alpha} \quad \text{for } z \geq 0.$$

It is easy to verify by direct calculation that

$$\ell'(z) \geq 0 \quad \text{for } z \geq \left(\frac{\omega}{1+\alpha}\right)^{\alpha}, \qquad \ell'(z) \leq 0 \quad \text{for } z \leq \left(\frac{\omega}{1+\alpha}\right)^{\alpha}$$

and $\ell\left(\left(\frac{\omega}{1+\alpha}\right)^{\alpha}\right) = 0$, which proves the desired assertion. □

Lemma 3.2. *Let $\alpha > 0$. Then*

$$\alpha|x+y|^{\frac{1+\alpha}{\alpha}} \geq \alpha|y|^{\frac{1+\alpha}{\alpha}} + (1+\alpha)x|y|^{\frac{1}{\alpha}}\operatorname{sgn} y \quad \text{for } x, y \in \mathbb{R}.$$

Proof. Let $x, y \in \mathbb{R}$ be arbitrary. It follows from Lemma 3.1 with $\omega = (1+\alpha)|y|^{\frac{1}{\alpha}}$ and $z = x + y$ that

$$\alpha|x+y|^{\frac{1+\alpha}{\alpha}} \geq (1+\alpha)|y|^{\frac{1}{\alpha}}|x+y| - |y|^{\frac{1+\alpha}{\alpha}}. \tag{3.2}$$

Moreover, we have

$$|x+y| \geq \begin{cases} x+y = |y| + x\operatorname{sgn} y & \text{if } y > 0, \\ 0 = |y| + x\operatorname{sgn} y & \text{if } y = 0, \\ -x-y = |y| + x\operatorname{sgn} y & \text{if } y < 0 \end{cases}$$

and thus, the assertion of the lemma follows from inequality (3.2). □

Lemma 3.3 (*[2, Theorem 27]*)**.** *If $r \geq 1$, then*

$$a^r + b^r \leq (a+b)^r \quad \text{for } a, b \geq 0,$$

and if $0 < r \leq 1$, then

$$a^r + b^r \geq (a+b)^r \quad \text{for } a, b \geq 0.$$

Lemma 3.4. *Let $\alpha \in]0, 1]$. Then*

$$(x-y)^{\frac{1+\alpha}{\alpha}} \geq x^{\frac{1+\alpha}{\alpha}} + y^{\frac{1+\alpha}{\alpha}} - \frac{1+\alpha}{\alpha} x^{\frac{1}{\alpha}} y \quad \text{for } x \geq y \geq 0 \tag{3.3}$$

and

$$(x+y)^{\frac{1+\alpha}{\alpha}} \geq x^{\frac{1+\alpha}{\alpha}} + y^{\frac{1+\alpha}{\alpha}} + \frac{1+\alpha}{\alpha} x^{\frac{1}{\alpha}} y \quad \text{for } x, y \geq 0. \tag{3.4}$$

Proof. Let $x \geq y \geq 0$ be arbitrary and

$$\ell(t) := (x - ty)^{\frac{1+\alpha}{\alpha}} \quad \text{for } t \in [0, 1].$$

The function ℓ is absolutely continuous on $[0, 1]$ and thus we get

$$(x-y)^{\frac{1+\alpha}{\alpha}} - x^{\frac{1+\alpha}{\alpha}} = \int_0^1 \ell'(s)\mathrm{d}s = -\frac{1+\alpha}{\alpha} y \int_0^1 (x - sy)^{\frac{1}{\alpha}} \mathrm{d}s. \tag{3.5}$$

It follows from Lemma 3.3 that

$$(x - sy)^{\frac{1}{\alpha}} \leq x^{\frac{1}{\alpha}} - (sy)^{\frac{1}{\alpha}} \quad \text{for } s \in [0, 1].$$

Consequently, relation (3.5) yields that

$$(x-y)^{\frac{1+\alpha}{\alpha}} - x^{\frac{1+\alpha}{\alpha}} \geq -\frac{1+\alpha}{\alpha} y \int_0^1 \left[x^{\frac{1}{\alpha}} - (sy)^{\frac{1}{\alpha}}\right] \mathrm{d}s = -\frac{1+\alpha}{\alpha} x^{\frac{1}{\alpha}} y + y^{\frac{1+\alpha}{\alpha}}$$

which proves desired relation (3.3).

Now let $x, y \geq 0$ be arbitrary and

$$m(t) := (x + ty)^{\frac{1+\alpha}{\alpha}} \quad \text{for } t \in [0, 1].$$

Using Lemma 3.3, we prove in a similar manner as above that relation (3.4) holds. □

The following lemma can be proved analogously to Lemma 3.4.

Lemma 3.5. *Let* $\alpha \geq 1$. *Then*

$$(x-y)^{\frac{1+\alpha}{\alpha}} \leq x^{\frac{1+\alpha}{\alpha}} + y^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha} x^{\frac{1}{\alpha}} y \quad \text{for } x \geq y \geq 0 \tag{3.6}$$

and

$$(x+y)^{\frac{1+\alpha}{\alpha}} \leq x^{\frac{1+\alpha}{\alpha}} + y^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha} x^{\frac{1}{\alpha}} y \quad \text{for } x, y \geq 0. \tag{3.7}$$

Lemma 3.6. *Let* $\alpha \in]0, 1]$. *Then the inequality*

$$|z+\gamma|^{\frac{1+\alpha}{\alpha}} \geq |z|^{\frac{1+\alpha}{\alpha}} + |\gamma|^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}} \operatorname{sgn}\gamma - \tfrac{1+\alpha}{\alpha} |z|^{\frac{1}{\alpha}} |\gamma| \tag{3.8}$$

holds for all $z, \gamma \in \mathbb{R}$.

Proof. Let $z, \gamma \in \mathbb{R}$ be arbitrary.

First suppose that $\gamma \leq 0$. Then, by using Lemma 3.4, we get:

(a) If $z \geq |\gamma|$, then relation (3.3) yields that

$$\begin{aligned} |z+\gamma|^{\frac{1+\alpha}{\alpha}} = (z - |\gamma|)^{\frac{1+\alpha}{\alpha}} &\geq z^{\frac{1+\alpha}{\alpha}} + |\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha} z^{\frac{1}{\alpha}} |\gamma| \\ &\geq |z|^{\frac{1+\alpha}{\alpha}} + |\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha} |z|^{\frac{1}{\alpha}} |\gamma| - \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}}. \end{aligned}$$

(b) If $|\gamma| > z \geq 0$, then relation (3.3) yields that

$$\begin{aligned} |z+\gamma|^{\frac{1+\alpha}{\alpha}} = (|\gamma| - z)^{\frac{1+\alpha}{\alpha}} &\geq |\gamma|^{\frac{1+\alpha}{\alpha}} + z^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha} |\gamma|^{\frac{1}{\alpha}} z \\ &\geq |z|^{\frac{1+\alpha}{\alpha}} + |\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}} - \tfrac{1+\alpha}{\alpha} |z|^{\frac{1}{\alpha}} |\gamma|. \end{aligned}$$

(c) If $z < 0$, then relation (3.4) yields that

$$\begin{aligned} |z+\gamma|^{\frac{1+\alpha}{\alpha}} = (|\gamma| + |z|)^{\frac{1+\alpha}{\alpha}} &\geq |\gamma|^{\frac{1+\alpha}{\alpha}} + |z|^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha} |\gamma|^{\frac{1}{\alpha}} |z| \\ &\geq |z|^{\frac{1+\alpha}{\alpha}} + |\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}} - \tfrac{1+\alpha}{\alpha} |z|^{\frac{1}{\alpha}} |\gamma|. \end{aligned}$$

Consequently, in all cases (a)–(c), inequality (3.8) is satisfied.

Now suppose that $\gamma > 0$. Then, by using Lemma 3.4 again, we obtain:

(A) If $z > 0$, then relation (3.4) yields that

$$\begin{aligned} |z+\gamma|^{\frac{1+\alpha}{\alpha}} = (\gamma + z)^{\frac{1+\alpha}{\alpha}} &\geq \gamma^{\frac{1+\alpha}{\alpha}} + z^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha} \gamma^{\frac{1}{\alpha}} z \\ &\geq |z|^{\frac{1+\alpha}{\alpha}} + |\gamma|^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}} - \tfrac{1+\alpha}{\alpha} |z|^{\frac{1}{\alpha}} |\gamma|. \end{aligned}$$

(B) If $\gamma \geq -z \geq 0$, then relation (3.3) yields that

$$\begin{aligned}|z+\gamma|^{\frac{1+\alpha}{\alpha}} = (\gamma-|z|)^{\frac{1+\alpha}{\alpha}} &\geq \gamma^{\frac{1+\alpha}{\alpha}} + |z|^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha}\gamma^{\frac{1}{\alpha}}|z| \\ &\geq |z|^{\frac{1+\alpha}{\alpha}} + |\gamma|^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}} - \tfrac{1+\alpha}{\alpha}|z|^{\frac{1}{\alpha}}|\gamma|.\end{aligned}$$

(C) If $-z > \gamma$, then relation (3.3) yields that

$$\begin{aligned}|z+\gamma|^{\frac{1+\alpha}{\alpha}} = (|z|-\gamma)^{\frac{1+\alpha}{\alpha}} &\geq |z|^{\frac{1+\alpha}{\alpha}} + \gamma^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha}|z|^{\frac{1}{\alpha}}\gamma \\ &\geq |z|^{\frac{1+\alpha}{\alpha}} + |\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha}|z|^{\frac{1}{\alpha}}|\gamma| + \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}}.\end{aligned}$$

Consequently, in all cases (A)–(C), inequality (3.8) is satisfied. □

Lemma 3.7. *Let $\alpha \geq 1$. Then the inequality*

$$|z+\gamma|^{\frac{1+\alpha}{\alpha}} \geq |z|^{\frac{1+\alpha}{\alpha}} + \tfrac{1}{\alpha}|\gamma|^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}}\operatorname{sgn}\gamma - \tfrac{1+\alpha}{\alpha}|z|\,|\gamma|^{\frac{1}{\alpha}} \tag{3.9}$$

holds for all $z, \gamma \in \mathbb{R}$.

Proof. Let $z, \gamma \in \mathbb{R}$ be arbitrary.

First suppose that $\gamma \leq 0$. Then, by using Lemma 3.5, we get:

(a) If $z \geq |\gamma|$, then relation (3.7) yields that

$$\begin{aligned}|z|^{\frac{1+\alpha}{\alpha}} = \big(|\gamma|+(z-|\gamma|)\big)^{\frac{1+\alpha}{\alpha}} &\leq |\gamma|^{\frac{1+\alpha}{\alpha}} + (z-|\gamma|)^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha}|\gamma|^{\frac{1}{\alpha}}(z-|\gamma|) \\ &\leq |z+\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1}{\alpha}|\gamma|^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}} + \tfrac{1+\alpha}{\alpha}|z|\,|\gamma|^{\frac{1}{\alpha}}.\end{aligned}$$

(b) If $|\gamma| > z \geq 0$, then relation (3.6) yields that

$$\begin{aligned}|z|^{\frac{1+\alpha}{\alpha}} = \big(|\gamma|-(|\gamma|-z)\big)^{\frac{1+\alpha}{\alpha}} &\leq |\gamma|^{\frac{1+\alpha}{\alpha}} + (|\gamma|-z)^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha}|\gamma|^{\frac{1}{\alpha}}(|\gamma|-z) \\ &\leq |z+\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1}{\alpha}|\gamma|^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}} + \tfrac{1+\alpha}{\alpha}|z|\,|\gamma|^{\frac{1}{\alpha}}.\end{aligned}$$

(c) If $z < 0$, then relation (3.6) yields that

$$\begin{aligned}|z|^{\frac{1+\alpha}{\alpha}} = \big((|z|+|\gamma|)-|\gamma|\big)^{\frac{1+\alpha}{\alpha}} &\leq (|z|+|\gamma|)^{\frac{1+\alpha}{\alpha}} + |\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha}(|z|+|\gamma|)^{\frac{1}{\alpha}}|\gamma| \\ &\leq |z+\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1}{\alpha}|\gamma|^{\frac{1+\alpha}{\alpha}} \\ &= |z+\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1}{\alpha}|\gamma|^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}} + \tfrac{1+\alpha}{\alpha}|z|\,|\gamma|^{\frac{1}{\alpha}}.\end{aligned}$$

Consequently, in all cases (a)–(c), inequality (3.9) is satisfied.

Now suppose that $\gamma > 0$. Then, by using Lemma 3.5 again, we obtain:

(A) If $z > 0$, then relation (3.6) yields that

$$\begin{aligned}|z|^{\frac{1+\alpha}{\alpha}} = \big((z+\gamma)-\gamma\big)^{\frac{1+\alpha}{\alpha}} &\leq (z+\gamma)^{\frac{1+\alpha}{\alpha}} + \gamma^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha}(z+\gamma)^{\frac{1}{\alpha}}\gamma \\ &\leq |z+\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1}{\alpha}|\gamma|^{\frac{1+\alpha}{\alpha}} \\ &= |z+\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1}{\alpha}|\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}} + \tfrac{1+\alpha}{\alpha}|z|\,|\gamma|^{\frac{1}{\alpha}}.\end{aligned}$$

(B) If $\gamma \geq -z \geq 0$, then relation (3.6) yields that

$$\begin{aligned}|z|^{\frac{1+\alpha}{\alpha}} = \big(\gamma-(\gamma-|z|)\big)^{\frac{1+\alpha}{\alpha}} &\leq \gamma^{\frac{1+\alpha}{\alpha}} + (\gamma-|z|)^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha}\gamma^{\frac{1}{\alpha}}(\gamma-|z|) \\ &\leq |z+\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1}{\alpha}|\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}} + \tfrac{1+\alpha}{\alpha}|z|\,|\gamma|^{\frac{1}{\alpha}}.\end{aligned}$$

(C) If $-z > \gamma$, then relation (3.7) yields that

$$\begin{aligned}|z|^{\frac{1+\alpha}{\alpha}} = \big(\gamma+(|z|-\gamma)\big)^{\frac{1+\alpha}{\alpha}} &\leq \gamma^{\frac{1+\alpha}{\alpha}} + (|z|-\gamma)^{\frac{1+\alpha}{\alpha}} + \tfrac{1+\alpha}{\alpha}\gamma^{\frac{1}{\alpha}}(|z|-\gamma) \\ &\leq |z+\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1}{\alpha}|\gamma|^{\frac{1+\alpha}{\alpha}} - \tfrac{1+\alpha}{\alpha} z|\gamma|^{\frac{1}{\alpha}} + \tfrac{1+\alpha}{\alpha}|z|\,|\gamma|^{\frac{1}{\alpha}}.\end{aligned}$$

Consequently, in all cases (A)–(C), inequality (3.9) is satisfied. □

4. Proofs of main results

To prove Theorem 2.1 we need the following two lemmas.

Lemma 4.1. *Let $m \in \{1, 2\}$ and conditions (1.2), (1.3), and (1.8_m) hold. Let, moreover, $\beta, \lambda \in \mathbb{R}$, $\kappa > \alpha$, and (u, v) be a solution to system (1.1) such that*

$$(-1)^{m-1}\beta > 0, \qquad (-1)^{m-1}(\alpha - \lambda) > 0,$$

and

$$u(t) \neq 0 \quad \text{for } t \geq t_u \tag{4.1}$$

with $t_u > 0$. Then

$$\limsup_{t\to+\infty} k_m(t; \kappa, \beta, \lambda) < +\infty, \tag{4.2}$$

where the function k_m is defined by formula (2.2) (resp. (2.10)). If, in addition, the inequality

$$\limsup_{t\to+\infty} k_m(t; \kappa, \beta, \lambda) > -\infty \tag{4.3}$$

is satisfied, then

$$\int_{t_u}^{+\infty} g(s) f_m^{\lambda}(s) |\varrho(s)|^{\frac{1+\alpha}{\alpha}} \mathrm{d}s < +\infty, \tag{4.4}$$

where the function f_m is defined by formula (2.1) (resp. (2.9)) and

$$\varrho(t) := \frac{v(t)}{|u(t)|^{\alpha} \operatorname{sgn} u(t)} + \frac{(-1)^m}{f_m^{\alpha}(t)} \left(\frac{|\lambda|}{1+\alpha}\right)^{\alpha} \operatorname{sgn}\lambda \quad \text{for } t \geq t_u. \tag{4.5}$$

Proof. Put

$$\gamma := (-1)^{m-1} \left(\frac{2|\lambda|}{1+\alpha}\right)^{\alpha} \operatorname{sgn}\lambda, \tag{4.6}$$

and

$$\sigma(t) := \frac{v(t)}{|u(t)|^{\alpha} \operatorname{sgn} u(t)}, \qquad \varphi(t) := \sigma(t) - \frac{\gamma}{f_m^{\alpha}(t)} \quad \text{for } t \geq t_u. \tag{4.7}$$

Then the functions σ, φ are absolutely continuous on every compact interval contained in $[t_u, +\infty[$ and, in view of (1.1), relations (4.7) yield that

$$\varphi'(t) = -p(t) - \alpha g(t)|\sigma(t)|^{\frac{1+\alpha}{\alpha}} + (-1)^{m-1}\alpha\gamma \frac{g(t)}{f_m^{1+\alpha}(t)} \quad \text{for a.e. } t \geq t_u. \tag{4.8}$$

Let

$$F_m(t, s) := f_m^{\beta}(t) - f_m^{\beta}(s) \quad \text{for } t \geq s \geq t_u.$$

Then it follows from equality (4.8) that

$$\begin{aligned}\int_{t_u}^{t} F_m^{\kappa}(t, s) f_m^{\lambda}(s)\varphi'(s)\mathrm{d}s = &- \int_{t_u}^{t} F_m^{\kappa}(t, s) f_m^{\lambda}(s) p(s)\mathrm{d}s - \alpha \int_{t_u}^{t} g(s) F_m^{\kappa}(t, s) f_m^{\lambda}(s) |\sigma(s)|^{\frac{1+\alpha}{\alpha}} \mathrm{d}s \\ &+ (-1)^{m-1}\alpha\gamma \int_{t_u}^{t} g(s) F_m^{\kappa}(t, s) f_m^{\lambda-1-\alpha}(s)\mathrm{d}s\end{aligned} \tag{4.9}$$

for $t \geq t_u$. Put

$$A_m := \left\{ t > t_u : (-1)^m \big(f_m(s) - f_m(t)\big) > 0 \text{ for all } s \in [t_u, t[\right\}. \tag{4.10}$$

Observe that for any $t \in A_m$ and $\zeta > 0$,

the function $s \mapsto F_m^{\zeta}(t, s)$ is absolutely continuous on $[t_u, t]$

and thus, we obtain

$$\int_{t_u}^{t} F_m^{\kappa}(t,s) f_m^{\lambda}(s)\varphi'(s)\mathrm{d}s = -F_m^{\kappa}(t,t_u) f_m^{\lambda}(t_u)\varphi(t_u) + (-1)^{m-1}\kappa\beta \int_{t_u}^{t} g(s)F_m^{\kappa-1}(t,s) f_m^{\lambda+\beta-1}(s)\varphi(s)\mathrm{d}s + (-1)^m\lambda \int_{t_u}^{t} g(s)F_m^{\kappa}(t,s) f_m^{\lambda-1}(s)\varphi(s)\mathrm{d}s$$

for $t \in A_m$. Therefore, using (4.7), from relation (4.9) we get

$$\int_{t_u}^{t} F_m^{\kappa}(t,s) f_m^{\lambda}(s)p(s)\mathrm{d}s = F_m^{\kappa}(t,t_u) f_m^{\lambda}(t_u)\varphi(t_u) - \frac{1}{2}\int_{t_u}^{t} g(s)F_m^{\kappa}(t,s) f_m^{\lambda-1-\alpha}(s) \times \Big[\alpha |f_m^{\alpha}(s)\varphi(s)+\gamma|^{\frac{1+\alpha}{\alpha}} + (-1)^m 2\lambda f_m^{\alpha}(s)\varphi(s)\Big]\mathrm{d}s + \frac{1}{2}\int_{t_u}^{t} g(s)F_m^{\kappa-1-\alpha}(t,s) f_m^{\lambda+(1+\alpha)(\beta-1)}(s) \times \Big[-\alpha |F_m^{\alpha}(t,s) f_m^{\alpha(1-\beta)}(s)\sigma(s)|^{\frac{1+\alpha}{\alpha}} + (-1)^m 2\kappa\beta F_m^{\alpha}(t,s) f_m^{\alpha(1-\beta)}(s)\sigma(s)\Big]\mathrm{d}s + (-1)^{m-1}\kappa\beta\gamma \int_{t_u}^{t} g(s)F_m^{\kappa-1}(t,s) f_m^{\beta-1+\lambda-\alpha}(s)\mathrm{d}s + (-1)^{m-1}\alpha\gamma \int_{t_u}^{t} g(s)F_m^{\kappa}(t,s) f_m^{\lambda-1-\alpha}(s)\mathrm{d}s \quad \text{for } t \in A_m$$

which, by virtue of Lemma 3.1 and notation (4.6), yields that

$$\int_{t_u}^{t} F_m^{\kappa}(t,s) f_m^{\lambda}(s)p(s)\mathrm{d}s \le F_m^{\kappa}(t,t_u) f_m^{\lambda}(t_u)\varphi(t_u) - \frac{1}{2}\int_{t_u}^{t} g(s)F_m^{\kappa}(t,s) f_m^{\lambda-1-\alpha}(s)\Big[\alpha |f_m^{\alpha}(s)\varphi(s)+\gamma|^{\frac{1+\alpha}{\alpha}} - (1+\alpha) f_m^{\alpha}(s)\varphi(s)|\gamma|^{\frac{1}{\alpha}}\operatorname{sgn}\gamma - \min\{\alpha,1\}|\gamma|^{\frac{1+\alpha}{\alpha}}\Big]\mathrm{d}s + \frac{1}{2}\left(\frac{2\kappa|\beta|}{1+\alpha}\right)^{1+\alpha}\int_{t_u}^{t} g(s)F_m^{\kappa-1-\alpha}(t,s) f_m^{\lambda+(1+\alpha)(\beta-1)}(s)\mathrm{d}s + \kappa|\beta|\gamma\int_{t_u}^{t} g(s)F_m^{\kappa-1}(t,s) f_m^{\beta-1+\lambda-\alpha}(s)\mathrm{d}s + \frac{2\alpha|\gamma| - \min\{\alpha,1\}|\gamma|^{\frac{1+\alpha}{\alpha}}}{2}\int_{t_u}^{t} g(s)F_m^{\kappa}(t,s) f_m^{\lambda-1-\alpha}(s)\mathrm{d}s \tag{4.11}$$

for $t \in A_m$. Now observe that

$$\frac{1}{f_m^{\kappa\beta}(t)} F_m^{\kappa}(t,t_u) f_m^{\lambda}(t_u)\varphi(t_u) \le f_m^{\lambda}(t_u)|\varphi(t_u)| \quad \text{for } t \in A_m \tag{4.12}$$

and

$$\frac{1}{f_m^{\kappa\beta}(t)}\int_{t_u}^{t} g(s)F_m^{\kappa-1-\alpha}(t,s) f_m^{\lambda+(1+\alpha)(\beta-1)}(s)\mathrm{d}s \le \frac{1}{(\kappa-\alpha)|\beta|}\,\frac{1}{f_m^{\alpha-\lambda}(t_u)} \quad \text{for } t \in A_m, \tag{4.13}$$

because:

- If $(-1)^m\big[\lambda+\alpha(\beta-1)\big] \ge 0$, then

$$\frac{1}{f_m^{\kappa\beta}(t)}\int_{t_u}^{t}\big[f_m^{\beta}(t)-f_m^{\beta}(s)\big]^{\kappa-1-\alpha} g(s) f_m^{\beta-1}(s) f_m^{\lambda+\alpha(\beta-1)}(s)\mathrm{d}s \le \frac{1}{f_m^{\kappa\beta}(t)}\,\frac{1}{f_m^{\alpha(1-\beta)-\lambda}(t_u)}\int_{t_u}^{t}\mathrm{d}_s\left(\frac{(-1)^m}{(\kappa-\alpha)\beta}\big[f_m^{\beta}(t)-f_m^{\beta}(s)\big]^{\kappa-\alpha}\right) \le \frac{1}{(\kappa-\alpha)|\beta|}\,\frac{1}{f_m^{\alpha-\lambda}(t_u)} \quad \text{for } t \in A_m.$$

- If $(-1)^m\big[\lambda+\alpha(\beta-1)\big] < 0$, then

$$\frac{1}{f_m^{\kappa\beta}(t)}\int_{t_u}^{t}\big[f_m^{\beta}(t)-f_m^{\beta}(s)\big]^{\kappa-1-\alpha} g(s) f_m^{\beta-1}(s) f_m^{\lambda+\alpha(\beta-1)}(s)\mathrm{d}s \le \frac{f_m^{\lambda+\alpha(\beta-1)}(t)}{f_m^{\kappa\beta}(t)}\int_{t_u}^{t}\mathrm{d}_s\left(\frac{(-1)^m}{(\kappa-\alpha)\beta}\big[f_m^{\beta}(t)-f_m^{\beta}(s)\big]^{\kappa-\alpha}\right) \le \frac{1}{(\kappa-\alpha)|\beta|}\,\frac{1}{f_m^{\alpha-\lambda}(t_u)} \quad \text{for } t \in A_m.$$

Moreover, we have

$$\frac{1}{f_m^{\kappa\beta}(t)}\int_{t_u}^{t} g(s)F_m^{\kappa-1}(t,s)f_m^{\beta-1+\lambda-\alpha}(s)\mathrm{d}s \le \frac{1}{f_m^{\kappa\beta}(t)}\,\frac{1}{f_m^{\alpha-\lambda}(t_u)}\int_{t_u}^{t}\mathrm{d}_s\left(\frac{(-1)^m}{\kappa\beta}\left[f_m^{\beta}(t)-f_m^{\beta}(s)\right]^{\kappa}\right) \le \frac{1}{\kappa|\beta|}\,\frac{1}{f_m^{\alpha-\lambda}(t_u)} \quad \text{for } t\in A_m \tag{4.14}$$

and

$$\frac{1}{f_m^{\kappa\beta}(t)}\int_{t_u}^{t} g(s)F_m^{\kappa}(t,s)f_m^{\lambda-1-\alpha}(s)\mathrm{d}s \le \int_{t_u}^{t} g(s)f_m^{\lambda-1-\alpha}(s)\mathrm{d}s \le \frac{1}{|\alpha-\lambda|}\,\frac{1}{f_m^{\alpha-\lambda}(t_u)} \quad \text{for } t\in A_m. \tag{4.15}$$

According to Lemma 3.2, it is clear that

$$\alpha|f_m^{\alpha}(s)\varphi(s)+\gamma|^{\frac{1+\alpha}{\alpha}}-(1+\alpha)f_m^{\alpha}(s)\varphi(s)|\gamma|^{\frac{1}{\alpha}}\operatorname{sgn}\gamma-\alpha|\gamma|^{\frac{1+\alpha}{\alpha}}\ge 0 \quad \text{for } s\ge t_u \tag{4.16}$$

and thus, using inequalities (4.12)–(4.16), it follows from relation (4.11) that the inequality

$$\frac{1}{f_m^{\kappa\beta}(t)}\int_{t_u}^{t}\left[f_m^{\beta}(t)-f_m^{\beta}(s)\right]^{\kappa}f_m^{\lambda}(s)p(s)\mathrm{d}s \le \delta_m(t_u) \tag{4.17}$$

holds for $t\in A_m$, where

$$\delta_m(t_u) := f_m^{\lambda}(t_u)|\varphi(t_u)|+\frac{1}{2}\left(\frac{2\kappa|\beta|}{1+\alpha}\right)^{1+\alpha}\frac{1}{(\kappa-\alpha)|\beta|}\,\frac{1}{f_m^{\alpha-\lambda}(t_u)}+\frac{|\gamma|}{f_m^{\alpha-\lambda}(t_u)}+\frac{\alpha|\gamma|}{|\alpha-\lambda|}\,\frac{1}{f_m^{\alpha-\lambda}(t_u)}. \tag{4.18}$$

However, in view of definition (4.10) of the set A_m and the non-negativity of the function g, we see that inequality (4.17) is satisfied, in fact, for all $t\ge t_u$, whence we get the desired relation (4.2).

Let, in addition, the function k_m satisfy inequality (4.3). Obviously, either

$$\int_{t_u}^{+\infty} g(s)f_m^{\lambda}(s)|\varphi(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s=+\infty \tag{4.19}$$

or

$$\int_{t_u}^{+\infty} g(s)f_m^{\lambda}(s)|\varphi(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s<+\infty. \tag{4.20}$$

Assume that equality (4.19) holds. We divide the proof into two cases:

Case (1): $\alpha\in]0,1]$. It follows from Lemma 3.6 that

$$\alpha|f_m^{\alpha}(s)\varphi(s)+\gamma|^{\frac{1+\alpha}{\alpha}} \ge \alpha|f_m^{\alpha}(s)\varphi(s)|^{\frac{1+\alpha}{\alpha}}+(1+\alpha)f_m^{\alpha}(s)\varphi(s)|\gamma|^{\frac{1}{\alpha}}\operatorname{sgn}\gamma +\alpha|\gamma|^{\frac{1+\alpha}{\alpha}}-(1+\alpha)|f_m^{\alpha}(s)\varphi(s)|^{\frac{1}{\alpha}}|\gamma| \quad \text{for } s\ge t_u. \tag{4.21}$$

Put

$$\widetilde{F}_m(t,s) := \left[1-\left(\frac{f_m(s)}{f_m(t)}\right)^{\beta}\right]^{\kappa} \quad \text{for } t\ge s\ge t_u. \tag{4.22}$$

Then, by using inequalities (4.12)–(4.15) and (4.21), from relation (4.11) one gets for any $t\in A_m$ the inequality

$$\frac{1}{f_m^{\kappa\beta}(t)}\int_{t_u}^{t} F_m^{\kappa}(t,s)f_m^{\lambda}(s)p(s)\mathrm{d}s \le -\frac{\alpha}{2}\int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda}(s)|\varphi(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s +\frac{(1+\alpha)|\gamma|}{2}\int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda-\alpha}(s)|\varphi(s)|^{\frac{1}{\alpha}}\mathrm{d}s+\delta_m(t_u), \tag{4.23}$$

where the number $\delta_m(t_u)$ is defined by formula (4.18). However, in view of definition (4.10) of the set A_m and the non-negativity of the function g, we see that inequality (4.23) is satisfied, in fact, for all $t\ge t_u$.

Let

$$I(t) := \int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda}(s)|\varphi(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \text{for } t\ge t_u. \tag{4.24}$$

Observe that for any $\tau \geq t_u$ fixed, we have

$$\int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda}(s)|\varphi(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \geq \widetilde{F}_m(t,\tau)\int_{t_u}^{\tau} g(s)f_m^{\lambda}(s)|\varphi(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \text{for } t \geq \tau$$

and thus

$$\liminf_{t\to+\infty} I(t) \geq \int_{t_u}^{\tau} g(s)f_m^{\lambda}(s)|\varphi(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \quad \text{for } \tau \geq t_u.$$

Since we suppose that equality (4.19) is satisfied, the last relation guarantees that

$$\lim_{t\to+\infty} I(t) = +\infty. \tag{4.25}$$

On the other hand, the Hölder inequality yields that

$$\begin{aligned}\int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda-\alpha}(s)|\varphi(s)|^{\frac{1}{\alpha}}\mathrm{d}s &\leq \left(\int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda-1-\alpha}(s)\mathrm{d}s\right)^{\frac{\alpha}{1+\alpha}}\left(\int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda}(s)|\varphi(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\right)^{\frac{1}{1+\alpha}}\\ &\leq \left(\frac{f_m^{\lambda-\alpha}(t_u)}{|\alpha-\lambda|}\right)^{\frac{\alpha}{1+\alpha}} I^{\frac{1}{1+\alpha}}(t) \quad \text{for } t \geq t_u.\end{aligned}$$

Using this inequality in (4.23), we get

$$\frac{1}{f_m^{\kappa\beta}(t)}\int_{t_u}^{t}\left[f_m^{\beta}(t)-f_m^{\beta}(s)\right]^{\kappa} f_m^{\lambda}(s)p(s)\mathrm{d}s \leq -I^{\frac{1}{1+\alpha}}(t)\left[\frac{\alpha}{2}I^{\frac{\alpha}{1+\alpha}}(t) - \frac{(1+\alpha)|\gamma|}{2}\left(\frac{f_m^{\lambda-\alpha}(t_u)}{|\alpha-\lambda|}\right)^{\frac{\alpha}{1+\alpha}}\right] + \delta_m(t_u)$$

for $t \geq t_u$ which, in view of the above-proved relation (4.25), contradicts our additional assumption (4.3).

Case (2): $\alpha > 1$. It follows from Lemma 3.7 that

$$\begin{aligned}\alpha|f_m^{\alpha}(s)\varphi(s)+\gamma|^{\frac{1+\alpha}{\alpha}} &\geq \alpha|f_m^{\alpha}(s)\varphi(s)|^{\frac{1+\alpha}{\alpha}} + (1+\alpha)f_m^{\alpha}(s)\varphi(s)|\gamma|^{\frac{1}{\alpha}}\operatorname{sgn}\gamma\\ &\quad + |\gamma|^{\frac{1+\alpha}{\alpha}} - (1+\alpha)|f_m^{\alpha}(s)\varphi(s)|\,|\gamma|^{\frac{1}{\alpha}} \quad \text{for } s \geq t_u.\end{aligned} \tag{4.26}$$

Then, by using inequalities (4.12)–(4.15) and (4.26), from relation (4.11) one gets for any $t \in A_m$ the inequality

$$\begin{aligned}\frac{1}{f_m^{\kappa\beta}(t)}\int_{t_u}^{t} F_m^{\kappa}(t,s)f_m^{\lambda}(s)p(s)\mathrm{d}s &\leq -\frac{\alpha}{2}\int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda}(s)|\varphi(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\\ &\quad + \frac{(1+\alpha)|\gamma|^{\frac{1}{\alpha}}}{2}\int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda-1}(s)|\varphi(s)|\mathrm{d}s + \delta_m(t_u),\end{aligned} \tag{4.27}$$

where the function $\widetilde{F}_m$ is defined by formula (4.22) and the number $\delta_m(t_u)$ is given by relation (4.18). However, in view of definition (4.10) of the set A_m and the non-negativity of the function g, we see that inequality (4.27) is satisfied, in fact, for all $t \geq t_u$. Since we suppose that equality (4.19) holds, one can show analogously to the previous case that relation (4.25) is satisfied, where the function I is defined by formula (4.24).

On the other hand, the Hölder inequality yields that

$$\begin{aligned}\int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda-1}(s)|\varphi(s)|\mathrm{d}s &\leq \left(\int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda-1-\alpha}(s)\mathrm{d}s\right)^{\frac{1}{1+\alpha}}\left(\int_{t_u}^{t} g(s)\widetilde{F}_m(t,s)f_m^{\lambda}(s)|\varphi(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\right)^{\frac{\alpha}{1+\alpha}}\\ &\leq \left(\frac{f_m^{\lambda-\alpha}(t_u)}{|\alpha-\lambda|}\right)^{\frac{1}{1+\alpha}} I^{\frac{\alpha}{1+\alpha}}(t) \quad \text{for } t \geq t_u.\end{aligned}$$

Using this inequality in (4.27), we get

$$\frac{1}{f_m^{\kappa\beta}(t)}\int_{t_u}^{t}\left[f_m^{\beta}(t)-f_m^{\beta}(s)\right]^{\kappa} f_m^{\lambda}(s)p(s)\mathrm{d}s \leq -I^{\frac{\alpha}{1+\alpha}}(t)\left[\frac{\alpha}{2}I^{\frac{1}{1+\alpha}}(t) - \frac{(1+\alpha)|\gamma|^{\frac{1}{\alpha}}}{2}\left(\frac{f_m^{\lambda-\alpha}(t_u)}{|\alpha-\lambda|}\right)^{\frac{1}{1+\alpha}}\right] + \delta_m(t_u)$$

for $t \geq t_u$ which, in view of the above-proved relation (4.25), contradicts our additional assumption (4.3).

Consequently, the contradictions obtained prove that for any $\alpha > 0$, inequality (4.20) holds, i.e.,

$$\int_{t_u}^{+\infty}\left|g^{\frac{\alpha}{1+\alpha}}(s)f_m^{\frac{\lambda\alpha}{1+\alpha}}(s)\varrho(s) + \widetilde{\gamma}\, g^{\frac{\alpha}{1+\alpha}}(s)f_m^{\frac{\lambda\alpha}{1+\alpha}-\alpha}(s)\right|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s < +\infty, \tag{4.28}$$

where

$$\widetilde{\gamma} := (-1)^m \big(2^\alpha - 1\big) \left(\tfrac{|\lambda|}{1+\alpha}\right)^\alpha \operatorname{sgn} \lambda.$$

Moreover, we have

$$\int_{t_u}^{+\infty} \left| g^{\frac{\alpha}{1+\alpha}}(s) f_m^{\frac{\lambda\alpha}{1+\alpha}-\alpha}(s) \right|^{\frac{1+\alpha}{\alpha}} \mathrm{d}s = \int_{t_u}^{+\infty} g(s) f_m^{\lambda-1-\alpha}(s) \mathrm{d}s < +\infty. \tag{4.29}$$

Since the space $L^{\frac{1+\alpha}{\alpha}}([t_u, +\infty[; \mathbb{R})$ is linear, inequalities (4.28) and (4.29) guarantee the desired relation (4.4). □

Lemma 4.2. *Let $m \in \{1, 2\}$, conditions (1.2), (1.3), and (1.8$_m$) hold, and $\lambda \in \mathbb{R}$ be such that $(-1)^{m-1}(\alpha-\lambda) > 0$. Let, moreover, (u, v) be a solution to system (1.1) fulfilling relation (4.1) with $t_u > 0$. If inequality (4.4) is satisfied, where the functions f_m and ϱ are defined by formulas (2.1) (resp. (2.9)) and (4.5), then the function c_m given by relation (2.3) (resp. (2.11)) has a finite limit*

$$\lim_{t\to+\infty} c_m(t; \lambda). \tag{4.30}$$

Proof. Put

$$\gamma := (-1)^{m-1} \left(\frac{|\lambda|}{1+\alpha} \right)^\alpha \operatorname{sgn} \lambda \tag{4.31}$$

and

$$\sigma(t) := \frac{v(t)}{|u(t)|^\alpha \operatorname{sgn} u(t)} \quad \text{for } t \geq t_u. \tag{4.32}$$

Then the functions ϱ, σ are absolutely continuous on every compact interval contained in $[t_u, +\infty[$ and, in view of (1.1), relations (4.5) and (4.32) yield that

$$\varrho'(t) = -p(t) - \alpha g(t) |\sigma(t)|^{\frac{1+\alpha}{\alpha}} + (-1)^{m-1} \alpha\gamma \frac{g(t)}{f_m^{1+\alpha}(t)} \quad \text{for a.e. } t \geq t_u,$$

whence we obtain

$$\begin{aligned} \int_{t_u}^t f_m^\lambda(s) \varrho'(s) \mathrm{d}s = &- \int_{t_u}^t f_m^\lambda(s) p(s) \mathrm{d}s - \alpha \int_{t_u}^t g(s) f_m^\lambda(s) |\sigma(s)|^{\frac{1+\alpha}{\alpha}} \mathrm{d}s \\ &+ (-1)^{m-1} \alpha\gamma \int_{t_u}^t g(s) f_m^{\lambda-1-\alpha}(s) \mathrm{d}s \quad \text{for } t \geq t_u. \end{aligned} \tag{4.33}$$

Using the integration by parts on the left-hand side of (4.33), one gets

$$\begin{aligned} f_m^\lambda(t)\varrho(t) = &f_m^\lambda(t_u)\varrho(t_u) - \int_{t_u}^t f_m^\lambda(s) p(s) \mathrm{d}s + (-1)^{m-1}\alpha\gamma \int_{t_u}^t g(s) f_m^{\lambda-1-\alpha}(s) \mathrm{d}s \\ &- \int_{t_u}^t g(s) f_m^{\lambda-1-\alpha}(s) \Big[\alpha |f_m^\alpha(s)\varrho(s) + \gamma|^{\frac{1+\alpha}{\alpha}} + (-1)^m \lambda f_m^\alpha(s) \varrho(s) \Big] \mathrm{d}s \end{aligned}$$

for $t \geq t_u$ which, in view of (4.31), yields that

$$\begin{aligned} f_m^\lambda(t)\varrho(t) = &- \int_{t_u}^t f_m^\lambda(s) p(s) \mathrm{d}s - \frac{\alpha\big((-1)^{m-1}\gamma - |\gamma|^{\frac{1+\alpha}{\alpha}}\big)}{|\alpha-\lambda|} \frac{1}{f_m^{\alpha-\lambda}(t)} - \int_{t_u}^t g(s) f_m^{\lambda-1-\alpha}(s) \Big[\alpha |f_m^\alpha(s)\varrho(s) + \gamma|^{\frac{1+\alpha}{\alpha}} \\ &- (1+\alpha) f_m^\alpha(s)\varrho(s)|\gamma|^{\frac{1}{\alpha}} \operatorname{sgn}\gamma - \alpha|\gamma|^{\frac{1+\alpha}{\alpha}} \Big] \mathrm{d}s + f_m^\lambda(t_u)\varrho(t_u) + \frac{\alpha\big((-1)^{m-1}\gamma - |\gamma|^{\frac{1+\alpha}{\alpha}}\big)}{|\alpha-\lambda|} \frac{1}{f_m^{\alpha-\lambda}(t_u)} \quad \text{for } t \geq t_u. \end{aligned} \tag{4.34}$$

Now we put

$$\ell_m(t) := \alpha |f_m^\alpha(t)\varrho(t) + \gamma|^{\frac{1+\alpha}{\alpha}} - (1+\alpha) f_m^\alpha(t)\varrho(t)|\gamma|^{\frac{1}{\alpha}} \operatorname{sgn}\gamma - \alpha|\gamma|^{\frac{1+\alpha}{\alpha}} \quad \text{for } t \geq t_u. \tag{4.35}$$

According to Lemma 3.2, we have

$$\ell_m(t) \geq 0 \quad \text{for } t \geq t_u. \tag{4.36}$$

Observe that inequality (4.29) holds and thus, assumption (4.4) guarantees that

$$\int_{t_u}^{+\infty} g(s) f_m^{\lambda-1-\alpha}(s) |f_m^{\alpha}(s)\varrho(s)+\gamma|^{\frac{1+\alpha}{\alpha}} \mathrm{d}s < +\infty. \tag{4.37}$$

Moreover, by using the Hölder inequality, we get

$$\begin{aligned}\int_{t_u}^{t} g(s) f_m^{\lambda-1-\alpha}(s) |f_m^{\alpha}(s)\varrho(s)| \mathrm{d}s &\le \left(\int_{t_u}^{t} g(s) f_m^{\lambda-1-\alpha}(s)\mathrm{d}s\right)^{\frac{1}{1+\alpha}} \left(\int_{t_u}^{t} g(s) f_m^{\lambda}(s) |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\right)^{\frac{\alpha}{1+\alpha}} \\ &\le \left(\frac{f_m^{\lambda-\alpha}(t_u)}{|\alpha-\lambda|}\right)^{\frac{1}{1+\alpha}} \left(\int_{t_u}^{t} g(s) f_m^{\lambda}(s) |\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\right)^{\frac{\alpha}{1+\alpha}} \quad \text{for } t \ge t_u.\end{aligned}$$

Therefore, in view of relations (4.4), (4.29) and (4.37), it follows from (4.35) and (4.36) that

$$\int_{t_u}^{+\infty} g(s) f_m^{\lambda-1-\alpha}(s) \ell_m(s)\mathrm{d}s < +\infty. \tag{4.38}$$

On the other hand, using notation (4.35) in equality (4.34), we obtain

$$\begin{aligned} f_m^{\lambda}(t)\varrho(t) &= \delta_m(t_u) - \int_0^t f_m^{\lambda}(s)p(s)\mathrm{d}s - \int_{t_u}^{t} g(s) f_m^{\lambda-1-\alpha}(s)\ell_m(s)\mathrm{d}s \\ &\quad - \frac{\alpha\left(\gamma(-1)^{m-1} - |\gamma|^{\frac{1+\alpha}{\alpha}}\right)}{|\alpha-\lambda|} \frac{1}{f_m^{\alpha-\lambda}(t)} \quad \text{for } t \ge t_u \end{aligned}$$

with

$$\delta_m(t_u) := f_m^{\lambda}(t_u)\varrho(t_u) + \int_0^{t_u} f_m^{\lambda}(s)p(s)\mathrm{d}s + \frac{\alpha\left(\gamma(-1)^{m-1} - |\gamma|^{\frac{1+\alpha}{\alpha}}\right)}{|\alpha-\lambda|} \frac{1}{f_m^{\alpha-\lambda}(t_u)},$$

whence we get

$$\begin{aligned} \int_{t_u}^{t} \frac{g(s)}{f_m^{1-\alpha}(s)} \varrho(s)\mathrm{d}s &= \frac{\delta_m(t_u)}{|\alpha-\lambda|}\left[f_m^{\alpha-\lambda}(t) - f_m^{\alpha-\lambda}(t_u)\right] - \int_{t_u}^{t} \frac{g(s)}{f_m^{\lambda+1-\alpha}(s)}\left(\int_0^s f_m^{\lambda}(\xi)p(\xi)\mathrm{d}\xi\right)\mathrm{d}s \\ &\quad - \int_{t_u}^{t} \frac{g(s)}{f_m^{\lambda+1-\alpha}(s)}\left(\int_{t_u}^{s} g(\xi) f_m^{\lambda-1-\alpha}(\xi)\ell_m(\xi)\mathrm{d}\xi\right)\mathrm{d}s - \frac{\alpha\left(\gamma+(-1)^m|\gamma|^{\frac{1+\alpha}{\alpha}}\right)}{|\alpha-\lambda|} \ln\frac{f_m(t)}{f_m(t_u)} \quad \text{for } t \ge t_u. \end{aligned} \tag{4.39}$$

Observe that, in view of inequality (4.36), we have

$$\begin{aligned} \int_{t_u}^{t} \frac{g(s)}{f_m^{\lambda+1-\alpha}(s)}\left(\int_{t_u}^{s} g(\xi) f_m^{\lambda-1-\alpha}(\xi)\ell_m(\xi)\mathrm{d}\xi\right)\mathrm{d}s &\ge \int_{\tau}^{t} \frac{g(s)}{f_m^{\lambda+1-\alpha}(s)}\mathrm{d}s \int_{t_u}^{\tau} g(s) f_m^{\lambda-1-\alpha}(s)\ell_m(s)\mathrm{d}s \\ &= \left[\frac{f_m^{\alpha-\lambda}(t)}{|\alpha-\lambda|} - \frac{f_m^{\alpha-\lambda}(\tau)}{|\alpha-\lambda|}\right] \int_{t_u}^{\tau} g(s) f_m^{\lambda-1-\alpha}(s)\ell_m(s)\mathrm{d}s \quad \text{for } t \ge \tau \ge t_u \end{aligned}$$

and thus

$$\liminf_{t\to+\infty} \frac{|\alpha-\lambda|}{f_m^{\alpha-\lambda}(t)} \int_{t_u}^{t} \frac{g(s)}{f_m^{\lambda+1-\alpha}(s)}\left(\int_{t_u}^{s} g(\xi) f_m^{\lambda-1-\alpha}(\xi)\ell_m(\xi)\mathrm{d}\xi\right)\mathrm{d}s \ge \int_{t_u}^{+\infty} g(s) f_m^{\lambda-1-\alpha}(s)\ell_m(s)\mathrm{d}s, \tag{4.40}$$

because inequality (4.38) holds. On the other hand, in view of inequalities (4.36) and (4.38), it is clear that

$$\begin{aligned} \int_{t_u}^{t} \frac{g(s)}{f_m^{\lambda+1-\alpha}(s)}\left(\int_{t_u}^{s} g(\xi) f_m^{\lambda-1-\alpha}(\xi)\ell_m(\xi)\mathrm{d}\xi\right)\mathrm{d}s &\le \int_{t_u}^{t} \frac{g(s)}{f_m^{\lambda+1-\alpha}(s)}\mathrm{d}s \int_{t_u}^{+\infty} g(s) f_m^{\lambda-1-\alpha}(s)\ell_m(s)\mathrm{d}s \\ &= \left[\frac{f_m^{\alpha-\lambda}(t)}{|\alpha-\lambda|} - \frac{f_m^{\alpha-\lambda}(t_u)}{|\alpha-\lambda|}\right] \int_{t_u}^{+\infty} g(s) f_m^{\lambda-1-\alpha}(s)\ell_m(s)\mathrm{d}s \quad \text{for } t \ge t_u. \end{aligned}$$

Consequently, by virtue of the above-proved relation (4.40), one gets

$$\lim_{t\to+\infty} \frac{|\alpha-\lambda|}{f_m^{\alpha-\lambda}(t)} \int_{t_u}^{t} \frac{g(s)}{f_m^{\lambda+1-\alpha}(s)}\left(\int_{t_u}^{s} g(\xi) f_m^{\lambda-1-\alpha}(\xi)\ell_m(\xi)\mathrm{d}\xi\right)\mathrm{d}s = \int_{t_u}^{+\infty} g(s) f_m^{\lambda-1-\alpha}(s)\ell_m(s)\mathrm{d}s. \tag{4.41}$$

Furthermore, the Hölder inequality yields that

$$\begin{aligned}\left|\int_{t_u}^{t}\frac{g(s)}{f_m^{1-\alpha}(s)}\varrho(s)\mathrm{d}s\right| &\le \int_{t_u}^{t} g(s)f_m^{\alpha-1}(s)|\varrho(s)|\mathrm{d}s\\ &\le \left(\int_{t_u}^{t} g(s)f_m^{\alpha(\alpha-\lambda)-1}(s)\mathrm{d}s\right)^{\frac{1}{1+\alpha}}\left(\int_{t_u}^{t} g(s)f_m^{\lambda}(s)|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\right)^{\frac{\alpha}{1+\alpha}}\\ &\le \left(\frac{1}{\alpha|\alpha-\lambda|}\right)^{\frac{1}{1+\alpha}} f_m^{\frac{\alpha(\alpha-\lambda)}{1+\alpha}}(t)\left(\int_{t_u}^{+\infty} g(s)f_m^{\lambda}(s)|\varrho(s)|^{\frac{1+\alpha}{\alpha}}\mathrm{d}s\right)^{\frac{\alpha}{1+\alpha}}\end{aligned}$$

for $t \ge t_u$, which guarantees the equality

$$\lim_{t\to+\infty}\frac{|\alpha-\lambda|}{f_m^{\alpha-\lambda}(t)}\int_{t_u}^{t}\frac{g(s)}{f_m^{1-\alpha}(s)}\varrho(s)\mathrm{d}s = 0.$$

Finally, it is clear that

$$\lim_{t\to+\infty}\frac{1}{f_m^{\alpha-\lambda}(t)}\ln\frac{f_m(t)}{f_m(t_u)} = 0. \tag{4.42}$$

Therefore, using relations (2.3), (4.41)–(4.42), from equality (4.39) we get

$$\lim_{t\to+\infty} c_m(t;\lambda) = \delta_m(t_u) - \int_{t_u}^{+\infty} g(s)f_m^{\lambda-1-\alpha}(s)\ell_m(s)\mathrm{d}s$$

and thus the lemma is proved. □

Proof of Theorem 2.1. Assume on the contrary that system (1.1) is not oscillatory, i.e., there exists a solution (u, v) to system (1.1) fulfilling relation (4.1) with $t_u > 0$.

If assumption (2.4) of the theorem is satisfied, then Lemma 4.1 with $m = 1$ immediately leads to a contradiction.

If assumptions (2.5) hold, by using Lemma 4.1 with $m = 1$, we get the validity of inequality (4.4) with $m = 1$, where the functions f_1 and ϱ are defined by formulas (2.1) and (4.5) with $m = 1$, respectively. However, Lemma 4.2 with $m = 1$ then guarantees that there exists a finite limit (4.30) with $m = 1$, which is in a contradiction with the second condition in (2.5). □

Proof of Corollary 2.2. By using integration by parts, one gets

$$\begin{aligned}\int_0^t\left[f_1^{\beta}(t)-f_1^{\beta}(s)\right]^{\alpha_*}f_1^{\lambda}(s)p(s)\mathrm{d}s &= \alpha_*(\alpha_*-1)\cdots(\alpha_*-m+1)\beta^m\int_0^t\left[f_1^{\beta}(t)-f_1^{\beta}(s_{\alpha_*})\right]^{\alpha_*-m}\frac{g(s_{\alpha_*})}{f_1^{1-\beta}(s_{\alpha_*})}\\ &\times\left(\int_0^{s_{\alpha_*}}\frac{g(s_{\alpha_*-1})}{f_1^{1-\beta}(s_{\alpha_*-1})}\left(\int_0^{s_{\alpha_*-1}}\cdots\left(\int_0^{s_{\alpha_*-m+1}}f_1^{\lambda}(\xi)p(\xi)\mathrm{d}\xi\right)\mathrm{d}s_{\alpha_*-m+1}\cdots\right)\mathrm{d}s_{\alpha_*-1}\right)\mathrm{d}s_{\alpha_*}\end{aligned}$$

for $t > 0$, $m = 2, \ldots, \alpha_*$. Therefore, we have

$$k_1(t;\alpha_*,\alpha-\lambda,\lambda) = h_1(t;\lambda) \quad \text{for } t > 0$$

and thus the assertion of the corollary follows from Theorem 2.1 with $\kappa = \alpha_*$ and $\beta = \alpha - \lambda$. □

Proof of Corollary 2.5. Let $n \in \mathbb{N}$ be fixed such that $n > \max\{1, \alpha\}$. By using integration by parts, for any $t > 0$ one gets

$$\begin{aligned}k_1(t;n,1-\nu,\lambda) &= \frac{1}{f_1^{n(1-\nu)}(t)}\int_0^t\left[f_1^{1-\nu}(t)-f_1^{1-\nu}(s)\right]^n f_1^{\lambda}(s)p(s)\mathrm{d}s\\ &= \frac{n(1-\nu)}{f_1^{n(1-\nu)}(t)}\int_0^t\left[f_1^{1-\nu}(t)-f_1^{1-\nu}(s)\right]^{n-1}\frac{g(s)}{f_1^{\nu}(s)}\left(\int_0^s f_1^{\lambda}(\xi)p(\xi)\mathrm{d}\xi\right)\mathrm{d}s\\ &= \frac{n(n-1)(1-\nu)}{f_1^{n(1-\nu)}(t)}\int_0^t\left[f_1^{1-\nu}(t)-f_1^{1-\nu}(s)\right]^{n-2}\frac{g(s)}{f_1^{2\nu-1}(s)}\widetilde{c}_1(s;\lambda,\nu)\mathrm{d}s.\end{aligned}$$

Assume that there are $A \in \mathbb{R}$ and $t_0 > 0$ such that

$$\widetilde{c}_1(t;\lambda,\nu) \ge A \quad \text{for } t \ge t_0.$$

Then we have

$$k_1(t; n, 1-\nu, \lambda) \geq \frac{n(n-1)(1-\nu)}{f_1^{n(1-\nu)}(t)} \int_0^{t_0} \left[f_1^{1-\nu}(t) - f_1^{1-\nu}(s)\right]^{n-2} \frac{g(s)}{f_1^{2\nu-1}(s)} \left[\widetilde{c}_1(s; \lambda, \nu) - A\right] ds$$
$$+ A \frac{n(n-1)(1-\nu)}{f_1^{n(1-\nu)}(t)} \int_0^{t} \left[f_1^{1-\nu}(t) - f_1^{1-\nu}(s)\right]^{n-2} g(s) f_1^{1-2\nu}(s) ds \tag{4.43}$$

for $t \geq t_0$. It is clear that

$$\left| \frac{1}{f_1^{n(1-\nu)}(t)} \int_0^{t_0} \left[f_1^{1-\nu}(t) - f_1^{1-\nu}(s)\right]^{n-2} \frac{g(s)}{f_1^{2\nu-1}(s)} \left[\widetilde{c}_1(s; \lambda, \nu) - A\right] ds \right|$$
$$\leq \frac{1}{f_1^{2(1-\nu)}(t)} \int_0^{t_0} \left[1 - \left(\frac{f_1(s)}{f_1(t)}\right)^{1-\nu}\right]^{n-2} \frac{g(s)}{f_1^{2\nu-1}(s)} \left|\widetilde{c}_1(s; \lambda, \nu) - A\right| ds$$
$$\leq \frac{1}{f_1^{2(1-\nu)}(t)} \int_0^{t_0} \frac{g(s)}{f_1^{2\nu-1}(s)} \left|\widetilde{c}_1(s; \lambda, \nu) - A\right| ds \quad \text{for } t \geq t_0$$

and thus

$$\lim_{t\to+\infty} \frac{1}{f_1^{n(1-\nu)}(t)} \int_0^{t_0} \left[f_1^{1-\nu}(t) - f_1^{1-\nu}(s)\right]^{n-2} \frac{g(s)}{f_1^{2\nu-1}(s)} \left[\widetilde{c}_1(s; \lambda, \nu) - A\right] ds = 0.$$

On the other hand, by using integration by parts, one gets

$$(n-1) \int_0^{t} \left[f_1^{1-\nu}(t) - f_1^{1-\nu}(s)\right]^{n-2} g(s) f_1^{1-2\nu}(s) ds$$
$$= \frac{(n-1)\cdots(n-m+1)}{(m-1)!} \int_0^{t} \left[f_1^{1-\nu}(t) - f_1^{1-\nu}(s)\right]^{n-m} g(s) f_1^{m(1-\nu)-1}(s) ds$$

for $t \geq 0$, $m = 2, \dots, n$, which yields that

$$\frac{n(n-1)(1-\nu)}{f_1^{n(1-\nu)}(t)} \int_0^{t} \left[f_1^{1-\nu}(t) - f_1^{1-\nu}(s)\right]^{n-2} g(s) f_1^{1-2\nu}(s) ds = 1 \quad \text{for } t > 0.$$

Consequently from inequality (4.43) we get

$$\liminf_{t\to+\infty} k_1(t; n, 1-\nu, \lambda) \geq A.$$

Therefore, we have proved that

$$\lim_{t\to+\infty} \widetilde{c}_1(t; \lambda, \nu) = +\infty \implies \liminf_{t\to+\infty} k_1(t; n, 1-\nu, \lambda) \geq A \quad \text{for every } A > 0$$
$$\implies \lim_{t\to+\infty} k_1(t; n, 1-\nu, \lambda) = +\infty$$

and

$$\liminf_{t\to+\infty} \widetilde{c}_1(t; \lambda, \nu) > -\infty \implies \liminf_{t\to+\infty} k_1(t; n, 1-\nu, \lambda) > -\infty.$$

Moreover, according to Remark 2.6(i), there exists a finite limit $\lim_{t\to+\infty} c_1(t; \lambda)$ if and only if there exists a finite limit $\lim_{t\to+\infty} \widetilde{c}_1(t; \lambda, \nu)$ because we have $c_1(t; \lambda) = \widetilde{c}_1(t; \lambda, \lambda + 1 - \alpha)$ for $t > 0$.

Consequently, the assertion of the corollary follows from Theorem 2.1 with $\kappa = n$ and $\beta = 1 - \nu$. □

Proof of Corollary 2.7. It is clear that half-linear equation (1.4) is a particular case of system (1.1) in which $g \equiv r^{\frac{1}{1-q}}$ and $\alpha = q - 1$. Therefore, the assertion of the corollary follows immediately from Corollary 2.5 with $\nu = 0$. □

Proof of Theorem 2.8. Assume on the contrary that system (1.1) is not oscillatory, i.e., there exists a solution (u, v) to system (1.1) fulfilling relation (4.1) with $t_u \geq 0$.

If assumption (2.12) of the theorem is satisfied, then Lemma 4.1 with $m = 2$ immediately leads to a contradiction.

If assumptions (2.13) hold, by using Lemma 4.1 with $m = 2$, we get the validity of inequality (4.4) with $m = 2$, where the functions f_2 and ϱ are defined by formulas (2.9) and (4.5) with $m = 2$, respectively. However, Lemma 4.2 with $m = 2$ then guarantees that there exists a finite limit (4.30) with $m = 2$, which is in a contradiction with the second condition (2.13). □

Proof of Corollary 2.9. By using integration by parts, one gets

$$\int_0^t \left[f_2^\beta(t) - f_2^\beta(s)\right]^{\alpha_*} f_2^\lambda(s)p(s)\mathrm{d}s = \alpha_*(\alpha_*-1)\cdots(\alpha_*-m+1)|\beta|^m \int_0^t \left[f_2^\beta(t) - f_2^\beta(s_{\alpha_*})\right]^{\alpha_*-m} \frac{g(s_{\alpha_*})}{f_2^{1-\beta}(s_{\alpha_*})} \times \left(\int_0^{s_{\alpha_*}} \frac{g(s_{\alpha_*-1})}{f_2^{1-\beta}(s_{\alpha_*-1})}\left(\int_0^{s_{\alpha_*-1}} \cdots \left(\int_0^{s_{\alpha_*-m+1}} f_2^\lambda(\xi)p(\xi)\mathrm{d}\xi\right)\mathrm{d}s_{\alpha_*-m+1}\cdots\right)\mathrm{d}s_{\alpha_*-1}\right)\mathrm{d}s_{\alpha_*}$$

for $t \geq 0$, $m = 2, \ldots, \alpha_*$. Therefore, we have

$$k_2(t; \alpha_*, \alpha-\lambda, \lambda) = h_2(t;\lambda) \quad \text{for } t \geq 0$$

and thus the assertion of the corollary follows from Theorem 2.8 with $\kappa = \alpha_*$ and $\beta = \alpha - \lambda$. □

Proof of Corollary 2.11. Let $n \in \mathbb{N}$ be fixed such that $n > \max\{1, \alpha\}$. By using integration by parts, for any $t \geq 0$ one gets

$$\begin{aligned} k_2(t; n, 1-\nu, \lambda) &= f_2^{n(\nu-1)}(t) \int_0^t \left[f_2^{1-\nu}(t) - f_2^{1-\nu}(s)\right]^n f_2^\lambda(s)p(s)\mathrm{d}s \\ &= n(\nu-1)f_2^{n(\nu-1)}(t)\int_0^t \left[f_2^{1-\nu}(t) - f_2^{1-\nu}(s)\right]^{n-1} \frac{g(s)}{f_2^\nu(s)}\left(\int_0^s f_2^\lambda(\xi)p(\xi)\mathrm{d}\xi\right)\mathrm{d}s \\ &= n(n-1)(\nu-1)f_2^{n(\nu-1)}(t)\int_0^t \left[f_2^{1-\nu}(t) - f_2^{1-\nu}(s)\right]^{n-2} \frac{g(s)}{f_2^{2\nu-1}(s)}\widetilde{c}_2(s;\lambda,\nu)\mathrm{d}s. \end{aligned}$$

Assume that there are $A \in \mathbb{R}$ and $t_0 \geq 0$ such that

$$\widetilde{c}_2(t;\lambda,\nu) \geq A \quad \text{for } t \geq t_0.$$

Then we have

$$\begin{aligned} k_2(t; n, 1-\nu, \lambda) \geq{}& n(n-1)(\nu-1)f_2^{n(\nu-1)}(t)\int_0^{t_0} \left[f_2^{1-\nu}(t) - f_2^{1-\nu}(s)\right]^{n-2} \frac{g(s)}{f_2^{2\nu-1}(s)}\left[\widetilde{c}_2(s;\lambda,\nu) - A\right]\mathrm{d}s \\ &+ An(n-1)(\nu-1)f_2^{n(\nu-1)}(t)\int_0^t \left[f_2^{1-\nu}(t) - f_2^{1-\nu}(s)\right]^{n-2} \frac{g(s)}{f_2^{2\nu-1}(s)}\mathrm{d}s \end{aligned} \tag{4.44}$$

for $t \geq t_0$. It is clear that

$$\begin{aligned} &\left| f_2^{n(\nu-1)}(t)\int_0^{t_0} \left[f_2^{1-\nu}(t) - f_2^{1-\nu}(s)\right]^{n-2} \frac{g(s)}{f_2^{2\nu-1}(s)}\left[\widetilde{c}_2(s;\lambda,\nu) - A\right]\mathrm{d}s \right| \\ &\leq f_2^{2(\nu-1)}(t)\int_0^{t_0}\left[1 - \left(\frac{f_2(t)}{f_2(s)}\right)^{\nu-1}\right]^{n-2} \frac{g(s)}{f_2^{2\nu-1}(s)}\left|\widetilde{c}_2(s;\lambda,\nu) - A\right|\mathrm{d}s \\ &\leq f_2^{2(\nu-1)}(t)\int_0^{t_0} \frac{g(s)}{f_2^{2\nu-1}(s)}\left|\widetilde{c}_2(s;\lambda,\nu) - A\right|\mathrm{d}s \quad \text{for } t \geq t_0 \end{aligned}$$

and thus

$$\lim_{t\to+\infty} f_2^{n(\nu-1)}(t)\int_0^{t_0} \left[f_2^{1-\nu}(t) - f_2^{1-\nu}(s)\right]^{n-2} \frac{g(s)}{f_2^{2\nu-1}(s)}\left[\widetilde{c}_1(s;\lambda,\nu) - A\right]\mathrm{d}s = 0.$$

On the other hand, by using integration by parts, one gets

$$\begin{aligned} (n-1)\int_0^t \left[f_2^{1-\nu}(t) - f_2^{1-\nu}(s)\right]^{n-2} \frac{g(s)}{f_2^{2\nu-1}(s)}\mathrm{d}s ={}& \frac{(n-1)\cdots(n-m+1)}{(m-1)!}\int_0^t \left[f_2^{1-\nu}(t) - f_2^{1-\nu}(s)\right]^{n-m} \frac{g(s)}{f_2^{m(\nu-1)+1}(s)}\mathrm{d}s \\ &- \sum_{\ell=2}^{m-1} \frac{(n-1)\cdots(n-\ell+1)}{\ell!\,(\nu-1)f_2^{\ell(\nu-1)}(0)}\left[f_2^{1-\nu}(t) - f_2^{1-\nu}(0)\right]^{n-\ell} \end{aligned}$$

for $t \geq 0$, $m = 2, \ldots, n$, which yields that

$$\begin{aligned} & n(n-1)(\nu-1)f_2^{n(\nu-1)}(t)\int_0^t \left[f_2^{1-\nu}(t) - f_2^{1-\nu}(s)\right]^{n-2} \frac{g(s)}{f_2^{2\nu-1}(s)}\mathrm{d}s \\ &= 1 - \left(\frac{f_2(t)}{f_2(0)}\right)^{n(\nu-1)} - f_2^{2(\nu-1)}(t)\sum_{\ell=2}^{n-1} \frac{n\cdots(n-\ell+1)f_2^{(\ell-2)(\nu-1)}(t)}{\ell!\,f_2^{\ell(\nu-1)}(0)}\left[1 - \left(\frac{f_2(t)}{f_2(0)}\right)^{\nu-1}\right]^{n-\ell} \end{aligned}$$

for $t \geq 0$ (note that we set $\sum_{\ell=2}^{1} = 0$). Consequently from inequality (4.44) we get

$$\liminf_{t\to+\infty} k_2(t; n, 1-\nu, \lambda) \geq A.$$

Therefore, we have proved that

$$\lim_{t\to+\infty} \widetilde{c}_2(t; \lambda, \nu) = +\infty \Longrightarrow \liminf_{t\to+\infty} k_2(t; n, 1-\nu, \lambda) \geq A \quad \text{for every } A > 0$$
$$\Longrightarrow \lim_{t\to+\infty} k_2(t; n, 1-\nu, \lambda) = +\infty$$

and

$$\liminf_{t\to+\infty} \widetilde{c}_2(t; \lambda, \nu) > -\infty \Longrightarrow \liminf_{t\to+\infty} k_2(t; n, 1-\nu, \lambda) > -\infty.$$

Moreover, according to Remark 2.12(i), there exists a finite limit $\lim_{t\to+\infty} c_2(t; \lambda)$ if and only if there exists a finite limit $\lim_{t\to+\infty} \widetilde{c}_2(t; \lambda, \nu)$ because we have $c_2(t; \lambda) = \widetilde{c}_2(t; \lambda, \lambda + 1 - \alpha)$ for $t \geq 0$.

Consequently, the assertion of the corollary follows from Theorem 2.8 with $\kappa = n$ and $\beta = 1 - \nu$. □

Proof of Corollary 2.13. It is clear that half-linear equation (1.4) is a particular case of system (1.1) in which $g \equiv r^{\frac{1}{1-q}}$ and $\alpha = q - 1$. Therefore, the assertion of the corollary follows immediately from Corollary 2.11 with $\nu = 2$. □

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