SOME BOUNDARY VALUE PROBLEMS FOR FIRST ORDER SCALAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Preface

Most of the material included in this monograph was gained during our discussions in the framework of the Working Seminar on Boundary Value Problems at Masaryk University in Brno in 2000/2001.

In the school year 2001/2002, the lectures in this topics were delivered for the under–graduate and post–graduate students on the Department of Mathematical Analysis at Masaryk University in Brno. At the same period the work took its final form.

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Authors

Introduction

Functional differential equations (FDE) have already appeared in the 18th century as mathematical formulations of certain problems in physics and geometry. We can find them especially in the works of Euler and Condorcet. However, until the end of the 19th century, FDE were investigated only in connection with particular applications and we cannot speak about their systematic study.

Only in the works of E. Schmidt, F. Schürer and E. Hilb (see [32,33,64–70]) from the first quarter of the twentieth century, first attempts of a systematic study of special equations with delayed argument appeared. The interest in this type of FDE grew in the 1930s, especially in connection with extensive applications in mechanics, biology, and economy. At that time, the basics of the qualitative theory of equations with delayed argument and of the so–called integrodifferential equations were put in the works of A. Myshkis and R. Bellman (see [55]). They and a number of other mathematicians (Elsgolc, Norkin, Hale, Halanay, Kolmanovskii, Razumikhin, Azbelev, etc.) who followed this direction are to be credited for building up the extensive qualitative theory of FDE that exists nowadays. This theory is not only important in applications, but influences also wide areas of pure mathematics (see, e.g., [1, 2, 30, 55]).

In the 1970s, great deal of attention was devoted to the construction of the theory of boundary value problems (BVP) for FDE. Various methods were proposed to be used in these problems, e.g., the theory of Fredholm operators, method of small parameters, topological methods, theory of integral manifolds and so on (detailed survey of these methods and corresponding results is, e.g., in [1–3, 30, 55, 58, 62, 63, 71–87]). From the contemporary viewpoint, it can be said that the methods of functional analysis and topological methods proved to be the most useful ones. By systematic application of these methods, the foundations of the theory of BVP for a large class of FDE were constructed (see [1, 2, 30, 55, 72], etc.).

However, until now, concrete BVP for FDE were studied only with partial success. The difficulties arising in the study of FDE lie in the nonlocal character of the equation and they appear even for the linear equation. For example, the question of solvability of the simplest BVP, the so–called initial value problem

$$u'(t) = p(t)u(\tau(t)) + q(t), \qquad u(a) = 0,$$

where $p, q : [a, b] \to R$ are Lebesgue integrable functions and $\tau : [a, b] \to [a, b]$ is a measurable function, is far from being so trivial as for the ordinary differential equations (ODE), i.e., for the case when $\tau(t) = t$ for $t \in [a, b]$. Therefore we cannot be surprised by the fact that in the large monographs [1,2,30,55,72], we cannot find detailed information on the solvability of the initial value problem.

On the other hand, if the deviation $\tau(t) - t$ is "small", i.e., if the equation is "close" to the ODE, we intuitively expect that the given problem possesses a unique solution. In simple cases, the validity of such hypothesis can be verified directly. With more complicated problems, where global methods do not provide sufficient accuracy, natural need for finding a more precise technique for the investigation of the FDE arose.

As for the ODE, a sufficiently complete theory of BVP was already built up, using namely the methods whose basis is laid in mathematical analysis (see [35, 38, 49]). Last, but not least, this fact corresponded to the efforts to modify the methods of mathematical analysis for the investigation of FDE. In the last couple of years, these efforts were successful in the case of some BVP for FDE. Especially in the works of I. Kiguradze and B. Půža (see [12, 36, 37, 39–48, 50–54]), sophisticated conditions for the solvability and unique solvability of a quite wide class of BVP for FDE in both linear and nonlinear cases were found (see also [9–11, 13–29]).

Inspired by these results we decided to use the methods of mathematical analysis and investigation technique of BVP for ODE with appropriate modifications for FDE. Mainly the method of a priori estimates and technique of differential inequalities. The method of a priori estiamtes is widely used in the theory of BVP both for ODE and FDE. The basis of this method was laid down in the beginning of 20th century. Later this method was succesfully developed in [35,38,49] even for singular ODE. Importance of theorems on differential inequalities in connection with study of Cauchy problem, resp. two–point BVP, was observed in the beginning of 20th, as well (see [7, 8, 34, 59] and references therein). Further this technique was extended and generalized for BVP of various other types (see [1, 14, 35, 38, 49, 56, 57]).

The present work deals with the questions of solvability and unique solvability of BVP

$$u'(t) = F(u)(t), \tag{0.1}$$

$$\lambda u(a) + \mu u(b) = h(u), \tag{0.2}$$

where $F : C\big([a,b];R\big) \to L\big([a,b];R\big)$, $h : C\big([a,b];R\big) \to R$ are continuous operators satisfying the Carathèodory conditions, $\lambda, \mu \in R$ and $|\lambda| + |\mu| \neq 0$. The particular cases of the boundary condition (0.2) are the initial conditions

$$u(a) = c$$

and

$$u(b) = c,$$

the periodic condition

$$u(a) = u(b)$$

and the antiperiodic condition

$$u(a) = -u(b).$$

A special case of the equation (0.1) is, for example, the equation with deviating arguments

$$u'(t) = f(t, u(t), u(\tau_1(t)), \dots, u(\tau_n(t))),$$

where $f : [a,b] \times R^{n+1} \to R$ is a Carathèodory function and $\tau_k : [a,b] \to [a,b]$ $(k = 1, \dots, n)$ are measurable functions.

The work is divided into two chapters. In Chapter I, the question of the unique solvability of the linear problem, i.e., of the problem

$$\begin{aligned} u'(t) &= \ell(u)(t) + q(t), \\ \lambda u(a) + \mu u(b) &= c, \end{aligned} \tag{0.3}$$

where $\ell : C\big([a,b];R\big) \to L\big([a,b];R\big)$ is a linear bounded operator, $q \in L\big([a,b];R\big)$, $\lambda, \mu, c \in R$, and $|\lambda| + |\mu| \neq 0$, is investigated. §§2, 4, and 7 contain the main results that are further expanded and detailed in §§3, 5, 6, 8, and 9 for the equation with deviating arguments of the form

$$u'(t) = \sum_{k=1}^{m} \Big(p_k(t) u(\tau_k(t)) - g_k(t) u(\nu_k(t)) \Big) + q(t),$$

where $p_k, g_k \in L([a,b]; R_+)$, $q \in L([a,b]; R)$, and $\tau_k, \nu_k : [a,b] \to [a,b]$ $(k = 1, \dots, m)$ are measurable functions.

§2 is devoted to the question on the validity of a theorem on differential inequalities. The results obtained here have an independent character since they give the information on the sign of the solution of the problem (0.3) (under certain natural sign assumptions imposed on the function q and the number c). On the other hand, these results are used later for studying the question on the solvability in both linear and nonlinear problems. In §§4 and 7, sufficient conditions for unique solvability of the BVP of periodic (i.e., when $\lambda\mu \le 0$) and antiperiodic (i.e., when $\lambda\mu > 0$) type are established. The presented results are optimal, which is demonstrated by the appropriated examples.

Chapter II deals with the nonlinear problem and is arranged in a similar way. On the basis of the technique developed in Chapter I, nonimprovable sufficient conditions of the solvability and unique solvability of the problem (0.1), (0.2) are established in §§12 and 14. In §§13 and 15, these results are specified for an equation with deviating argument of the form

$$u'(t) = \sum_{k=1}^{m} \Big(p_k(t) u(\tau_k(t)) - g_k(t) u(\nu_k(t)) \Big) + \\ + f(t, u(t), u(\zeta_1(t)), \dots, u(\zeta_n(t))),$$

where $f : [a,b] \times R^{n+1} \to R$ is a Carathèodory function, $p_k, g_k \in L([a,b]; R_+)$, and $\tau_k, \nu_k, \zeta_j : [a,b] \to [a,b]$ $(k = 1, \dots, m;\ j = 1, \dots, n)$ are measurable functions.

Notation

N is the set of all natural numbers;

R is the set of all real numbers;

$R_+ = [0, +\infty[\ ,\ \ R_- =]-\infty, 0]\,;$

$\overline{A}$ is the closure of the set A;

$C\big([a,b];R\big)$ is the Banach space of continuous functions $v : [a,b] \to R$ with the norm

$$\|v\|_C = \max\{|v(t)| : a \le t \le b\};$$

$C\big([a,b];D\big) = \{v \in C\big([a,b];R\big) : v : [a,b] \to D\}$, where $D \subseteq R$;

$C_{\lambda\mu}\big([a,b];D\big) = \{v \in C\big([a,b];D\big) : \lambda v(a) + \mu v(b) = 0\}$, where $D \subseteq R$;

$\widetilde{C}\big([a,b];D\big)$, where $D \subseteq R$, is the set of absolutely continuous functions $v : [a,b] \to D$;

$B^i_{\lambda\mu c}\big([a,b];R\big)$, where $\lambda, \mu, c \in R$ and $i \in \{1,2\}$, is the set of functions $v \in C\big([a,b];R\big)$ satisfying

$$\big[\lambda v(a) + \mu v(b)\big]\,\mathrm{sgn}\,\big((2-i)\lambda v(a) + (i-1)\mu v(b)\big) \le c\,;$$

$L\big([a,b];R\big)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm

$$\|p\|_L = \int_a^b |p(s)|ds;$$

$L\big([a,b];D\big) = \{p \in L\big([a,b];R\big) : p : [a,b] \to D\}$, where $D \subseteq R$;

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$;

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C\big([a,b];R\big) \to L\big([a,b];R\big)$ for each of them there exists $\eta \in L\big([a,b];R_+\big)$ such that

$$|\ell(v)(t)| \le \eta(t)\|v\|_C \quad \text{for almost all} \quad t \in [a,b], \quad v \in C\big([a,b];R\big);$$

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C\big([a,b];R_+\big)$ into the set $L\big([a,b];R_+\big)$;

K_{ab} is the set of continuous operators $F : C\big([a,b];R\big) \to L\big([a,b];R\big)$ satisfying the Caratheodory conditions, i.e., for every $r>0$ there exists $q_r \in L\big([a,b];R_+\big)$ such that

$$|F(v)(t)| \le q_r(t) \quad \text{for almost all} \quad t \in [a,b], \quad \|v\|_C \le r;$$

$K([a,b] \times A; B)$, where $A \subseteq R^n$, $B \subseteq R$, $n \in N$, is the set of functions $f : [a,b] \times A \to B$ satisfying the Caratheodory conditions, i.e., $f(\cdot,x) : [a,b] \to B$ is a measurable function for all $x \in A$, $f(t,\cdot) : A \to B$ is a continuous function for almost all $t \in [a,b]$, and for every $r>0$ there exists $q_r \in L\big([a,b];R_+\big)$ such that

$$|f(t,x)| \le q_r(t) \quad \text{for almost all} \quad t \in [a,b],\ x \in A,\ \|x\| \le r;$$

$[x]_+ = \frac{1}{2}(|x|+x)$, $[x]_- = \frac{1}{2}(|x|-x)$.

We will say that $\ell \in \mathcal{L}_{ab}$ is a t_0-Volterra operator, where $t_0 \in [a,b]$, if for arbitrary $a_1 \in [a,t_0]$, $b_1 \in [t_0,b]$, $a_1 \neq b_1$, and $v \in C\big([a,b];R\big)$ satisfying the condition

$$v(t) = 0 \quad \text{for} \quad t \in [a_1,b_1],$$

we have

$$\ell(v)(t) = 0 \quad \text{for almost all} \quad t \in [a_1,b_1].$$

An operator $\ell \in \mathcal{L}_{ab}$ is said to be nontrivial, if $\ell(1) \not\equiv 0$.

CHAPTER I

Linear Problem

§1. Statement of the Problem

Consider the problem on the existence and uniqueness of a solution of the equation

$$u'(t) = \ell(u)(t) + q(t) \tag{1.1}$$

satisfying the boundary condition

$$\lambda u(a) + \mu u(b) = c, \tag{1.2}$$

where $\ell \in \mathcal{L}_{ab}$, $q \in L\big([a,b];R\big)$, $\lambda, \mu, c \in R$, and $|\lambda| + |\mu| \neq 0$. By a solution of the equation (1.1) we understand a function $u \in \widetilde{C}\big([a,b];R\big)$ satisfying this equation almost everywhere in $[a,b]$. Note also that the equalities and inequalities with integrable functions are understood almost everywhere.

Along with the problem (1.1), (1.2) we consider the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \tag{1.1$_0$}$$

$$\lambda u(a) + \mu u(b) = 0. \tag{1.2$_0$}$$

All results will be concretized for the differential equation with deviating arguments (EDA), i.e., for the case, when the equation (1.1) has the form

$$u'(t) = \sum_{k=1}^{m} \Big(p_k(t)u(\tau_k(t)) - g_k(t)u(\nu_k(t))\Big) + q(t), \tag{1.1$'$}$$

where $p_k, g_k \in L\big([a,b];R_+\big)$, $q \in L\big([a,b];R\big)$, $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, and $m \in N$.

The following result is well–known from the general theory of the boundary value problems for functional differential equations (see, e.g., [1–3, 42, 72].

Theorem 1.1. *The problem* (1.1), (1.2) *is uniquely solvable iff the corresponding homogeneous problem* (1.1$_0$), (1.2$_0$) *has only the trivial solution.*

Remark 1.1. It follows from the Riesz–Schauder theory that if the problem (1.1$_0$), (1.2$_0$) has a nontrivial solution, then there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (1.1), (1.2) has no solution.

§2. On Differential Inequalities

Throughout this section we will assume that $|\lambda| + |\mu| \neq 0$ and

$$\lambda\mu \leq 0. \tag{2.1}$$

Furthermore, if $\lambda = -\mu$, then the operator $\ell \in \mathcal{L}_{ab}$ is supposed to be nontrivial, i.e., $\ell(1) \not\equiv 0$.

Definition 2.1. We will say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $V_{ab}^{+}(\lambda, \mu)$ (resp. $V_{ab}^{-}(\lambda, \mu)$), if the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution and for every $q \in L\big([a, b]; R_+\big)$ and $c \in R$ satisfying

$$\big(\operatorname{sgn}\lambda - \operatorname{sgn}\mu\big)\, c \geq 0, \tag{2.2}$$

the solution of the problem (1.1), (1.2) is nonnegative (resp. nonpositive).

Remark 2.1. According to Theorem 1.1, it is clear that if $\ell \in V_{ab}^{+}(\lambda, \mu)$, resp. $\ell \in V_{ab}^{-}(\lambda, \mu)$, then the problem (1.1), (1.2) is uniquely solvable for any $c \in R$ and $q \in L\big([a, b]; R\big)$.

Note also that if $\ell \in \mathcal{P}_{ab}$ and $\ell \in V_{ab}^{+}(\lambda, \mu)$, then $|\mu| < |\lambda|$, and if $-\ell \in \mathcal{P}_{ab}$ and $\ell \in V_{ab}^{-}(\lambda, \mu)$, then $|\mu| > |\lambda|$.

Remark 2.2. Furthermore, $V_{ab}^{-}(\lambda, 0) = \emptyset$ for every $\lambda \neq 0$. Indeed, suppose on the contrary that $\ell \in V_{ab}^{-}(\lambda, 0)$ for some $\lambda \neq 0$. Then, according to Remark 2.1, the problem (1.1), (1.2) with $\mu = 0$ and $c = 0$ has a unique solution for every $q \in L\big([a, b]; R\big)$. Let Ω be an operator, which assigns to every $q \in L\big([a, b]; R\big)$ the solution of the problem (1.1), (1.2) with $\mu = 0$ and $c = 0$. In view of Theorem 1.4 in [42], $\Omega : L\big([a, b]; R\big) \to C\big([a, b]; R\big)$ is a linear bounded operator. Moreover, since $\ell \in \mathcal{L}_{ab}$, there exists a function $\eta \in L\big([a, b]; R_+\big)$ such that

$$|\ell(v)(t)| \leq \eta(t)\|v\|_C \quad \text{for} \quad t \in [a, b], \quad v \in C\big([a, b]; R\big). \tag{2.3}$$

Choose $t_0 \in \,]a, b[$ satisfying

$$\|\Omega\| \int_a^{t_0} \eta(s)ds < 1 \tag{2.4}$$

and let $q \in L\big([a, b]; R_+\big)$ be such that

$$q(t) = 0 \quad \text{for} \quad t \in [t_0, b], \qquad q \not\equiv 0. \tag{2.5}$$

Furthermore, let u be a solution of the problem (1.1), (1.2) with $\mu = 0$ and $c = 0$. Obviously, the inequality (2.2) is satisfied and

$$u(t) = \int_a^t \ell(u)(s)ds + \int_a^t q(s)ds \quad \text{for} \quad t \in [a,b]. \tag{2.6}$$

On the other hand, according to the definition of the operator Ω, we have

$$u(t) = \Omega(q)(t) \quad \text{for} \quad t \in [a,b]$$

and thus,

$$\|u\|_C \le \|\Omega\| \|q\|_L. \tag{2.7}$$

By virtue of (2.3) and (2.7), (2.6) yields

$$u(t) \ge \int_a^t q(s)ds - \int_a^t |\ell(u)(s)|ds \ge$$
$$\ge \int_a^t q(s)ds - \|\Omega\| \|q\|_L \int_a^t \eta(s)ds \quad \text{for} \quad t \in [a,b].$$

Hence, with respect to (2.4) and (2.5), we obtain

$$u(t_0) \ge \|q\|_L \left(1 - \|\Omega\| \int_a^{t_0} \eta(s)ds \right) > 0,$$

which, according to Definition 2.1, contradicts the assumption $\ell \in V_{ab}^-(\lambda, 0)$.

In a similar manner it can be shown that $V_{ab}^+(0,\mu) = \emptyset$ for every $\mu \ne 0$.

Remark 2.3. It follows from Definition 2.1 that $\ell \in V_{ab}^+(\lambda,\mu)$ (resp. $\ell \in V_{ab}^-(\lambda,\mu)$) iff for the problem (1.1), (1.2) a certain theorem on differential inequalities holds, i.e., whenever $u, v \in \widetilde{C}([a,b];R)$ satisfy the inequalities

$$u'(t) \le \ell(u)(t) + q(t), \qquad v'(t) \ge \ell(v)(t) + q(t) \quad \text{for} \quad t \in [a,b],$$

$$|\lambda| u(a) - |\mu| u(b) \le |\lambda| v(a) - |\mu| v(b),$$

then $u(t) \le v(t)$ (resp. $u(t) \ge v(t)$) for $t \in [a,b]$.

In this section, we will establish sufficient conditions for an operator ℓ to belong to the sets $V_{ab}^+(\lambda,\mu)$ and $V_{ab}^-(\lambda,\mu)$. These results have an independent character in the sense that they give us the information about sign of the solution of the problem (1.1), (1.2) (under certain natural sign assumptions imposed on the function q and the number c). On the other hand, these results play an important role in the following investigation of the solvability of considered problem both in linear and nonlinear cases.

2.1. On the Set $V_{ab}^+(\lambda,\mu)$

In this subsection, nonimprovable, in a certain sense, sufficient conditions guaranteeing the inclusion $\ell \in V_{ab}^+(\lambda,\mu)$ are established. First, in Theorems 2.1–2.5, we consider the case $|\mu| \le |\lambda|$. Theorems 2.6–2.8 concern the case $|\mu| \ge |\lambda|$.

In the case, where $|\mu| \le |\lambda|$, the following assertions hold.

Proposition 2.1. *Let $|\mu| < |\lambda|$ and*

$$\ell \in \mathcal{P}_{ab}. \tag{2.8}$$

Then $\ell \in V_{ab}^+(\lambda,\mu)$ iff the problem

$$u'(t) \le \ell(u)(t), \qquad \lambda u(a) + \mu u(b) = 0 \tag{2.9}$$

has no nontrivial nonnegative solution.

Theorem 2.1. *Let $|\mu| < |\lambda|$ and $\ell \in \mathcal{P}_{ab}$. Then the operator ℓ belongs to the set $V_{ab}^+(\lambda,\mu)$ iff there exists a function $\gamma \in \widetilde{C}\big([a,b];]0,+\infty[\big)$ satisfying the inequalities*

$$\gamma'(t) \ge \ell(\gamma)(t) \quad \text{for} \quad t \in [a,b], \tag{2.10}$$

$$|\lambda|\gamma(a) > |\mu|\gamma(b). \tag{2.11}$$

Corollary 2.1. *Let $|\mu| < |\lambda|$, $\ell \in \mathcal{P}_{ab}$, and let at least one of the following items be fulfilled:*

a) ℓ is an $a-$Volterra operator and

$$|\mu| \exp\left(\int_a^b \ell(1)(s)ds\right) < |\lambda|\,; \tag{2.12}$$

b) there exist $m, k \in N$ and a constant $\alpha \in]0,1[$ such that $m > k$ and

$$\rho_m(t) \le \alpha \rho_k(t) \quad \textit{for} \quad t \in [a,b], \tag{2.13}$$

where $\rho_1 \equiv 1$ and

$$\rho_{i+1}(t) \stackrel{\text{def}}{=} \frac{|\mu|}{|\lambda| - |\mu|} \int_a^b \ell(\rho_i)(s)ds + \int_a^t \ell(\rho_i)(s)ds \tag{2.14}$$

for $t \in [a,b]$, $i \in N$;

c) there exists $\overline{\ell} \in \mathcal{P}_{ab}$ such that

$$\begin{gathered}|\mu| \exp\left(\int_a^b \ell(1)(s)ds\right) + \\ +|\lambda| \int_a^b \overline{\ell}(1)(s) \exp\left(\int_s^b \ell(1)(\xi)d\xi\right) ds < |\lambda|,\end{gathered} \tag{2.15}$$

and on the set $C_{\lambda\mu}\big([a,b]; R_+\big)$ the inequality

$$\ell(\vartheta(v))(t) - \ell(1)(t)\vartheta(v)(t) \le \overline{\ell}(v)(t) \quad \textit{for} \quad t \in [a,b] \tag{2.16}$$

holds, where

$$\vartheta(v)(t) \stackrel{\text{def}}{=} \frac{|\mu|}{|\lambda| - |\mu|} \int_a^b \ell(v)(s)ds + \int_a^t \ell(v)(s)ds \quad \textit{for} \quad t \in [a,b]. \tag{2.17}$$

Then the operator ℓ belongs to the set $V_{ab}^+(\lambda, \mu)$.

Remark 2.4. Let $|\mu| < |\lambda|$, $\ell \in \mathcal{P}_{ab}$, ℓ be an $a-$Volterra operator, and the problem (1.1_0), (1.2_0) has only the trivial solution. If, moreover, (instead of (2.12)) the equality

$$|\mu| \exp\left(\int_a^b \ell(1)(s)ds\right) = |\lambda| \tag{2.18}$$

holds, then $\ell \in V_{ab}^+(\lambda, \mu)$ again (see On Remark 2.4, p. 50).

On the other hand, for every $\varepsilon > 0$ there exists an $a-$Volterra operator $\ell \in \mathcal{P}_{ab}$ such that the problem (1.1_0), (1.2_0) has only the trivial solution,

$$|\mu| \exp\left(\int_a^b \ell(1)(s)ds\right) = |\lambda| + \varepsilon,$$

and $\ell \notin V_{ab}^{+}(\lambda,\mu)$ (see Example 2.1, p. 51).

Remark 2.5. It follows from Corollary 2.1 b) (for $k = 1$ and $m = 2$) that if $|\mu| < |\lambda|$, $\ell \in \mathcal{P}_{ab}$, and

$$|\lambda| \int_a^b \ell(1)(s)ds < |\lambda| - |\mu|,$$

then $\ell \in V_{ab}^{+}(\lambda,\mu)$. Note that if the problem (1.1_0), (1.2_0) has only the trivial solution and

$$|\lambda| \int_a^b \ell(1)(s)ds = |\lambda| - |\mu|, \tag{2.19}$$

then $\ell \in V_{ab}^{+}(\lambda,\mu)$ again (see On Remark 2.5, p. 52).

On the other hand, for every $\varepsilon > 0$ there exists an operator $\ell \in \mathcal{P}_{ab}$ such that the problem (1.1_0), (1.2_0) has only the trivial solution,

$$|\lambda| \int_a^b \ell(1)(s)ds = |\lambda| - |\mu| + \varepsilon,$$

and $\ell \notin V_{ab}^{+}(\lambda,\mu)$ (see Example 2.2, p. 53).

Remark 2.6. Corollary 2.1 is nonimprovable in a certain sense. More precisely, the assumption $\alpha \in]0,1[$ cannot be replaced by the assumption $\alpha \in]0,1]$, and the strict inequalities (2.12) and (2.15) cannot be replaced by the nonstrict ones (see Examples 2.3 and 2.4, p. 54).

Theorem 2.2. *Let $|\mu| \le |\lambda|$, $-\ell \in \mathcal{P}_{ab}$, ℓ be an $a-$Volterra operator, and let there exist a function $\gamma \in \widetilde{C}\big([a,b];R_+\big)$ satisfying*

$$\gamma'(t) \le \ell(\gamma)(t) \quad \textit{for} \quad t \in [a,b], \tag{2.20}$$

$$\gamma(t) > 0 \quad \textit{for} \quad t \in [a,b[\,. \tag{2.21}$$

Then the operator ℓ belongs to the set $V_{ab}^{+}(\lambda,\mu)$.

Remark 2.7. Theorem 2.2 is nonimprovable in a certain sense. More precisely, the condition (2.21) cannot be replaced by the condition

$$\gamma(t) > 0 \quad \text{for} \quad t \in [a, b_1[\,, \tag{2.22}$$

where $b_1 \in]a, b[$ is an arbitrarily fixed point (see Example 2.5, p. 55).

Theorem 2.3. *Let $|\mu| \le |\lambda|$, $-\ell \in \mathcal{P}_{ab}$, ℓ be an $a-$Volterra operator, and*

$$\int_a^b |\ell(1)(s)| ds \le 1. \tag{2.23}$$

Then the operator ℓ belongs to the set $V_{ab}^{+}(\lambda, \mu)$.

Remark 2.8. Theorem 2.3 is nonimprovable in the sense that the inequality (2.23) cannot be replaced by the inequality

$$\int_a^b |\ell(1)(s)| ds \le 1 + \varepsilon, \tag{2.24}$$

no matter how small $\varepsilon > 0$ would be (see Example 2.5, p. 55).

Corollary 2.2. *Let $|\mu| \le |\lambda|$, $-\ell \in \mathcal{P}_{ab}$, ℓ be an $a-$Volterra operator, and*

$$\int_a^b \big|\widetilde{\ell}(1)(s)\big| \exp\left(\int_a^s |\ell(1)(\xi)| d\xi \right) ds \le 1, \tag{2.25}$$

where

$$\begin{gathered} \widetilde{\ell}(v)(t) \overset{\text{def}}{=} \ell\big(\widetilde{\theta}(v)\big)(t) - \ell(1)(t)\widetilde{\theta}(v)(t) \quad \textit{for} \quad t \in [a, b], \\ \widetilde{\theta}(v)(t) \overset{\text{def}}{=} \int_a^t \ell(\widetilde{v})(s) ds \quad \textit{for} \quad t \in [a, b], \\ \widetilde{v}(t) \overset{\text{def}}{=} v(t) \exp\left(\int_a^t \ell(1)(s) ds \right) \quad \textit{for} \quad t \in [a, b]. \end{gathered} \tag{2.26}$$

Then the operator ℓ belongs to the set $V_{ab}^{+}(\lambda, \mu)$.

Remark 2.9. Corollary 2.2 is nonimprovable in the sense that the inequality (2.25) cannot be replaced by the inequality

$$\int_a^b \left|\widetilde{\ell}(1)(s)\right| \exp\left(\int_a^s |\ell(1)(\xi)|d\xi\right) ds \le 1+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be (see Example 2.5, p. 55).

Theorem 2.4. *Let $0 \neq |\mu| \le |\lambda|$ and the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where*

$$\ell_0, \ell_1 \in \mathcal{P}_{ab}. \tag{2.27}$$

Let, moreover,

$$\|\ell_0(1)\|_L < 1, \tag{2.28}$$

$$\frac{\|\ell_0(1)\|_L}{1-\|\ell_0(1)\|_L} - \frac{|\lambda|-|\mu|}{|\mu|} < \|\ell_1(1)\|_L \le \left|\frac{\mu}{\lambda}\right|. \tag{2.29}$$

Then the operator ℓ belongs to the set $V_{ab}^{+}(\lambda,\mu)$.

Remark 2.10. Let $0 \neq |\mu| \le |\lambda|$ and

$$A \stackrel{\text{def}}{=} \left\{(x,y) \in R_+ \times R_+ \;:\; x<1,\; \frac{x}{1-x} - \frac{|\lambda|-|\mu|}{|\mu|} < y \le \left|\frac{\mu}{\lambda}\right|\right\}$$

(see Fig. 2.1).

According to Theorem 2.4, if $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and

$$\Big(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L\Big) \in A,$$

then $\ell \in V_{ab}^{+}(\lambda,\mu)$. Below we will show (see On Remark 2.10, p. 56) that for every $x_0, y_0 \in R_+$, $(x_0,y_0) \notin A$ there exists $\ell \in \mathcal{L}_{ab}$ such that $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$,

$$x_0 = \|\ell_0(1)\|_L, \qquad y_0 = \|\ell_1(1)\|_L, \tag{2.30}$$

and $\ell \notin V_{ab}^{+}(\lambda,\mu)$. In particular, neither one of the inequalities in (2.28) and (2.29) can be weakened.

Remark 2.11. In [6], there is proved that if $-\ell \in \mathcal{P}_{ab}$, then the condition imposed on an operator ℓ to be of $a-$Volterra type is necessary for ℓ to

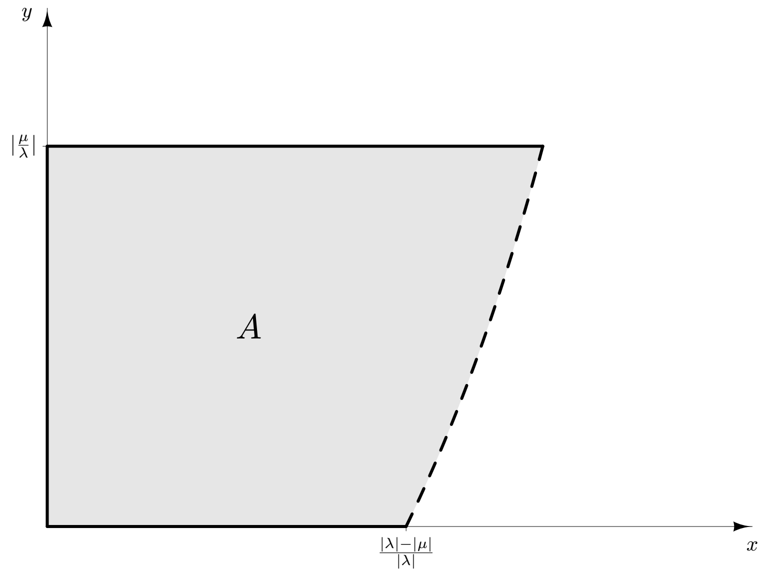

Fig. 2.1.

belong to the set $V_{ab}^{+}(1,0)$. On the other hand, it follows from Theorem 2.4 that if $\mu \neq 0$, $-\ell \in \mathcal{P}_{ab}$, and

$$|\lambda| \int_a^b |\ell(1)(s)|ds \leq |\mu|,$$

then $\ell \in V_{ab}^{+}(\lambda,\mu)$. Therefore, the condition imposed on an operator ℓ to be of $a-$Volterra type is not necessary for ℓ to belong to the set $V_{ab}^{+}(\lambda,\mu)$ with $\mu \neq 0$.

Theorem 2.5. *Let $|\mu| < |\lambda|$ and the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. If, moreover,*

$$\ell_0 \in V_{ab}^{+}(\lambda,\mu), \qquad -\ell_1 \in V_{ab}^{+}(\lambda,\mu),$$

then the operator ℓ belongs to the set $V_{ab}^{+}(\lambda,\mu)$.

Remark 2.12. Theorem 2.5 is nonimprovable in the sense that the assumption

$$\ell_0 \in V_{ab}^{+}(\lambda,\mu), \qquad -\ell_1 \in V_{ab}^{+}(\lambda,\mu)$$

can be replaced neither by the assumption

$$(1-\varepsilon)\ell_0 \in V_{ab}^{+}(\lambda,\mu), \qquad -\ell_1 \in V_{ab}^{+}(\lambda,\mu)$$

nor by the assumption

$$\ell_0 \in V_{ab}^{+}(\lambda,\mu), \qquad -(1-\varepsilon)\ell_1 \in V_{ab}^{+}(\lambda,\mu),$$

no matter how small $\varepsilon > 0$ would be (see Examples 2.6 and 2.7, p. 57).

In the case, where $|\mu| \geq |\lambda|$, the following statements hold.

Theorem 2.6. *Let $|\mu| \geq |\lambda| \neq 0$, $-\ell \in \mathcal{P}_{ab}$, and let there exist a function $\gamma \in \widetilde{C}([a,b];R_+)$ satisfying the inequalities* (2.10) *and* (2.11)*. If, moreover, the inequality* (2.23) *holds, then the operator ℓ belongs to the set $V_{ab}^{+}(\lambda,\mu)$.*

Remark 2.13. Theorem 2.6 is nonimprovable in the sense that the inequality (2.23) cannot be replaced by the inequality (2.24), no matter how small $\varepsilon > 0$ would be (see Example 2.8, p. 59).

Note also that if $|\mu| = |\lambda|$ and $-\ell \in \mathcal{P}_{ab}$, then there exists a function $\gamma \in \widetilde{C}([a,b];R_+)$ satisfying (2.10) and (2.11). Indeed, in this case the operator ℓ is considered to be nontrivial and thus, the function

$$\gamma(t) = 1 + \int_t^b |\ell(1)(s)|ds \quad \text{for} \quad t \in [a,b]$$

satisfies (2.10) and (2.11).

Nevertheless, if $|\mu| > |\lambda| \neq 0$, then the strict inequality (2.11) cannot be replaced by the nonstrict inequality

$$|\lambda|\gamma(a) \geq |\mu|\gamma(b) \tag{2.31}$$

(see Example 2.9, p. 59).

Theorem 2.7. *Let $|\mu| \geq |\lambda| \neq 0$, $-\ell \in \mathcal{P}_{ab}$, ℓ be an $a-$Volterra operator, and let there exist a function $\gamma \in \widetilde{C}([a,b];R_+)$ satisfying the inequalities* (2.10) *and* (2.11)*. If, moreover, there exists a function $\beta \in \widetilde{C}([a,b];R_+)$ satisfying the inequalities*

$$\beta(t) > 0 \quad \textit{for} \quad t \in [a,b[\,, \tag{2.32}$$

$$\beta'(t) \leq \ell(\beta)(t) \quad \textit{for} \quad t \in [a,b], \tag{2.33}$$

then the operator ℓ belongs to the set $V_{ab}^{+}(\lambda,\mu)$.

Remark 2.14. Theorem 2.7 is nonimprovable in the sense that the assumption (2.32) cannot be replaced by the assumption

$$\beta(t) > 0 \quad \text{for} \quad t \in [a, b_1[\,, \tag{2.34}$$

where $b_1 \in]a, b[$ is an arbitrarily fixed point (see Example 2.10, p. 60).

Note also that if $|\mu| = |\lambda|$ and $-\ell \in \mathcal{P}_{ab}$, then there exists a function $\gamma \in \widetilde{C}([a, b]; R_+)$ satisfying (2.10) and (2.11) (see Remark 2.13).

Nevertheless, if $|\mu| > |\lambda| \neq 0$, then the inequality (2.11) cannot be replaced by the inequality (2.31) (see Example 2.9, p. 59).

Theorem 2.8. *Let $|\mu| \geq |\lambda| \neq 0$ and the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. Let, moreover,*

$$\|\ell_0(1)\|_L < \left|\frac{\lambda}{\mu}\right|, \tag{2.35}$$

$$\frac{|\mu|}{|\lambda| - |\mu| \|\ell_0(1)\|_L} - 1 < \|\ell_1(1)\|_L \leq 1. \tag{2.36}$$

Then the operator ℓ belongs to the set $V_{ab}^+(\lambda, \mu)$.

Remark 2.15. Let $|\mu| \geq |\lambda| \neq 0$ and

$$B \stackrel{\text{def}}{=} \left\{ (x, y) \in R_+ \times R_+ \;:\; x < \left|\frac{\lambda}{\mu}\right|, \; \frac{|\mu|}{|\lambda| - |\mu| x} - 1 < y \leq 1. \right\}$$

(see Fig. 2.2; note also that if $|\mu| \geq 2|\lambda|$, then $B = \emptyset$).

According to Theorem 2.8, if $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and

$$\Big(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L \Big) \in B,$$

then $\ell \in V_{ab}^+(\lambda, \mu)$. Below we will show (see On Remark 2.15, p. 61) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin B$ there exists $\ell \in \mathcal{L}_{ab}$ such that $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, (2.30) holds, and $\ell \notin V_{ab}^+(\lambda, \mu)$. In particular, neither one of the inequalities in (2.35) and (2.36) can be weakened.

2.2. On the Set $V_{ab}^-(\lambda, \mu)$

In this subsection, nonimprovable, in a certain sense, sufficient conditions guaranteeing the inclusion $\ell \in V_{ab}^-(\lambda, \mu)$ are established. First, in Theorems 2.9–2.11, we consider the case $|\mu| \leq |\lambda|$. Theorems 2.12–2.16 concern the case $|\mu| \geq |\lambda|$.

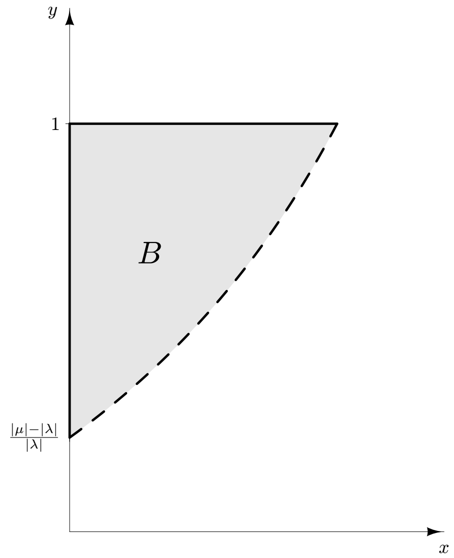

Fig. 2.2.

In the case, where $|\mu| \le |\lambda|$, the following statements hold.

Theorem 2.9. *Let* $0 \ne |\mu| \le |\lambda|$, $\ell \in \mathcal{P}_{ab}$, *and let there exist a function* $\gamma \in \widetilde{C}\big([a,b];R_+\big)$ *satisfying the inequalities* (2.20) *and*

$$|\lambda|\gamma(a) < |\mu|\gamma(b). \tag{2.37}$$

If, moreover,

$$\int_a^b \ell(1)(s)ds \le 1, \tag{2.38}$$

then the operator ℓ *belongs to the set* $V_{ab}^{-}(\lambda,\mu)$.

Theorem 2.10. *Let* $0 \ne |\mu| \le |\lambda|$, $\ell \in \mathcal{P}_{ab}$, ℓ *be a* $b-$*Volterra operator, and let there exist a function* $\gamma \in \widetilde{C}\big([a,b];R_+\big)$ *satisfying the inequalities* (2.20) *and* (2.37). *If, moreover, there exists a function* $\beta \in \widetilde{C}\big([a,b];R_+\big)$ *satisfying*

$$\beta(t) > 0 \quad for \quad t \in \,]a,b], \tag{2.39}$$

$$\beta'(t) \ge \ell(\beta)(t) \quad for \quad t \in [a,b], \tag{2.40}$$

then the operator ℓ *belongs to the set* $V_{ab}^{-}(\lambda,\mu)$.

Theorem 2.11. *Let* $0 \neq |\mu| \leq |\lambda|$ *and the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. *Let, moreover,*

$$\|\ell_1(1)\|_L < \left|\frac{\mu}{\lambda}\right|,$$

$$\frac{|\lambda|}{|\mu| - |\lambda| \|\ell_1(1)\|_L} - 1 < \|\ell_0(1)\|_L \leq 1.$$

Then the operator ℓ *belongs to the set* $V_{ab}^-(\lambda, \mu)$.

In the case, where $|\mu| \geq |\lambda|$, the following assertions hold.

Proposition 2.2. *Let* $|\mu| > |\lambda|$ *and* $-\ell \in \mathcal{P}_{ab}$. *Then* $\ell \in V_{ab}^-(\lambda, \mu)$ *iff the problem*

$$u'(t) \geq \ell(u)(t), \qquad \lambda u(a) + \mu u(b) = 0$$

has no nontrivial nonnegative solution.

Theorem 2.12. *Let* $|\mu| > |\lambda|$ *and* $-\ell \in \mathcal{P}_{ab}$. *Then the operator* ℓ *belongs to the set* $V_{ab}^-(\lambda, \mu)$ *iff there exists a function* $\gamma \in \widetilde{C}([a, b];]0, +\infty[)$ *satisfying the inequalities* (2.20) *and* (2.37).

Corollary 2.3. *Let* $|\mu| > |\lambda|$, $-\ell \in \mathcal{P}_{ab}$, *and let at least one of the following items be fulfilled:*

a) ℓ *is a* $b-$ *Volterra operator and*

$$|\lambda| \exp\left(\int_a^b |\ell(1)(s)| ds\right) < |\mu|;$$

b) there exist $m, k \in N$ *and a constant* $\alpha \in]0, 1[$ *such that* $m > k$ *and the inequality* (2.13) *is fulfilled, where* $\rho_1 \equiv 1$ *and*

$$\rho_{i+1}(t) \stackrel{\text{def}}{=} -\frac{|\lambda|}{|\mu| - |\lambda|} \int_a^b \ell(\rho_i)(s) ds - \int_t^b \ell(\rho_i)(s) ds$$

for $t \in [a, b]$, $i \in N$;

c) there exists $\overline{\ell} \in \mathcal{P}_{ab}$ *such that*

$$|\lambda| \exp\left(\int\limits_a^b |\ell(1)(s)| ds\right) +$$

$$+|\mu| \int\limits_a^b \overline{\ell}(1)(s) \exp\left(\int\limits_a^s |\ell(1)(\xi)| d\xi\right) ds < |\mu|,$$

and on the set $C_{\lambda\mu}\big([a,b];R_+\big)$ *the inequality*

$$\ell(1)(t)\vartheta(v)(t) - \ell(\vartheta(v))(t) \le \overline{\ell}(v)(t) \quad \textit{for} \quad t \in [a,b]$$

holds, where

$$\vartheta(v)(t) \stackrel{\text{def}}{=} -\frac{|\lambda|}{|\mu|-|\lambda|} \int\limits_a^b \ell(v)(s) ds - \int\limits_t^b \ell(v)(s) ds \quad \textit{for} \quad t \in [a,b].$$

Then the operator ℓ *belongs to the set* $V_{ab}^{-}(\lambda,\mu)$.

Theorem 2.13. *Let* $|\mu| \ge |\lambda|$, $\ell \in \mathcal{P}_{ab}$, ℓ *be a* $b-$ *Volterra operator, and let there exist a function* $\gamma \in \widetilde{C}\big([a,b];R_+\big)$ *satisfying* (2.10) *and*

$$\gamma(t) > 0 \quad \textit{for} \quad t \in \,]a,b]\,. \tag{2.41}$$

Then the operator ℓ *belongs to the set* $V_{ab}^{-}(\lambda,\mu)$.

Theorem 2.14. *Let* $|\mu| \ge |\lambda|$, $\ell \in \mathcal{P}_{ab}$, ℓ *be a* $b-$ *Volterra operator, and let the inequality* (2.38) *be satisfied. Then the operator* ℓ *belongs to the set* $V_{ab}^{-}(\lambda,\mu)$.

Corollary 2.4. *Let* $|\mu| \ge |\lambda|$, $\ell \in \mathcal{P}_{ab}$, ℓ *be a* $b-$ *Volterra operator, and*

$$\int\limits_a^b \widetilde{\ell}(1)(s) \exp\left(\int\limits_s^b \ell(1)(\xi) d\xi\right) ds \le 1,$$

where

$$\widetilde{\ell}(v)(t) \overset{\text{def}}{=} \ell\big(\widetilde{\theta}(v)\big)(t) - \ell(1)(t)\widetilde{\theta}(v)(t) \quad \textit{for} \quad t \in [a,b],$$

$$\widetilde{\theta}(v)(t) \overset{\text{def}}{=} -\int_t^b \ell(\widetilde{v})(s)ds \quad \textit{for} \quad t \in [a,b],$$

$$\widetilde{v}(t) \overset{\text{def}}{=} v(t)\exp\left(-\int_t^b \ell(1)(s)ds\right) \quad \textit{for} \quad t \in [a,b].$$

Then the operator ℓ belongs to the set $V_{ab}^-(\lambda,\mu)$.

Theorem 2.15. *Let $|\mu| \geq |\lambda| \neq 0$ and the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. Let, moreover,*

$$\|\ell_1(1)\|_L < 1,$$

$$\frac{\|\ell_1(1)\|_L}{1 - \|\ell_1(1)\|_L} - \frac{|\mu| - |\lambda|}{|\lambda|} < \|\ell_0(1)\|_L \leq \left|\frac{\lambda}{\mu}\right|.$$

Then the operator ℓ belongs to the set $V_{ab}^-(\lambda,\mu)$.

Theorem 2.16. *Let $|\mu| > |\lambda|$ and the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. If, moreover,*

$$\ell_0 \in V_{ab}^-(\lambda,\mu), \qquad -\ell_1 \in V_{ab}^-(\lambda,\mu),$$

then the operator ℓ belongs to the set $V_{ab}^-(\lambda,\mu)$.

Remark 2.16. Let $\ell \in \mathcal{L}_{ab}$, $q \in L\big([a,b];R\big)$, and $c \in R$. Define the operator $\psi : L\big([a,b];R\big) \to L\big([a,b];R\big)$ by

$$\psi(w)(t) \overset{\text{def}}{=} w(a+b-t) \quad \text{for} \quad t \in [a,b].$$

Let, moreover, φ be a restriction of ψ to the space $C\big([a,b];R\big)$ and

$$\widehat{\ell}(w)(t) \overset{\text{def}}{=} -\psi(\ell(\varphi(w)))(t), \qquad \widehat{q}(t) \overset{\text{def}}{=} -\psi(q)(t) \quad \text{for} \quad t \in [a,b].$$

It is clear that if u is a solution of the problem (1.1), (1.2), then the function $v \overset{\text{def}}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{\ell}(v)(t) + \widehat{q}(t), \qquad \mu v(a) + \lambda v(b) = c, \tag{2.42}$$

and vice versa, if v is a solution of the problem (2.42), then the function $u \overset{\text{def}}{=} \varphi(v)$ is a solution of the problem (1.1), (1.2).

Therefore, $\ell \in V_{ab}^{-}(\lambda, \mu)$ (resp. $\ell \in V_{ab}^{+}(\lambda, \mu)$) if and only if $\widehat{\ell} \in V_{ab}^{+}(\mu, \lambda)$ (resp. $\widehat{\ell} \in V_{ab}^{-}(\mu, \lambda)$).

It is also evident that if $\alpha \in \widetilde{C}\big([a,b]; R\big)$ satisfies the inequality

$$\alpha'(t) \leq \ell(\alpha)(t), \quad \big(\text{resp.} \quad \alpha'(t) \geq \ell(\alpha)(t)\big) \quad \text{for} \quad t \in [a,b], \tag{2.43}$$

then the function $\beta \overset{\text{def}}{=} \varphi(\alpha)$ satisfies the inequality

$$\beta'(t) \geq \widehat{\ell}(\beta)(t), \quad \big(\text{resp.} \quad \beta'(t) \leq \widehat{\ell}(\beta)(t)\big) \quad \text{for} \quad t \in [a,b], \tag{2.44}$$

and vice versa, if $\beta \in \widetilde{C}\big([a,b]; R\big)$ satisfies the inequality (2.44), then the function $\alpha \overset{\text{def}}{=} \varphi(\beta)$ satisfies the inequality (2.43).

Remark 2.17. According to Remark 2.16, Theorems 2.9–2.16, Proposition 2.2, and Corollaries 2.3 and 2.4 can be immediately derived from Theorems 2.1–2.8, Proposition 2.1, and Corollaries 2.1 and 2.2. Moreover, by virtue of Remarks 2.4–2.15, the results guaranteeing the inclusion $\ell \in V_{ab}^{-}(\lambda, \mu)$ are nonimprovable in an appropriate sense.

2.3. Proofs

Proof of Proposition 2.1. First suppose that $\ell \in V_{ab}^{+}(\lambda, \mu)$. If u is a solution of the problem (2.9), then, according to (2.1), the assumption $\ell \in V_{ab}^{+}(\lambda, \mu)$, and Remark 2.3 (see p. 16), we obtain $u(t) \leq 0$ for $t \in [a,b]$. Therefore, the problem (2.9) has no notrivial nonnegative solution.

Now suppose that the problem (2.9) has no nontrivial nonnegative solution. Let u_0 be a solution of the problem (1.1_0), (1.2_0). According to (2.1) and (2.8), we obtain

$$|u_0(t)|' = \ell(u_0)(t) \operatorname{sgn} u_0(t) \leq \ell(|u_0|)(t) \quad \text{for} \quad t \in [a,b],$$

$$\lambda |u_0(a)| + \mu |u_0(b)| = 0.$$

Therefore, $|u_0|$ is a solution of the problem (2.9). Hence, $|u_0| \equiv 0$, i.e., the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution.

Let u be a solution of the problem (1.1), (1.2) with $q \in L\big([a,b]; R_+\big)$ and $c \in R$ such that (2.2) is fulfilled. It easily follows from (2.2) that

$$\frac{c \operatorname{sgn} \lambda}{|\lambda| - |\mu|} \geq 0. \tag{2.45}$$

Taking now into account (1.1), (2.8), (2.45), and the assumption $q \in L\big([a,b];R_+\big)$, we get that on $[a,b]$ the inequality

$$[v(t)]'_- \le \ell([v]_-)(t) + \frac{\operatorname{sgn} v(t) - 1}{2}\left(q(t) + \frac{c \operatorname{sgn}\lambda}{|\lambda| - |\mu|}\,\ell(1)(t)\right) \le \ell([v]_-)(t)$$

holds, where

$$v(t) = u(t) - \frac{c \operatorname{sgn}\lambda}{|\lambda| - |\mu|} \quad \text{for} \quad t \in [a,b]. \tag{2.46}$$

On the other hand, by virtue of (1.2) and (2.1),

$$\lambda v(a) + \mu v(b) = 0.$$

This equality, together with (2.1), yields

$$\lambda [v(a)]_- + \mu [v(b)]_- = 0.$$

Thus, $[v]_-$ is a solution of the problem (2.9). Hence, $[v]_- \equiv 0$. Taking now into account (2.45) and (2.46), we get $u(t) \ge 0$ for $t \in [a,b]$ and so $\ell \in V^+_{ab}(\lambda,\mu)$. □

Proof of Theorem 2.1. First suppose that there exists a function $\gamma \in \widetilde{C}\big([a,b];]0,+\infty[\big)$ satisfying the inequalities (2.10) and (2.11).

Let u be a solution of the problem (1.1), (1.2), where $q \in L\big([a,b];R_+\big)$ and $c \in R$ is such that the inequality (2.2) is fulfilled. It easily follows from (1.2), (2.1), and (2.2) that

$$|\lambda| u(a) \ge |\mu| u(b). \tag{2.47}$$

We will show that

$$u(t) \ge 0 \quad \text{for} \quad t \in [a,b]. \tag{2.48}$$

Assume the contrary that (2.48) is not valid. Then there exists $t_0 \in [a,b]$ such that

$$u(t_0) < 0. \tag{2.49}$$

Put

$$r = \max\left\{-\frac{u(t)}{\gamma(t)} : t \in [a,b]\right\}$$

and

$$w(t) = r\gamma(t) + u(t) \quad \text{for} \quad t \in [a,b]. \tag{2.50}$$

According to (2.49),

$$r > 0. \tag{2.51}$$

It is clear that

$$w(t) \geq 0 \quad \text{for} \quad t \in [a,b] \tag{2.52}$$

and there exists $t_* \in [a,b]$ such that

$$w(t_*) = 0. \tag{2.53}$$

By virtue of (1.1), (2.8), (2.10), (2.51), (2.52), and the assumption $q \in L\big([a,b]; R_+\big)$, we get

$$w'(t) \geq \ell(w)(t) + q(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

From the last inequality, (2.52), and (2.53), we obtain $w(a) = 0$ and, in view of (2.11), (2.50), (2.51), and (2.52), we get

$$|\lambda| u(a) = -r|\lambda|\gamma(a) < -r|\mu|\gamma(b) = |\mu|(u(b) - w(b)) \leq |\mu| u(b),$$

which contradicts (2.47).

We have proved that if u is a solution of the problem (1.1), (1.2), where $q \in L\big([a,b]; R_+\big)$ and $c \in R$ is such that the inequality (2.2) holds, then the inequality (2.48) is satisfied. Now we will show that the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution. Indeed, let u_0 be a solution of the problem (1.1_0), (1.2_0). Obviously, $-u_0$ is a solution of the problem (1.1_0), (1.2_0), as well, and, according to the above–proved, we have

$$u_0(t) \geq 0, \qquad -u_0(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

Therefore, $u_0 \equiv 0$.

Now suppose that $\ell \in V_{ab}^+(\lambda,\mu)$. According to Definition 2.1 (see p. 15) and Theorem 1.1 (see p. 14), the problem

$$\gamma'(t) = \ell(\gamma)(t), \tag{2.54}$$

$$\lambda\gamma(a) + \mu\gamma(b) = \operatorname{sgn}\lambda \tag{2.55}$$

has a unique solution γ and

$$\gamma(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{2.56}$$

By virtue of (2.1), (2.55), and (2.56), it is clear that (2.11) holds and

$$\gamma(a) > 0. \tag{2.57}$$

On account of (2.8), (2.56), and (2.57), it follows from (2.54) that $\gamma(t) > 0$ for $t \in [a,b]$. □

Proof of Corollary 2.1. a) Let ℓ be an $a-$Volterra operator. It is not difficult to verify that the function

$$\gamma(t)=\exp\left(\int_a^t \ell(1)(s)ds\right) \quad \text{for} \quad t\in[a,b]$$

satisfies the inequality (2.10). On the other hand, on account of (2.12), the condition (2.11) is fulfilled. Therefore, the assumptions of Theorem 2.1 (see p. 17) are satisfied.

b) It can be easily verified that the function

$$\gamma(t)\stackrel{\text{def}}{=}(1-\alpha)\sum_{j=1}^{k}\rho_j(t)+\sum_{j=k+1}^{m}\rho_j(t) \quad \text{for} \quad t\in[a,b]$$

satisfies the assumptions of Theorem 2.1 (see p. 17).

c) According to (2.15), there exists $\varepsilon>0$ such that

$$\begin{gathered}\varepsilon\gamma_0\exp\left(\int_a^b \ell(1)(s)ds\right)+\\ +|\lambda|\gamma_0\int_a^b \overline{\ell}(1)(s)\exp\left(\int_s^b \ell(1)(\xi)d\xi\right)ds\le 1,\end{gathered} \tag{2.58}$$

where

$$\gamma_0=\frac{1}{|\lambda|-|\mu|\exp\left(\int\limits_a^b \ell(1)(s)ds\right)}.$$

Put

$$\begin{aligned}\gamma(t)=\gamma_0\Bigg[\varepsilon\exp\left(\int_a^t \ell(1)(s)ds\right)+|\lambda|\int_a^t \overline{\ell}(1)(s)\exp\left(\int_s^t \ell(1)(\xi)d\xi\right)ds+\\ +|\mu|\exp\left(\int_a^b \ell(1)(\xi)d\xi\right)\int_t^b \overline{\ell}(1)(s)\exp\left(\int_s^t \ell(1)(\xi)d\xi\right)ds\Bigg] \quad \text{for } t\in[a,b].\end{aligned}$$

Obviously, $\gamma \in \widetilde{C}\big([a,b];\,]0,+\infty[\big)$ and γ is a solution of the problem

$$\gamma'(t) = \ell(1)(t)\gamma(t) + \overline{\ell}(1)(t), \tag{2.59}$$

$$\lambda\gamma(a) + \mu\gamma(b) = \varepsilon \operatorname{sgn} \lambda. \tag{2.60}$$

Since $\ell, \overline{\ell} \in \mathcal{P}_{ab}$ and $\gamma(t) > 0$ for $t \in [a,b]$, the inequality (2.59) yields $\gamma'(t) \geq 0$ for $t \in [a,b]$ and, in view of (2.58), we have $\gamma(t) \leq 1$ for $t \in [a,b]$. Therefore, from (2.1), (2.59), and (2.60) we obtain

$$\gamma'(t) \geq \ell(1)(t)\gamma(t) + \overline{\ell}(\gamma)(t) \quad \text{for} \quad t \in [a,b], \qquad |\lambda|\gamma(a) > |\mu|\gamma(b).$$

Consequently, by Theorem 2.1 (see p. 17) we find

$$\widetilde{\ell} \in V_{ab}^{+}(\lambda,\mu), \tag{2.61}$$

where

$$\widetilde{\ell}(v)(t) \overset{\text{def}}{=} \ell(1)(t)v(t) + \overline{\ell}(v)(t) \quad \text{for} \quad t \in [a,b]. \tag{2.62}$$

According to Proposition 2.1 (see p. 17), it is sufficient to show that the problem (2.9) has no nontrivial nonnegative solution. Let $u \in \widetilde{C}\big([a,b];R_+\big)$ satisfy (2.9). Put

$$w(t) = \vartheta(u)(t) \quad \text{for} \quad t \in [a,b], \tag{2.63}$$

where ϑ is defined by (2.17). Obviously,

$$w'(t) = \ell(u)(t) \geq u'(t) \quad \text{for} \quad t \in [a,b]$$

and

$$0 \leq u(t) \leq w(t) \quad \text{for} \quad t \in [a,b], \qquad \lambda w(a) + \mu w(b) = 0. \tag{2.64}$$

On the other hand, in view of (2.16), (2.62)–(2.64), and the assumptions $\ell, \overline{\ell} \in \mathcal{P}_{ab}$, we get

$$w'(t) = \ell(u)(t) \leq \ell(1)(t)w(t) + \ell(w)(t) - \ell(1)(t)w(t) =$$

$$= \ell(1)(t)w(t) + \ell(\vartheta(u))(t) - \ell(1)(t)\vartheta(u)(t) \leq \ell(1)(t)w(t) + \overline{\ell}(u)(t) \leq$$

$$\leq \ell(1)(t)w(t) + \overline{\ell}(w)(t) = \widetilde{\ell}(w)(t) \quad \text{for} \quad t \in [a,b].$$

Now, by (2.61), (2.64), and Proposition 2.1 (see p. 17) we obtain $w \equiv 0$. Consequently, $u \equiv 0$. □

To prove Theorems 2.2 and 2.3 we need the following lemma.

Lemma 2.1. *Let* $|\mu| \le |\lambda|$, $-\ell \in \mathcal{P}_{ab}$, *and* ℓ *be an* $a-$*Volterra operator. Let, moreover,* u *be a nontrivial solution of the problem* (1.1), (1.2), *where* $q \in L\big([a,b];R_+\big)$ *and* $c \in R$ *is such that the inequality* (2.2) *holds, satisfying*

$$\min\{u(t) : t \in [a,b]\} < 0. \tag{2.65}$$

Then there exist $t_* \in \,]a,b]$ *and* $t^* \in [a,t_*[$ *such that*

$$\begin{aligned} u(t_*) &= \min\{u(t) : t \in [a,b]\}, \\ u(t^*) &= \max\{u(t) : t \in [a,t_*]\} > 0. \end{aligned} \tag{2.66}$$

Proof. Put

$$m = -\min\{u(t) : t \in [a,b]\}, \tag{2.67}$$
$$I = \{t \in [a,b] : u(t) = -m\}, \qquad t_* = \sup I.$$

Obviously, $m > 0$ and

$$u(t_*) = -m. \tag{2.68}$$

In view of (1.2), (2.1) and (2.2), it is clear that

$$\text{if} \quad a \in I, \quad \text{then} \quad |\lambda| = |\mu|, \ c = 0, \ \text{ and } \ t_* = b. \tag{2.69}$$

Therefore, $t_* \in \,]a,b]$.

We will show that

$$\max\{u(t) : t \in [a,t_*]\} > 0.$$

Assume the contrary that

$$u(t) \le 0 \quad \text{for} \quad t \in [a,t_*]. \tag{2.70}$$

Since ℓ is an $a-$Volterra operator, the integration of (1.1) from a to t_*, on account of (2.70) and the assumptions $-\ell \in \mathcal{P}_{ab}$ and $q \in L\big([a,b];R_+\big)$, results in

$$u(t_*) - u(a) = \int_a^{t_*} |\ell(u)(s)|ds + \int_a^{t_*} q(s)ds \ge 0. \tag{2.71}$$

From the last inequality, in view of (2.67) and (2.68), we obtain $a \in I$ and thus, it follows from (2.69) that $|\lambda| = |\mu|$, $c = 0$, and $t_* = b$. According to (2.71) we find $q \equiv 0$ and $\ell(u) \equiv 0$, i.e.,

$$u(t) = u(a) = -m \quad \text{for} \quad t \in [a, b].$$

Hence, (2.71) implies

$$0 = m\|\ell(1)\|_L.$$

Since we suppose that for $|\lambda| = |\mu|$ the operator ℓ is nontrivial, the last equality yields $m = 0$, a contradiction. □

Proof of Theorem 2.2. Let u be a nontrivial solution of the problem (1.1), (1.2), where $q \in L([a,b]; R_+)$ and $c \in R$ is such that the inequality (2.2) holds. We will show that (2.48) is fulfilled. Assume the contrary that the inequality (2.65) holds. According to Lemma 2.1 (see p. 34), there exist $t_* \in \,]a, b]$ and $t^* \in [a, t_*[$ such that (2.66) is valid. It is clear that there exists $t_0 \in \,]t^*, t_*[$ such that

$$u(t_0) = 0. \tag{2.72}$$

Put

$$w(t) = r\gamma(t) - u(t) \quad \text{for} \quad t \in [a, b],$$

where

$$r = \max\left\{ \frac{u(t)}{\gamma(t)} : t \in [a, t_0] \right\}.$$

Obviously,

$$r > 0 \tag{2.73}$$

and there exists $t_1 \in [a, t_0[$ such that

$$w(t_1) = 0. \tag{2.74}$$

It is also evident that

$$w(t) \geq 0 \quad \text{for} \quad t \in [a, t_0]. \tag{2.75}$$

Due to (1.1), (2.20), and (2.73), we get

$$w'(t) \leq \ell(w)(t) - q(t) \quad \text{for} \quad t \in [a, b].$$

Hence, by virtue of (2.75), the assumptions $-\ell \in \mathcal{P}_{ab}$ and $q \in L([a,b]; R_+)$, and the fact that ℓ is an $a-$Volterra operator, we obtain

$$w'(t) \leq 0 \quad \text{for} \quad t \in [a, t_0].$$

Thus, in view of (2.74),

$$w(t) \le 0 \quad \text{for} \quad t \in [t_1, t_0],$$

whence, together with (2.21), (2.72), and (2.73), we find $0 < w(t_0) \le 0$, a contradiction.

We have proved that if u is a nontrivial solution of the problem (1.1), (1.2), where $q \in L\big([a,b]; R_+\big)$ and $c \in R$ is such that the inequality (2.2) holds, then the inequality (2.48) is satisfied. Now suppose that the homogeneous problem (1.1_0), (1.2_0) has a nontrivial solution u_0. Obviously, $-u_0$ is a nontrivial solution of the problem (1.1_0), (1.2_0), as well, and, according to the above–proved, we have

$$u_0(t) \ge 0, \qquad -u_0(t) \ge 0 \quad \text{for} \quad t \in [a,b],$$

a contradiction. □

Proof of Theorem 2.3. Let u be a nontrivial solution of the problem (1.1), (1.2), where $q \in L\big([a,b]; R_+\big)$ and $c \in R$ is such that the inequality (2.2) holds. We will show that (2.48) is fulfilled. Assume the contrary that the inequality (2.65) holds. According to Lemma 2.1 (see p. 34), there exist $t_* \in\,]a,b]$ and $t^* \in [a, t_*[$ such that (2.66) is valid. The integration of (1.1) from t^* to t_* yields

$$u(t^*) - u(t_*) = -\int_{t^*}^{t_*} \ell(u)(s)ds - \int_{t^*}^{t_*} q(s)ds.$$

Hence, in view of (2.66), the assumptions $-\ell \in \mathcal{P}_{ab}$, $q \in L\big([a,b]; R_+\big)$, and the fact that ℓ is an $a-$Volterra operator, we find

$$u(t^*) < u(t^*) + |u(t_*)| \le u(t^*) \int_a^b |\ell(1)(s)|ds.$$

The last inequality, together with (2.23), implies the contradiction $u(t^*) < u(t^*)$.

We have proved that if u is a nontrivial solution of the problem (1.1), (1.2), where $q \in L\big([a,b]; R_+\big)$ and $c \in R$ is such that the inequality (2.2) holds, then the inequality (2.48) is satisfied. Now suppose that the homogeneous problem (1.1_0), (1.2_0) has a nontrivial solution u_0. Obviously, $-u_0$

is a nontrivial solution of the problem (1.1_0), (1.2_0), as well, and, according to the above–proved, we have

$$u_0(t) \geq 0, \qquad -u_0(t) \geq 0 \quad \text{for} \quad t \in [a,b],$$

a contradiction. □

Proof of Corollary 2.2. Let u be a solution of the problem (1.1), (1.2), where $q \in L\big([a,b];R_+\big)$ and $c \in R$ is such that the inequality (2.2) is fulfilled. We will show that (2.48) holds. From (1.1) we get

$$u'(t) = \ell(1)(t)u(t) + \ell(u)(t) - \ell(1)(t)u(t) + q(t) \quad \text{for} \quad t \in [a,b]. \tag{2.76}$$

On the other hand, the integration of (1.1) from a to t yields

$$u(t) = u(a) + \int_a^t \ell(u)(s)ds + \int_a^t q(s)ds \quad \text{for} \quad t \in [a,b]. \tag{2.77}$$

By virtue of (2.77), from (2.76) we obtain

$$\begin{gathered} u'(t) = \ell(1)(t)u(t) + \ell(\theta(u))(t) - \ell(1)(t)\theta(u)(t)+ \\ +q_0(t) \quad \text{for} \quad t \in [a,b], \end{gathered} \tag{2.78}$$

where

$$q_0(t) = \ell(q^*)(t) - \ell(1)(t)q^*(t) + q(t) \quad \text{for} \quad t \in [a,b], \tag{2.79}$$

$$\theta(v)(t) = \int_a^t \ell(v)(s)ds, \qquad q^*(t) = \int_a^t q(s)ds \quad \text{for} \quad t \in [a,b].$$

In view of the conditions $-\ell \in \mathcal{P}_{ab}$, $q \in L\big([a,b];R_+\big)$, and the fact that ℓ is an $a-$Volterra operator, we have

$$\ell(q^*)(t) - \ell(1)(t)q^*(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

Thus, due to the condition $q \in L\big([a,b];R_+\big)$, (2.79) yields

$$q_0(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{2.80}$$

Put

$$w(t) = u(t)\exp\left(-\int_a^t \ell(1)(s)ds\right) \quad \text{for} \quad t \in [a,b]. \tag{2.81}$$

Then

$$\lambda w(a) = \lambda u(a) = c - \mu u(b) = c - \widetilde{\mu} w(b), \tag{2.82}$$

where

$$\widetilde{\mu} = \mu \exp\left(\int\limits_a^b \ell(1)(s)ds\right),$$

and since $-\ell \in \mathcal{P}_{ab}$, we have $|\widetilde{\mu}| \leq |\mu|$. Clearly, $\lambda\widetilde{\mu} \leq 0$.

Due to (2.78), it is evident that

$$w'(t) = \exp\left(\int\limits_a^t |\ell(1)(s)|ds\right) \widetilde{\ell}(w)(t) + \widetilde{q}(t) \quad \text{for} \quad t \in [a,b], \tag{2.83}$$

where $\widetilde{\ell}$ is defined by (2.26) and

$$\widetilde{q}(t) = q_0(t) \exp\left(-\int\limits_a^t \ell(1)(s)ds\right) \quad \text{for} \quad t \in [a,b]. \tag{2.84}$$

It is easy to verify that $-\widetilde{\ell} \in \mathcal{P}_{ab}$ and $\widetilde{\ell}$ is an $a-$Volterra operator. Thus, in view of (2.25), the conditions $\lambda\widetilde{\mu} \leq 0$ and $|\widetilde{\mu}| \leq |\lambda|$, and Theorem 2.3 (see p. 20), the operator T defined by

$$T(v)(t) \overset{\text{def}}{=} \widetilde{\ell}(v)(t) \exp\left(\int\limits_a^t |\ell(1)(s)|ds\right) \quad \text{for} \quad t \in [a,b],$$

belongs to the set $V_{ab}^{+}(\lambda, \widetilde{\mu})$. Therefore, by virtue of (2.2), (2.80), (2.82)–(2.84), and the condition $\lambda\widetilde{\mu} \leq 0$, we have $w(t) \geq 0$ for $t \in [a,b]$. Consequently, in view of (2.81), the inequality (2.48) is satisfied.

We have proved that if u is a solution of the problem (1.1), (1.2), where $q \in L\big([a,b]; R_+\big)$ and $c \in R$ is such that the inequality (2.2) holds, then the inequality (2.48) is satisfied. Now we will show that the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution. Indeed, let u_0 be a solution of the problem (1.1_0), (1.2_0). Obviously, $-u_0$ is a solution of the problem (1.1_0), (1.2_0), as well, and, according to the above–proved, we have

$$u_0(t) \geq 0, \qquad -u_0(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

Therefore, $u_0 \equiv 0$. □

To prove Theorem 2.4 we need the following lemma.

Lemma 2.2. *Let* $0 \neq |\mu| \leq |\lambda|$, $q \in L\big([a,b];R_-\big)$, *and* $c \in R$ *be such that the inequality*

$$c \operatorname{sgn} \lambda \leq 0 \tag{2.85}$$

holds. Further, let the operator ℓ *admit the representation* $\ell = \ell_0 - \ell_1$, *where* ℓ_0 *and* ℓ_1 *satisfy the condition* (2.27). *If, moreover,*

$$\|\ell_0(1)\|_L < 1, \qquad \frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{|\lambda| - |\mu|}{|\mu|} < \|\ell_1(1)\|_L, \tag{2.86}$$

then the problem (1.1), (1.2) *has no nontrivial solution* u *satisfying the inequality*

$$u(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{2.87}$$

Proof. Assume the contrary that the problem (1.1), (1.2) has a nontrivial solution u satisfying (2.87). Put

$$M = \max\{u(t) : t \in [a,b]\}, \qquad m = \min\{u(t) : t \in [a,b]\} \tag{2.88}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = m. \tag{2.89}$$

Obviously, $M > 0$, $m \geq 0$, and either

$$t_M < t_m \tag{2.90}$$

or

$$t_M > t_m. \tag{2.91}$$

First suppose that (2.91) holds. The integration of (1.1) from t_m to t_M, on account of (2.27), (2.87)–(2.89), and the assumption $q \in L\big([a,b];R_-\big)$, results in

$$M - m = \int_{t_m}^{t_M} [\ell_0(u)(s) - \ell_1(u)(s) + q(s)]ds \leq M \int_{t_m}^{t_M} \ell_0(1)(s)ds \leq M\|\ell_0(1)\|_L.$$

Hence, by virtue of the first inequality in (2.86), we get

$$0 < M(1 - \|\ell_0(1)\|_L) \leq m. \tag{2.92}$$

Now suppose that (2.90) is fulfilled. Clearly, the condition (1.2), in view of (2.1), (2.85), (2.87), and the assumption $\left|\frac{\mu}{\lambda}\right| \in]0,1]$, implies

$$u(b) - u(a) \geq \left|\frac{\mu}{\lambda}\right| u(b) - u(a) = -\frac{c \operatorname{sgn} \lambda}{|\lambda|} \geq 0. \tag{2.93}$$

The integration of (1.1) from a to t_M and from t_m to b, in view of (2.27), (2.87)–(2.89), and the assumption $q \in L\big([a,b]; R_-\big)$, yields

$$M - u(a) \leq M \int_a^{t_M} \ell_0(1)(s)ds, \qquad u(b) - m \leq M \int_{t_m}^b \ell_0(1)(s)ds.$$

Summing the last two inequalities and taking into account (2.27), (2.86), and (2.93), we find that the inequality (2.92) is satisfied.

Therefore, in both cases (2.90) and (2.91), the inequality (2.92) is valid.

On the other hand, the integration of (1.1) from a to b, in view of (2.27), (2.88), and the assumption $q \in L\big([a,b]; R_-\big)$, implies

$$u(b) - u(a) = \int_a^b [\ell_0(u)(s) - \ell_1(u)(s) + q(s)]ds \leq M\|\ell_0(1)\|_L - m\|\ell_1(1)\|_L.$$

Hence, by (1.2), (2.1), (2.85), (2.88), and the assumption $0 \neq |\mu| \leq |\lambda|$, we have

$$m\|\ell_1(1)\|_L \leq M\|\ell_0(1)\|_L + u(a)\left(1 + \frac{\lambda}{\mu}\right) - \frac{c}{\mu} \leq M\|\ell_0(1)\|_L + m\left(1 - \left|\frac{\lambda}{\mu}\right|\right).$$

Thus,

$$m\left(\|\ell_1(1)\|_L + \frac{|\lambda| - |\mu|}{|\mu|}\right) \leq M\|\ell_0(1)\|_L.$$

This inequality, together with (2.92), results in

$$\|\ell_1(1)\|_L \leq \frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{|\lambda| - |\mu|}{|\mu|},$$

which contradicts the second inequality in (2.86). □

Proof of Theorem 2.4. Let u be a nontrivial solution of the problem (1.1), (1.2), where $q \in L\big([a,b]; R_+\big)$ and $c \in R$ is such that the inequality (2.2) is fulfilled. We will show that (2.48) is satisfied.

Assume that u changes its sign. Put

$$M = \max\{u(t) : t \in [a,b]\}, \qquad m = -\min\{u(t) : t \in [a,b]\} \tag{2.94}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{2.95}$$

Obviously,

$$M > 0, \qquad m > 0, \tag{2.96}$$

and either (2.90) or (2.91) is valid.

First suppose that (2.90) is fulfilled. The integration of (1.1) from t_M to t_m, in view of (2.27), (2.94), (2.95), and the assumption $q \in L\big([a,b]; R_+\big)$, results in

$$M + m = \int_{t_M}^{t_m} \ell_1(u)(s)ds - \int_{t_M}^{t_m} \ell_0(u)(s)ds - \int_{t_M}^{t_m} q(s)ds \le$$

$$\le M \int_{t_M}^{t_m} \ell_1(1)(s)ds + m \int_{t_M}^{t_m} \ell_0(1)(s)ds \le M\|\ell_1(1)\|_L + m\|\ell_0(1)\|_L.$$

Hence, according to (2.28), (2.29), (2.96), and the assumption $\left|\frac{\mu}{\lambda}\right| \le 1$, we get the contradiction $M + m < M + m$.

Now suppose that (2.91) holds. Clearly, the condition (1.2), in view of (2.1) and (2.2), implies

$$u(a) - \left|\frac{\mu}{\lambda}\right| u(b) = \frac{c}{\lambda} = \frac{c}{|\lambda|} \operatorname{sgn} \lambda \ge 0. \tag{2.97}$$

The integration of (1.1) from a to t_m and from t_M to b, on account of (2.27), (2.94), (2.95), and the assumptions $\left|\frac{\mu}{\lambda}\right| \in \,]0,1]$ and $q \in L\big([a,b]; R_+\big)$, yields

$$m + u(a) \le M \int_a^{t_m} \ell_1(1)(s)ds + m \int_a^{t_m} \ell_0(1)(s)ds,$$

$$\left|\frac{\mu}{\lambda}\right| \big(M - u(b)\big) \le M - u(b) \le M \int_{t_M}^{b} \ell_1(1)(s)ds + m \int_{t_M}^{b} \ell_0(1)(s)ds.$$

Summing these inequalities and taking into account (2.97), we obtain

$$\left|\frac{\mu}{\lambda}\right| M + m \le M\|\ell_1(1)\|_L + m\|\ell_0(1)\|_L,$$

which, according to (2.28), (2.29), and (2.96), yields the contradiction $\left|\frac{\mu}{\lambda}\right| M + m < \left|\frac{\mu}{\lambda}\right| M + m$.

Therefore, u does not change its sign, and, by virtue of Lemma 2.2 (see p. 39), the inequality (2.48) is valid.

We have proved that if u is a nontrivial solution of the problem (1.1), (1.2), where $q \in L\big([a,b]; R_+\big)$ and $c \in R$ is such that the inequality (2.2) holds, then the inequality (2.48) is satisfied. Now suppose that the homogeneous problem (1.1_0), (1.2_0) has a nontrivial solution u_0. Obviously, $-u_0$ is a nontrivial solution of the problem (1.1_0), (1.2_0), as well, and, according to the above–proved, we have

$$u_0(t) \ge 0, \qquad -u_0(t) \ge 0 \quad \text{for} \quad t \in [a,b],$$

a contradiction. □

Proof of Theorem 2.5. Let u be a solution of the problem (1.1), (1.2), where $q \in L\big([a,b]; R_+\big)$ and $c \in R$ is such that the inequality (2.2) is fulfilled. Since $-\ell_1 \in V_{ab}^+(\lambda,\mu)$, the problem

$$\alpha'(t) = -\ell_1(\alpha)(t) - \ell_0([u]_-)(t), \tag{2.98}$$

$$\lambda\alpha(a) + \mu\alpha(b) = 0 \tag{2.99}$$

has a unique solution α and

$$\alpha(t) \le 0 \quad \text{for} \quad t \in [a,b]. \tag{2.100}$$

In view of (1.1), (2.98), and (2.99), we get

$$\big(u(t) - \alpha(t)\big)' = -\ell_1(u-\alpha) + \ell_0([u]_+) + q(t) \quad \text{for} \quad t \in [a,b],$$

$$\lambda\big(u(a) - \alpha(a)\big) + \mu\big(u(b) - \alpha(b)\big) = c.$$

According to (2.2) and the assumptions $\ell_0 \in \mathcal{P}_{ab}$, $q \in L\big([a,b]; R_+\big)$, and $-\ell_1 \in V_{ab}^+(\lambda,\mu)$, it is obvious that

$$\alpha(t) \le u(t) \quad \text{for} \quad t \in [a,b]. \tag{2.101}$$

Now, (2.100) and (2.101) imply

$$\alpha(t) \leq -[u(t)]_- \quad \text{for} \quad t \in [a,b]. \tag{2.102}$$

On the other hand, according to (2.98), (2.100), (2.102), and the assumptions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, we have

$$\alpha'(t) \geq \ell_0(\alpha)(t) - \ell_1(\alpha)(t) \geq \ell_0(\alpha)(t) \quad \text{for} \quad t \in [a,b].$$

Hence, the inclusion $\ell_0 \in V_{ab}^+(\lambda, \mu)$, on account of (2.1), (2.99), and Remark 2.3 (see p. 16), implies

$$\alpha(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

It follows from this inequality and (2.101) that (2.48) holds.

We have proved that if u is a solution of the problem (1.1), (1.2), where $q \in L\big([a,b]; R_+\big)$ and $c \in R$ is such that the inequality (2.2) holds, then the inequality (2.48) is satisfied. Now we will show that the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution. Indeed, let u_0 be a solution of the problem (1.1_0), (1.2_0). Obviously, $-u_0$ is a solution of the problem (1.1_0), (1.2_0), as well, and, according to the above–proved, we have

$$u_0(t) \geq 0, \qquad -u_0(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

Therefore, $u_0 \equiv 0$. □

Proof of Theorem 2.6. Let u be a solution of the problem (1.1), (1.2), where $q \in L\big([a,b]; R_+\big)$ and $c \in R$ is such that the inequality (2.2) holds. We will show that (2.48) is fulfilled.

Suppose that u changes its sign. Define the numbers M and m by (2.94) and choose $t_M, t_m \in [a,b]$ such that (2.95) is fulfilled. Obviously, (2.96) holds and either (2.90) or (2.91) is satisfied.

First assume that (2.90) is fulfilled. The integration of (1.1) from t_M to t_m, in view of (2.94), (2.95), and the assumptions $-\ell \in \mathcal{P}_{ab}$ and $q \in L\big([a,b]; R_+\big)$, results in

$$M + m = -\int_{t_M}^{t_m} \ell(u)(s)ds - \int_{t_M}^{t_m} q(s)ds \leq M \int_a^b |\ell(1)(s)|ds.$$

Hence, according to (2.23) and (2.96), we obtain $M + m \leq M$, which contradicts (2.96).

Now suppose that (2.91) holds. The integration of (1.1) from a to t_m and from t_M to b, on account of (2.94), (2.95), and the assumptions $-\ell \in \mathcal{P}_{ab}$ and $q \in L\big([a,b];R_+\big)$, yields

$$u(a) + m = -\int_a^{t_m} \ell(u)(s)ds - \int_a^{t_m} q(s)ds \leq M \int_a^{t_m} |\ell(1)(s)|ds, \tag{2.103}$$

$$M - u(b) = -\int_{t_M}^{b} \ell(u)(s)ds - \int_{t_M}^{b} q(s)ds \leq M \int_{t_M}^{b} |\ell(1)(s)|ds. \tag{2.104}$$

Multiplying both sides of (2.103) by $\left|\frac{\lambda}{\mu}\right|$ and taking into account the assumptions $\left|\frac{\lambda}{\mu}\right| \in \,]0,1]$ and $M > 0$, we get

$$\left|\frac{\lambda}{\mu}\right| u(a) + \left|\frac{\lambda}{\mu}\right| m \leq M \int_a^{t_m} |\ell(1)(s)|ds.$$

Summing the last inequality and (2.104), on account of (1.2), (2.1), (2.2), and the condition $M > 0$, we find

$$\left|\frac{\lambda}{\mu}\right| m + M \leq \left|\frac{\lambda}{\mu}\right| m + M + \frac{c}{|\mu|} \operatorname{sgn} \lambda \leq M \int_a^{b} |\ell(1)(s)|ds.$$

Hence, according to (2.23) and (2.96), we reach the contradiction $\left|\frac{\lambda}{\mu}\right| m + M \leq M$.

Therefore, u does not change its sign. Now assume on the contrary that (2.48) is not valid. Due to the above–proved we have

$$u(t) \leq 0 \quad \text{for} \quad t \in [a,b], \qquad u \not\equiv 0. \tag{2.105}$$

It follows from (1.1), (2.105), and the assumptions $-\ell \in \mathcal{P}_{ab}$ and $q \in L\big([a,b];R_+\big)$ that

$$u'(t) \geq \ell(u)(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{2.106}$$

Clearly, (2.105) and (2.106) imply $u(a) < 0$. Further, by virtue of (1.2), (2.1), and (2.2), we have

$$|\mu| u(b) = |\lambda| u(a) + c \operatorname{sgn} \mu \leq |\lambda| u(a). \tag{2.107}$$

Hence, with respect to the condition $u(a) < 0$ and the assumptions $\lambda \neq 0$ and $\mu \neq 0$, we get $u(b) < 0$. Thus, (2.106) implies

$$u(t) < 0 \quad \text{for} \quad t \in [a, b]. \tag{2.108}$$

Put

$$r = \max\left\{ -\frac{\gamma(t)}{u(t)} : t \in [a, b] \right\}$$

and

$$v(t) = ru(t) + \gamma(t) \quad \text{for} \quad t \in [a, b]. \tag{2.109}$$

According to (2.11), (2.108), and the assumptions $\gamma \in \widetilde{C}([a, b]; R_+)$ and $\lambda \neq 0$, we have

$$r > 0. \tag{2.110}$$

It is clear that

$$v(t) \leq 0 \quad \text{for} \quad t \in [a, b] \tag{2.111}$$

and there exists $t_0 \in [a, b]$ such that

$$v(t_0) = 0. \tag{2.112}$$

By virtue of (1.1), (2.10), (2.110), (2.111), and the assumptions $-\ell \in \mathcal{P}_{ab}$ and $q \in L([a, b]; R_+)$, we get

$$v'(t) \geq \ell(v)(t) + rq(t) \geq 0 \quad \text{for} \quad t \in [a, b].$$

From the last inequality, (2.111), and (2.112), we obtain $v(b) = 0$ and, in view of (2.11) and (2.109)–(2.111), we find

$$\begin{aligned} |\mu|u(b) = -\frac{|\mu|}{r}\gamma(b) > -\frac{|\lambda|}{r}\gamma(a) = \\ = |\lambda|\left(u(a) - \frac{v(a)}{r}\right) \geq |\lambda|u(a), \end{aligned} \tag{2.113}$$

which contradicts (2.107).

We have proved that if u is a solution of the problem (1.1), (1.2), where $q \in L([a, b]; R_+)$ and $c \in R$ is such that the inequality (2.2) holds, then the inequality (2.48) is satisfied. Now we will show that the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution. Indeed, let u_0 be a solution of the problem (1.1_0), (1.2_0). Obviously, $-u_0$ is a solution of the problem (1.1_0), (1.2_0), as well, and, according to the above–proved, we have

$$u_0(t) \geq 0, \qquad -u_0(t) \geq 0 \quad \text{for} \quad t \in [a, b].$$

Therefore, $u_0 \equiv 0$. □

Proof of Theorem 2.7. Let u be a solution of the problem (1.1), (1.2), where $q \in L\big([a,b];R_+\big)$ and $c \in R$ is such that the inequality (2.2) holds. We will show that (2.48) is fulfilled.

Suppose that $u(b) < 0$. Then there exists $t_0 \in \,]a,b[$ such that

$$u(t) < 0 \quad \text{for} \quad t \in [t_0, b]. \tag{2.114}$$

Since ℓ is an $a-$Volterra operator, the restriction of u to the interval $[a,t_0]$ is a solution of the equation (1.1) with the condition $u(t_0) < 0$. Moreover, the restriction of β to the interval $[a,t_0]$ is a positive absolutely continuous function satisfying

$$\beta'(t) \leq \ell(\beta)(t) \quad \text{for} \quad t \in [a,t_0].$$

According to Theorem 2.12 (for $\lambda = 0$, $\mu = 1$, and $b = t_0$, see p. 26), the condition $u(t_0) < 0$, and the assumptions $-\ell \in \mathcal{P}_{ab}$ and $q \in L\big([a,b];R_+\big)$, we get

$$u(t) < 0 \quad \text{for} \quad t \in [a,t_0].$$

This inequality, together with (2.114), yields (2.108). By the same arguments as in the proof of Theorem 2.6 it can be shown that the inequality (2.113) holds. On the other hand, by virtue of (1.2), (2.1), and (2.2), we get (2.107), which contradicts (2.113).

Therefore, $u(b) \geq 0$. In view of (1.2), (2.1), and (2.2), the inequality

$$u(a) \geq 0 \tag{2.115}$$

holds. Since ℓ is an $a-$Volterra operator, with respect to (2.32), (2.33), and Theorem 2.2 (see p. 19), we get $\ell \in V_{ab}^{+}(1,0)$, which, by virtue of (2.115) and the assumption $q \in L\big([a,b];R_+\big)$, implies (2.48).

We have proved that if u is a solution of the problem (1.1), (1.2), where $q \in L\big([a,b];R_+\big)$ and $c \in R$ is such that the inequality (2.2) holds, then the inequality (2.48) is satisfied. Now we will show that the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution. Indeed, let u_0 be a solution of the problem (1.1_0), (1.2_0). Obviously, $-u_0$ is a solution of the problem (1.1_0), (1.2_0), as well, and, according to the above–proved, we have

$$u_0(t) \geq 0, \qquad -u_0(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

Therefore, $u_0 \equiv 0$. □

To prove Theorem 2.8 we need the following lemma.

Lemma 2.3. *Let* $|\mu| \geq |\lambda| \neq 0$, $q \in L\big([a,b];R_-\big)$, $c \in R$ *be such that* (2.85) *holds, and let the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$, *where* ℓ_0 *and* ℓ_1 *satisfy the condition* (2.27). *If, moreover,*

$$\|\ell_0(1)\|_L < \left|\frac{\lambda}{\mu}\right|, \qquad \frac{|\mu|}{|\lambda| - |\mu|\|\ell_0(1)\|_L} - 1 < \|\ell_1(1)\|_L, \tag{2.116}$$

then the problem (1.1), (1.2) *has no nontrivial solution* u *satisfying the inequality* (2.87).

Proof. Assume the contrary that the problem (1.1), (1.2) has a nontrivial solution u satisfying the condition (2.87). Define the numbers M and m by (2.88) and choose $t_M, t_m \in [a,b]$ such that (2.89) is satisfied. Obviously, $M > 0$, $m \geq 0$, and either (2.90) or (2.91) is valid.

First suppose that (2.91) holds. The integration of (1.1) from t_m to t_M, on account of (2.27), (2.87)–(2.89), and the assumptions $\left|\frac{\lambda}{\mu}\right| \in \,]0,1]$ and $q \in L\big([a,b];R_-\big)$, results in

$$\left|\frac{\lambda}{\mu}\right| M - m \leq M - m = \int_{t_m}^{t_M} [\ell_0(u)(s) - \ell_1(u)(s) + q(s)]\, ds \leq$$

$$\leq M \int_{t_m}^{t_M} \ell_0(1)(s) ds \leq M\|\ell_0(1)\|_L.$$

Hence, by virtue of the first inequality in (2.116), we get

$$0 < M\left(\left|\frac{\lambda}{\mu}\right| - \|\ell_0(1)\|_L\right) \leq m. \tag{2.117}$$

Now suppose that (2.90) is fulfilled. Clearly, the condition (1.2), on account of (2.1) and (2.85), implies

$$u(b) - \left|\frac{\lambda}{\mu}\right| u(a) = \frac{c}{\mu} = -\frac{c}{|\mu|}\operatorname{sgn}\lambda \geq 0. \tag{2.118}$$

The integration of (1.1) from a to t_M and from t_m to b, in view of (2.27), (2.87)–(2.89), and the assumptions $\left|\frac{\lambda}{\mu}\right| \in]0,1]$ and $q \in L\big([a,b];R_-\big)$, yields

$$\left|\frac{\lambda}{\mu}\right| \big(M - u(a)\big) \le M - u(a) \le \int\limits_a^{t_M} \ell_0(u)(s)ds \le M \int\limits_a^{t_M} \ell_0(1)(s)ds,$$

$$u(b) - m \le \int\limits_{t_m}^{b} \ell_0(u)(s)ds \le M \int\limits_{t_m}^{b} \ell_0(1)(s)ds.$$

Summing the last two inequalities and taking into account (2.118) and the first inequality in (2.116), we find that the inequlity (2.117) is satisfied.

Therefore, in both cases (2.90) and (2.91), the inequality (2.117) is valid.

On the other hand, the integration of (1.1) from a to b, in view of (2.27), (2.88), and the assumption $q \in L\big([a,b];R_-\big)$, results in

$$u(b) - u(a) = \int\limits_a^b \big[\ell_0(u)(s) - \ell_1(u)(s) + q(s)\big]ds \le M\|\ell_0(1)\|_L - m\|\ell_1(1)\|_L.$$

Hence, by (1.2), (2.1), (2.2), (2.85), (2.88), and the assumption $\left|\frac{\lambda}{\mu}\right| \in]0,1]$, we have

$$m\|\ell_1(1)\|_L \le M\|\ell_0(1)\|_L + u(a)\left(1 - \left|\frac{\lambda}{\mu}\right|\right) + \frac{c}{|\mu|}\operatorname{sgn}\lambda \le$$
$$\le M\|\ell_0(1)\|_L + M\left(1 - \left|\frac{\lambda}{\mu}\right|\right) = M\left(\|\ell_0(1)\|_L - \left|\frac{\lambda}{\mu}\right| + 1\right).$$

This inequality, together with (2.117), yields

$$\|\ell_1(1)\|_L \le \frac{|\mu|}{|\lambda| - |\mu|\|\ell_0(1)\|_L} - 1,$$

which contradicts the second inequality in (2.116). □

In a similar manner one can prove the following assertion (we will need it in §4).

Lemma 2.4. *Let* $0 \ne |\mu| \le |\lambda|$, $q \in L\big([a,b];R_+\big)$, *and* $c \in R$ *be such that*

$$c\operatorname{sgn}\lambda \ge 0.$$

Further, let the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 satisfy the condition (2.27)*. If, moreover,*

$$\|\ell_1(1)\|_L < \left|\frac{\mu}{\lambda}\right|, \qquad \frac{|\lambda|}{|\mu| - |\lambda|\|\ell_1(1)\|_L} - 1 < \|\ell_0(1)\|_L,$$

then the problem (1.1), (1.2) *has no nontrivial solution u satisfying the inequality* (2.87).

Proof of Theorem 2.8. Let u be a nontrivial solution of the problem (1.1), (1.2), where $q \in L\big([a,b];R_+\big)$ and $c \in R$ is such that the inequality (2.2) is fulfilled. We will show that (2.48) is satisfied.

Assume that u changes its sign. Define the numbers M and m by (2.94) and choose $t_M, t_m \in [a,b]$ such that (2.95) is satisfied. Obviously, (2.96) holds, and either (2.90) or (2.91) is fulfilled.

First suppose that (2.90) is valid. The integration of (1.1) from t_M to t_m, in view of (2.27), (2.94), (2.95), and the assumption $q \in L\big([a,b];R_+\big)$, results in

$$M + m = \int\limits_{t_M}^{t_m} \big[\ell_1(u)(s) - \ell_0(u)(s)ds - q(s)\big]ds \leq$$

$$\leq M \int\limits_{t_M}^{t_m} \ell_1(1)(s)ds + m \int\limits_{t_M}^{t_m} \ell_0(1)(s)ds \leq M\|\ell_1(1)\|_L + m\|\ell_0(1)\|_L.$$

Hence, according to (2.35), (2.36), (2.96), and the assumption $\left|\frac{\lambda}{\mu}\right| \leq 1$, we get the contradiction $M + m < M + m$.

Now suppose that (2.91) holds. Clearly, the condition (1.2), in view of (2.1) and (2.2), implies

$$\left|\frac{\lambda}{\mu}\right| u(a) - u(b) = -\frac{c}{\mu} = \frac{c}{|\mu|} \operatorname{sgn} \lambda \geq 0. \tag{2.119}$$

The integration of (1.1) from a to t_m and from t_M to b, on account of (2.27), (2.94), (2.95), and the assumptions $\left|\frac{\lambda}{\mu}\right| \in \,]0,1]$ and $q \in L\big([a,b];R_+\big)$, yields

$$\left|\frac{\lambda}{\mu}\right|(m+u(a)) \le m+u(a) \le M\int_a^{t_m}\ell_1(1)(s)ds + m\int_a^{t_m}\ell_0(1)(s)ds,$$

$$M-u(b) \le M\int_{t_M}^{b}\ell_1(1)(s)ds + m\int_{t_M}^{b}\ell_0(1)(s)ds.$$

Summing these inequalities and taking into account (2.119), we obtain

$$M + \left|\frac{\lambda}{\mu}\right| m \le M\|\ell_1(1)\|_L + m\|\ell_0(1)\|_L,$$

which, according to (2.35), (2.36), and (2.96), yields the contradiction $M + \left|\frac{\lambda}{\mu}\right| m < M + \left|\frac{\lambda}{\mu}\right| m$.

Therefore, u does not change its sign, and, by virtue of Lemma 2.3 (see p. 47), the inequality (2.48) is valid.

We have proved that if u is a nontrivial solution of the problem (1.1), (1.2), where $q \in L\big([a,b];R_+\big)$ and $c \in R$ is such that the inequality (2.2) holds, then the inequality (2.48) is satisfied. Now suppose that the homogeneous problem (1.1_0), (1.2_0) has a nontrivial solution u_0. Obviously, $-u_0$ is a nontrivial solution of the problem (1.1_0), (1.2_0), as well, and, according to the above–proved, we have

$$u_0(t) \ge 0, \qquad -u_0(t) \ge 0 \quad \text{for} \quad t \in [a,b],$$

a contradiction. □

2.4. Comments and Examples

On Remark 2.4. Suppose that $|\mu| < |\lambda|$, $\ell \in \mathcal{P}_{ab}$, ℓ is an $a-$Volterra operator, (2.18) holds, and the problem (1.1_0), (1.2_0) has only the trivial solution. According to Theorem 1.1 (see p. 14), the equation (1.1_0) has a unique solution u satisfying the condition

$$\lambda u(a) + \mu u(b) = \operatorname{sgn}\lambda. \tag{2.120}$$

Evidently, $u \not\equiv 0$. According to Corollary 2.1 a) (see p. 17), we have

$$\ell \in V_{ab}^{+}(1,0).$$

Therefore,

$$u(a) \neq 0 \tag{2.121}$$

and, moreover, on account of the assumption $\ell \in \mathcal{P}_{ab}$, either

$$u(t) > 0 \quad \text{for} \quad t \in [a,b] \tag{2.122}$$

or

$$u(t) < 0 \quad \text{for} \quad t \in [a,b].$$

Thus, from (1.1_0), (2.1), and (2.120) we have

$$|u(t)|' = \ell(|u|)(t) \quad \text{for} \quad t \in [a,b], \tag{2.123}$$

$$|\lambda u(a)| - |\mu u(b)| = \operatorname{sgn} u(a). \tag{2.124}$$

It follows from (2.123) that $|u(t)|' \geq 0$ for $t \in [a,b]$ and therefore, since ℓ is an $a-$Volterra operator, we have

$$|u(t)|' \leq \ell(1)(t)|u(t)| \quad \text{for} \quad t \in [a,b].$$

The last inequality yields

$$|u(t)| \leq |u(a)| \exp\left(\int_a^t \ell(1)(s)ds\right) \quad \text{for} \quad t \in [a,b],$$

whence, in view of (2.18), we get

$$|\mu u(b)| \leq |\lambda u(a)|.$$

This inequality, together with (2.121) and (2.124), implies $u(a) > 0$ and so (2.122) holds. Therefore, u is a positive solution of (1.1_0) and $|\lambda|u(a) > |\mu|u(b)$, thus, according to Theorem 2.1 (see p. 17), we have $\ell \in V_{ab}^{+}(\lambda,\mu)$.

Example 2.1. Let $0 \neq |\mu| < |\lambda|$, $\varepsilon > 0$, and let $p \in L\big([a,b]; R_+\big)$ be such that

$$|\mu| \exp\left(\int_a^b p(s)ds\right) = |\lambda| + \varepsilon. \tag{2.125}$$

It is clear that the operator ℓ defined by

$$\ell(v)(t) \stackrel{\text{def}}{=} p(t)v(t) \quad \text{for} \quad t \in [a,b] \tag{2.126}$$

is an $a-$Volterra operator and satisfies

$$|\mu| \exp\left(\int_a^b \ell(1)(s)ds\right) = |\lambda| + \varepsilon.$$

According to (2.125), the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution. Obviously, the function

$$u(t) = -\exp\left(\int_a^t p(s)ds\right) \quad \text{for} \quad t \in [a,b]$$

is a solution of the problem

$$u'(t) = \ell(u)(t), \qquad \lambda u(a) + \mu u(b) = \varepsilon \operatorname{sgn} \lambda.$$

On the other hand, $u(t) < 0$ for $t \in [a,b]$, and so $\ell \notin V_{ab}^+(\lambda,\mu)$.

On Remark 2.5. Suppose that $|\mu| < |\lambda|$, $\ell \in \mathcal{P}_{ab}$, (2.19) holds, and the problem (1.1_0), (1.2_0) has only the trivial solution. According to Theorem 1.1 (see p. 14), the problem (1.1_0), (2.120) has a unique solution u. Assume that u admits negative values. Put

$$m = \max\{-u(t) : t \in [a,b]\} \tag{2.127}$$

and choose $t_0 \in [a,b]$ such that $u(t_0) = -m$. The integration of (1.1_0) from a to t_0 and from t_0 to b, in view of (2.8), (2.127) and the assumption $|\mu| < |\lambda|$, yields

$$m + u(a) = -\int_a^{t_0} \ell(u)(s)ds \leq m\int_a^{t_0} \ell(1)(s)ds,$$

$$-\left|\frac{\mu}{\lambda}\right| u(b) - \left|\frac{\mu}{\lambda}\right| m = -\left|\frac{\mu}{\lambda}\right| \int_{t_0}^b \ell(u)(s)ds \leq m\int_{t_0}^b \ell(1)(s)ds.$$

Summing the last two inequalities and taking into account (2.1) and (2.120), we obtain

$$m\Big(1 - \Big|\frac{\mu}{\lambda}\Big|\Big) + \frac{1}{|\lambda|} \leq m \int_a^b \ell(1)(s)ds.$$

The last inequality, together with (2.19), yields the contradiction $m < m$.

Consequently, $u(t) \geq 0$ for $t \in [a,b]$ and, in view of (1.1_0) and (2.8), $u'(t) \geq 0$ for $t \in [a,b]$. On the other hand, since $\lambda \neq 0$, it follows from (2.1) and (2.120) that

$$u(a) > 0, \qquad |\lambda| u(a) = 1 + |\mu| u(b).$$

Hence, $u(t) > 0$ for $t \in [a,b]$ and $|\lambda| u(a) > |\mu| u(b)$. Thus, according to Theorem 2.1 (see p. 17), we have $\ell \in V_{ab}^+(\lambda, \mu)$.

Example 2.2. Let $|\mu| < |\lambda|$, $\varepsilon > 0$, and let $p \in L\big([a,b]; R_+\big)$ be such that

$$|\lambda| \int_a^b p(s)ds = |\lambda| - |\mu| + \varepsilon.$$

It is clear that the operator ℓ defined by

$$\ell(v)(t) \stackrel{\text{def}}{=} p(t)v(b) \quad \text{for} \quad t \in [a,b] \tag{2.128}$$

satisfies

$$|\lambda| \int_a^b \ell(1)(s)ds = |\lambda| - |\mu| + \varepsilon.$$

Moreover, the problem (1.1_0), (1.2_0) has only the trivial solution. Indeed, the integration of (1.1_0) from a to b, in view of (1.2_0) and (2.1), implies $|\lambda| u(b) = u(b)(|\lambda| + \varepsilon)$, i.e., $u(b) = 0$. Hence, by (1.1_0) we get $u'(t) = 0$ for $t \in [a,b]$ and so $u \equiv 0$.

On the other hand, the function

$$u(t) = \varepsilon - |\mu| - |\lambda| \int_a^t p(s)ds \quad \text{for} \quad t \in [a,b]$$

is a solution of the problem

$$u'(t) = \ell(u)(t), \qquad \lambda u(a) + \mu u(b) = \lambda\varepsilon$$

with $u(b) = -|\lambda|$. Therefore, $\ell \notin V_{ab}^+(\lambda, \mu)$.

Example 2.3. Let $0 \neq |\mu| < |\lambda|$ and let $p \in L\big([a,b];R_+\big)$ be such that

$$|\mu| \exp\left(\int_a^b p(s)ds\right) = |\lambda|.$$

It is clear that the operator ℓ defined by (2.126) is an $a-$Volterra operator and satisfies

$$|\mu| \exp\left(\int_a^b \ell(1)(s)ds\right) = |\lambda|.$$

On the other hand, the function

$$u(t) = \exp\left(\int_a^t p(s)ds\right) \quad \text{for} \quad t \in [a,b]$$

is a nontrivial solution of the problem (1.1_0), (1.2_0), and so $\ell \notin V_{ab}^{+}(\lambda,\mu)$.

Example 2.4. Let $|\mu| < |\lambda|$, and let $p \in L\big([a,b];R_+\big)$ be such that

$$|\lambda| \int_a^b p(s)ds = |\lambda| - |\mu|.$$

Obviously,

$$|\mu| \exp\left(\int_a^b \ell(1)(s)ds\right) + |\lambda| \int_a^b \overline{\ell}(1)(s) \exp\left(\int_s^b \ell(1)(\xi)d\xi\right) ds = |\lambda|,$$

where ℓ is defined by (2.128) and

$$\overline{\ell}(v)(t) \stackrel{\text{def}}{=} p(t)v(b) \int_t^b p(s)ds \quad \text{for} \quad t \in [a,b].$$

It is also evident that the inequalities (2.13) (with $\alpha = 1$, $k = 2$, and $m = 3$) and (2.16) are fulfilled.

On the other hand, the function

$$u(t) = |\mu| + |\lambda| \int_a^t p(s)ds \quad \text{for} \quad t \in [a,b]$$

is a nontrivial solution of the problem (1.1_0), (1.2_0) and so $\ell \notin V_{ab}^{+}(\lambda,\mu)$.

Example 2.5. Let $|\mu| \le |\lambda|$, $b_1 \in]a,b[$, and let $\varepsilon \in]0,2[$. Choose $\delta \in]0,\varepsilon]$ such that

$$\frac{\delta}{2}\, e^{\frac{\delta}{2}} + \frac{\delta}{2}\, e^{1+\frac{\delta}{2}} + 1 - e^{\frac{\delta}{2}} \le \varepsilon,$$

and let $g \in L\big([a,b];R_+\big)$ be such that $g \not\equiv 0$ in $[b_1-\delta_0, b_1[$ for some $\delta_0 > 0$, and

$$\int_a^{b_1} g(s)ds = \frac{\delta}{2}, \qquad \int_{b_1}^{b} g(s)ds = 1 + \frac{\delta}{2}.$$

Put

$$\nu(t) = \begin{cases} a & \text{for } \ t \in [a,b_1[\\ b_1 & \text{for } \ t \in [b_1,b] \end{cases}, \quad \gamma(t) = \begin{cases} \frac{\delta}{2} - \int\limits_a^t g(s)ds & \text{for } \ t \in [a,b_1[\\ 0 & \text{for } \ t \in [b_1,b] \end{cases}.$$

Obviously, the assumptions of Theorem 2.2 are fulfilled except of (2.21), instead of which the condition (2.22) is satisfied, where ℓ is defined by

$$\ell(v)(t) \overset{\text{def}}{=} -g(t)v(\nu(t)) \quad \text{for} \quad t \in [a,b]. \tag{2.129}$$

Evidently,

$$\int_a^b |\ell(1)(s)|ds = 1 + \delta \le 1 + \varepsilon$$

and, moreover,

$$\int_a^b \big|\widetilde{\ell}(1)(s)\big| \exp\left(\int_a^s |\ell(1)(\xi)|d\xi\right) ds = 1 + \frac{\delta}{2}\, e^{\frac{\delta}{2}} + \frac{\delta}{2}\, e^{1+\frac{\delta}{2}} + 1 - e^{\frac{\delta}{2}} \le 1 + \varepsilon,$$

where $\widetilde{\ell}$ is defined by (2.26).

On the other hand, the function

$$u(t) = \begin{cases} 1 - \int\limits_a^t g(s)ds & \text{for } \ t \in [a,b_1[\\ \left(1 - \frac{\delta}{2}\right)\left(1 - \int\limits_{b_1}^t g(s)ds\right) & \text{for } \ t \in [b_1,b] \end{cases}$$

is a solution of the problem

$$u'(t) = \ell(u)(t), \qquad \lambda u(a) + \mu u(b) = \lambda - \frac{\mu\delta}{2}\left(1 - \frac{\delta}{2}\right)$$

with $u(b) = -\frac{\delta}{2}(1 - \frac{\delta}{2}) < 0$. Therefore, $\ell \notin V_{ab}^+(\lambda,\mu)$.

On Remark 2.10. Let $0 \neq |\mu| \leq |\lambda|$. Below, for every $x_0, y_0 \in R_+$ such that $(x_0, y_0) \notin A$ the functions $p \in L\big([a,b];R\big)$, $q \in L\big([a,b];R_+\big)$ and $\tau \in \mathcal{M}_{ab}$ are constructed such that

$$x_0 = \int\limits_a^b [p(s)]_+ ds, \qquad y_0 = \int\limits_a^b [p(s)]_- ds, \tag{2.130}$$

and the problem

$$u'(t) = p(t)u(\tau(t)) + q(t), \qquad \lambda u(a) + \mu u(b) = 0 \tag{2.131}$$

has a solution, which is not nonnegative. Thus, according to Definition 2.1 (see p. 15), we have $\ell \notin V_{ab}^+(\lambda, \mu)$, where $\ell = \ell_0 - \ell_1$ with

$$\ell_0(v)(t) \overset{\text{def}}{=} [p(t)]_+ v(\tau(t)), \qquad \ell_1(v)(t) \overset{\text{def}}{=} [p(t)]_- v(\tau(t)). \tag{2.132}$$

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin A$, then (x_0, y_0) belongs at least to one of the following sets:

$$A_1 = \left\{(x,y) \in R_+ \times R_+ \ : \ \left|\frac{\mu}{\lambda}\right| < y\right\},$$
$$A_2 = \left\{(x,y) \in R_+ \times R_+ \ : \ y \leq \left|\frac{\mu}{\lambda}\right|, \ 1 - \frac{|\mu|}{|\mu| y + |\lambda|} \leq x\right\}.$$

Let $(x_0, y_0) \in A_1$. Put $a = 0$, $b = 3$,

$$p(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,2[\\ x_0 & \text{for } t \in [2,3] \end{cases}, \qquad \tau(t) = \begin{cases} 2 & \text{for } t \in [0,2[\\ 1 & \text{for } t \in [2,3] \end{cases},$$

$$q(t) = \begin{cases} 0 & \text{for } t \in [0,1[\\ |\lambda|(1+y_0) - |\mu| & \text{for } t \in [1,2[\\ x_0(|\lambda| y_0 - |\mu|) & \text{for } t \in [2,3] \end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (2.131) has the solution

$$u(t) = \begin{cases} |\mu| - |\lambda| y_0 t & \text{for } t \in [0,1[\\ \big(|\lambda|(1+y_0) - |\mu|\big)(t-2) + |\lambda| & \text{for } t \in [1,2[\\ |\lambda| & \text{for } t \in [2,3] \end{cases}$$

with $u(1) = -(|\lambda| y_0 - |\mu|) < 0$.

Let $(x_0, y_0) \in A_2$. Put $a = 0$, $b = 3$,

$$p(t) = \begin{cases} 1 - \frac{|\mu|}{|\mu| y_0 + |\lambda|} & \text{for } t \in [0,1[\\ x_0 - 1 + \frac{|\mu|}{|\mu| y_0 + |\lambda|} & \text{for } t \in [1,2[\\ -y_0 & \text{for } t \in [2,3] \end{cases}, \quad \tau(t) = \begin{cases} 1 & \text{for } t \in [0,1[\\ 3 & \text{for } t \in [1,2[\\ 0 & \text{for } t \in [2,3] \end{cases},$$

$$q(t) = \begin{cases} 0 & \text{for } t \in [0,1[\cup [2,3] \\ |\lambda| \left(x_0 - 1 + \frac{|\mu|}{|\mu| y_0 + |\lambda|} \right) & \text{for } t \in [1,2[\end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (2.131) has the solution

$$u(t) = \begin{cases} -(|\mu| y_0 + |\lambda| - |\mu|)t - |\mu| & \text{for } t \in [0,1[\\ -(|\mu| y_0 + |\lambda|) & \text{for } t \in [1,2[\\ |\mu| y_0 (t-3) - |\lambda| & \text{for } t \in [2,3] \end{cases}$$

with $u(3) = -|\lambda| < 0$.

Example 2.6. Let $|\mu| < |\lambda|$, $\varepsilon \in]0,1[$, and let $p, g \in L([a,b]; R_+)$ be such that

$$\int_a^b p(s)ds = (1+\varepsilon) \frac{|\lambda| - |\mu|}{|\lambda|}, \tag{2.133}$$

$$\int_a^b g(s)ds < 1, \qquad |\mu| \int_a^b g(s)ds < \varepsilon\, (|\lambda| - |\mu|). \tag{2.134}$$

Let $\ell = \ell_0 - \ell_1$, where

$$\ell_0(v)(t) \stackrel{\text{def}}{=} p(t)v(b), \qquad \ell_1(v)(t) \stackrel{\text{def}}{=} g(t)v(a) \quad \text{for} \quad t \in [a,b]. \tag{2.135}$$

According to (2.133), (2.134), Remark 2.5 (see p. 19), and Theorem 2.3 (see p. 20), we find

$$(1-\varepsilon)\ell_0 \in V_{ab}^+(\lambda, \mu), \qquad -\ell_1 \in V_{ab}^+(\lambda, \mu).$$

Note also that the problem (1.1_0), (1.2_0) has only the trivial solution. Indeed, the integration of (1.1_0) from a to b, on account of (1.2_0), (2.1), and (2.133), yields

$$u(b) \left(\varepsilon \frac{|\lambda| - |\mu|}{|\lambda|} - \|g\|_L \left| \frac{\mu}{\lambda} \right| \right) = 0.$$

Now, by (2.134) we get $u(b) = 0$. Consequently, $u(a) = 0$, $u'(t) = 0$ for $t \in [a,b]$ and so $u \equiv 0$. Therefore, the problem (1.1_0), (2.120) has a unique solution u.

On the other hand, the integration of (1.1_0) from a to b, by virtue of (2.133), implies

$$u(b) - u(a) = u(b)(1+\varepsilon)\,\frac{|\lambda| - |\mu|}{|\lambda|} - u(a)\|g\|_L,$$

whence, in view of (2.1) and (2.120), we get

$$u(b)\left(\varepsilon\,\frac{|\lambda| - |\mu|}{|\lambda|} - \|g\|_L\left|\frac{\mu}{\lambda}\right|\right) = \frac{1}{|\lambda|}\big(\|g\|_L - 1\big).$$

By (2.134) we obtain $u(b) < 0$ and so $\ell \notin V_{ab}^{+}(\lambda,\mu)$.

Example 2.7. Let $|\mu| < |\lambda|$, $\varepsilon \in \,]0,1[\,$, and let $p, g \in L\big([a,b];R_+\big)$ be such that

$$\int_a^b p(s)ds < \frac{|\lambda| - |\mu|}{|\lambda|}, \qquad \int_a^b g(s)ds = 1+\varepsilon. \tag{2.136}$$

Let $\ell = \ell_0 - \ell_1$, where ℓ_0, ℓ_1 are defined by (2.135). According to (2.136), Remark 2.5 (see p. 19), and Theorem 2.3 (see p. 20), we find

$$\ell_0 \in V_{ab}^{+}(\lambda,\mu), \qquad -(1-\varepsilon)\ell_1 \in V_{ab}^{+}(\lambda,\mu).$$

Note also that the problem (1.1_0), (1.2_0) has only the trivial solution. Indeed, the integration of (1.1_0) from a to b, on account of (1.2_0), (2.1), and (2.136), yields

$$u(b)\left(\|p\|_L - 1 - \varepsilon\left|\frac{\mu}{\lambda}\right|\right) = 0$$

Now, by (2.136) we get $u(b) = 0$. Consequently, $u(a) = 0$, $u'(t) = 0$ for $t \in [a,b]$ and so $u \equiv 0$. Therefore, the problem (1.1_0), (2.120) has a unique solution u.

On the other hand, the integration of (1.1_0) from a to b, by virtue of (2.136), implies

$$u(b) - u(a) = u(b)\|p\|_L - u(a)(1+\varepsilon),$$

whence, in view of (2.1) and (2.120), we get

$$\frac{\varepsilon}{|\lambda|} = u(b)\left(\|p\|_L - 1 - \varepsilon\left|\frac{\mu}{\lambda}\right|\right).$$

By (2.136) we obtain $u(b) < 0$ and so $\ell \notin V_{ab}^{+}(\lambda,\mu)$.

Example 2.8. Let $|\mu| \geq |\lambda| \neq 0$, $\varepsilon > 0$ and let $g \in L\big([a,b]; R_+\big)$ be such that

$$\int_a^b g(s)ds = 1 + \varepsilon.$$

It is clear that the operator ℓ defined by

$$\ell(v)(t) \overset{\text{def}}{=} -g(t)v(a) \quad \text{for} \quad t \in [a,b] \tag{2.137}$$

satisfies

$$\int_a^b |\ell(1)(s)|ds = 1 + \varepsilon.$$

Put

$$u(t) = -\varepsilon + \int_t^b g(s)ds \quad \text{for} \quad t \in [a,b].$$

Obviously, there exists $t_0 \in]a,b[$ such that $u(t_0) = 0$. Define the function $\gamma \in \widetilde{C}\big([a,b]; R_+\big)$ by

$$\gamma(t) = \begin{cases} -\varepsilon + \int_t^b g(s)ds & \text{for} \quad t \in [a, t_0[\\ 0 & \text{for} \quad t \in [t_0, b] \end{cases}.$$

It is not difficult to verify that γ satisfies the inequalities (2.10) and (2.11).

On the other hand, u is a solution of the problem

$$u'(t) = \ell(u)(t), \qquad \lambda u(a) + \mu u(b) = \lambda - \mu\varepsilon$$

with $(\operatorname{sgn}\lambda - \operatorname{sgn}\mu)(\lambda - \mu\varepsilon) \geq 0$. However, $u(b) = -\varepsilon < 0$ and therefore, $\ell \notin V_{ab}^+(\lambda, \mu)$.

Example 2.9. Let $|\mu| > |\lambda| \neq 0$ and let $g \in L\big([a,b]; R_+\big)$ be such that

$$|\mu| \int_a^b g(s)ds = |\mu| - |\lambda|.$$

It is clear that the condition (2.23) is fulfilled, where ℓ is defined by (2.137). Moreover, the function

$$\beta(t) = |\lambda| + |\mu| \int_t^b g(s)ds \quad \text{for} \quad t \in [a,b]$$

satisfies the inequalities (2.32) and (2.33) and the function $\gamma \equiv \beta$ satisfies the inequalities (2.10) and (2.31).

On the other hand, β is a nontrivial solution of the problem (1.1_0), (1.2_0). Therefore, $\ell \notin V_{ab}^{+}(\lambda, \mu)$.

Example 2.10. Let $|\mu| \geq |\lambda| \neq 0$, $\varepsilon > 0$ and let $b_1 \in \,]a, b[$ be an arbitrarily fixed point. Choose $g, q \in L\big([a, b]; R_+\big)$ and $c \in R$ such that (2.2) holds, $g \not\equiv 0$ in $[b_1 - \delta_0, b_1[$ for some $\delta_0 > 0$, and

$$\int_a^{b_1} g(s)ds = 1, \qquad \int_{b_1}^{b} g(s)ds > 1,$$

$$q(t) = 0 \quad \text{for} \quad t \in [b_1, b], \qquad \int_a^{b_1} q(s)ds = \frac{\varepsilon}{\int_{b_1}^{b} g(s)ds - 1}.$$

Put

$$\beta(t) = \begin{cases} \int_t^{b_1} g(s)ds & \text{for } t \in [a, b_1[\\ 0 & \text{for } t \in [b_1, b] \end{cases}, \qquad \nu(t) = \begin{cases} a & \text{for } t \in [a, b_1[\\ b_1 & \text{for } t \in [b_1, b] \end{cases},$$

$$\gamma(t) = \begin{cases} \int_{b_1}^{b} g(s)ds & \text{for } t \in [a, b_1[\\ \int_t^{b} g(s)ds & \text{for } t \in [b_1, b] \end{cases}.$$

Let the operator ℓ be defined by (2.129). Obviously, the function γ satisfies the inequalities (2.10) and (2.11). It is also evident that ℓ is an $a-$Volterra operator and the function β satisfies the inequalities (2.33) and (2.34).

On the other hand, the function

$$u(t) = \begin{cases} \frac{\varepsilon}{\int_{b_1}^{b} g(s)ds - 1} + \frac{c + \mu\varepsilon}{\lambda} \int_t^{b_1} g(s)ds - \int_t^{b_1} q(s)ds & \text{for } t \in [a, b_1[\\ -\varepsilon + \frac{\varepsilon}{\int_{b_1}^{b} g(s)ds - 1} \int_t^{b} g(s)ds & \text{for } t \in [b_1, b] \end{cases}$$

is a solution of the problem (1.1), (1.2) with $u(b) = -\varepsilon < 0$. Consequently, $\ell \notin V_{ab}^{+}(\lambda, \mu)$.

On Remark 2.15. Let $|\mu| \geq |\lambda| \neq 0$. Below, for every $x_0, y_0 \in R_+$ such that $(x_0, y_0) \notin B$ the functions $p \in L\big([a,b]; R\big)$, $q \in L\big([a,b]; R_+\big)$, and $\tau \in \mathcal{M}_{ab}$ are constructed such that (2.130) holds and the problem (2.131) has a solution, which is not nonnegative. Thus, according to Definition 2.1 (see p. 15), we find $\ell \notin V_{ab}^{+}(\lambda, \mu)$, where $\ell = \ell_0 - \ell_1$ with ℓ_0, ℓ_1 given by (2.132).

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin B$, then (x_0, y_0) belongs at least to one of the following sets:

$$
\begin{aligned}
B_1 &= \{(x,y) \in R_+ \times R_+ \ : \ 1 < y\}, \\
B_2 &= \left\{(x,y) \in R_+ \times R_+ \ : \ \left|\frac{\lambda}{\mu}\right| - \frac{1}{1+y} \leq x, \ \frac{|\mu| - |\lambda|}{|\lambda|} < y \leq 1\right\}, \\
B_3 &= \left\{(x,y) \in R_+ \times R_+ \ : \ y \leq \frac{|\mu| - |\lambda|}{|\lambda|}\right\}.
\end{aligned}
$$

Let $(x_0, y_0) \in B_1$. Put $a = 0$, $b = 3$,

$$
p(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,2[\\ x_0 & \text{for } t \in [2,3] \end{cases}, \qquad \tau(t) = \begin{cases} 0 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,3] \end{cases},
$$

$$
q(t) = \begin{cases} 0 & \text{for } t \in [0,1[\\ |\lambda| + |\mu|(y_0 - 1) & \text{for } t \in [1,2[\\ |\mu| x_0 (y_0 - 1) & \text{for } t \in [2,3] \end{cases}.
$$

It is not difficult to verify that (2.130) holds, and the problem (2.131) has the solution

$$
u(t) = \begin{cases} |\mu| - |\mu| y_0 t & \text{for } t \in [0,1[\\ \big(|\lambda| + |\mu|(y_0 - 1)\big)(t-2) + |\lambda| & \text{for } t \in [1,2[\\ |\lambda| & \text{for } t \in [2,3] \end{cases}
$$

with $u(1) = -|\mu|(y_0 - 1) < 0$.

Let $2|\lambda| > |\mu| \geq |\lambda|$, $(x_0, y_0) \in B_2$. Put $a = 0$, $b = 3$,

$$
p(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[\\ \left|\frac{\lambda}{\mu}\right| - \frac{1}{1+y_0} & \text{for } t \in [1,2[\\ x_0 - \left|\frac{\lambda}{\mu}\right| + \frac{1}{1+y_0} & \text{for } t \in [2,3] \end{cases}, \qquad \tau(t) = \begin{cases} 1 & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,2[\\ 2 & \text{for } t \in [2,3] \end{cases},
$$

$$q(t) = \begin{cases} 0 & \text{for } t \in [0,2[\\ |\lambda| \left(x_0 - \left|\frac{\lambda}{\mu}\right| + \frac{1}{1+y_0} \right) & \text{for } t \in [2,3[\end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (2.131) has the solution

$$u(t) = \begin{cases} \frac{|\mu| y_0}{1+y_0}\, t - |\mu| & \text{for } t \in [0,1[\\ \left(|\lambda| - \frac{|\mu|}{1+y_0} \right)(2-t) - |\lambda| & \text{for } t \in [1,2[\\ -|\lambda| & \text{for } t \in [2,3] \end{cases}$$

with $u(0) = -|\mu| < 0$.

Let $(x_0, y_0) \in B_3$. Put $a = 0$, $b = 2$, $\tau \equiv 2$,

$$p(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[\\ x_0 & \text{for } t \in [1,2] \end{cases}, \quad q(t) = \begin{cases} |\mu| - |\lambda| - |\lambda| y_0 & \text{for } t \in [0,1[\\ |\lambda| x_0 & \text{for } t \in [1,2] \end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (2.131) has the solution

$$u(t) = \begin{cases} (|\mu| - |\lambda|)t - |\mu| & \text{for } t \in [0,1[\\ -|\lambda| & \text{for } t \in [1,2] \end{cases}$$

with $u(0) = -|\mu| < 0$.

§3. Differential Inequalities for EDA

In this section, the results from §2 will be concretized for the case, when the operator $\ell \in \mathcal{L}_{ab}$ has one of the following forms:

$$\ell(v)(t) \stackrel{\text{def}}{=} \sum_{k=1}^{m} p_k(t)v(\tau_k(t)) \quad \text{for} \quad t \in [a,b], \tag{3.1}$$

$$\ell(v)(t) \stackrel{\text{def}}{=} -\sum_{k=1}^{m} g_k(t)v(\nu_k(t)) \quad \text{for} \quad t \in [a,b], \tag{3.2}$$

$$\ell(v)(t) \stackrel{\text{def}}{=} \sum_{k=1}^{m} \Big(p_k(t)v(\tau_k(t)) - g_k(t)v(\nu_k(t))\Big) \quad \text{for} \quad t \in [a,b], \tag{3.3}$$

where $p_k, g_k \in L\big([a,b]; R_+\big)$, $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, and $m \in N$.

We will also assume that the inequality (2.1) holds and $|\lambda| + |\mu| \neq 0$. Furthermore, if $\lambda = -\mu$, then the operator $\ell \in \mathcal{L}_{ab}$ is supposed to be nontrivial, i.e., $\ell(1) \not\equiv 0$.

In what follows we will use the notation

$$p_0(t) = \sum_{j=1}^{m} p_j(t), \qquad g_0(t) = \sum_{j=1}^{m} g_j(t) \quad \text{for} \quad t \in [a,b].$$

3.1. On the Set $V_{ab}^{+}(\lambda, \mu)$

In the case, where $|\mu| \leq |\lambda|$, the following assertions hold.

Theorem 3.1. *Let $|\mu| < |\lambda|$, $p_k \in L\big([a,b]; R_+\big)$, $\tau_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, and let at least one of the following items be fulfilled:*

a) $\tau_k(t) \leq t$ for $t \in [a,b]$ $(k = 1, \dots, m)$ and

$$|\mu| \exp\left(\int_a^b p_0(s)ds\right) < |\lambda|; \tag{3.4}$$

b) there exists $\alpha \in \,]0,1[$ such that on $[a,b]$ the inequality

$$\frac{|\mu|}{|\lambda| - |\mu|} \int_a^b \widetilde{p}(s)ds + \int_a^t \widetilde{p}(s)ds \leq (\alpha - \widetilde{p}_0)\left(\widetilde{p}_0 + \int_a^t p_0(s)ds\right) \tag{3.5}$$

holds, where

$$\widetilde{p}(t)=\sum_{k=1}^{m}p_k(t)\left(\int_a^{\tau_k(t)}p_0(s)ds\right)\quad \text{for}\quad t\in[a,b],$$

$$\widetilde{p}_0=\frac{|\mu|}{|\lambda|-|\mu|}\int_a^b p_0(s)ds;$$

c)

$$|\mu|\exp\left(\int_a^b p_0(s)ds\right)+ \\ +|\lambda|\int_a^b\sum_{k=1}^{m}p_k(s)\sigma_k(s)\left(\int_s^{\tau_k(s)}p_0(\xi)d\xi\right)\exp\left(\int_s^b p_0(\eta)d\eta\right)ds<|\lambda|, \tag{3.6}$$

where

$$\sigma_k(t)=\frac{1}{2}(1+\operatorname{sgn}(\tau_k(t)-t))\quad \text{for}\quad t\in[a,b]\qquad (k=1,\dots,m).$$

Then the operator ℓ *defined by* (3.1) *belongs to the set* $V_{ab}^{+}(\lambda,\mu)$.

Remark 3.1. Examples 2.3 and 2.4 (see p. 54) also show that the assumption $\alpha\in\,]0,1[$ in Theorem 3.1 b) cannot be replaced by the assumption $\alpha\in\,]0,1]$ and the strict inequalities (3.4) and (3.6) cannot be replaced by the nonstrict ones.

Theorem 3.2. *Let* $0\neq|\mu|<|\lambda|$, $p_k\in L\big([a,b];R_+\big)$, $\tau_k\in\mathcal{M}_{ab}$ $(k=1,\dots,m)$, $p_0\not\equiv0$, *and let there exist* $x\in\left]0,\ln\left|\frac{\lambda}{\mu}\right|\right[$ *such that*

$$\operatorname{ess\,sup}\left\{\int_t^{\tau_k(t)}p_0(s)ds: t\in[a,b]\right\}<\eta(x)\qquad (k=1,\dots,m), \tag{3.7}$$

where

$$\eta(x)=\frac{\int_a^b p_0(s)ds}{x}\left(x+\ln\frac{\big(|\lambda|-|\mu|\big)x}{|\lambda|(e^x-1)\int_a^b p_0(s)ds}\right). \tag{3.8}$$

Then the operator ℓ defined by (3.1) *belongs to the set* $V_{ab}^{+}(\lambda,\mu)$.

Corollary 3.1. *Let* $0 \neq |\mu| < |\lambda|$, $p_k \in L\big([a,b];R_+\big)$, $\tau_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$, $p_0 \not\equiv 0$, *and*

$$\operatorname{ess\,sup}\left\{\int\limits_{t}^{\tau_k(t)} p_0(s)ds : t \in [a,b]\right\} < \frac{\int_a^b p_0(s)ds}{\ln\left|\frac{\lambda}{\mu}\right|} \ln \frac{\ln\left|\frac{\lambda}{\mu}\right|}{\int_a^b p_0(s)ds}$$

for $k = 1,\dots,m$. *Then the operator ℓ defined by* (3.1) *belongs to the set* $V_{ab}^{+}(\lambda,\mu)$.

Theorem 3.3. *Let* $|\mu| \leq |\lambda|$, $g_k \in L\big([a,b];R_+\big)$, $\nu_k \in \mathcal{M}_{ab}$, $\nu_k(t) \leq t$ *for* $t \in [a,b]$ $(k = 1,\dots,m)$, *and let at least one of the following items be fulfilled:*

a)

$$\int\limits_a^b g_0(s)ds \leq 1; \tag{3.9}$$

b)

$$\int\limits_a^b \sum_{k=1}^m g_k(s) \int\limits_{\nu_k(s)}^{s} \sum_{i=1}^m g_i(\xi) \exp\left(\int\limits_{\nu_i(\xi)}^{s} g_0(\eta)d\eta\right) d\xi ds \leq 1; \tag{3.10}$$

c) $g_0 \not\equiv 0$ *and*

$$\operatorname{ess\,sup}\left\{\int\limits_{\nu_k(t)}^{t} g_0(s)ds : t \in [a,b]\right\} < \eta^* \qquad (k = 1,\dots,m), \tag{3.11}$$

where

$$\eta^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp\left(x\int_a^b g_0(s)ds\right) - 1}\right) : x > 0\right\}. \tag{3.12}$$

Then the operator ℓ *defined by* (3.2) *belongs to the set* $V_{ab}^{+}(\lambda,\mu)$.

Remark 3.2. Example 2.5 (see p. 55) also shows that the inequalities (3.9) and (3.10) in Theorem 3.3 cannot be replaced by the inequalities

$$\int_a^b g_0(s)ds \le 1+\varepsilon$$

and

$$\int_a^b \sum_{k=1}^m g_k(s) \int_{\nu_k(s)}^s \sum_{i=1}^m g_i(\xi) \exp\left(\int_{\nu_i(\xi)}^s g_0(\eta)d\eta\right) d\xi ds \le 1+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

Theorem 3.4. *Let* $0 \ne |\mu| \le |\lambda|$ *and* $p_k, g_k \in L\big([a,b];R_+\big)$ $(k=1,\dots,m)$. *Let, moreover,*

$$\int_a^b p_0(s)ds < 1, \tag{3.13}$$

$$\frac{\int_a^b p_0(s)ds}{1-\int_a^b p_0(s)ds} - \frac{|\lambda|-|\mu|}{|\mu|} < \int_a^b g_0(s)ds \le \left|\frac{\mu}{\lambda}\right|. \tag{3.14}$$

Then the operator ℓ *defined by* (3.3) *belongs to the set* $V_{ab}^{+}(\lambda,\mu)$.

Remark 3.3. The examples constructed in Subsection 2.4 (see On Remark 2.10, p. 56) also show that neither one of the inequalities in (3.13) and (3.14) can be weakened.

Theorem 3.5. *Let* $|\mu| < |\lambda|$, $p_k, g_k \in L\big([a,b];R_+\big)$, *and* $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k=1,\dots,m)$. *Let, moreover, the functions* p_k, τ_k $(k=1,\dots,m)$ *satisfy at least one of the conditions a), b) or c) in Theorem 3.1 or the assumptions of Theorem 3.2, while the functions* g_k, ν_k $(k=1,\dots,m)$ *satisfy* $\nu_k(t) \le t$ *for* $t \in [a,b]$ $(k=1,\dots,m)$ *and at least one of the conditions a), b) or c) in Theorem 3.3. Then the operator* ℓ *defined by* (3.3) *belongs to the set* $V_{ab}^{+}(\lambda,\mu)$.

Remark 3.4. According to the optimality of Theorems 2.5, 3.1, and 3.3, Theorem 3.5 is also nonimprovable in a certain sense.

In the case, where $|\mu| \geq |\lambda|$, the following statements hold.

Theorem 3.6. *Let* $|\mu| > |\lambda| \neq 0$, $g_k \in L\big([a,b];R_+\big)$, $\nu_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$, $g_0 \not\equiv 0$,

$$\int_a^b g_0(s)ds \leq 1, \tag{3.15}$$

and let there exist $x \in \big[\ln\big|\frac{\mu}{\lambda}\big|, +\infty\big[$ *such that*

$$\operatorname{ess\,inf}\left\{\int_{\nu_k(t)}^{t} g_0(s)ds : t \in [a,b]\right\} > \omega(x) \qquad (k = 1,\dots,m), \tag{3.16}$$

where

$$\omega(x) = \frac{\int_a^b g_0(s)ds}{x}\left(x + \ln\frac{\big(|\mu|-|\lambda|\big)x}{|\mu|(e^x-1)\int_a^b g_0(s)ds}\right). \tag{3.17}$$

Then the operator ℓ *defined by* (3.2) *belongs to the set* $V_{ab}^{+}(\lambda,\mu)$.

Corollary 3.2. *Let* $|\mu| > |\lambda| \neq 0$, $g_k \in L\big([a,b];R_+\big)$, $\nu_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$, $g_0 \not\equiv 0$, *the inequality* (3.15) *hold, and*

$$\operatorname{ess\,inf}\left\{\int_{\nu_k(t)}^{t} g_0(s)ds : t \in [a,b]\right\} > \frac{\int_a^b g_0(s)ds}{\ln\big|\frac{\mu}{\lambda}\big|}\ln\frac{\ln\big|\frac{\mu}{\lambda}\big|}{\int_a^b g_0(s)ds}$$

for $k = 1,\dots,m$. *Then the operator* ℓ *defined by* (3.2) *belongs to the set* $V_{ab}^{+}(\lambda,\mu)$.

Remark 3.5. Example 2.8 (see p. 59) also shows that the inequality (3.15) in Theorem 3.6 and Corollary 3.2 cannot be replaced by the inequality

$$\int_a^b g_0(s)ds \leq 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

Theorem 3.7. *Let $|\mu| > |\lambda| \neq 0$, $g_k \in L\big([a,b];R_+\big)$, $\nu_k \in \mathcal{M}_{ab}$,*

$$\nu_k(t) \leq t \quad \text{for} \quad t \in [a,b] \qquad (k = 1,\dots,m), \tag{3.18}$$

$g_0 \not\equiv 0$, and let the condition (3.11) hold, where η^* is defined by (3.12). If, moreover,

$$|\lambda| \exp\left(\int_a^b g_0(s)ds\right) > |\mu|\,, \tag{3.19}$$

then the operator ℓ defined by (3.2) belongs to the set $V_{ab}^+(\lambda,\mu)$.

Corollary 3.3. *Let $|\mu| > |\lambda| \neq 0$, $g_k \in L\big([a,b];R_+\big)$, $\nu_k \in \mathcal{M}_{ab}$, $\nu_k(t) \leq t$ for $t \in [a,b]$ $(k = 1,\dots,m)$, $g_0 \not\equiv 0$, the inequality (3.19) hold, and*

$$\int_{\nu_k(t)}^t g_0(s)ds \leq \frac{1}{e} \quad \text{for} \quad t \in [a,b] \qquad (k = 1,\dots,m). \tag{3.20}$$

Then the operator ℓ defined by (3.2) belongs to the set $V_{ab}^+(\lambda,\mu)$.

Remark 3.6. It is clear that for the ordinary differential equations, i.e., if ℓ is defined by

$$\ell(v)(t) \overset{\text{def}}{=} -\sum_{k=1}^m g_k(t)v(t) \quad \text{for} \quad t \in [a,b], \tag{3.21}$$

where $g_k \in L\big([a,b];R_+\big)$ $(k = 1,\dots,m)$, the conditions (3.11) and (3.20) are fulfilled, and the condition (3.19) is sufficient and necessary for the operator ℓ given by (3.21) to belong to the set $V_{ab}^+(\lambda,\mu)$ with $|\mu| \geq |\lambda| \neq 0$. Thus, the inequality (3.19) in Theorem 3.7 and Corollary 3.3 cannot be weakened.

Theorem 3.8. *Let $|\mu| \geq |\lambda| \neq 0$ and $p_k, g_k \in L\big([a,b];R_+\big)$ $(k = 1,\dots,m)$. If, moreover,*

$$\int_a^b p_0(s)ds < \left|\frac{\lambda}{\mu}\right|, \tag{3.22}$$

$$\frac{|\mu|}{|\lambda| - |\mu| \int_a^b p_0(s)ds} - 1 < \int_a^b g_0(s)ds \leq 1, \tag{3.23}$$

then the operator ℓ defined by (3.3) belongs to the set $V_{ab}^+(\lambda,\mu)$.

Remark 3.7. The examples constructed in Subsection 2.4 (see On Remark 2.15, p. 61) also show that neither one of the inequalities in (3.22) and (3.23) can be weakened.

3.2. On the Set $V_{ab}^{-}(\lambda,\mu)$

In the case, where $|\mu| \le |\lambda|$, the following assertions hold.

Theorem 3.9. *Let* $0 \ne |\mu| < |\lambda|$, $p_k \in L\big([a,b];R_+\big)$, $\tau_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$, $p_0 \not\equiv 0$,

$$\int_a^b p_0(s)ds \le 1, \tag{3.24}$$

and let there exist $x \in \Big[\ln\Big|\frac{\lambda}{\mu}\Big|, +\infty\Big[$ *such that*

$$\operatorname{ess\,inf}\left\{\int_t^{\tau_k(t)} p_0(s)ds : t \in [a,b]\right\} > \eta(x) \qquad (k = 1,\dots,m), \tag{3.25}$$

where η *is defined by* (3.8). *Then the operator* ℓ *defined by* (3.1) *belongs to the set* $V_{ab}^{-}(\lambda,\mu)$.

Corollary 3.4. *Let* $0 \ne |\mu| < |\lambda|$, $p_k \in L\big([a,b];R_+\big)$, $\tau_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$, $p_0 \not\equiv 0$, *the inequality* (3.24) *hold, and*

$$\operatorname{ess\,inf}\left\{\int_t^{\tau_k(t)} p_0(s)ds : t \in [a,b]\right\} > \frac{\int_a^b p_0(s)ds}{\ln\Big|\frac{\lambda}{\mu}\Big|} \ln \frac{\ln\Big|\frac{\lambda}{\mu}\Big|}{\int_a^b p_0(s)ds}$$

for $k = 1,\dots,m$. *Then the operator* ℓ *defined by* (3.1) *belongs to the set* $V_{ab}^{-}(\lambda,\mu)$.

Theorem 3.10. *Let* $0 \ne |\mu| < |\lambda|$, $p_k \in L\big([a,b];R_+\big)$, $\tau_k \in \mathcal{M}_{ab}$, $\tau_k(t) \ge t$ *for* $t \in [a,b]$ $(k = 1,\dots,m)$, $p_0 \not\equiv 0$, *and*

$$\operatorname{ess\,sup}\left\{\int_t^{\tau_k(t)} p_0(s)ds : t \in [a,b]\right\} < \omega^* \qquad (k = 1,\dots,m), \tag{3.26}$$

where

$$\omega^* = \sup\left\{ \frac{1}{x} \ln \left(x + \frac{x}{\exp\left(x \int\limits_a^b p_0(s)ds \right) - 1} \right) : x > 0 \right\}. \tag{3.27}$$

If, moreover,

$$|\mu| \exp\left(\int\limits_a^b p_0(s)ds \right) > |\lambda|, \tag{3.28}$$

then the operator ℓ *defined by* (3.1) *belongs to the set* $V_{ab}^-(\lambda,\mu)$.

Corollary 3.5. *Let* $0 \neq |\mu| < |\lambda|$, $p_k \in L([a,b];R_+)$, $\tau_k \in \mathcal{M}_{ab}$, $\tau_k(t) \geq t$ *for* $t \in [a,b]$ $(k = 1,\dots,m)$, $p_0 \not\equiv 0$, *the inequality* (3.28) *hold, and*

$$\int\limits_t^{\tau_k(t)} p_0(s)ds \leq \frac{1}{e} \quad \textit{for} \quad t \in [a,b] \qquad (k = 1,\dots,m).$$

Then the operator ℓ *defined by* (3.1) *belongs to the set* $V_{ab}^-(\lambda,\mu)$.

Theorem 3.11. *Let* $0 \neq |\mu| \leq |\lambda|$ *and let* $p_k, g_k \in L([a,b];R_+)$ $(k = 1,\dots,m)$. *If, moreover,*

$$\int\limits_a^b g_0(s)ds < \left|\frac{\mu}{\lambda}\right|,$$

$$\frac{|\lambda|}{|\mu| - |\lambda| \int\limits_a^b g_0(s)ds} - 1 < \int\limits_a^b p_0(s)ds \leq 1,$$

then the operator ℓ *defined by* (3.3) *belongs to the set* $V_{ab}^-(\lambda,\mu)$.

In the case, where $|\mu| \geq |\lambda|$, the following statements hold.

Theorem 3.12. *Let* $|\mu| > |\lambda|$, $g_k \in L([a,b];R_+)$, $\nu_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$, *and let at least one of the following items be fulfilled:*

a) $\nu_k(t) \geq t$ *for* $t \in [a,b]$ $(k=1,\dots,m)$ *and*

$$|\lambda| \exp\left(\int_a^b g_0(s)ds\right) < |\mu|;$$

b) there exists $\alpha \in]0,1[$ *such that on* $[a,b]$ *the inequality*

$$\frac{|\lambda|}{|\mu|-|\lambda|}\int_a^b \widetilde{g}(s)ds + \int_t^b \widetilde{g}(s)ds \leq (\alpha - \widetilde{g}_0)\left(\widetilde{g}_0 + \int_t^b g_0(s)ds\right),$$

holds, where

$$\widetilde{g}(t) = \sum_{k=1}^m g_k(t)\left(\int_{\nu_k(t)}^b g_0(s)ds\right) \quad \textit{for} \quad t \in [a,b],$$

$$\widetilde{g}_0 = \frac{|\lambda|}{|\mu|-|\lambda|}\int_a^b g_0(s)ds;$$

c)

$$|\lambda| \exp\left(\int_a^b g_0(s)ds\right) +$$

$$+|\mu|\int_a^b \sum_{k=1}^m g_k(s)\sigma_k(s)\left(\int_{\nu_k(s)}^s g_0(\xi)d\xi\right)\exp\left(\int_a^s g_0(\eta)d\eta\right)ds < |\mu|,$$

where

$$\sigma_k(t) = \frac{1}{2}(1+\operatorname{sgn}(t-\nu_k(t))) \quad \textit{for} \quad t \in [a,b] \qquad (k=1,\dots,m).$$

Then the operator ℓ *defined by* (3.2) *belongs to the set* $V_{ab}^{-}(\lambda,\mu)$.

Theorem 3.13. *Let* $|\mu| > |\lambda| \neq 0$*,* $g_k \in L\big([a,b];R_+\big)$*,* $\nu_k \in \mathcal{M}_{ab}$ *(*$k = 1,\dots,m$*),* $g_0 \not\equiv 0$*, and let there exist* $x \in \big]0, \ln\big|\frac{\mu}{\lambda}\big|\big[$ *such that*

$$\operatorname{ess\,sup}\left\{\int\limits_{\nu_k(t)}^{t} g_0(s)ds : t \in [a,b]\right\} < \omega(x) \qquad (k = 1,\dots,m),$$

where ω *is defined by* (3.17)*. Then the operator* ℓ *defined by* (3.2) *belongs to the set* $V_{ab}^{-}(\lambda,\mu)$*.*

Corollary 3.6. *Let* $|\mu| > |\lambda| \neq 0$*,* $g_k \in L\big([a,b];R_+\big)$*,* $\nu_k \in \mathcal{M}_{ab}$ *(*$k = 1,\dots,m$*),* $g_0 \not\equiv 0$*, and*

$$\operatorname{ess\,sup}\left\{\int\limits_{\nu_k(t)}^{t} g_0(s)ds : t \in [a,b]\right\} < \frac{\int_a^b g_0(s)ds}{\ln\left|\frac{\mu}{\lambda}\right|} \ln \frac{\ln\left|\frac{\mu}{\lambda}\right|}{\int_a^b g_0(s)ds}$$

for $k = 1,\dots,m$*. Then the operator* ℓ *defined by* (3.2) *belongs to the set* $V_{ab}^{-}(\lambda,\mu)$*.*

Theorem 3.14. *Let* $|\mu| \geq |\lambda|$*,* $p_k \in L\big([a,b];R_+\big)$*,* $\tau_k \in \mathcal{M}_{ab}$*,* $\tau_k(t) \geq t$ *for* $t \in [a,b]$ *(*$k = 1,\dots,m$*), and let at least one of the following items be fulfilled:*

a)

$$\int\limits_a^b p_0(s)ds \leq 1;$$

b)

$$\int\limits_a^b \sum_{k=1}^{m} p_k(s) \int\limits_s^{\tau_k(s)} \sum_{i=1}^{m} p_i(\xi) \exp\left(\int\limits_s^{\tau_i(\xi)} p_0(\eta)d\eta\right) d\xi ds \leq 1;$$

c) $p_0 \not\equiv 0$ *and the condition* (3.26) *holds, where* ω^* *is defined by* (3.27)*.*

Then the operator ℓ *defined by* (3.1) *belongs to the set* $V_{ab}^{-}(\lambda,\mu)$*.*

Theorem 3.15. *Let $|\mu| \geq |\lambda| \neq 0$ and let $p_k, g_k \in L([a,b];R_+)$ $(k = 1,\dots,m)$. If, moreover,*

$$\int_a^b g_0(s)ds < 1,$$

$$\frac{\int_a^b g_0(s)ds}{1 - \int_a^b g_0(s)ds} - \frac{|\mu| - |\lambda|}{|\lambda|} < \int_a^b p_0(s)ds \leq \left|\frac{\lambda}{\mu}\right|,$$

then the operator ℓ defined by (3.3) *belongs to the set $V_{ab}^-(\lambda,\mu)$.*

Theorem 3.16. *Let $|\mu| > |\lambda|$, $p_k, g_k \in L([a,b];R_+)$, and $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$. Let, moreover, the functions p_k, τ_k $(k = 1,\dots,m)$ satisfy $\tau_k(t) \geq t$ for $t \in [a,b]$ $(k = 1,\dots,m)$ and at least one of the conditions a), b) or c) in Theorem 3.14, while the functions g_k, ν_k $(k = 1,\dots,m)$ satisfy at least one of the conditions a), b) or c) in Theorem 3.12 or the assumptions of Theorem 3.13. Then the operator ℓ defined by* (3.3) *belongs to the set $V_{ab}^-(\lambda,\mu)$.*

Remark 3.8. Similarly as in Subsection 3.1 one can show that Theorems 3.9–3.12 and 3.14–3.16 and Corollaries 3.4 and 3.5 are also nonimprovable in a certain sense.

3.3. Proofs

Proof of Theorem 3.1. a) The validity of the theorem immediately follows from Corollary 2.1 a) (see p. 17).

b) According to (3.5) we have

$$\rho_3(t) \leq \alpha\rho_2(t) \quad \text{for} \quad t \in [a,b],$$

where ρ_2, ρ_3 are defined by (2.14) and ℓ is given by (3.1). Therefore, the assumptions of Corollary 2.1 b) (see p. 17) are fulfilled for $k = 2$ and $m = 3$.

c) Let $\overline{\ell}$ be an operator defined by

$$\overline{\ell}(v)(t) \stackrel{\text{def}}{=} \sum_{k=1}^m p_k(t)\sigma_k(t) \int_t^{\tau_k(t)} \sum_{i=1}^m p_i(s)v(\tau_i(s))ds \quad \text{for} \quad t \in [a,b],$$

where $\sigma_k(t) = \frac{1}{2}(1 + \operatorname{sgn}(\tau_k(t) - t))$ for $t \in [a, b]$ $(k = 1, \dots, m)$. Obviously, $\overline{\ell} \in \mathcal{P}_{ab}$ and

$$\ell(\vartheta(v))(t) - \ell(1)(t)\vartheta(v)(t) = \sum_{k=1}^{m} p_k(t) \int_t^{\tau_k(t)} \sum_{i=1}^{m} p_i(s) v(\tau_i(s)) ds \le$$

$$\le \overline{\ell}(v)(t) \quad \text{for} \quad t \in [a, b], \ v \in C\big([a, b]; R_+\big),$$

where ϑ is defined by (2.17). On the other hand, according to (3.6), the inequality (2.15) holds. Hence, the assumptions of Corollary 2.1 c) (see p. 17) are fulfilled. □

Proof of Theorem 3.2. According to (3.7), there exists $\varepsilon \in \,]0, 1[$ such that

$$\int_t^{\tau_k(t)} p_0(s) ds \le \frac{\int_a^b p_0(s) ds}{\varepsilon x} \ln \frac{\varepsilon x e^{\varepsilon x}}{\int_a^b p_0(s) ds \left(e^{\varepsilon x} - \frac{|\lambda| - |\mu| e^x}{|\lambda| - |\mu|} \right)} \tag{3.29}$$

for $t \in [a, b]$ $(k = 1, \dots, m)$. Put

$$x_0 = \frac{\varepsilon x}{\int_a^b p_0(s) ds} \,. \tag{3.30}$$

Obviously, $x_0 > 0$. By virtue of (3.29), (3.30), and the assumption $x \in \left]0, \ln\left|\frac{\lambda}{\mu}\right|\right]$, we obtain

$$x_0 \, e^{x_0 \int_a^t p_0(s) ds} \ge e^{x_0 \int_a^{\tau_k(t)} p_0(s) ds} - \frac{|\lambda| - |\mu| e^x}{|\lambda| - |\mu|} \tag{3.31}$$

for $t \in [a, b]$ $(k = 1, \dots, m)$, and

$$x_0 \int_a^b p_0(s) ds < x. \tag{3.32}$$

Put

$$\gamma(t) = e^{x_0 \int_a^t p_0(s) ds} - \frac{|\lambda| - |\mu| e^x}{|\lambda| - |\mu|} \quad \text{for} \quad t \in [a, b].$$

According to the assumption $x \in \left]0, \ln\left|\frac{\lambda}{\mu}\right|\right]$, it is clear that $\gamma(t) > 0$ for $t \in [a,b]$ and, on account of (3.31), we obtain

$$\ell(\gamma)(t) = \sum_{k=1}^{m} p_k(t)\left(e^{x_0 \int\limits_a^{\tau_k(t)} p_0(s)ds} - \frac{|\lambda| - |\mu|e^x}{|\lambda| - |\mu|}\right) \leq$$

$$\leq p_0(t)x_0\, e^{x_0 \int\limits_a^{t} p_0(s)ds} = \gamma'(t) \quad \text{for} \quad t \in [a,b],$$

where ℓ is defined by (3.1), i.e., the inequality (2.10) is fulfilled. On the other hand, it follows from (3.32) that the inequality (2.11) holds. Thus, the assumptions of Theorem 2.1 (see p. 17) are fulfilled. □

Proof of Corollary 3.1. The validity of the corollary immediately follows from Theorem 3.2 for $x = \ln\left|\frac{\lambda}{\mu}\right|$. □

Proof of Theorem 3.3. a) The validity of the theorem immediately follows from Theorem 2.3 (see p. 20).

b) If (3.10) holds, then the operator ℓ defined by (3.2) satisfies the condition (2.25), where $\widetilde{\ell}$ is given by (2.26), and thus, the assumptions of Corollary 2.2 (see p. 20) are satisfied.

c) According to (3.11), there exists $\varepsilon > 0$ such that

$$\int\limits_{\nu_k(t)}^{t} g_0(s)ds < \eta^* - \varepsilon \quad \text{for} \quad t \in [a,b] \qquad (k = 1, \dots, m). \tag{3.33}$$

Choose $x_0 > 0$ and $\delta \in \,]0,1[$ such that

$$\frac{1}{x_0}\ln\left(x_0 + \frac{x_0(1-\delta)}{\exp\left(x_0 \int\limits_a^b g_0(s)ds\right) - (1-\delta)}\right) > \eta^* - \varepsilon, \tag{3.34}$$

and put

$$\gamma(t) = e^{x_0 \int\limits_t^b g_0(s)ds} - (1-\delta) \quad \text{for} \quad t \in [a,b].$$

The inequalities (3.33) and (3.34) imply

$$x_0\, e^{x_0 \int\limits_t^b g_0(s)ds} \geq e^{x_0 \int\limits_{\nu_k(t)}^b g_0(s)ds} - (1-\delta) \quad \text{for} \quad t \in [a,b]. \tag{3.35}$$

Hence, we obtain

$$\ell(\gamma)(t) = -\sum_{k=1}^m g_k(t) \left(e^{x_0 \int\limits_{\nu_k(t)}^b g_0(s)ds} - (1-\delta) \right) \geq$$

$$\geq -g_0(t) x_0\, e^{x_0 \int\limits_t^b g_0(s)ds} = \gamma'(t) \quad \text{for} \quad t \in [a,b],$$

where ℓ is defined by (3.2), i.e., the inequality (2.20) is fulfilled. Obviously, (2.21) holds and thus, the assumptions of Theorem 2.2 (see p. 19) are satisfied. □

Proof of Theorem 3.4. The validity of the theorem immediately follows from Theorem 2.4 (see p. 21). □

Proof of Theorem 3.5. The validity of the theorem follows from Theorem 2.5 (see p. 22) and Theorems 3.1, 3.2, and 3.3. □

Proof of Theorem 3.6. According to (3.16), there exists $\varepsilon \in \,]1,+\infty[$ such that

$$\int\limits_{\nu_k(t)}^t g_0(s)ds \geq \frac{\int\limits_a^b g_0(s)ds}{\varepsilon x} \ln \frac{\varepsilon x e^{\varepsilon x}}{\int\limits_a^b g_0(s)ds \left(e^{\varepsilon x} + \frac{|\lambda| e^x - |\mu|}{|\mu| - |\lambda|} \right)} \tag{3.36}$$

for $t \in [a,b]$ $(k = 1, \dots, m)$. Put

$$x_0 = \frac{\varepsilon x}{\int\limits_a^b g_0(s)ds}. \tag{3.37}$$

Obviously, $x_0 > 0$. By (3.36), (3.37), and the assumption $x \in \left[\ln \left|\frac{\mu}{\lambda}\right|, +\infty\right[$, we obtain

$$x_0\, e^{x_0 \int\limits_t^b g_0(s)ds} \leq e^{x_0 \int\limits_{\nu_k(t)}^b g_0(s)ds} + \frac{|\lambda| e^x - |\mu|}{|\mu| - |\lambda|} \tag{3.38}$$

for $t \in [a,b]$ $(k = 1, \dots, m)$, and

$$x_0 \int_a^b g_0(s)ds > x. \tag{3.39}$$

Define the function $\gamma \in \widetilde{C}\big([a,b];R\big)$ by

$$\gamma(t) = e^{x_0 \int_t^b g_0(s)ds} + \frac{|\lambda|e^x - |\mu|}{|\mu| - |\lambda|} \quad \text{for} \quad t \in [a,b]. \tag{3.40}$$

Obviously, if ℓ is defined by (3.2), then (3.15) implies (2.23), and by virtue of (3.39) and (3.40), the function γ satisfies (2.11). Moreover, in view of (3.38), we obtain

$$\ell(\gamma)(t) = -\sum_{k=1}^{m} g_k(t) \left(e^{x_0 \int_{\nu_k(t)}^b g_0(s)ds} + \frac{|\lambda|e^x - |\mu|}{|\mu| - |\lambda|} \right) \leq$$

$$\leq -g_0(t)x_0\, e^{x_0 \int_t^b g_0(s)ds} = \gamma'(t) \quad \text{for} \quad t \in [a,b],$$

i.e., the inequality (2.10) is fulfilled. Therefore, according to Theorem 2.6 (see p. 23), $\ell \in V_{ab}^+(\lambda,\mu)$. □

Proof of Corollary 3.2. The validity of the corollary immediately follows from Theorem 3.6 for $x = \ln\left|\frac{\mu}{\lambda}\right|$. □

Proof of Theorem 3.7. According to (3.11), there exists $\varepsilon > 0$ such that (3.33) holds. In view of (3.12), we can choose $x_0 > 0$ and $\delta \in\,]0,1[$ such that (3.34) is fulfilled. Put

$$\beta(t) = e^{x_0 \int_t^b g_0(s)ds} - (1-\delta) \quad \text{for} \quad t \in [a,b].$$

The inequalities (3.33) and (3.34) imply (3.35). Hence, we obtain

$$\ell(\beta)(t) = -\sum_{k=1}^{m} g_k(t) \left(e^{x_0 \int_{\nu_k(t)}^b g_0(s)ds} - (1-\delta) \right) \geq$$

$$\geq -g_0(t)x_0\, e^{x_0 \int_t^b g_0(s)ds} = \beta'(t) \quad \text{for} \quad t \in [a,b],$$

where ℓ is defined by (3.2), i.e., the inequality (2.33) is fulfilled. Obviously, $\beta(t) > 0$ for $t \in [a,b]$, i.e., the condition (2.32) holds.

It is not difficult to verify that, according to (3.18) and (3.19), the condition (3.16) holds for $x = \ln\left|\frac{\mu}{\lambda}\right|$, where ω is defined by (3.17). Therefore, analogously to the proof of Theorem 3.6 it can be shown that there exists a function $\gamma \in \widetilde{C}\big([a,b]; R_+\big)$, which satisfies the inequalities (2.10) and (2.11).

Consequently, the assumptions of Theorem 2.7 (see p. 23) are fulfilled. □

Proof of Corollary 3.3. The validity of the corollary immediately follows from Theorem 3.7. □

Proof of Theorem 3.8. The validity of the theorem immediately follows from Theorem 2.8 (see p. 24). □

Proof of Theorem 3.9. Similarly to the proof of Theorem 3.6 one can show that there exists a function $\gamma \in \widetilde{C}\big([a,b]; R_+\big)$ satisfying (2.20) and (2.37), where ℓ is defined by (3.1), and thus, the assumptions of Theorem 2.9 (see p. 25) are satisfied. □

Proof of Corollary 3.4. It follows immediately from Theorem 3.9 for $x = \ln\left|\frac{\lambda}{\mu}\right|$. □

Proof of Theorem 3.10. Similarly to the proof of Theorem 3.7 one can show that there exists a function $\gamma \in \widetilde{C}\big([a,b]; R_+\big)$ satisfying (2.20) and (2.37), and that there exists a function $\beta \in \widetilde{C}\big([a,b]; R_+\big)$ satisfying (2.39) and (2.40), where ℓ is defined by (3.1). Thus, the assumptions of Theorem 2.10 (see p. 25) are satisfied. □

Proof of Corollary 3.5. The validity of the corollary immediately follows from Theorem 3.10. □

Proof of Theorem 3.11. The validity of the theorem immediately follows from Theorem 2.11 (see p. 26). □

Proof of Theorem 3.12. a) The validity of the theorem immediately follows from Corollary 2.3 a) (see p. 26).

b) Similarly to the proof of Theorem 3.1 b) one can show that the assumptions of Corollary 2.3 b) (see p. 26) are satisfied.

c) Similarly to the proof of Theorem 3.1 c) one can show that the assumptions of Corollary 2.3 c) (see p. 26) are satisfied. □

Proof of Theorem 3.13. Similarly to the proof of Theorem 3.2 one can show that there exists a function $\gamma \in \widetilde{C}\big([a,b];]0,+\infty[\big)$ satisfying (2.20) and (2.37), and thus, the assumptions of Theorem 2.12 (see p. 26) are fulfilled. $\square$

Proof of Corollary 3.6. The validity of the corollary immediately follows from Theorem 3.13 for $x = \ln\left|\frac{\mu}{\lambda}\right|$. $\square$

Proof of Theorem 3.14. a) The validity of the theorem immediately follows from Theorem 2.14 (see p. 27).

b) Similarly to the proof of Theorem 3.3 b) one can show that the assumptions of Corollary 2.4 (see p. 27) are satisfied.

c) Similarly to the proof of Theorem 3.3 c) one can show that there exists a function $\gamma \in \widetilde{C}\big([a,b];R_+\big)$ satisfying (2.10) and (2.41) and thus, the assumptions of Theorem 2.13 (see p. 27) are satisfied. $\square$

Proof of Theorem 3.15. The validity of the theorem immediately follows from Theorem 2.15 (see p. 28). $\square$

Proof of Theorem 3.16. The validity of the theorem follows from Theorem 2.16 (see p. 28) and Theorems 3.12, 3.13, and 3.14. $\square$

§4. Periodic Type BVP

In this section, we will establish nonimprovable, in a certain sense, sufficient conditions for unique solvability of the problem (1.1), (1.2), where the boundary condition (1.2) is of a periodic type, i.e., when the inequality (2.1) is satisfied. In Subsection 4.1, the main results are formulated. Theorems 4.1–4.5 deal with the case $|\mu| \leq |\lambda|$, while the case $|\mu| \geq |\lambda|$ is considered in Theorems 4.7–4.11. Moreover, Theorems 4.6 and 4.12 are valid for the case $\lambda \neq 0$ and $\mu \neq 0$, respectively. The proofs of the main results can be found in Subsection 4.2. Subsection 4.3 is devoted to the examples verifying the optimality of the main results.

As above, throughout this section, if $\lambda = -\mu$, then the operator ℓ is supposed to be nontrivial, i.e., $\ell(1) \not\equiv 0$.

4.1. Existence and Uniqueness Theorems

In the case, where $|\mu| \leq |\lambda|$, the following statements hold.

Theorem 4.1. *Let $0 \neq |\mu| \leq |\lambda|$, the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and let either*

$$\|\ell_0(1)\|_L < 1, \tag{4.1}$$

$$\frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{|\lambda| - |\mu|}{|\mu|} < \|\ell_1(1)\|_L < 1 + \left|\frac{\mu}{\lambda}\right| + 2\sqrt{1 - \|\ell_0(1)\|_L} \tag{4.2}$$

or

$$\|\ell_1(1)\|_L < \left|\frac{\mu}{\lambda}\right|, \tag{4.3}$$

$$\frac{|\lambda|}{|\mu| - |\lambda|\|\ell_1(1)\|_L} - 1 < \|\ell_0(1)\|_L < 2 + 2\sqrt{\left|\frac{\mu}{\lambda}\right| - \|\ell_1(1)\|_L}\,. \tag{4.4}$$

Then the problem (1.1), (1.2) *has a unique solution.*

Remark 4.1. Let $0 \neq |\mu| \leq |\lambda|$. Denote by H^+, resp. H^- the set of pairs $(x, y) \in R_+ \times R_+$ such that

$$x < 1, \qquad \frac{x}{1 - x} - \frac{|\lambda| - |\mu|}{|\mu|} < y < 1 + \left|\frac{\mu}{\lambda}\right| + 2\sqrt{1 - x}\,,$$

resp.

$$y < \left|\frac{\mu}{\lambda}\right|, \qquad \frac{|\lambda|}{|\mu| - |\lambda| y} - 1 < x < 2 + 2\sqrt{\left|\frac{\mu}{\lambda}\right| - y}$$

(see Fig. 4.1; note that if $|\lambda| \geq 4|\mu|$, then $H^- = \emptyset$).

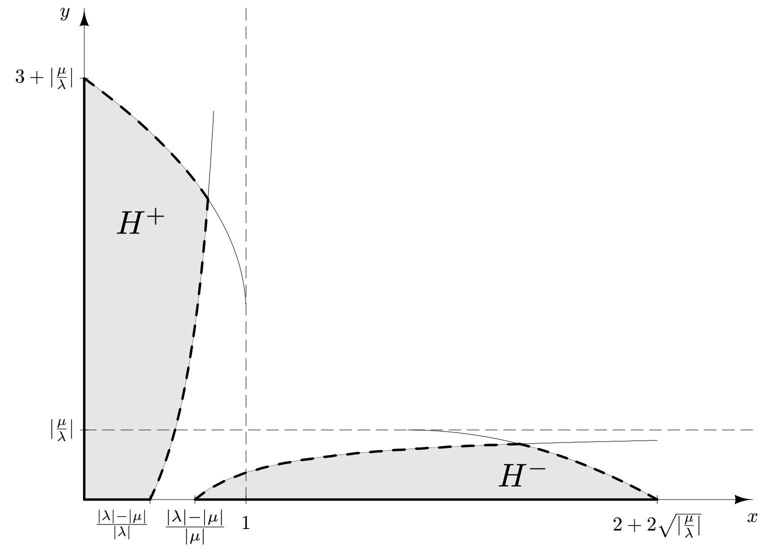

Fig. 4.1.

According to Theorem 4.1, if $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and

$$\Big(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L\Big) \in H^+ \cup H^-,$$

then the problem (1.1), (1.2) has a unique solution. Below we will show (see On Remark 4.1, p. 94) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin H^+ \cup H^-$ there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $q \in L\big([a,b]; R\big)$, and $c \in R$ such that (2.30) holds, and the problem (1.1), (1.2) with $\ell = \ell_0 - \ell_1$ has no solution. In particular, neither one of the strict inequalities in (4.1)–(4.4) can be replaced by the nonstrict one.

The next theorem can be understood as a supplement of the previous one for the case $\mu = 0$.

Theorem 4.2. *Let $\mu = 0$, the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and let*

$$\|\ell_0(1)\|_L < 1, \tag{4.5}$$

$$\|\ell_1(1)\|_L < 1 + 2\sqrt{1 - \|\ell_0(1)\|_L}\,. \tag{4.6}$$

Then the problem (1.1), (1.2) *has a unique solution.*

Remark 4.2. Let $\mu = 0$. Denote by H the set of pairs $(x, y) \in R_+ \times R_+$ such that

$$x < 1, \qquad y < 1 + 2\sqrt{1 - x}$$

(see Fig. 4.2).

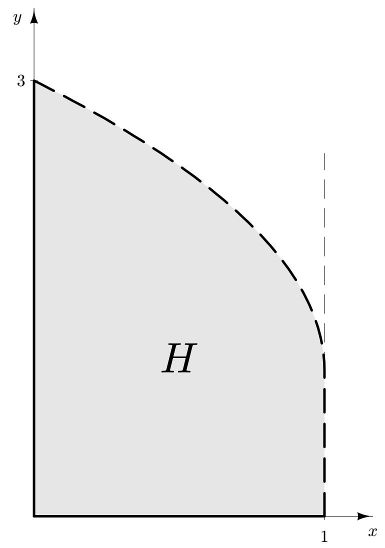

Fig. 4.2.

According to Theorem 4.2, if $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and

$$\Big(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L\Big) \in H,$$

then the problem (1.1), (1.2) has a unique solution. Below we will show (see On Remark 4.2, p. 97) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin H$ there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $q \in L\big([a, b]; R\big)$, and $c \in R$ such that (2.30) holds, and the problem (1.1), (1.2) with $\ell = \ell_0 - \ell_1$ has no solution. In particular, the strict inequalities (4.5) and (4.6) cannot be replaced by the nonstrict ones.

Theorem 4.3. *Let* $|\mu| < |\lambda|$ *and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $C_{\lambda\mu}\big([a,b];R\big)$ *the inequality*

$$|\ell(v)(t) + \ell_1(v)(t)| \leq \ell_0(|v|)(t) \quad \textit{for} \quad t \in [a,b] \tag{4.7}$$

holds. If, moreover,

$$\ell_0 \in V_{ab}^+(\lambda,\mu), \qquad -\frac{1}{2}\ell_1 \in V_{ab}^+(\lambda,\mu), \tag{4.8}$$

then the problem (1.1), (1.2) *has a unique solution.*

Remark 4.3. The inequality (4.7) in Theorem 4.3 cannot be replaced by the inequality

$$|\ell(v)(t) + \ell_1(v)(t)| \leq (1+\varepsilon)\ell_0(|v|)(t) \quad \text{for} \quad t \in [a,b], \tag{4.9}$$

no matter how small $\varepsilon > 0$ would be (see Example 4.1, p. 98). Moreover, the assumption (4.8) can be replaced neither by the assumption

$$(1-\varepsilon)\ell_0 \in V_{ab}^+(\lambda,\mu), \qquad -\frac{1}{2}\ell_1 \in V_{ab}^+(\lambda,\mu) \tag{4.10}$$

nor by the assumption

$$\ell_0 \in V_{ab}^+(\lambda,\mu), \qquad -\frac{1}{2+\varepsilon}\ell_1 \in V_{ab}^+(\lambda,\mu),$$

no matter how small $\varepsilon > 0$ would be (see Examples 4.2 and 4.3, p. 98).

Theorem 4.4. *Let* $|\mu| < |\lambda|$, *the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, *and let there exist* $\gamma \in \widetilde{C}\big([a,b];]0,+\infty[\big)$ *satisfying*

$$\gamma'(t) \geq \ell_0(\gamma)(t) + \ell_1(1)(t) \quad \textit{for} \quad t \in [a,b], \tag{4.11}$$

$$|\lambda|\gamma(a) > |\mu|\gamma(b), \tag{4.12}$$

$$\gamma(b) - \gamma(a) < 3 + \left|\frac{\mu}{\lambda}\right|. \tag{4.13}$$

Then the problem (1.1), (1.2) *has a unique solution.*

Remark 4.4. Theorem 4.4 is nonimprovable in the sense that the strict inequality (4.13) cannot be replaced by the nonstrict one (see Example 4.4, p. 100).

Theorem 4.5. *Let* $0 \neq |\mu| \leq |\lambda|$, $\ell \in \mathcal{P}_{ab}$, *and let there exist a function* $\gamma \in \widetilde{C}\big([a,b];R_+\big)$ *such that*

$$\gamma'(t) \leq \ell(\gamma)(t) \quad \textit{for} \quad t \in [a,b], \tag{4.14}$$

$$|\lambda|\gamma(a) < |\mu|\gamma(b). \tag{4.15}$$

Let, moreover, at least one of the following items be fulfilled:

a)
$$\|\ell(1)\|_L < 2 + 2\sqrt{\left|\frac{\mu}{\lambda}\right|}; \tag{4.16}$$

b)
$$\ell \in V_{ab}^{+}(1,0); \tag{4.17}$$

c)
$$\ell \in V_{ab}^{-}(0,1). \tag{4.18}$$

Then the problem (1.1), (1.2) *has a unique solution.*

Remark 4.5. Theorem 4.5 is nonimprovable in the sense that the strict inequality (4.16) cannot be replaced by the nonstrict one (see Example 4.5, p. 101).

Note also that if $|\lambda| = |\mu|$ and $\ell \in \mathcal{P}_{ab}$, then there exists a function $\gamma \in \widetilde{C}\big([a,b];R_+\big)$ satisfying (4.14) and (4.15). Indeed, in this case the operator ℓ is supposed to be nontrivial and thus, the function

$$\gamma(t) = 1 + \int_a^t \ell(1)(s)ds \quad \text{for} \quad t \in [a,b]$$

satisfies (4.14) and (4.15).

Nevertheless, if $0 \neq |\mu| < |\lambda|$, then the inequality (4.15) cannot be replaced by the inequality

$$|\lambda|\gamma(a) \leq |\mu|\gamma(b) \tag{4.19}$$

(see Example 4.6, p. 102).

The following theorem does not deal only with the case $|\mu| \leq |\lambda|$. On the other hand, the assumption $\lambda \neq 0$ is necessary (see Remark 2.2, p. 15).

Theorem 4.6. *Let $\lambda \neq 0$ and let there exist an operator*

$$\ell_0 \in V_{ab}^{+}(\lambda, \mu) \tag{4.20}$$

such that on the set $C_{\lambda\mu}\big([a,b]; R\big)$ the inequality

$$\ell(v)(t)\operatorname{sgn} v(t) \le \ell_0(|v|)(t) \quad \textit{for} \quad t \in [a,b] \tag{4.21}$$

holds. Then the problem (1.1), (1.2) *has a unique solution.*

Remark 4.6. Examples 4.1 and 4.2 (see p. 98) also show that if $|\mu| < |\lambda|$, then the assumption (4.20) in Theorem 4.6 cannot be replaced by the assumption

$$(1-\varepsilon)\ell_0 \in V_{ab}^{+}(\lambda, \mu), \tag{4.22}$$

and the inequality (4.21) cannot be replaced by the inequality

$$\ell(v)(t)\operatorname{sgn} v(t) \le (1+\varepsilon)\ell_0(|v|)(t) \quad \text{for} \quad t \in [a,b], \tag{4.23}$$

no matter how small $\varepsilon > 0$ would be.

Furthermore, if $|\mu| > |\lambda| \neq 0$, then the inequality (4.21) in Theorem 4.6 cannot be replaced by the inequality

$$\ell(v)(t)\operatorname{sgn} v(t) \le (1-\varepsilon)\ell_0(|v|)(t) \quad \text{for} \quad t \in [a,b] \tag{4.24}$$

and the condition (4.20) cannot be replaced by the condition

$$(1+\varepsilon)\ell_0 \in V_{ab}^{+}(\lambda, \mu), \tag{4.25}$$

no matter how small $\varepsilon > 0$ would be (see Example 4.7, p. 103).

In the case, where $|\mu| \ge |\lambda|$, the following assertions hold.

Theorem 4.7. *Let $|\mu| \ge |\lambda| \neq 0$, the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and let either*

$$\|\ell_1(1)\|_L < 1,$$

$$\frac{\|\ell_1(1)\|_L}{1 - \|\ell_1(1)\|_L} - \frac{|\mu| - |\lambda|}{|\lambda|} < \|\ell_0(1)\|_L < 1 + \left|\frac{\lambda}{\mu}\right| + 2\sqrt{1 - \|\ell_1(1)\|_L}$$

or

$$\|\ell_0(1)\|_L < \left|\frac{\lambda}{\mu}\right|,$$

$$\frac{|\mu|}{|\lambda| - |\mu|\|\ell_0(1)\|_L} - 1 < \|\ell_1(1)\|_L < 2 + 2\sqrt{\left|\frac{\lambda}{\mu}\right| - \|\ell_0(1)\|_L}\,.$$

Then the problem (1.1), (1.2) *has a unique solution.*

The next theorem can be understood as a supplement of the previous one for the case $\lambda = 0$.

Theorem 4.8. *Let* $\lambda = 0$, *the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, *and let*

$$\|\ell_1(1)\|_L < 1,$$

$$\|\ell_0(1)\|_L < 1 + 2\sqrt{1 - \|\ell_1(1)\|_L}\,.$$

Then the problem (1.1), (1.2) *has a unique solution.*

Theorem 4.9. *Let* $|\mu| > |\lambda|$ *and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $C_{\lambda\mu}\big([a,b];R\big)$ *the inequality*

$$|\ell(v)(t) - \ell_1(v)(t)| \le \ell_0(|v|)(t) \quad \textit{for} \quad t \in [a,b]$$

holds. If, moreover,

$$-\ell_0 \in V_{ab}^-(\lambda,\mu), \qquad \frac{1}{2}\ell_1 \in V_{ab}^-(\lambda,\mu),$$

then the problem (1.1), (1.2) *has a unique solution.*

Theorem 4.10. *Let* $|\mu| > |\lambda|$, *the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, *and let there exist* $\gamma \in \widetilde{C}\big([a,b];]0,+\infty[\big)$ *satisfying*

$$-\gamma'(t) \ge \ell_1(\gamma)(t) + \ell_0(1)(t) \quad \textit{for} \quad t \in [a,b], \tag{4.26}$$

$$|\lambda|\gamma(a) < |\mu|\gamma(b), \tag{4.27}$$

$$\gamma(a) - \gamma(b) < 3 + \left|\frac{\lambda}{\mu}\right|. \tag{4.28}$$

Then the problem (1.1), (1.2) *has a unique solution.*

Theorem 4.11. *Let* $|\mu| \geq |\lambda| \neq 0$, $-\ell \in \mathcal{P}_{ab}$, *and let there exist a function* $\gamma \in \widetilde{C}\big([a,b];R_+\big)$ *such that*

$$\gamma'(t) \geq \ell(\gamma)(t) \quad \textit{for} \quad t \in [a,b],$$
$$|\lambda|\gamma(a) > |\mu|\gamma(b).$$

Let, moreover, at least one of the following items be fulfilled:

a)
$$\|\ell(1)\|_L < 2 + 2\sqrt{\left|\frac{\lambda}{\mu}\right|};$$

b) the condition (4.18) *is fulfilled;*

c) the condition (4.17) *is fulfilled.*

Then the problem (1.1), (1.2) *has a unique solution.*

The last theorem does not deal only with the case $|\mu| \geq |\lambda|$. On the other hand, the assumption $\mu \neq 0$ is necessary (see Remark 2.2, p. 15).

Theorem 4.12. *Let* $\mu \neq 0$ *and let there exist an operator* $\ell_0 \in V_{ab}^-(\lambda,\mu)$ *such that on the set* $C_{\lambda\mu}\big([a,b];R\big)$ *the inequality*

$$\ell(v)(t)\operatorname{sgn} v(t) \geq \ell_0(|v|)(t) \quad \textit{for} \quad t \in [a,b]$$

holds. Then the problem (1.1), (1.2) *has a unique solution.*

Remark 4.7. According to Remark 2.16 (see p. 28), Theorems 4.7–4.12 can be immediately derived from Theorems 4.1–4.6. Moreover, by virtue of Remarks 4.1–4.6, Theorems 4.7–4.12 are nonimprovable in an appropriate sense.

4.2. Proofs

According to Theorem 1.1 (see p. 14), it is sufficient to show that the homogeneous problem (1.1_0), (1.2_0) has no nontrivial solution.

Proof of Theorem 4.1. First suppose that (4.1) and (4.2) hold. Assume that the problem (1.1_0), (1.2_0) has a nontrivial solution u. According to Lemma 2.2 (see p. 39), u changes its sign. Define numbers M and m by (2.94) and choose $t_M, t_m \in [a,b]$ such that (2.95) is fulfilled. Obviously, $M > 0$, $m > 0$, and without loss of generality we can assume that $t_m < t_M$.

The integration of (1.1_0) from a to t_m, from t_m to t_M, and from t_M to b, by virtue of (2.94), (2.95), and the assumptions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, results in

$$u(a) + m \le M \int_a^{t_m} \ell_1(1)(s)ds + m \int_a^{t_m} \ell_0(1)(s)ds, \tag{4.29}$$

$$M + m \le M \int_{t_m}^{t_M} \ell_0(1)(s)ds + m \int_{t_m}^{t_M} \ell_1(1)(s)ds, \tag{4.30}$$

$$M - u(b) \le M \int_{t_M}^{b} \ell_1(1)(s)ds + m \int_{t_M}^{b} \ell_0(1)(s)ds. \tag{4.31}$$

Multiplying both sides of (4.31) by $\left|\frac{\mu}{\lambda}\right|$ and taking into account (2.1), (2.94), and the assumption $\left|\frac{\mu}{\lambda}\right| \in \,]0,1]$, we get

$$\left|\frac{\mu}{\lambda}\right| M + \frac{\mu}{\lambda} u(b) \le M \int_{t_M}^{b} \ell_1(1)(s)ds + m \int_{t_M}^{b} \ell_0(1)(s)ds.$$

Summing the last inequality and (4.29), by virtue of (1.2_0), we obtain

$$\left|\frac{\mu}{\lambda}\right| M + m \le M \int_J \ell_1(1)(s)ds + m \int_J \ell_0(1)(s)ds, \tag{4.32}$$

where $J = [a, t_m] \cup [t_M, b]$. It follows from (4.30) and (4.32) that

$$M(1-D) \le m(B-1), \qquad m(1-C) \le M\left(A - \left|\frac{\mu}{\lambda}\right|\right), \tag{4.33}$$

where

$$\begin{aligned} A &= \int_J \ell_1(1)(s)ds, \qquad & B &= \int_{t_m}^{t_M} \ell_1(1)(s)ds, \\ C &= \int_J \ell_0(1)(s)ds, \qquad & D &= \int_{t_m}^{t_M} \ell_0(1)(s)ds. \end{aligned} \tag{4.34}$$

Due to (4.34) and (4.1), $C < 1$ and $D < 1$. Consequently, (4.33) implies $A > \left|\frac{\mu}{\lambda}\right|$, $B > 1$, and

$$0 < (1 - C)(1 - D) \leq \left(A - \left|\frac{\mu}{\lambda}\right|\right)(B - 1). \tag{4.35}$$

Obviously,

$$(1 - C)(1 - D) \geq 1 - (C + D) = 1 - \|\ell_0(1)\|_L > 0,$$

$$4\left(A - \left|\frac{\mu}{\lambda}\right|\right)(B - 1) \leq \left[A + B - 1 - \left|\frac{\mu}{\lambda}\right|\right]^2 = \left[\|\ell_1(1)\|_L - 1 - \left|\frac{\mu}{\lambda}\right|\right]^2.$$

By the last inequalities, (4.35) results in

$$0 < 4(1 - \|\ell_0(1)\|_L) \leq \left[\|\ell_1(1)\|_L - \left(1 + \left|\frac{\mu}{\lambda}\right|\right)\right]^2,$$

which contradicts the second inequality in (4.2).

Now suppose that (4.3) and (4.4) are fulfilled. Assume that the problem (1.1_0), (1.2_0) has a nontrivial solution u. According to Lemma 2.4 (see p. 48), u changes its sign. Define numbers M and m by (2.94) and choose $t_M, t_m \in [a, b]$ such that (2.95) is fulfilled. Obviously, $M > 0$, $m > 0$, and without loss of generality we can assume that $t_m < t_M$. In a similar manner as above, one can show that the inequalities (4.29)–(4.32) hold, where $J = [a, t_m] \cup [t_M, b]$. It follows from (4.30) and (4.32) that

$$m(1 - B) \leq M(D - 1), \qquad M\left(\left|\frac{\mu}{\lambda}\right| - A\right) \leq m(C - 1), \tag{4.36}$$

where A, B, C, D are defined by (4.34). According to (4.3) and (4.34), $A < \left|\frac{\mu}{\lambda}\right|$ and $B < \left|\frac{\mu}{\lambda}\right| \leq 1$. Consequently, (4.36) implies $C > 1$, $D > 1$, and

$$0 < \left(\left|\frac{\mu}{\lambda}\right| - A\right)(1 - B) \leq (C - 1)(D - 1). \tag{4.37}$$

Obviously,

$$\left(\left|\frac{\mu}{\lambda}\right| - A\right)(1 - B) \geq \left|\frac{\mu}{\lambda}\right| - (A + B) = \left|\frac{\mu}{\lambda}\right| - \|\ell_1(1)\|_L > 0,$$

$$4(C - 1)(D - 1) \leq (C + D - 2)^2 = (\|\ell_0(1)\|_L - 2)^2.$$

By the last inequalities, (4.37) results in

$$0 < 4\left(\left|\frac{\mu}{\lambda}\right| - \|\ell_1(1)\|_L\right) \leq (\|\ell_0(1)\|_L - 2)^2,$$

which contradicts the second inequality in (4.4). □

Proof of Theorem 4.2. It can be proved in a similar manner as Theorem 4.1. Moreover, the proof of Theorem 4.2 can be found in [5]. □

Proof of Theorem 4.3. Let u be a solution of the problem (1.1_0), (1.2_0). Then, in view of (1.1_0), u satisfies

$$u'(t) = -\frac{1}{2}\ell_1(u)(t) + \ell(u)(t) + \frac{1}{2}\ell_1(u)(t) \quad \text{for} \quad t \in [a,b]. \tag{4.38}$$

By virtue of the assumption $-\frac{1}{2}\ell_1 \in V_{ab}^+(\lambda,\mu)$ and Theorem 1.1 (see p. 14), the problem

$$\alpha'(t) = -\frac{1}{2}\ell_1(\alpha)(t) + \ell_0(|u|)(t) + \frac{1}{2}\ell_1(|u|)(t), \tag{4.39}$$

$$\lambda\alpha(a) + \mu\alpha(b) = 0 \tag{4.40}$$

has a unique solution α. Moreover, since $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and $-\frac{1}{2}\ell_1 \in V_{ab}^+(\lambda,\mu)$,

$$\alpha(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

The equality (4.39), in view of (4.7) and the condition $\ell_1 \in \mathcal{P}_{ab}$, yields

$$\alpha'(t) \geq -\frac{1}{2}\ell_1(\alpha)(t) + \ell(u)(t) + \frac{1}{2}\ell_1(u)(t) \quad \text{for} \quad t \in [a,b],$$
$$(-\alpha(t))' \leq -\frac{1}{2}\ell_1(-\alpha)(t) + \ell(u)(t) + \frac{1}{2}\ell_1(u)(t) \quad \text{for} \quad t \in [a,b].$$

From the last two inequalities and (4.38), on account of (2.1), (4.40), the assumption $-\frac{1}{2}\ell_1 \in V_{ab}^+(\lambda,\mu)$, and Remark 2.3 (see p. 16), we get

$$|u(t)| \leq \alpha(t) \quad \text{for} \quad t \in [a,b]. \tag{4.41}$$

On the other hand, due to (4.41) and the conditions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, the equality (4.39) results in

$$\alpha'(t) \leq \ell_0(\alpha)(t) \quad \text{for} \quad t \in [a,b].$$

Since $\ell_0 \in V_{ab}^+(\lambda,\mu)$, the last inequality, together with (4.40), yields $\alpha(t) \leq 0$ for $t \in [a,b]$. Consequently, it follows from (4.41) that $u \equiv 0$. □

Proof of Theorem 4.4. Assume that the problem (1.1_0), (1.2_0) possesses a nontrivial solution u.

According to Theorem 2.1 (see p. 17) and the assumptions (4.11), (4.12), and $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, it is clear that $\ell_0 \in V_{ab}^+(\lambda,\mu)$. It follows easily from

Definition 2.1 (see p. 15) and the assumptions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ that u changes its sign. Define numbers M and m by (2.94) and choose $t_M, t_m \in [a,b]$ such that (2.95) holds. Obviously,

$$M > 0, \qquad m > 0, \tag{4.42}$$

and without loss of generality we can assume that $t_M < t_m$. From (1.1_0), (1.2_0), (2.1), (4.11), and (4.12), with respect to (2.94) and (4.42), we find

$$\begin{gathered}(M\gamma(t) + u(t))' \geq \ell_0(M\gamma + u)(t) + \ell_1(M - u)(t) \geq \\ \geq \ell_0(M\gamma + u)(t) \quad \text{for} \quad t \in [a,b],\end{gathered} \tag{4.43}$$

$$|\lambda|(M\gamma(a) + u(a)) - |\mu|(M\gamma(b) + u(b)) \geq 0, \tag{4.44}$$

and

$$\begin{gathered}(m\gamma(t) - u(t))' \geq \ell_0(m\gamma - u)(t) + \ell_1(m + u)(t) \geq \\ \geq \ell_0(m\gamma - u)(t) \quad \text{for} \quad t \in [a,b],\end{gathered} \tag{4.45}$$

$$|\lambda|(m\gamma(a) - u(a)) - |\mu|(m\gamma(b) - u(b)) \geq 0. \tag{4.46}$$

Hence, according to the condition $\ell_0 \in V_{ab}^+(\lambda, \mu)$ and Remark 2.3 (p. 16), we get

$$M\gamma(t) + u(t) \geq 0, \qquad m\gamma(t) - u(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

By virtue of the last two inequalities and the assumption $\ell_0 \in \mathcal{P}_{ab}$, it follows from (4.43) and (4.45) that

$$(M\gamma(t) + u(t))' \geq 0, \qquad (m\gamma(t) - u(t))' \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{4.47}$$

The integration of the first inequality in (4.47) from t_M to t_m, in view of (2.95) and (4.42), results in

$$M\gamma(t_m) - m - M\gamma(t_M) - M \geq 0,$$

i.e.,

$$\gamma(t_m) - \gamma(t_M) \geq 1 + \frac{m}{M}\,. \tag{4.48}$$

On the other hand, the integration of the second inequality in (4.47) from a to t_M and from t_m to b, on account of (2.95), yields

$$m\gamma(t_M) - M - m\gamma(a) + u(a) \geq 0,$$

$$m\gamma(b) - u(b) - m\gamma(t_m) - m \geq 0.$$

Summing these two inequalities and taking into account (4.42) and

$$u(b) - u(a) = u(b)\left(1 - \left|\frac{\mu}{\lambda}\right|\right) \geq -m\left(1 - \left|\frac{\mu}{\lambda}\right|\right),$$

we get

$$\gamma(t_M) - \gamma(t_m) + \gamma(b) - \gamma(a) \geq \left|\frac{\mu}{\lambda}\right| + \frac{M}{m}\,. \tag{4.49}$$

Now, from (4.48) and (4.49) we have

$$\gamma(b) - \gamma(a) \geq 1 + \left|\frac{\mu}{\lambda}\right| + \frac{m}{M} + \frac{M}{m} \geq 3 + \left|\frac{\mu}{\lambda}\right|,$$

which contradicts (4.13). □

Proof of Theorem 4.5. Let u be a solution of the problem (1.1_0), (1.2_0).

First we will show that each of the assumptions (4.16), (4.17) or (4.18) ensure u not to assume both positive and negative values. Suppose on the contrary that u changes its sign. Define numbers M and m by (2.94) and choose $t_M, t_m \in [a,b]$ such that (2.95) holds. Obviously, (2.96) is satisfied.

If (4.16) is fulfilled, then analogously to the proof of Theroem 4.1 (with $\ell_0 \equiv \ell$ and $\ell_1 \equiv 0$), it can be shown that

$$0 < 4\left|\frac{\mu}{\lambda}\right| \leq \left(\|\ell(1)\|_L - 2\right)^2,$$

which contradicts (4.16).

If (4.17) holds, then, in view of Definition 2.1 (see p. 15), the assumption $u(a) \geq 0$ (resp. $u(a) < 0$) implies $u(t) \geq 0$ (resp. $u(t) \leq 0$) for $t \in [a,b]$, which contradicts (2.96).

If (4.18) holds, then, in view of Definition 2.1 (see p. 15), the assumption $u(b) \geq 0$ (resp. $u(b) < 0$) implies $u(t) \geq 0$ (resp. $u(t) \leq 0$) for $t \in [a,b]$, which contradicts (2.96).

Therefore, u does not change its sign and without loss of generality we can assume that

$$u(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{4.50}$$

It follows from (1.1_0), (4.50), and the assumption $\ell \in \mathcal{P}_{ab}$ that

$$u'(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{4.51}$$

Suppose that $u(a) > 0$. Then, in view of (4.51), we have

$$u(t) > 0 \quad \text{for} \quad t \in [a,b]. \tag{4.52}$$

Put

$$r = \max\left\{\frac{\gamma(t)}{u(t)} : t \in [a,b]\right\} \tag{4.53}$$

and

$$v(t) = ru(t) - \gamma(t) \quad \text{for} \quad t \in [a,b]. \tag{4.54}$$

According to (4.15), (4.52), (4.53), and the assumptions $0 \neq |\mu| \leq |\lambda|$ and $\gamma \in \widetilde{C}\big([a,b]; R_+\big)$, we get

$$r > 0. \tag{4.55}$$

It is clear that

$$v(t) \geq 0 \quad \text{for} \quad t \in [a,b] \tag{4.56}$$

and there exists $t_0 \in [a,b]$ such that

$$v(t_0) = 0. \tag{4.57}$$

By virtue of (1.1_0), (1.2_0), (2.1), (4.14), (4.15), (4.54)–(4.56), and the assumption $\ell \in \mathcal{P}_{ab}$, we have

$$v'(t) \geq \ell(v)(t) \geq 0 \quad \text{for} \quad t \in [a,b], \qquad |\lambda| v(a) > |\mu| v(b).$$

From the last two inequalities, (4.56), and the assumption $\lambda \neq 0$, we get

$$v(t_0) \geq v(a) > \left|\frac{\mu}{\lambda}\right| v(b) \geq 0,$$

which contradicts (4.57). Therefore, $u(a) = 0$ and, on account of (1.2_0), (4.51), and the assumption $\mu \neq 0$, we find $u \equiv 0$. □

Proof of Theorem 4.6. Let u be a solution of the problem (1.1_0), (1.2_0). Then, in view of (2.1) and (4.21), we get

$$|u(t)|' = \ell(u)(t) \operatorname{sgn} u(t) \leq \ell_0(|u|)(t) \quad \text{for} \quad t \in [a,b],$$

$$|\lambda u(a)| - |\mu u(b)| = 0.$$

Now, according to (4.20) and Remark 2.3 (see p. 16), we obtain $|u(t)| \leq 0$ for $t \in [a,b]$, i.e., $u \equiv 0$. □

4.3. Comments and Examples

On Remark 4.1. Let $0 \neq |\mu| \leq |\lambda|$. Below, for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin H^+ \cup H^-$ the functions $p \in L\big([a,b];R\big)$ and $\tau \in \mathcal{M}_{ab}$ are constructed such that (2.130) holds, and the problem

$$u'(t) = p(t)u(\tau(t)), \qquad \lambda u(a) + \mu u(b) = 0 \tag{4.58}$$

has a nontrivial solution. Then by Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (1.1), (1.2), where $\ell = \ell_0 - \ell_1$ and ℓ_0, ℓ_1 are defined by (2.132), has no solution.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin H^+ \cup H^-$, then (x_0, y_0) belongs at least to one of the following sets:

$$H_1 = \left\{(x,y) \in R_+ \times R_+ \ : \ 1 \leq x, \ \left|\frac{\mu}{\lambda}\right| \leq y\right\},$$
$$H_2 = \left\{(x,y) \in R_+ \times R_+ \ : \ x < 1, \ 1 + \left|\frac{\mu}{\lambda}\right| + 2\sqrt{1-x} \leq y\right\},$$
$$H_3 = \left\{(x,y) \in R_+ \times R_+ \ : \ y < \left|\frac{\mu}{\lambda}\right|, \ 2 + 2\sqrt{\left|\frac{\mu}{\lambda}\right| - y} \leq x\right\},$$
$$H_4 = \left\{(x,y) \in R_+ \times R_+ \ : \ y < \left|\frac{\mu}{\lambda}\right|, \ y + 1 - \left|\frac{\mu}{\lambda}\right| \leq \right.$$
$$\left. \leq x \leq \frac{|\lambda|(y+1) - |\mu|}{|\mu| - |\lambda| y}\right\},$$
$$H_5 = \left\{(x,y) \in R_+ \times R_+ \ : \ 1 - \left|\frac{\mu}{\lambda}\right| < x < 1, \ \left|\frac{\lambda}{\mu}\right|(x-1) + 1 \leq \right.$$
$$\left. \leq y \leq \frac{|\lambda|(x-1) + |\mu|}{|\mu|(1-x)}\right\},$$
$$H_6 = \left\{(x,y) \in R_+ \times R_+ \ : \ 1 - \left|\frac{\mu}{\lambda}\right| < x < 1, \ x - 1 + \left|\frac{\mu}{\lambda}\right| \leq \right.$$
$$\left. \leq y \leq \left|\frac{\lambda}{\mu}\right|(x-1) + 1\right\}.$$

Let $(x_0, y_0) \in H_1$. Put $a = 0$, $b = 4$,

$$p(t) = \begin{cases} -\left|\frac{\mu}{\lambda}\right| & \text{for } t \in [0,1[\\ x_0 - 1 & \text{for } t \in [1,2[\\ \left|\frac{\mu}{\lambda}\right| - y_0 & \text{for } t \in [2,3[\\ 1 & \text{for } t \in [3,4] \end{cases}, \qquad \tau(t) = \begin{cases} 4 & \text{for } t \in [0,1[\cup [3,4] \\ 1 & \text{for } t \in [1,3[\end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t)=\begin{cases}|\mu|(1-t) & \text{for } t\in[0,1[\\ 0 & \text{for } t\in[1,3[\\ |\lambda|(t-3) & \text{for } t\in[3,4]\end{cases}.$$

Let $(x_0,y_0)\in H_2$. Put $a=0$, $b=6$, $\alpha=\sqrt{1-x_0}$, $\beta=y_0-1-\left|\frac{\mu}{\lambda}\right|-2\alpha$,

$$p(t)=\begin{cases}-\left|\frac{\mu}{\lambda}\right| & \text{for } t\in[0,1[\\ -\beta & \text{for } t\in[1,2[\\ -\alpha & \text{for } t\in[2,4[\\ -1 & \text{for } t\in[4,5[\\ x_0 & \text{for } t\in[5,6]\end{cases},\quad \tau(t)=\begin{cases}6 & \text{for } t\in[0,1[\,\cup\,[2,3[\,\cup\,[5,6]\\ 1 & \text{for } t\in[1,2[\\ 3 & \text{for } t\in[3,5[\end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t)=\begin{cases}\left|\frac{\mu}{\lambda}\right|(1-t) & \text{for } t\in[0,1[\\ 0 & \text{for } t\in[1,2[\\ \alpha(2-t) & \text{for } t\in[2,3[\\ \alpha^2(t-3)-\alpha & \text{for } t\in[3,4[\\ \alpha(t-5)+\alpha^2 & \text{for } t\in[4,5[\\ x_0(t-6)+1 & \text{for } t\in[5,6]\end{cases}.$$

Let $(x_0,y_0)\in H_3$. Put $a=0$, $b=6$, $\alpha=\sqrt{\left|\frac{\mu}{\lambda}\right|-y_0}$, $\beta=x_0-2-2\alpha$,

$$p(t)=\begin{cases}\alpha & \text{for } t\in[0,1[\\ -y_0 & \text{for } t\in[1,2[\\ \beta & \text{for } t\in[2,3[\\ 1 & \text{for } t\in[3,4[\\ \alpha & \text{for } t\in[4,5[\\ 1 & \text{for } t\in[5,6]\end{cases},\quad \tau(t)=\begin{cases}4 & \text{for } t\in[0,1[\,\cup\,[3,4[\\ 6 & \text{for } t\in[1,2[\,\cup\,[4,6]\\ 2 & \text{for } t\in[2,3[\end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t) = \begin{cases} -\alpha^2 t + \left|\frac{\mu}{\lambda}\right| & \text{for } t \in [0,1[\\ y_0(2-t) & \text{for } t \in [1,2[\\ 0 & \text{for } t \in [2,3[\\ \alpha(3-t) & \text{for } t \in [3,4[\\ \alpha(t-5) & \text{for } t \in [4,5[\\ t-5 & \text{for } t \in [5,6] \end{cases}.$$

Let $(x_0, y_0) \in H_4$. Put $a = 0$, $b = 2$, $\alpha = |\lambda|(1+y_0) - |\mu|$,

$$t_0 = \begin{cases} 2 & \text{if } |\lambda| = |\mu|,\ x_0 = 0,\ y_0 = 0 \\ \frac{1}{x_0} - \frac{|\lambda|}{\alpha} + 2 & \text{otherwise} \end{cases},$$

$$p(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[\\ x_0 & \text{for } t \in [1,2] \end{cases}, \quad \tau(t) = \begin{cases} 2 & \text{for } t \in [0,1[\\ t_0 & \text{for } t \in [1,2] \end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t) = \begin{cases} -y_0|\lambda|t + |\mu| & \text{for } t \in [0,1[\\ \alpha(t-2) + |\lambda| & \text{for } t \in [1,2] \end{cases}.$$

Let $(x_0, y_0) \in H_5$. Put $a = 0$, $b = 2$, $\alpha = \frac{|\mu|+|\lambda|(x_0-1)}{1-x_0}$, $\beta = \frac{|\mu|x_0}{1-x_0}$, $t_0 = \left(\frac{\alpha}{y_0} - |\mu|\right)\frac{1}{\beta}$,

$$p(t) = \begin{cases} x_0 & \text{for } t \in [0,1[\\ -y_0 & \text{for } t \in [1,2] \end{cases}, \quad \tau(t) = \begin{cases} 1 & \text{for } t \in [0,1[\\ t_0 & \text{for } t \in [1,2] \end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t) = \begin{cases} \beta t + |\mu| & \text{for } t \in [0,1[\\ \alpha(2-t) + |\lambda| & \text{for } t \in [1,2] \end{cases}.$$

Let $(x_0, y_0) \in H_6$. Put $a = 0$, $b = 2$, $\alpha = |\mu| + |\lambda|(x_0 - 1)$, $t_0 = \frac{\alpha - y_0|\lambda|}{|\lambda|x_0 y_0} + 2$,

$$p(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[\\ x_0 & \text{for } t \in [1,2] \end{cases}, \quad \tau(t) = \begin{cases} t_0 & \text{for } t \in [0,1[\\ 2 & \text{for } t \in [1,2] \end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t) = \begin{cases} -\alpha t + |\mu| & \text{for } t \in [0,1[\\ x_0|\lambda|(t-2) + |\lambda| & \text{for } t \in [1,2] \end{cases}.$$

On Remark 4.2. Let $\mu = 0$. Below, for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin H$ the functions $p \in L([a,b];R)$ and $\tau \in \mathcal{M}_{ab}$ are constructed such that (2.130) holds, and the problem (4.58) has a nontrivial solution. Then by Remark 1.1 (see p. 14), there exist $q \in L([a,b];R)$ and $c \in R$ such that the problem (1.1), (1.2), where $\ell = \ell_0 - \ell_1$ and ℓ_0, ℓ_1 are defined by (2.132), has no solution.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin H$, then (x_0, y_0) belongs at least to one of the following sets:

$$\widetilde{H}_1 = \{(x,y) \in R_+ \times R_+ \ : \ 1 \le x\},$$
$$\widetilde{H}_2 = \left\{(x,y) \in R_+ \times R_+ \ : \ x < 1, \ 1 + 2\sqrt{1-x} \le y\right\}.$$

Let $(x_0, y_0) \in \widetilde{H}_1$. Put $a = 0$, $b = 2$, $t_0 = \frac{1}{x_0}$,

$$p(t) = \begin{cases} x_0 & \text{for } t \in [0,1[\\ -y_0 & \text{for } t \in [1,2] \end{cases}, \quad \tau(t) = \begin{cases} t_0 & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,2] \end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t) = \begin{cases} t & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,2] \end{cases}.$$

Let $(x_0, y_0) \in \widetilde{H}_2$. Put $a = 0$, $b = 5$, $\alpha = \sqrt{1 - x_0}$, $\beta = y_0 - 1 - 2\alpha$,

$$p(t) = \begin{cases} -\beta & \text{for } t \in [0,1[\\ -\alpha & \text{for } t \in [1,2[\cup [3,4[\\ -1 & \text{for } t \in [2,3[\\ x_0 & \text{for } t \in [4,5] \end{cases}, \quad \tau(t) = \begin{cases} 0 & \text{for } t \in [0,1[\\ 5 & \text{for } t \in [1,2[\cup [4,5] \\ 2 & \text{for } t \in [2,4[\end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t) = \begin{cases} 0 & \text{for } t \in [0,1[\\ \alpha(1-t) & \text{for } t \in [1,2[\\ \alpha(t-3) & \text{for } t \in [2,3[\\ \alpha^2(t-3) & \text{for } t \in [3,4[\\ x_0(t-5)+1 & \text{for } t \in [4,5] \end{cases} .$$

Example 4.1. Let $|\mu| < |\lambda|$, $\varepsilon > 0$, and let the operators $\ell, \ell_0, \ell_1 \in \mathcal{L}_{ab}$ be defined as follows:

$$\begin{gathered} \ell(v)(t) \stackrel{\text{def}}{=} (1+\varepsilon)p(t)v(b) \quad \text{for} \quad t \in [a,b], \\ \ell_0(v)(t) \stackrel{\text{def}}{=} p(t)v(b) \quad \text{for} \quad t \in [a,b], \qquad \ell_1 \equiv 0, \end{gathered} \tag{4.59}$$

where $p \in L\big([a,b]; R_+\big)$ is such that

$$\int_a^b p(s)ds = \frac{|\lambda| - |\mu|}{(1+\varepsilon)|\lambda|} . \tag{4.60}$$

According to Remark 2.5 (see p. 19), we have

$$\ell_0 \in V_{ab}^+(\lambda,\mu), \qquad -\frac{1}{2}\ell_1 \in V_{ab}^+(\lambda,\mu).$$

Therefore, the assumptions of Theorem 4.3 are fulfilled except of the inequality (4.7), instead of which the inequality (4.9) is satisfied.

On the other hand, the problem (1.1_0), (1.2_0) has the nontrivial solution

$$u(t) = |\mu| + (1+\varepsilon)|\lambda| \int_a^t p(s)ds \quad \text{for} \quad t \in [a,b].$$

Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b]; R\big)$ and $c \in R$ such that the problem (1.1), (1.2) has no solution.

Example 4.2. Let $|\mu| < |\lambda|$, $\varepsilon \in \,]0,1[\,$, and let $\ell \in \mathcal{L}_{ab}$ be defined by

$$\ell(v)(t) \stackrel{\text{def}}{=} p(t)v(b) \quad \text{for} \quad t \in [a,b], \tag{4.61}$$

where $p \in L\big([a,b]; R_+\big)$ is such that

$$\int_a^b p(s)ds = \frac{|\lambda| - |\mu|}{|\lambda|}\,. \tag{4.62}$$

Put $\ell_0 \equiv \ell$, $\ell_1 \equiv 0$. Evidently, the inequality (4.7) is fulfilled and, according to Remark 2.5 (see p. 19), we have

$$(1-\varepsilon)\ell_0 \in V_{ab}^+(\lambda,\mu), \qquad -\frac{1}{2}\ell_1 \in V_{ab}^+(\lambda,\mu).$$

On the other hand, the problem (1.1_0), (1.2_0) has the nontrivial solution

$$u(t) = |\mu| + |\lambda| \int_a^t p(s)ds \quad \text{for} \quad t \in [a,b].$$

Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b]; R\big)$ and $c \in R$ such that the problem (1.1), (1.2) has no solution.

Example 4.3. Let $|\mu| < |\lambda|$, $a = 0$, $b = 3$, $\varepsilon > 0$, $\delta = \frac{\varepsilon(|\lambda|-|\mu|)}{(1+\varepsilon)|\lambda|}$, and $\ell \in \mathcal{L}_{ab}$ be an operator defined by

$$\ell(v)(t) \overset{\text{def}}{=} p(t)v(\tau(t)) \quad \text{for} \quad t \in [a,b], \tag{4.63}$$

where

$$p(t) = \begin{cases} \frac{|\lambda|-|\mu|}{|\lambda|} - \delta & \text{for } t \in [0,1[\\ -\frac{2-\delta}{1-\delta} & \text{for } t \in [1,2[\\ -2 & \text{for } t \in [2,3] \end{cases}, \qquad \tau(t) = \begin{cases} 3 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,2[\\ 2 & \text{for } t \in [2,3] \end{cases}.$$

Let, moreover,

$$\ell_0(v)(t) \overset{\text{def}}{=} [p(t)]_+ v(\tau_0(t)), \qquad \ell_1(v)(t) \overset{\text{def}}{=} [p(t)]_- v(\tau_1(t)) \quad \text{for} \quad t \in [a,b], \tag{4.64}$$

where $\tau_0 \equiv 3$ and

$$\tau_1(t) = \begin{cases} 0 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,2[\\ 2 & \text{for } t \in [2,3] \end{cases}.$$

It is clear that $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and the condition (4.7) is fulfilled. Moreover,

$$\int_0^3 \ell_0(1)(s)ds = \int_0^1 p_0(s)ds = \frac{|\lambda| - |\mu|}{|\lambda|} - \delta < \frac{|\lambda| - |\mu|}{|\lambda|} .$$

Consequently, according to Remark 2.5 (see p. 19), we have $\ell_0 \in V_{ab}^{+}(\lambda, \mu)$. It is not difficult to verify that the homogeneous problem

$$u'(t) = -\frac{1}{2+\varepsilon}\,\ell_1(u)(t), \qquad \lambda u(0) + \mu u(3) = 0$$

has only the trivial solution, and for arbitrary $q \in L\big([0,3]; R_+\big)$ and $c \in R$ satisfying (2.2) the solution of the problem

$$u'(t) = -\frac{1}{2+\varepsilon}\,\ell_1(u)(t) + q(t), \qquad \lambda u(0) + \mu u(3) = c$$

is nonnegative. Therefore, by Definition 2.1 (see p. 15), we obtain

$$-\frac{1}{2+\varepsilon}\,\ell_1 \in V_{ab}^{+}(\lambda, \mu).$$

On the other hand, the function

$$u(t) = \begin{cases} \left(\frac{|\lambda|-|\mu|}{|\lambda|} - \delta\right) t + \left|\frac{\mu}{\lambda}\right| & \text{for } t \in [0,1[\\ (2-\delta)(1-t) + 1 - \delta & \text{for } t \in [1,2[\\ 2t - 5 & \text{for } t \in [2,3] \end{cases}$$

is a nontrivial solution of the problem (1.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b]; R\big)$ and $c \in R$ such that the problem (1.1), (1.2) has no solution.

Example 4.4. Let $|\mu| < |\lambda|$, $a = 0$, $b = 4$, $\varepsilon \geq 0$,

$$\ell_0 \equiv 0, \qquad \ell_1(v)(t) \overset{\text{def}}{=} g(t)v(\nu(t)) \quad \text{for} \quad t \in [a,b], \tag{4.65}$$

where

$$g(t) = \begin{cases} 1 + \left|\frac{\mu}{\lambda}\right| & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,3[\\ \varepsilon & \text{for } t \in [3,4] \end{cases}, \qquad \nu(t) = \begin{cases} 3 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,3[\\ 2 & \text{for } t \in [3,4] \end{cases}.$$

Put

$$\gamma(t) = \delta + \int_a^t g(s)ds \quad \text{for} \quad t \in [a, b]. \tag{4.66}$$

where $\delta > \frac{|\mu|}{|\lambda|-|\mu|} \int_a^b g(s)ds$. It is clear that $\gamma \in \widetilde{C}\big([a,b];\,]0,+\infty[\big)$, the conditions (4.11) and (4.12) hold, and

$$\gamma(b) - \gamma(a) = 3 + \left|\frac{\mu}{\lambda}\right| + \varepsilon.$$

On the other hand, the problem

$$u'(t) = -g(t)u(\nu(t)), \qquad \lambda u(a) + \mu u(b) = 0$$

has the nontrivial solution

$$u(t) = \begin{cases} |\mu| - (|\mu| + |\lambda|)t & \text{for } t \in [0, 1[\\ |\lambda|(t-2) & \text{for } t \in [1, 3[\\ |\lambda| & \text{for } t \in [3, 4] \end{cases}.$$

Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (1.1), (1.2) with $\ell = \ell_0 - \ell_1$ has no solution.

Example 4.5. Let $0 \neq |\mu| \leq |\lambda|$, $\varepsilon \geq 0$, $a = 0$, $b = 5$, and let $\ell \in \mathcal{P}_{ab}$ be defined by (4.63), where

$$p(t) = \begin{cases} \sqrt{\left|\frac{\mu}{\lambda}\right|} & \text{for} \quad t \in [0,1[\cup [2,3[\\ 1 & \text{for} \quad t \in [1,2[\cup [3,4[\\ \varepsilon & \text{for} \quad t \in [4,5] \end{cases}, \qquad \tau(t) = \begin{cases} 2 & \text{for} \quad t \in [0,2[\\ 4 & \text{for} \quad t \in [2,4[\\ 3 & \text{for} \quad t \in [4,5] \end{cases}.$$

Let, moreover, the function $\gamma \in \widetilde{C}\big([a,b];R_+\big)$ be defined by

$$\gamma(t) = \begin{cases} 0 & \text{for} \quad t \in [0,3[\\ t-3 & \text{for} \quad t \in [3,4[\\ 1 & \text{for} \quad t \in [4,5] \end{cases}.$$

Obviously,

$$\int_a^b \ell(1)(s)ds = \int_0^5 p(s)ds = 2 + 2\sqrt{\left|\frac{\mu}{\lambda}\right|} + \varepsilon$$

and the function γ satisfies (4.14) and (4.15).

On the other hand, the problem (1.1_0), (1.2_0) has a nontrivial solution

$$u(t) = \begin{cases} |\mu|(1-t) & \text{for} \quad t \in [0,1[\\ \sqrt{|\lambda\mu|}(1-t) & \text{for} \quad t \in [1,2[\\ \sqrt{|\lambda\mu|}(t-3) & \text{for} \quad t \in [2,3[\\ |\lambda|(t-3) & \text{for} \quad t \in [3,4[\\ |\lambda| & \text{for} \quad t \in [4,5] \end{cases}.$$

Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (1.1), (1.2) has no solution.

Example 4.6. Let $0 \neq |\mu| < |\lambda|$ and let $\ell \in \mathcal{P}_{ab}$ be defined by

$$\ell(v)(t) \stackrel{\text{def}}{=} p(t)v(t) \quad \text{for} \quad t \in [a,b],$$

where $p \in L\big([a,b];R_+\big)$ is such that

$$\int_a^b p(s)ds = \ln\left|\frac{\lambda}{\mu}\right|.$$

Let, moreover, the function $\gamma \in L\big([a,b];R_+\big)$ be defined by

$$\gamma(t) = |\mu| \exp\left(\int_a^t p(s)ds\right) \quad \text{for} \quad t \in [a,b].$$

Obviously, γ satisfies (4.14) and (4.19). Furthermore, if $\left|\frac{\mu}{\lambda}\right| \in]\delta_0, 1[$, where $\delta_0 \in]0,1[$ is such that

$$\ln\frac{1}{\delta_0} = 2 + 2\sqrt{\delta_0}\,,$$

then the condition (4.16) is fulfilled. Moreover, according to Theorem 3.1 a) (see p. 63) and Theorem 3.14 b) (see p. 72), we have $\ell \in V_{ab}^{+}(1,0)$ and $\ell \in V_{ab}^{-}(0,1)$.

On the other hand, the function γ is a nontrivial solution of the problem (1.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (1.1), (1.2) has no solution.

Example 4.7. Let $|\mu| > |\lambda| \neq 0$. Below, the operator $\ell \in \mathcal{L}_{ab}$ is constructed in such a way that the homogeneous problem (1.1_0), (1.2_0) has a nontrivial solution. Then, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (1.1), (1.2) has no solution.

Let $\varepsilon \in \,]0,1[$ and $\ell, \ell_0 \in \mathcal{L}_{ab}$ be defined by

$$\ell(v)(t) \stackrel{\text{def}}{=} -g(t)v(t), \qquad \ell_0(v)(t) \stackrel{\text{def}}{=} -\frac{1}{1-\varepsilon}\, g(t)v(t),$$

where $g \in L\big([a,b];R_+\big)$ is such that

$$\int_a^b g(s)ds = \ln\left|\frac{\mu}{\lambda}\right| . \tag{4.67}$$

According to Corollary 3.3 (see p. 68), we have $\ell_0 \in V_{ab}^+(\lambda,\mu)$. Obviously, the assumptions of Theorem 4.6 are fulfilled except of the condition (4.21), instead of which the condition (4.24) is satisfied.

On the other hand, the problem (1.1_0), (1.2_0) has the nontrivial solution

$$u(t) = |\lambda| \exp\left(\int_t^b g(s)ds\right) \quad \text{for} \quad t \in [a,b]. \tag{4.68}$$

Thus, the inequality (4.21) in Theorem 4.6 cannot be replaced by the inequality (4.24), no matter how small $\varepsilon > 0$ would be.

Let $\varepsilon > 0$ and $\ell \in \mathcal{L}_{ab}$ be defined by

$$\ell(v)(t) \stackrel{\text{def}}{=} -g(t)v(t) \quad \text{for} \quad t \in [a,b],$$

where $g \in L\big([a,b];R_+\big)$ is such that (4.67) holds. Put $\ell_0 \equiv \ell$. Evidently, the condition (4.21) is fulfilled and, according to Corollary 3.3 (see p. 68), we have $(1+\varepsilon)\ell_0 \in V_{ab}^+(\lambda,\mu)$.

On the other hand, the problem (1.1_0), (1.2_0) has the nontrivial solution u given by (4.68). Thus, the assumption (4.20) in Theorem 4.6 cannot be replaced by the assumption (4.25), no matter how small $\varepsilon > 0$ would be.

§5. Periodic Type BVP for EDA

In this section, we will establish some consequences of the main results from §4 for the equation with deviating arguments (1.1′). Here we will also suppose that the inequality (2.1) is fulfilled.

In what follows we will use the notation

$$p_0(t) = \sum_{j=1}^{m} p_j(t), \qquad g_0(t) = \sum_{j=1}^{m} g_j(t) \quad \text{for} \quad t \in [a,b]$$

and we will suppose that if $\lambda = -\mu$, then $p_0 \not\equiv g_0$.

5.1. Existence and Uniqueness Theorems

In the case, where $|\mu| \le |\lambda|$, the following statements hold.

Theorem 5.1. *Let* $0 \ne |\mu| \le |\lambda|$, $p_k, g_k \in L\big([a,b]; R_+\big)$ $(k = 1, \dots, m)$, *and let either*

$$\|p_0\|_L < 1, \tag{5.1}$$

$$\frac{\|p_0\|_L}{1 - \|p_0\|_L} - \frac{|\lambda| - |\mu|}{|\mu|} < \|g_0\|_L < 1 + \left|\frac{\mu}{\lambda}\right| + 2\sqrt{1 - \|p_0\|_L} \tag{5.2}$$

or

$$\|g_0\|_L < \left|\frac{\mu}{\lambda}\right|, \tag{5.3}$$

$$\frac{|\lambda|}{|\mu| - |\lambda| \|g_0\|_L} - 1 < \|p_0\|_L < 2 + 2\sqrt{\left|\frac{\mu}{\lambda}\right| - \|g_0\|_L}\,. \tag{5.4}$$

Then the problem (1.1′), (1.2) *has a unique solution.*

Remark 5.1. The examples constructed in Subsection 4.3 (see On Remark 4.1, p. 94) also show that neither one of the strict inequalities in (5.1)–(5.4) can be replaced by the nonstrict one.

The next theorem can be understood as a supplement of the previous one for the case $\mu = 0$.

Theorem 5.2. *Let* $\mu = 0$, $p_k, g_k \in L\big([a,b]; R_+\big)$ $(k = 1, \dots, m)$, *and let*

$$\|p_0\|_L < 1, \tag{5.5}$$

$$\|g_0\|_L < 1 + 2\sqrt{1 - \|p_0\|_L}\,. \tag{5.6}$$

Then the problem (1.1′), (1.2) *has a unique solution.*

Remark 5.2. The examples constructed in Subsection 4.3 (see On Remark 4.2, p. 97) also show that the strict inequalities (5.5) and (5.6) cannot be replaced by the nonstrict ones.

Theorem 5.3. *Let* $|\mu| < |\lambda|$, $p_k, g_k \in L\big([a,b]; R_+\big)$, *and* $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$. *Let, moreover, the functions* p_k, τ_k $(k = 1, \dots, m)$ *satisfy at least one of the conditions a), b) or c) in Theorem 3.1 (see p. 63) or the assumptions of Theorem 3.2 (see p. 64), while the functions* g_k, ν_k $(k = 1, \dots, m)$ *satisfy* $\nu_k(t) \le t$ *for* $t \in [a,b]$ $(k = 1, \dots, m)$ *and at least one of the following items:*

a)

$$\int\limits_a^b g_0(s)ds \le 2;$$

b)

$$\int\limits_a^b \sum_{k=1}^m g_k(s) \int\limits_{\nu_k(s)}^s \sum_{i=1}^m g_i(\xi) \exp\left(\frac{1}{2} \int\limits_{\nu_i(\xi)}^s g_0(\eta)d\eta\right) d\xi ds \le 4;$$

c) $g_0 \not\equiv 0$ *and*

$$\text{ess sup}\left\{ \int\limits_{\nu_k(t)}^t g_0(s)ds : t \in [a,b] \right\} < 2\eta^* \qquad (k = 1, \dots, m),$$

where

$$\eta^* = \sup\left\{ \frac{1}{x} \ln\left(x + \frac{x}{\exp\left(\frac{x}{2} \int\limits_a^b g_0(s)ds\right) - 1} \right) : x > 0 \right\}.$$

Then the problem (1.1′), (1.2) *has a unique solution.*

Theorem 5.4. *Let* $|\mu| < |\lambda|$, $p_k, g_k \in L([a,b];R_+)$, *and* $\tau_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$ *such that*

$$|\mu| \exp\left(\int_a^b p_0(s)ds\right) < |\lambda|, \tag{5.7}$$

$$\tau_k(t) \le t \quad for \quad t \in [a,b] \quad (k = 1,\dots,m), \tag{5.8}$$

and

$$\frac{|\lambda| - |\mu|}{|\lambda| - |\mu| \exp\left(\int_a^b p_0(s)ds\right)} \int_a^b g_0(s) \exp\left(\int_s^b p_0(\xi)d\xi\right) ds < 3 + \left|\frac{\mu}{\lambda}\right|. \tag{5.9}$$

Then the problem (1.1′), (1.2) *has a unique solution.*

Theorem 5.5. *Let* $|\mu| < |\lambda|$, $p_k, g_k \in L([a,b];R_+)$, *and* $\tau_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$ *such that*

$$\frac{|\lambda| - |\mu|}{|\lambda|} \left(\int_a^b g_0(s)ds + \alpha_1\right) + \left(3 + \left|\frac{\mu}{\lambda}\right|\right) \beta_1 < 3 + \left|\frac{\mu}{\lambda}\right|, \tag{5.10}$$

where

$$\alpha_1 = \int_a^b \sum_{k=1}^m p_k(s) \left(\int_a^{\tau_k(s)} g_0(\xi)d\xi\right) \exp\left(\int_s^b p_0(\xi)d\xi\right) ds, \tag{5.11}$$

$$\begin{gathered} \beta_1 = \left|\frac{\mu}{\lambda}\right| \exp\left(\int_a^b p_0(s)ds\right) + \\ + \int_a^b \sum_{k=1}^m p_k(s)\sigma_k(s) \left(\int_s^{\tau_k(s)} p_0(\xi)d\xi\right) \exp\left(\int_s^b p_0(\xi)d\xi\right) ds, \end{gathered} \tag{5.12}$$

$$\sigma_k(t) = \frac{1}{2}\left(1 + \operatorname{sgn}(\tau_k(t) - t)\right) \quad for \quad t \in [a,b] \quad (k = 1,\dots,m). \tag{5.13}$$

Then the problem (1.1′), (1.2) *has a unique solution.*

Remark 5.3. Example 4.4 also shows (see p. 100) that the strict inequality (5.9) in Theorem 5.4 and the strict inequality (5.10) in Theorem 5.5 cannot be replaced by the nonstrict ones.

Theorem 5.6. *Let* $0 \neq |\mu| \leq |\lambda|$, $p_k \in L\big([a,b];R_+\big)$ $(k=1,\dots,m)$, $p_0 \not\equiv 0$, *and let there exist* $x \in \left[\left|\frac{\lambda}{\mu}\right|, +\infty\right[$ *such that the condition* (3.25) *holds, where* η *is defined by* (3.8). *Let, moreover, at least one of the following items be fulfilled:*

a)

$$\int_a^b p_0(s) < 2 + 2\sqrt{\left|\frac{\mu}{\lambda}\right|}\;;$$

b) there exists $\alpha \in]0,1[$ *such that*

$$\int_a^t \sum_{k=1}^m p_k(s)\left(\int_a^{\tau_k(s)} p_0(\xi)d\xi\right) ds \leq \alpha \int_a^t p_0(s)ds \quad \textit{for} \quad t \in [a,b];$$

c)

$$\int_a^b \sum_{k=1}^m p_k(s)\sigma_k(s)\left(\int_s^{\tau_k(s)} p_0(\xi)d\xi\right) \exp\left(\int_s^b p_0(\eta)d\eta\right) ds < 1,$$

where $\sigma_k(t) = \frac{1}{2}(1+\operatorname{sgn}(\tau_k(t)-t))$ *for* $t \in [a,b]$ $(k=1,\dots,m)$*;*

d) $\int\limits_a^{\tau^*} p_0(s)ds \neq 0$ *and*

$$\operatorname{ess\,sup}\left\{\int_t^{\tau_k(t)} p_0(s)ds : t \in [a,b]\right\} < \eta^* \qquad (k=1,\dots,m), \tag{5.14}$$

where

$$\eta^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp\left(x\int\limits_a^{\tau^*} p_0(s)ds\right) - 1}\right) : x > 0\right\} \tag{5.15}$$

with $\tau^* = \max\big\{\operatorname{ess\,sup}\{\tau_k(t) : t \in [a,b]\} : k=1,\dots,m\big\}$*;*

e) $\tau_k(t) \geq t$ *for* $t \in [a,b]$ $(k=1,\dots,m)$ *and*

$$\int_a^b \sum_{k=1}^m p_k(s) \int_s^{\tau_k(s)} \sum_{i=1}^m p_i(\xi) \exp\left(\int_s^{\tau_i(\xi)} p_0(\eta)d\eta \right) d\xi ds \leq 1\,.$$

Then the problem (1.1′), (1.2) *with* $g_k \equiv 0$ $(k=1,\dots,m)$ *has a unique solution.*

In the case, where $|\mu| \geq |\lambda|$, the following assertions hold.

Theorem 5.7. *Let* $|\mu| \geq |\lambda| \neq 0$, $p_k, g_k \in L\big([a,b];R_+\big)$ $(k=1,\dots,m)$, *and let either*

$$\|g_0\|_L < 1,$$

$$\frac{\|g_0\|_L}{1-\|g_0\|_L} - \frac{|\mu|-|\lambda|}{|\lambda|} < \|p_0\|_L < 1 + \left|\frac{\lambda}{\mu}\right| + 2\sqrt{1-\|g_0\|_L}$$

or

$$\|p_0\|_L < \left|\frac{\lambda}{\mu}\right|,$$

$$\frac{|\mu|}{|\lambda|-|\mu|\|p_0\|_L} - 1 < \|g_0\|_L < 2 + 2\sqrt{\left|\frac{\lambda}{\mu}\right| - \|p_0\|_L}\,.$$

Then the problem (1.1′), (1.2) *has a unique solution.*

The next theorem can be understood as a supplement of the previous one for the case $\lambda = 0$.

Theorem 5.8. *Let* $\lambda = 0$, $p_k, g_k \in L\big([a,b];R_+\big)$ $(k=1,\dots,m)$, *and*

$$\|g_0\|_L < 1,$$

$$\|p_0\|_L < 1 + 2\sqrt{1-\|g_0\|_L}\,.$$

Then the problem (1.1′), (1.2) *has a unique solution.*

Theorem 5.9. *Let* $|\mu| > |\lambda|$, $p_k, g_k \in L\big([a,b]; R_+\big)$, *and* $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$. *Let, moreover, the functions* g_k, ν_k $(k = 1, \dots, m)$ *satisfy at least one of the conditions a), b) or c) in Theorem 3.12 (see p. 70) or the assumptions of Theorem 3.13 (see p. 71), while the functions* p_k, τ_k $(k = 1, \dots, m)$ *satisfy* $\tau_k(t) \geq t$ *for* $t \in [a,b]$ $(k = 1, \dots, m)$, *and at least one of the following items:*

a)

$$\int_a^b p_0(s)ds \leq 2;$$

b)

$$\int_a^b \sum_{k=1}^m p_k(s) \int_s^{\tau_k(s)} \sum_{i=1}^m p_i(\xi) \exp\left(\frac{1}{2}\int_s^{\tau_i(\xi)} p_0(\eta)d\eta\right) d\xi ds \leq 4;$$

c) $p_0 \not\equiv 0$ *and*

$$\operatorname{ess\,sup}\left\{\int_t^{\tau_k(t)} p_0(s)ds : t \in [a,b]\right\} < 2\omega^* \qquad (k = 1, \dots, m),$$

where

$$\omega^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp\left(\frac{x}{2}\int_a^b p_0(s)ds\right) - 1}\right) : x > 0\right\}.$$

Then the problem $(1.1')$, (1.2) *has a unique solution.*

Theorem 5.10. *Let* $|\mu| > |\lambda|$, $p_k, g_k \in L\big([a,b]; R_+\big)$, *and* $\nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$ *such that*

$$|\lambda| \exp\left(\int_a^b g_0(s)ds\right) < |\mu|,$$

$$\nu_k(t) \geq t \quad \textit{for} \quad t \in [a,b] \quad (k = 1, \dots, m),$$

and

$$\frac{|\mu|-|\lambda|}{|\mu|-|\lambda|\exp\left(\int\limits_a^b g_0(s)ds\right)}\int\limits_a^b p_0(s)\exp\left(\int\limits_a^s g_0(\xi)d\xi\right)ds<3+\left|\frac{\lambda}{\mu}\right|.$$

Then the problem (1.1′), (1.2) *has a unique solution.*

Theorem 5.11. *Let* $|\mu|>|\lambda|$, $p_k, g_k\in L\big([a,b];R_+\big)$, *and* $\nu_k\in\mathcal{M}_{ab}$ $(k=1,\dots,m)$ *such that*

$$\frac{|\mu|-|\lambda|}{|\mu|}\left(\int\limits_a^b p_0(s)ds+\alpha_2\right)+\left(3+\left|\frac{\lambda}{\mu}\right|\right)\beta_2<3+\left|\frac{\lambda}{\mu}\right|,$$

where

$$\alpha_2=\int\limits_a^b\sum_{k=1}^m g_k(s)\left(\int\limits_{\nu_k(s)}^b p_0(\xi)d\xi\right)\exp\left(\int\limits_a^s g_0(\xi)d\xi\right)ds,$$

$$\beta_2=\left|\frac{\lambda}{\mu}\right|\exp\left(\int\limits_a^b g_0(s)ds\right)+$$

$$+\int\limits_a^b\sum_{k=1}^m g_k(s)\sigma_k(s)\left(\int\limits_{\nu_k(s)}^s g_0(\xi)d\xi\right)\exp\left(\int\limits_a^s g_0(\xi)d\xi\right)ds,$$

$$\sigma_k(t)=\frac{1}{2}\left(1+\operatorname{sgn}(t-\nu_k(t))\right)\quad \textit{for}\quad t\in[a,b]\quad (k=1,\dots,m).$$

Then the problem (1.1′), (1.2) *has a unique solution.*

Theorem 5.12. *Let* $|\mu|\geq|\lambda|\neq 0$, $g_k\in L\big([a,b];R_+\big)$ $(k=1,\dots,m)$, $g_0\not\equiv 0$, *and let there exist* $x\in\big[\big|\frac{\mu}{\lambda}\big|,+\infty\big[$ *such that the condition* (3.16) *holds, where* ω *is defined by* (3.17). *Let, moreover, at least one of the following items be fulfilled:*

a)

$$\int\limits_a^b g_0(s)<2+2\sqrt{\left|\frac{\lambda}{\mu}\right|};$$

b) there exists $\alpha \in]0,1[$ *such that*

$$\int_t^b \sum_{k=1}^m g_k(s) \left(\int_{\nu_k(s)}^s g_0(\xi) d\xi \right) ds \leq \alpha \int_t^b g_0(s) ds \quad \textit{for} \quad t \in [a,b];$$

c)

$$\int_a^b \sum_{k=1}^m g_k(s)\sigma_k(s) \left(\int_{\nu_k(s)}^t g_0(\xi) d\xi \right) \exp\left(\int_a^s g_0(\eta) d\eta \right) ds < 1,$$

where $\sigma_k(t) = \frac{1}{2}(1 + \operatorname{sgn}(t - \nu_k(t)))$ *for* $t \in [a,b]$ $(k = 1, \ldots, m)$*;*

d) $\int_{\nu_*}^b g_0(s) ds \neq 0$ *and*

$$\operatorname{ess\,sup} \left\{ \int_{\nu_k(t)}^t g_0(s) ds : t \in [a,b] \right\} < \omega^* \qquad (k = 1, \ldots, m),$$

where

$$\omega^* = \sup \left\{ \frac{1}{x} \ln \left(x + \frac{x}{\exp\left(x \int_{\nu_*}^b g_0(s) ds \right) - 1} \right) : x > 0 \right\}$$

with $\nu_* = \min\big\{ \operatorname{ess\,inf}\{\nu_k(t) : t \in [a,b]\} : k = 1, \ldots, m \big\}$*;*

e) $\nu_k(t) \leq t$ *for* $t \in [a,b]$ $(k = 1, \ldots, m)$ *and*

$$\int_a^b \sum_{k=1}^m g_k(s) \int_{\nu_k(s)}^s \sum_{i=1}^m g_i(\xi) \exp\left(\int_{\nu_i(\xi)}^s g_0(\eta) d\eta \right) d\xi ds \leq 1\,.$$

Then the problem (1.1′), (1.2) *with* $p_k \equiv 0$ $(k = 1, \ldots, m)$ *has a unique solution.*

Remark 5.4. Similarly as in the case $|\mu| \leq |\lambda|$ one can show that Theorems 5.7, 5.10 and 5.11 are also nonimprovable in a certain sense.

5.2. Proofs

Proof of Theorem 5.1. The validity of the theorem immediately follows from Theorem 4.1 (see p. 80). □

Proof of Theorem 5.2. The validity of the theorem immediately follows from Theorem 4.2 (see p. 81). □

Proof of Theorem 5.3. It is a consequence of Theorem 4.3 (see p. 83) and Theorems 3.1–3.3 (see pp. 63–65). □

Proof of Theorem 5.4. According to (5.9), there exists $\varepsilon > 0$ such that

$$\frac{\varepsilon}{|\lambda| - |\mu| \exp\left(\int\limits_a^b p_0(s)ds\right)} \left(\exp\left(\int\limits_a^b p_0(s)ds\right) - 1\right) + \\ + \frac{|\lambda| - |\mu|}{|\lambda| - |\mu| \exp\left(\int\limits_a^b p_0(s)ds\right)} \int\limits_a^b g_0(s) \exp\left(\int\limits_s^b p_0(\xi)d\xi\right) ds < 3 + \left|\frac{\mu}{\lambda}\right|. \tag{5.16}$$

Put

$$\gamma(t) = \frac{\varepsilon}{|\lambda| - |\mu| \exp\left(\int\limits_a^b p_0(s)ds\right)} \exp\left(\int\limits_a^t p_0(s)ds\right) + \\ + \frac{|\lambda|}{|\lambda| - |\mu| \exp\left(\int\limits_a^b p_0(s)ds\right)} \int\limits_a^t g_0(s) \exp\left(\int\limits_s^t p_0(\xi)d\xi\right) ds + \\ + \frac{|\mu| \exp\left(\int\limits_a^b p_0(s)ds\right)}{|\lambda| - |\mu| \exp\left(\int\limits_a^b p_0(s)ds\right)} \int\limits_t^b g_0(s) \exp\left(\int\limits_s^t p_0(\xi)d\xi\right) ds \quad \text{for} \quad t \in [a,b].$$

Then γ is a solution of the problem

$$\gamma'(t) = p_0(t)\gamma(t) + g_0(t), \qquad \lambda\gamma(a) + \mu\gamma(b) = \varepsilon \operatorname{sgn} \lambda. \tag{5.17}$$

Since $\varepsilon > 0$, in view of (5.7), we have $\gamma(t) > 0$ for $t \in [a,b]$. Consequently, (5.17) implies $\gamma'(t) \geq 0$ for $t \in [a,b]$, and thus, (5.8) yields

$$p_k(t)\gamma(t) \geq p_k(t)\gamma(\tau_k(t)) \quad \text{for} \quad t \in [a,b] \quad (k = 1,\dots,m). \tag{5.18}$$

On account of (2.1) and (5.16)–(5.18), the function γ satisfies the inequalities (4.11), (4.12), and (4.13) with

$$\begin{aligned} \ell_0(v)(t) &\stackrel{\text{def}}{=} \sum_{k=1}^{m} p_k(t)v(\tau_k(t)) \quad \text{for} \quad t \in [a,b], \\ \ell_1(v)(t) &\stackrel{\text{def}}{=} \sum_{k=1}^{m} g_k(t)v(\nu_k(t)) \quad \text{for} \quad t \in [a,b]. \end{aligned} \tag{5.19}$$

Therefore, the assumptions of Theorem 4.4 (see p. 83) are satisfied. □

Proof of Theorem 5.5. Let the operators ℓ_0 and ℓ_1 be defined by (5.19). From (5.10) we obtain $\beta_1 < 1$. Consequently, the assumptions of Theorem 3.1 c) (see p. 63) are fulfilled, and thus, $\ell_0 \in V_{ab}^{+}(\lambda,\mu)$. Choose $\delta > 0$ and $\varepsilon > 0$ such that

$$\frac{|\lambda| - |\mu|}{|\lambda|}(1-\beta_1)^{-1}\left(\alpha_1 + \int_a^b g_0(s)ds\right) < 3 + \left|\frac{\mu}{\lambda}\right| - \delta, \tag{5.20}$$

$$\varepsilon < \frac{\delta|\lambda|(1-\beta_1)}{|\lambda| - |\mu|} \exp\left(-\int_a^b p_0(s)ds\right). \tag{5.21}$$

According to the condition $\ell_0 \in V_{ab}^{+}(\lambda,\mu)$ and Remark 2.1 (see p. 15), the problem

$$\gamma'(t) = \sum_{i=1}^{m} p_i(t)\gamma(\tau_i(t)) + g_0(t), \tag{5.22}$$

$$\lambda\gamma(a) + \mu\gamma(b) = \lambda\varepsilon \tag{5.23}$$

has a unique solution γ. It is clear that the conditions (4.11) and (4.12) are fulfilled. Due to the conditions $\ell_0 \in V_{ab}^{+}(\lambda,\mu)$ and $\varepsilon > 0$, we get $\gamma(t) \geq 0$

for $t \in [a,b]$. Hence, the condition (4.12) implies $\gamma(a) > 0$. Taking now into account (5.22), it is evident that $\gamma(t) > 0$ for $t \in [a,b]$. On the other hand, γ is a solution of the equation

$$\gamma'(t) = p_0(t)\gamma(t) + \sum_{k=1}^{m} p_k(t) \int_t^{\tau_k(t)} \sum_{i=1}^{m} p_i(s)\gamma(\tau_i(s))ds +$$

$$+ \sum_{k=1}^{m} p_k(t) \int_t^{\tau_k(t)} g_0(s)ds + g_0(t).$$

Hence, the Cauchy formula implies

$$\gamma(b) \le \beta_1\gamma(b) + \alpha_1 + \int_a^b g_0(s)ds + \varepsilon \exp\left(\int_a^b p_0(s)ds\right).$$

The last inequality results in

$$\gamma(b) \le (1-\beta_1)^{-1}\left(\alpha_1 + \int_a^b g_0(s)ds\right) + \varepsilon(1-\beta_1)^{-1}\exp\left(\int_a^b p_0(s)ds\right),$$

and thus, in view of (5.20), (5.21), and (5.23), we have

$$\gamma(b) - \gamma(a) \le \frac{|\lambda| - |\mu|}{|\lambda|}\,\gamma(b) < 3 + \left|\frac{\mu}{\lambda}\right|.$$

Therefore, the assumptions of Theorem 4.4 (see p. 83) are fulfilled. □

Proof of Theorem 5.6. To prove the corollary it is sufficient to show that the assumptions of Theorem 4.5 (see p. 84) are satisfied.

Let $\ell \in \mathcal{L}_{ab}$ be defined by (3.1). Obviously, $\ell \in \mathcal{P}_{ab}$. First we will show that, on account of (3.25) with η given by (3.8), there exists a function $\gamma \in \widetilde{C}\big([a,b]; R_+\big)$ satisfying (4.14) and (4.15). Indeed, according to (3.25), there exists $\varepsilon \in\,]1, +\infty[$ such that

$$\int_t^{\tau_k(t)} p_0(s)ds \ge \frac{\int_a^b p_0(s)ds}{\varepsilon x} \ln \frac{\varepsilon x e^{\varepsilon x}}{\int_a^b p_0(s)ds\left(e^{\varepsilon x} + \frac{|\mu|e^{x} - |\lambda|}{|\lambda| - |\mu|}\right)} \tag{5.24}$$

for $t \in [a,b]$ $(k = 1,\dots,m)$. Put

$$x_0 = \frac{\varepsilon x}{\int\limits_a^b p_0(s)ds}\,. \tag{5.25}$$

Obviously, $x_0 > 0$. By (5.24), (5.25), and the assumption $x \in \left[\ln\left|\frac{\lambda}{\mu}\right|, +\infty\right[$, we obtain that for $k = 1,\dots,m$ the inequality

$$x_0\, e^{x_0 \int\limits_a^t p_0(s)ds} \le e^{x_0 \int\limits_a^{\tau_k(t)} p_0(s)ds} + \frac{|\mu|e^x - |\lambda|}{|\lambda| - |\mu|} \quad \text{for} \quad t \in [a,b] \tag{5.26}$$

holds, and

$$x_0 \int\limits_a^b p_0(s)ds > x. \tag{5.27}$$

Define the function $\gamma \in \widetilde{C}\big([a,b]; R_+\big)$ by

$$\gamma(t) = e^{x_0 \int\limits_a^t p_0(s)ds} + \frac{|\mu|e^x - |\lambda|}{|\lambda| - |\mu|} \quad \text{for} \quad t \in [a,b]. \tag{5.28}$$

Obviously, by virtue of (5.27) and (5.28) the function γ satisfies (4.15). Moreover, in view of (5.26), we obtain

$$\ell(\gamma)(t) = \sum_{k=1}^m p_k(t) \left(e^{x_0 \int\limits_a^{\tau_k(t)} p_0(s)ds} + \frac{|\mu|e^x - |\lambda|}{|\lambda| - |\mu|} \right) \ge$$

$$\ge p_0(t) x_0\, e^{x_0 \int\limits_a^t p_0(s)ds} = \gamma'(t) \quad \text{for} \quad t \in [a,b],$$

i.e., the inequality (4.14) is fulfilled.

It remains to show that each of the assumptions a), b), c), d) or e) in Theorem 5.6 ensures that at least one of the assumptions a), b) or c) in Theorem 4.5 (see p. 84) is satisfied.

It is clear that the assumption a) implies the condition (4.16). Moreover, according to Theorem 3.1 b) and c) (with $\lambda = 1$ and $\mu = 0$, see p. 63), the assumptions b) and c) yield the condition (4.17). On the other hand, on account of Theorem 3.14 b) (with $\lambda = 0$ and $\mu = 1$, see p. 72), the assumption e) implies the condition (4.18).

Finally we will show that the condition d) yields the condition (4.17). Indeed, according to (5.14), there exists $\varepsilon > 0$ such that

$$\int_t^{\tau_k(t)} p_0(s)ds < \eta^* - \varepsilon \quad \text{for} \quad t \in [a,b] \qquad (k = 1,\dots,m). \tag{5.29}$$

Choose $x_1 > 0$ and $\delta \in]0,1[$ such that

$$\frac{1}{x_1} \ln \left(x_1 + \frac{x_1(1-\delta)}{\exp\left(x_1 \int\limits_a^{\tau^*} p_0(s)ds \right) - (1-\delta)} \right) > \eta^* - \varepsilon \tag{5.30}$$

and put

$$\gamma(t) = e^{x_1 \int\limits_a^t p_0(s)ds} - (1-\delta) \quad \text{for} \quad t \in [a,b].$$

Obviously, $\gamma \in \widetilde{C}\big([a,b];]0,+\infty[\big)$. Moreover, the inequalities (5.29) and (5.30) imply that for $k = 1,\dots,m$ the inequality

$$x_1 \, e^{x_1 \int\limits_a^t p_0(s)ds} \geq e^{x_1 \int\limits_a^{\tau_k(t)} p_0(s)ds} - (1-\delta) \quad \text{for} \quad t \in [a,b]$$

holds. Hence, we obtain

$$\ell(\gamma)(t) = \sum_{k=1}^m p_k(t) \left(e^{x_1 \int\limits_a^{\tau_k(t)} p_0(s)ds} - (1-\delta) \right) \leq$$

$$\leq p_0(t) x_1 \, e^{x_1 \int\limits_a^t p_0(s)ds} = \gamma'(t) \quad \text{for} \quad t \in [a,b],$$

i.e., the inequality (2.10) is fulfilled. Thus, according to Theorem 2.1 (with $\lambda = 1$ and $\mu = 0$, see p. 17), the condition (4.17) is satisfied. □

Proof of Theorem 5.7. The validity of the theorem immediately follows from Theorem 4.7 (see p. 85). □

Proof of Theorem 5.8. The validity of the theorem immediately follows from Theorem 4.8 (see p. 86). □

Proof of Theorem 5.9. It is a consequence of Theorem 4.9 (see p. 86) and Theorems 3.12–3.14 (see pp. 70–72). □

Proof of Theorem 5.10. Similarly to the proof of Theorem 5.4 one can show that there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying (4.26), (4.27), and (4.28), where ℓ_0 and ℓ_1 are defined by (5.19). Therefore, the assumptions of Theorem 4.10 (see p. 86) are satisfied. □

Proof of Theorem 5.11. Similarly to the proof of Theorem 5.5 one can show that there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying (4.26), (4.27), and (4.28), where ℓ_0 and ℓ_1 are given by (5.19). Therefore, the assumptions of Theorem 4.10 (see p. 86) are satisfied. □

Proof of Theorem 5.12. In a similar manner as in the proof of Theorem 5.6 one can show that the assumptions of Theorem 4.11 (see p. 87) are satisfied. □

§6. Periodic Type BVP for Two Terms EDA

This section deals with the special case of the equation $(1.1')$ with $m = 1$ and $\tau_1 \equiv \nu_1$. In that case the equation $(1.1')$ can be rewritten in the form

$$u'(t) = p(t)u\big(\tau(t)\big) + q(t), \tag{6.1}$$

where $p, q \in L\big([a,b]; R\big)$ and $\tau \in \mathcal{M}_{ab}$. Throughout the section we will also suppose that the inequality (2.1) is satisfied.

In §5, there were established effective sufficient conditions for unique solvability of the problem (6.1), (1.2). Although those results are, in general, nonimprovable, in the special case, when τ maps the segment $[a,b]$ into some subsegment $[\tau_0, \tau_1] \subseteq [a,b]$, some of them can be improved in a certain way.

Therefore, in the sequel we will assume that there exist $\tau_0, \tau_1 \in [a,b]$, $\tau_0 \le \tau_1$ such that $\tau(t) \in [\tau_0, \tau_1]$ for almost all $t \in [a,b]$. Thus, it will be supposed that

$$\tau_0 = \text{ess inf}\{\tau(t) : t \in [a,b]\}, \qquad \tau_1 = \text{ess sup}\{\tau(t) : t \in [a,b]\}.$$

Note also that if $\tau_0 = a$ and $\tau_1 = b$, then obtained results coincide with the appropriate ones from §5.

In Subsection 6.1, the main results are formulated, Subsection 6.2 is devoted to their proofs, and the examples verifying the optimality of the main results can be found in Subsection 6.3.

6.1. Existence and Uniqueness Theorems

In the case, where $|\mu| \le |\lambda|$, the following statements hold.

Theorem 6.1. *Let* $|\mu| \le |\lambda|$ *and*

$$A = \int_a^{\tau_1} [p(s)]_+ ds + \left|\frac{\mu}{\lambda}\right| \int_{\tau_1}^{b} [p(s)]_+ ds \,. \tag{6.2}$$

If

$$A < 1\,, \tag{6.3}$$

$$\begin{gathered}\left(\int_a^{\tau_0}[p(s)]_-ds + \left|\frac{\mu}{\lambda}\right|\int_{\tau_0}^{b}[p(s)]_-ds\right)\times\\ \times\left(1-\int_{\tau_0}^{\tau_1}[p(s)]_+ds\right) > \left|\frac{\mu}{\lambda}\right| - 1 + A\,,\end{gathered} \tag{6.4}$$

$$(1-A)\left(1+\int_{\tau_0}^{\tau_1}[p(s)]_-ds\right) > \left|\frac{\mu}{\lambda}\right| - \int_a^{\tau_0}[p(s)]_-ds - \left|\frac{\mu}{\lambda}\right|\int_{\tau_1}^{b}[p(s)]_-ds\,, \tag{6.5}$$

and either

$$\int_a^{\tau_0}[p(s)]_-ds + \left|\frac{\mu}{\lambda}\right|\int_{\tau_1}^{b}[p(s)]_-ds < \left|\frac{\mu}{\lambda}\right| + \sqrt{1-A}\,, \tag{6.6}$$

$$\int_a^{\tau_1}[p(s)]_-ds + \left|\frac{\mu}{\lambda}\right|\int_{\tau_1}^{b}[p(s)]_-ds < 1 + \left|\frac{\mu}{\lambda}\right| + 2\sqrt{1-A} \tag{6.7}$$

or

$$\int_a^{\tau_0}[p(s)]_-ds + \left|\frac{\mu}{\lambda}\right|\int_{\tau_1}^{b}[p(s)]_-ds \geq \left|\frac{\mu}{\lambda}\right| + \sqrt{1-A}\,, \tag{6.8}$$

$$\int_{\tau_0}^{\tau_1}[p(s)]_-ds < 1 + \frac{1-A}{\int_a^{\tau_0}[p(s)]_-ds + \left|\frac{\mu}{\lambda}\right|\int_{\tau_1}^{b}[p(s)]_-ds - \left|\frac{\mu}{\lambda}\right|}\,, \tag{6.9}$$

then the problem (6.1), (1.2) *has a unique solution.*

Remark 6.1. Theorem 6.1 is nonimprovable in the sense that neither one of the strict inequalities (6.4), (6.5), (6.7), and (6.9) can be replaced by the nonstrict one (see Examples 6.1–6.4, pp. 144–148).

Theorem 6.2. *Let* $0 \neq |\mu| \leq |\lambda|$ *and*

$$B = \int_a^{\tau_1} [p(s)]_- ds + \left|\frac{\mu}{\lambda}\right| \int_{\tau_1}^{b} [p(s)]_- ds \,. \tag{6.10}$$

If

$$B < \left|\frac{\mu}{\lambda}\right| \,, \tag{6.11}$$

$$\begin{gathered}\left(\left|\frac{\mu}{\lambda}\right| - B\right)\left(1 + \int_{\tau_0}^{\tau_1} [p(s)]_+ ds\right) > \\ > 1 - \int_a^{\tau_0} [p(s)]_+ ds - \left|\frac{\mu}{\lambda}\right| \int_{\tau_1}^{b} [p(s)]_+ ds \,,\end{gathered} \tag{6.12}$$

$$\begin{gathered}\left(\int_a^{\tau_0} [p(s)]_+ ds + \left|\frac{\mu}{\lambda}\right| \int_{\tau_0}^{b} [p(s)]_+ ds\right)\left(1 - \int_{\tau_0}^{\tau_1} [p(s)]_- ds\right) > \\ > 1 - \left|\frac{\mu}{\lambda}\right| + \int_a^{\tau_0} [p(s)]_- ds + \left|\frac{\mu}{\lambda}\right| \int_{\tau_0}^{b} [p(s)]_- ds \,,\end{gathered} \tag{6.13}$$

and either

$$\int_a^{\tau_0} [p(s)]_+ ds + \left|\frac{\mu}{\lambda}\right| \int_{\tau_1}^{b} [p(s)]_+ ds < 1 + \sqrt{\left|\frac{\mu}{\lambda}\right| - B} \,, \tag{6.14}$$

$$\int_a^{\tau_1} [p(s)]_+ ds + \left|\frac{\mu}{\lambda}\right| \int_{\tau_1}^{b} [p(s)]_+ ds < 2 + 2\sqrt{\left|\frac{\mu}{\lambda}\right| - B} \tag{6.15}$$

or

$$\int_a^{\tau_0} [p(s)]_+ ds + \left|\frac{\mu}{\lambda}\right| \int_{\tau_1}^{b} [p(s)]_+ ds \geq 1 + \sqrt{\left|\frac{\mu}{\lambda}\right| - B} \,, \tag{6.16}$$

$$\int_{\tau_0}^{\tau_1} [p(s)]_+ ds < 1 + \frac{\left|\frac{\mu}{\lambda}\right| - B}{\int_a^{\tau_0} [p(s)]_+ ds + \left|\frac{\mu}{\lambda}\right| \int_{\tau_1}^{b} [p(s)]_+ ds - 1} \,, \tag{6.17}$$

then the problem (6.1), (1.2) *has a unique solution.*

Remark 6.2. Theorem 6.2 is nonimprovable in the sense that neither one of the strict inequalities (6.12), (6.13), (6.15), and (6.17) can be replaced by the nonstrict one (see Examples 6.5–6.8, pp. 149–153).

Note also that if $\tau_0 = a$ and $\tau_1 = b$, then the assumptions of Theorems 6.1 and 6.2 coincide with the assumptions of Theorems 5.1 and 5.2 (see p. 104).

Theorem 6.3. *Let* $|\mu| \leq |\lambda|$,

$$\int_{\tau_0}^{\tau_1} [p(s)]_+ ds < 1\,, \qquad \int_{\tau_0}^{\tau_1} [p(s)]_- ds < 1\,, \tag{6.18}$$

and either

$$\begin{gathered} |\lambda| \int_a^{\tau_1} [p(s)]_+ ds + |\mu| \int_{\tau_1}^b [p(s)]_+ ds - \\ - \left(|\lambda| \int_a^{\tau_0} [p(s)]_- ds + |\mu| \int_{\tau_0}^b [p(s)]_- ds \right) \big(1 - T\big) < |\lambda| - |\mu| \end{gathered} \tag{6.19}$$

or

$$\begin{gathered} \left(|\lambda| \int_a^{\tau_0} [p(s)]_+ ds + |\mu| \int_{\tau_0}^b [p(s)]_+ ds \right) \big(1 - T\big) - \\ - |\lambda| \int_a^{\tau_1} [p(s)]_- ds - |\mu| \int_{\tau_1}^b [p(s)]_- ds > \big(|\lambda| - |\mu|\big)\big(1 - T\big)\,, \end{gathered} \tag{6.20}$$

where

$$T = \max\left\{ \int_{\tau_0}^{\tau_1} [p(s)]_+ ds,\ \int_{\tau_0}^{\tau_1} [p(s)]_- ds \right\}. \tag{6.21}$$

Then the problem (6.1), (1.2) *has a unique solution.*

Remark 6.3. Theorem 6.3 is nonimprovable in the sense that the strict inequalities (6.19) and (6.20) cannot be replaced by the nonstrict ones (see Examples 6.9–6.11, pp. 154–156).

Note also that if the segment $[\tau_0, \tau_1]$ is degenerated to a point $c \in [a, b]$, i.e., $\tau(t) = c$ for $t \in [a, b]$, then $T = 0$ and the inequalities (6.19) and (6.20) can be rewritten as

$$\lambda \int_a^c p(s)ds - \mu \int_c^b p(s)ds \neq \lambda + \mu \,,$$

which is sufficient and necessary for the unique solvability of the problem (6.1), (1.2) with $\tau(t) = c$ for $t \in [a, b]$.

Theorem 6.4. *Let $|\mu| < |\lambda|$ and let there exist $\gamma \in \widetilde{C}\big([a, b];]0, +\infty[\big)$ such that*

$$\gamma'(t) \geq [p(t)]_+ \gamma(\tau(t)) + [p(t)]_- \quad \textit{for} \quad t \in [a, b] \,, \tag{6.22}$$

$$|\lambda|\gamma(a) > |\mu|\gamma(b) \,, \tag{6.23}$$

and either

$$|\lambda|\big(\gamma(\tau_0) - \gamma(a)\big) + |\mu|\big(\gamma(b) - \gamma(\tau_1)\big) < |\lambda| + |\mu| \,, \tag{6.24}$$

$$\left|\frac{\mu}{\lambda}\right| \big(\gamma(b) - \gamma(\tau_1)\big) + \gamma(\tau_1) - \gamma(a) < 3 + \left|\frac{\mu}{\lambda}\right| \tag{6.25}$$

or

$$|\lambda|\big(\gamma(\tau_0) - \gamma(a)\big) + |\mu|\big(\gamma(b) - \gamma(\tau_1)\big) \geq |\lambda| + |\mu| \,, \tag{6.26}$$

$$\gamma(\tau_1) - \gamma(\tau_0) < 1 + \frac{|\lambda|}{|\lambda|\big(\gamma(\tau_0) - \gamma(a)\big) + |\mu|\big(\gamma(b) - \gamma(\tau_1)\big) - |\mu|} \,. \tag{6.27}$$

Then the problem (6.1), (1.2) *has a unique solution.*

Remark 6.4. Theorem 6.4 is nonimprovable in the sense that the strict inequalities (6.25) and (6.27) cannot be replaced by the nonstrict ones (see Examples 6.3 and 6.4, p. 146).

Note also that if $\tau_0 = a$ and $\tau_1 = b$, then from Theorem 6.4 we obtain Theorem 4.4 (see p. 83).

In the case, where $|\mu| \geq |\lambda|$, the following assertions hold.

Theorem 6.5. *Let* $|\mu| \geq |\lambda|$ *and*

$$\widetilde{B} = \left|\frac{\lambda}{\mu}\right| \int_a^{\tau_0} [p(s)]_- ds + \int_{\tau_0}^{b} [p(s)]_- ds \,.$$

If

$$\widetilde{B} < 1 \,,$$

$$\left(\left|\frac{\lambda}{\mu}\right| \int_a^{\tau_1} [p(s)]_+ ds + \int_{\tau_1}^{b} [p(s)]_+ ds \right) \left(1 - \int_{\tau_0}^{\tau_1} [p(s)]_- ds \right) > \left|\frac{\lambda}{\mu}\right| - 1 + \widetilde{B} \,,$$

$$\left(1 - \widetilde{B}\right) \left(1 + \int_{\tau_0}^{\tau_1} [p(s)]_+ ds \right) > \left|\frac{\lambda}{\mu}\right| - \left|\frac{\lambda}{\mu}\right| \int_a^{\tau_0} [p(s)]_+ ds - \int_{\tau_1}^{b} [p(s)]_+ ds \,,$$

and either

$$\left|\frac{\lambda}{\mu}\right| \int_a^{\tau_0} [p(s)]_+ ds + \int_{\tau_1}^{b} [p(s)]_+ ds < \left|\frac{\lambda}{\mu}\right| + \sqrt{1 - \widetilde{B}} \,,$$

$$\left|\frac{\lambda}{\mu}\right| \int_a^{\tau_0} [p(s)]_+ ds + \int_{\tau_0}^{b} [p(s)]_+ ds < 1 + \left|\frac{\lambda}{\mu}\right| + 2\sqrt{1 - \widetilde{B}}$$

or

$$\left|\frac{\lambda}{\mu}\right| \int_a^{\tau_0} [p(s)]_+ ds + \int_{\tau_1}^{b} [p(s)]_+ ds \geq \left|\frac{\lambda}{\mu}\right| + \sqrt{1 - \widetilde{B}} \,,$$

$$\int_{\tau_0}^{\tau_1} [p(s)]_+ ds < 1 + \frac{1 - \widetilde{B}}{\left|\frac{\lambda}{\mu}\right| \int_a^{\tau_0} [p(s)]_+ ds + \int_{\tau_1}^{b} [p(s)]_+ ds - \left|\frac{\lambda}{\mu}\right|} \,,$$

then the problem (6.1), (1.2) *has a unique solution.*

Theorem 6.6. *Let* $|\mu| \geq |\lambda| \neq 0$ *and*

$$\widetilde{A} = \left|\frac{\lambda}{\mu}\right| \int_a^{\tau_0} [p(s)]_+ ds + \int_{\tau_0}^{b} [p(s)]_+ ds \,.$$

If

$$\widetilde{A} < \left|\frac{\lambda}{\mu}\right| ,$$

$$\left(\left|\frac{\lambda}{\mu}\right| - \widetilde{A}\right)\left(1 + \int_{\tau_0}^{\tau_1}[p(s)]_- ds\right) > 1 - \left|\frac{\lambda}{\mu}\right|\int_a^{\tau_0}[p(s)]_- ds - \int_{\tau_1}^{b}[p(s)]_- ds,$$

$$\left(\left|\frac{\lambda}{\mu}\right|\int_a^{\tau_1}[p(s)]_- ds + \int_{\tau_1}^{b}[p(s)]_- ds\right)\left(1 - \int_{\tau_0}^{\tau_1}[p(s)]_+ ds\right) >$$

$$> 1 - \left|\frac{\lambda}{\mu}\right| + \left|\frac{\lambda}{\mu}\right|\int_a^{\tau_1}[p(s)]_+ ds + \int_{\tau_1}^{b}[p(s)]_+ ds\,,$$

and either

$$\left|\frac{\lambda}{\mu}\right|\int_a^{\tau_0}[p(s)]_- ds + \int_{\tau_1}^{b}[p(s)]_- ds < 1 + \sqrt{\left|\frac{\lambda}{\mu}\right| - \widetilde{A}},$$

$$\left|\frac{\lambda}{\mu}\right|\int_a^{\tau_0}[p(s)]_- ds + \int_{\tau_0}^{b}[p(s)]_- ds < 2 + 2\sqrt{\left|\frac{\lambda}{\mu}\right| - \widetilde{A}}$$

or

$$\left|\frac{\lambda}{\mu}\right|\int_a^{\tau_0}[p(s)]_- ds + \int_{\tau_1}^{b}[p(s)]_- ds \geq 1 + \sqrt{\left|\frac{\lambda}{\mu}\right| - \widetilde{A}},$$

$$\int_{\tau_0}^{\tau_1}[p(s)]_- ds < 1 + \frac{\left|\frac{\lambda}{\mu}\right| - \widetilde{A}}{\left|\frac{\lambda}{\mu}\right|\int\limits_a^{\tau_0}[p(s)]_- ds + \int\limits_{\tau_1}^{b}[p(s)]_- ds - 1}\,,$$

then the problem (6.1), (1.2) *has a unique solution.*

Theorem 6.7. *Let $|\mu| \geq |\lambda|$, the condition* (6.18) *be fulfilled, and let either*

$$|\lambda| \int_a^{\tau_0} [p(s)]_- ds + |\mu| \int_{\tau_0}^{b} [p(s)]_- ds -$$

$$- \left(|\lambda| \int_a^{\tau_1} [p(s)]_+ ds + |\mu| \int_{\tau_1}^{b} [p(s)]_+ ds \right) \big(1 - T\big) < |\mu| - |\lambda|$$

or

$$\left(|\lambda| \int_a^{\tau_1} [p(s)]_- ds + |\mu| \int_{\tau_1}^{b} [p(s)]_- ds \right) \big(1 - T\big) -$$

$$-|\lambda| \int_a^{\tau_0} [p(s)]_+ ds - |\mu| \int_{\tau_0}^{b} [p(s)]_+ ds > \big(|\mu| - |\lambda|\big)\big(1 - T\big),$$

where T is defined by (6.21)*. Then the problem* (6.1), (1.2) *has a unique solution.*

Theorem 6.8. *Let $|\mu| > |\lambda|$ and let there exist $\gamma \in \widetilde{C}\big([a,b];\,]0,+\infty[\big)$ such that*

$$-\gamma'(t) \geq [p(t)]_- \gamma(\tau(t)) + [p(t)]_+ \quad \textit{for} \quad t \in [a,b],$$

$$|\lambda| \gamma(a) < |\mu| \gamma(b),$$

and either

$$|\lambda|\big(\gamma(a) - \gamma(\tau_0)\big) + |\mu|\big(\gamma(\tau_1) - \gamma(b)\big) < |\lambda| + |\mu|,$$

$$\left|\frac{\lambda}{\mu}\right| \big(\gamma(a) - \gamma(\tau_0)\big) + \gamma(\tau_0) - \gamma(b) < 3 + \left|\frac{\lambda}{\mu}\right|$$

or

$$|\lambda|\big(\gamma(a) - \gamma(\tau_0)\big) + |\mu|\big(\gamma(\tau_1) - \gamma(b)\big) \geq |\lambda| + |\mu|,$$

$$\gamma(\tau_0) - \gamma(\tau_1) < 1 + \frac{|\mu|}{|\lambda|\big(\gamma(a) - \gamma(\tau_0)\big) + |\mu|\big(\gamma(\tau_1) - \gamma(b)\big) - |\lambda|}.$$

Then the problem (6.1), (1.2) *has a unique solution.*

Remark 6.5. Let $p, q \in L([a,b];R)$, $\tau \in \mathcal{M}_{ab}$, and $c \in R$. Put

$$\widetilde{p}(t) \overset{\text{def}}{=} -p(a+b-t), \qquad \widetilde{\tau}(t) \overset{\text{def}}{=} a+b-\tau(a+b-t),$$
$$\widetilde{q}(t) \overset{\text{def}}{=} -q(a+b-t) \quad \text{for} \quad t \in [a,b].$$

It is clear that if u is a solution of the problem (6.1), (1.2), then the function v, defined by $v(t) \overset{\text{def}}{=} u(a+b-t)$ for $t \in [a,b]$, is a solution of the problem

$$v'(t) = \widetilde{p}(t)v(\widetilde{\tau}(t)) + \widetilde{q}(t), \qquad \mu v(a) + \lambda v(b) = c\,, \tag{6.28}$$

and vice versa, if v is a solution of the problem (6.28), then the function u, defined by $u(t) \overset{\text{def}}{=} v(a+b-t)$ for $t \in [a,b]$, is a solution of the problem (6.1), (1.2).

Remark 6.6. According to Remark 6.5, Theorems 6.5–6.8 can be immediately derived from Theorems 6.1–6.4. Moreover, by virtue of Remarks 6.1–6.4, Theorems 6.5–6.8 are nonimprovable in an appropriate sense.

6.2. Proofs

According to Theorem 1.1 (see p. 14), to prove Theorems 6.1–6.4 it is sufficient to show that the homogeneous problem

$$u'(t) = p(t)u(\tau(t)), \tag{6.1_0}$$
$$\lambda u(a) + \mu u(b) = 0$$

has only the trivial solution.

First introduce the following notation

$$\begin{aligned} A_1 &= \int_a^{\tau_0} [p(s)]_+ ds, \qquad A_2 = \int_{\tau_0}^{\tau_1} [p(s)]_+ ds, \qquad A_3 = \int_{\tau_1}^{b} [p(s)]_+ ds, \\ B_1 &= \int_a^{\tau_0} [p(s)]_- ds, \qquad B_2 = \int_{\tau_0}^{\tau_1} [p(s)]_- ds, \qquad B_3 = \int_{\tau_1}^{b} [p(s)]_- ds. \end{aligned} \tag{6.29}$$

Proof of Theorem 6.1. Assume that the problem (6.1_0), (1.2_0) possesses a nontrivial solution u.

First suppose that u does not change its sign in $[\tau_0,\tau_1]$. Without loss of generality we can assume that

$$u(t) \geq 0 \quad \text{for} \quad t \in [\tau_0,\tau_1]. \tag{6.30}$$

Put

$$M = \max\{u(t) : t \in [\tau_0,\tau_1]\}, \qquad m = \min\{u(t) : t \in [\tau_0,\tau_1]\}, \tag{6.31}$$

and choose $t_M, t_m \in [\tau_0,\tau_1]$ such that

$$u(t_M) = M, \qquad u(t_m) = m. \tag{6.32}$$

Furthermore, let

$$\alpha_0 = \min\{t_M, t_m\}, \qquad \alpha_1 = \max\{t_M, t_m\}, \tag{6.33}$$

$$\begin{aligned} A_{21} &= \int_{\tau_0}^{\alpha_0} [p(s)]_+ ds, \quad A_{22} = \int_{\alpha_0}^{\alpha_1} [p(s)]_+ ds, \quad A_{23} = \int_{\alpha_1}^{\tau_1} [p(s)]_+ ds, \\ B_{21} &= \int_{\tau_0}^{\alpha_0} [p(s)]_- ds, \quad B_{22} = \int_{\alpha_0}^{\alpha_1} [p(s)]_- ds, \quad B_{23} = \int_{\alpha_1}^{\tau_1} [p(s)]_- ds. \end{aligned} \tag{6.34}$$

It is clear that

$$m \geq 0, \qquad M > 0, \tag{6.35}$$

since if $M = 0$, then, in view of (6.1_0), (6.30), and (6.31), we obtain $u(\tau_0) = 0$ and $u'(t) = 0$ for $t \in [a,b]$, i.e., $u \equiv 0$. Obviously, either

$$t_M < t_m \tag{6.36}$$

or

$$t_M \geq t_m. \tag{6.37}$$

First suppose that (6.36) holds. The integrations of (6.1_0) from a to t_M, from t_M to t_m, from t_m to τ_1, and from τ_1 to b, on account of (6.29), (6.31)–(6.34), and the assumption $\left|\frac{\mu}{\lambda}\right| \in [0,1]$, result in

$$\begin{aligned} M - u(a) = \int_a^{t_M} [p(s)]_+ u(\tau(s)) ds - \int_a^{t_M} [p(s)]_- u(\tau(s)) ds \leq \\ \leq M\big(A_1 + A_{21}\big) - m\big(B_1 + B_{21}\big), \end{aligned} \tag{6.38}$$

$$m - M = \int_{t_M}^{t_m} [p(s)]_+ u(\tau(s))ds - \int_{t_M}^{t_m} [p(s)]_- u(\tau(s))ds \leq \\ \leq MA_{22} - mB_{22}, \tag{6.39}$$

$$\left|\frac{\mu}{\lambda}\right| \big(u(\tau_1) - m\big) \leq u(\tau_1) - m = \int_{t_m}^{\tau_1} [p(s)]_+ u(\tau(s))ds - \\ - \int_{t_m}^{\tau_1} [p(s)]_- u(\tau(s))ds \leq MA_{23} - mB_{23}, \tag{6.40}$$

$$u(b) - u(\tau_1) = \int_{\tau_1}^{b} [p(s)]_+ u(\tau(s))ds - \int_{\tau_1}^{b} [p(s)]_- u(\tau(s))ds \leq \\ \leq MA_3 - mB_3. \tag{6.41}$$

Multiplying both sides of (6.41) by $\left|\frac{\mu}{\lambda}\right|$, summing with (6.38) and (6.40), and taking into account (1.2_0) and (2.1), we get

$$M - \left|\frac{\mu}{\lambda}\right| m \leq M\left(A_1 + A_{21} + A_{23} + \left|\frac{\mu}{\lambda}\right| A_3\right) - \\ -m\left(B_1 + B_{21} + B_{23} + \left|\frac{\mu}{\lambda}\right| B_3\right).$$

Hence, by virtue of (6.2), (6.3), (6.29), (6.34), and (6.35), the last inequality implies

$$0 < M\left(1 - A_1 - A_{21} - A_{23} - \left|\frac{\mu}{\lambda}\right| A_3\right) \leq \\ \leq m\left(\left|\frac{\mu}{\lambda}\right| - B_1 - B_{21} - B_{23} - \left|\frac{\mu}{\lambda}\right| B_3\right). \tag{6.42}$$

On the other hand, with recpect to (6.34) and (6.35), (6.39) results in

$$0 \leq m\big(1 + B_{22}\big) \leq M\big(1 + A_{22}\big). \tag{6.43}$$

Thus, it follows from (6.42) and (6.43) that

$$\left(1 - A_1 - A_{21} - A_{23} - \left|\frac{\mu}{\lambda}\right| A_3\right)\big(1 + B_{22}\big) \leq \\ \leq \left(\left|\frac{\mu}{\lambda}\right| - B_1 - B_{21} - B_{23} - \left|\frac{\mu}{\lambda}\right| B_3\right)\big(1 + A_{22}\big). \tag{6.44}$$

Obviously, on account of (6.2), (6.29), (6.34), and the assumption $\left|\frac{\mu}{\lambda}\right| \in [0,1]$, we find

$$\Big(1 - A_1 - A_{21} - A_{23} - \Big|\frac{\mu}{\lambda}\Big| A_{23}\Big)\big(1 + B_{22}\big) =$$

$$= \Big(1 - A_1 - A_2 - \Big|\frac{\mu}{\lambda}\Big| A_3\Big)\big(1 + B_2\big) -$$

$$- \Big(1 - A_1 - A_2 - \Big|\frac{\mu}{\lambda}\Big| A_3\Big)\big(B_{21} + B_{23}\big) + A_{22}\big(1 + B_{22}\big) \geq$$

$$\geq \big(1 - A\big)\big(1 + B_2\big) - \big(B_{21} + B_{23}\big) + A_{22}$$

and

$$\Big(\Big|\frac{\mu}{\lambda}\Big| - B_1 - B_{21} - B_{23} - \Big|\frac{\mu}{\lambda}\Big| B_3\Big)\big(1 + A_{22}\big) =$$

$$= \Big|\frac{\mu}{\lambda}\Big| + \Big|\frac{\mu}{\lambda}\Big| A_{22} - \Big(B_1 + \Big|\frac{\mu}{\lambda}\Big| B_3 + B_{21} + B_{23}\Big)\big(1 + A_{22}\big) \leq$$

$$\leq \Big|\frac{\mu}{\lambda}\Big| - B_1 - \Big|\frac{\mu}{\lambda}\Big| B_3 + A_{22} - \big(B_{21} + B_{23}\big).$$

By virtue of the last two inequalities, (6.44) yields

$$\big(1 - A\big)\big(1 + B_2\big) \leq \Big|\frac{\mu}{\lambda}\Big| - B_1 - \Big|\frac{\mu}{\lambda}\Big| B_3,$$

which, in view of (6.29), contradicts (6.5).

Now suppose that (6.37) is fulfilled. The integrations of (6.1_0) from a to t_m, from t_m to t_M, and from t_M to b, on account of (6.29) and (6.31)–(6.34), result in

$$\begin{aligned} m - u(a) = \int_a^{t_m} [p(s)]_+ u(\tau(s))ds - \int_a^{t_m} [p(s)]_- u(\tau(s))ds \leq \\ \leq M\big(A_1 + A_{21}\big) - m\big(B_1 + B_{21}\big), \end{aligned} \tag{6.45}$$

$$\begin{aligned} M - m = \int_{t_m}^{t_M} [p(s)]_+ u(\tau(s))ds - \int_{t_m}^{t_M} [p(s)]_- u(\tau(s))ds \leq \\ \leq M A_{22} - m B_{22}, \end{aligned} \tag{6.46}$$

$$\begin{aligned} u(b) - M = \int_{t_M}^{b} [p(s)]_+ u(\tau(s))ds - \int_{t_M}^{b} [p(s)]_- u(\tau(s))ds \leq \\ \leq M\big(A_{23} + A_3\big) - m\big(B_{23} + B_3\big). \end{aligned} \tag{6.47}$$

Multiplying both sides of (6.47) by $\left|\frac{\mu}{\lambda}\right|$, summing with (6.45), and taking into account (1.2_0) and (2.1), we get

$$\begin{gathered} m - \left|\frac{\mu}{\lambda}\right| M \leq M\left(A_1 + A_{21} + \left|\frac{\mu}{\lambda}\right| A_{23} + \left|\frac{\mu}{\lambda}\right| A_3\right) - \\ -m\left(B_1 + B_{21} + \left|\frac{\mu}{\lambda}\right| B_{23} + \left|\frac{\mu}{\lambda}\right| B_3\right). \end{gathered} \tag{6.48}$$

Hence, by virtue of (6.2), (6.3), (6.29), (6.34), and (6.35), it follows from (6.46) and (6.48) that

$$0 < M\big(1 - A_{22}\big) \leq m\big(1 - B_{22}\big),$$

$$\begin{gathered} 0 \leq m\bigg(1 + B_1 + B_{21} + \left|\frac{\mu}{\lambda}\right| B_{23} + \left|\frac{\mu}{\lambda}\right| B_3\bigg) \leq \\ \leq M\bigg(\left|\frac{\mu}{\lambda}\right| + A_1 + A_{21} + \left|\frac{\mu}{\lambda}\right| A_{23} + \left|\frac{\mu}{\lambda}\right| A_3\bigg). \end{gathered}$$

Thus,

$$\begin{gathered} \bigg(1 + B_1 + B_{21} + \left|\frac{\mu}{\lambda}\right| B_{23} + \left|\frac{\mu}{\lambda}\right| B_3\bigg)(1 - A_{22}) \leq \\ \leq \bigg(\left|\frac{\mu}{\lambda}\right| + A_1 + A_{21} + \left|\frac{\mu}{\lambda}\right| A_{23} + \left|\frac{\mu}{\lambda}\right| A_3\bigg)\big(1 - B_{22}\big). \end{gathered} \tag{6.49}$$

Obviously, in view of (6.2), (6.3), (6.29), (6.34), and the assumption $\left|\frac{\mu}{\lambda}\right| \in [0, 1]$, we obtain

$$\begin{gathered} \left(1 + B_1 + B_{21} + \left|\frac{\mu}{\lambda}\right| B_{23} + \left|\frac{\mu}{\lambda}\right| B_3\right)\big(1 - A_{22}\big) = 1 - A_{22} + \\ +\bigg(B_1 + B_{21} + \left|\frac{\mu}{\lambda}\right| B_{22} + \left|\frac{\mu}{\lambda}\right| B_{23} + \left|\frac{\mu}{\lambda}\right| B_3\bigg)\big(1 - A_{22}\big) - \left|\frac{\mu}{\lambda}\right| B_{22}\big(1 - A_{22}\big) \geq \\ \geq 1 - A_{22} + \left(B_1 + \left|\frac{\mu}{\lambda}\right| B_2 + \left|\frac{\mu}{\lambda}\right| B_3\right)\big(1 - A_2\big) - \left|\frac{\mu}{\lambda}\right| B_{22} \end{gathered}$$

and

$$\begin{gathered} \left(\left|\frac{\mu}{\lambda}\right| + A_1 + A_{21} + \left|\frac{\mu}{\lambda}\right| A_{23} + \left|\frac{\mu}{\lambda}\right| A_3\right)\big(1 - B_{22}\big) = \\ = \left|\frac{\mu}{\lambda}\right| - \left|\frac{\mu}{\lambda}\right| B_{22} + \left(A_1 + A_{21} + \left|\frac{\mu}{\lambda}\right| A_{23} + \left|\frac{\mu}{\lambda}\right| A_3\right)\big(1 - B_{22}\big) \leq \\ \leq \left|\frac{\mu}{\lambda}\right| - \left|\frac{\mu}{\lambda}\right| B_{22} + A_1 + A_{21} + A_{22} + \left|\frac{\mu}{\lambda}\right| A_{23} + \left|\frac{\mu}{\lambda}\right| A_3 - A_{22} \leq \\ \leq \left|\frac{\mu}{\lambda}\right| - \left|\frac{\mu}{\lambda}\right| B_{22} + A - A_{22}. \end{gathered}$$

By virtue of the last two inequalities, (6.49) implies

$$\left(B_1 + \left|\frac{\mu}{\lambda}\right| B_2 + \left|\frac{\mu}{\lambda}\right| B_3\right)\left(1 - A_2\right) \le \left|\frac{\mu}{\lambda}\right| - 1 + A,$$

which, in view of (6.29), contradicts (6.4).

Now suppose that u changes its sign in $[\tau_0, \tau_1]$. Put

$$m_0 = -\min\{u(t) : t \in [\tau_0, \tau_1]\}, \qquad M_0 = \max\{u(t) : t \in [\tau_0, \tau_1]\} \tag{6.50}$$

and choose $\alpha_0, \alpha_1 \in [\tau_0, \tau_1]$ such that

$$u(\alpha_0) = -m_0, \qquad u(\alpha_1) = M_0. \tag{6.51}$$

It is clear that

$$M_0 > 0, \qquad m_0 > 0, \tag{6.52}$$

and without loss of generality we can assume that $\alpha_0 < \alpha_1$. Furthermore, define numbers A_{2i}, B_{2i} $(i = 1, 2, 3)$ by (6.34) and put

$$g(x) \stackrel{\text{def}}{=} \frac{1 - A}{x + B_1 + \left|\frac{\mu}{\lambda}\right| B_3 - \left|\frac{\mu}{\lambda}\right|} + x \quad \text{for} \quad x > \left|\frac{\mu}{\lambda}\right| - B_1 - \left|\frac{\mu}{\lambda}\right| B_3, \tag{6.53}$$

where A is given by (6.2).

The integrations of (6.1_0) from a to α_0, from α_0 to α_1, and from α_1 to b, in view of (6.29), (6.34), (6.50), and (6.51), result in

$$\begin{aligned} u(a) + m_0 = \int_a^{\alpha_0} [p(s)]_- u(\tau(s))ds - \int_a^{\alpha_0} [p(s)]_+ u(\tau(s))ds \le \\ \le M_0\big(B_1 + B_{21}\big) + m_0\big(A_1 + A_{21}\big), \end{aligned} \tag{6.54}$$

$$\begin{aligned} M_0 + m_0 = \int_{\alpha_0}^{\alpha_1} [p(s)]_+ u(\tau(s))ds - \int_{\alpha_0}^{\alpha_1} [p(s)]_- u(\tau(s))ds \le \\ \le M_0 A_{22} + m_0 B_{22}, \end{aligned} \tag{6.55}$$

$$\begin{aligned} M_0 - u(b) = \int_{\alpha_1}^{b} [p(s)]_- u(\tau(s))ds - \int_{\alpha_1}^{b} [p(s)]_+ u(\tau(s))ds \le \\ \le M_0\big(B_{23} + B_3\big) + m_0\big(A_{23} + A_3\big). \end{aligned} \tag{6.56}$$

Multiplying both sides of (6.56) by $\left|\frac{\mu}{\lambda}\right|$, summing with (6.54), and taking into account (1.2_0), (2.1), (6.52), and the assumption $\left|\frac{\mu}{\lambda}\right| \in [0,1]$, we get

$$\left|\frac{\mu}{\lambda}\right| M_0 + m_0 \leq M_0 \left(B_1 + B_{21} + B_{23} + \left|\frac{\mu}{\lambda}\right| B_3\right) + \\ + m_0 \left(A_1 + A_{21} + A_{23} + \left|\frac{\mu}{\lambda}\right| A_3\right). \tag{6.57}$$

Due to (6.2), (6.3), (6.29), and (6.34), we have

$$A_1 + A_{21} + A_{23} + \left|\frac{\mu}{\lambda}\right| A_3 < 1, \qquad A_{22} < 1.$$

Thus, it follows from (6.52), (6.55), and (6.57) that

$$B_{22} > 1, \qquad B_1 + B_{21} + B_{23} + \left|\frac{\mu}{\lambda}\right| B_3 > \left|\frac{\mu}{\lambda}\right|, \tag{6.58}$$

and

$$B_{22} \geq 1 + \frac{M_0}{m_0}\big(1 - A_{22}\big), \tag{6.59}$$

$$\frac{M_0}{m_0} \geq \frac{1 - A_1 - A_{21} - A_{23} - \left|\frac{\mu}{\lambda}\right| A_3}{B_1 + B_{21} + B_{23} + \left|\frac{\mu}{\lambda}\right| B_3 - \left|\frac{\mu}{\lambda}\right|}. \tag{6.60}$$

According to (6.58) and the fact that

$$\big(1 - A_{22}\big)\left(1 - A_1 - A_{21} - A_{23} - \left|\frac{\mu}{\lambda}\right| A_3\right) \geq \\ \geq 1 - A_1 - A_{21} - A_{22} - A_{23} - \left|\frac{\mu}{\lambda}\right| A_3 = 1 - A,$$

from (6.59) and (6.60) we get

$$B_{22} \geq 1 + \frac{1 - A}{B_1 + B_{21} + B_{23} + \left|\frac{\mu}{\lambda}\right| B_3 - \left|\frac{\mu}{\lambda}\right|}. \tag{6.61}$$

First suppose that (6.6) and (6.7) are satisfied. By virtue of (6.58), from (6.61) we have

$$1 - A \leq \big(B_{22} - 1\big)\left(B_1 + B_{21} + B_{23} + \left|\frac{\mu}{\lambda}\right| B_3 - \left|\frac{\mu}{\lambda}\right|\right) \leq \\ \leq \frac{1}{4}\left(B_1 + B_{21} + B_{22} + B_{23} + \left|\frac{\mu}{\lambda}\right| B_3 - 1 - \left|\frac{\mu}{\lambda}\right|\right)^2 = \\ = \frac{1}{4}\left(B_1 + B_2 + \left|\frac{\mu}{\lambda}\right| B_3 - 1 - \left|\frac{\mu}{\lambda}\right|\right)^2,$$

which, in view of (6.3), (6.29), (6.34), and (6.58), contradicts (6.7).

Now suppose that (6.8) and (6.9) are fulfilled. It is not difficult to verify that, on account of (6.8) and (6.29), the function g defined by (6.53) is nondecreasing in $[0,+\infty[$. Therefore, from (6.61) we obtain

$$B_{21}+B_{22}+B_{23}\geq 1+\frac{1-A}{B_1+B_{21}+B_{23}+\left|\frac{\mu}{\lambda}\right|B_3-\left|\frac{\mu}{\lambda}\right|}+B_{21}+B_{23}=$$
$$=1+g\big(B_{21}+B_{23}\big)\geq 1+g(0)=1+\frac{1-A}{B_1+\left|\frac{\mu}{\lambda}\right|B_3-\left|\frac{\mu}{\lambda}\right|},$$

which, in view of (6.29) and (6.34), contradicts (6.9). □

Proof of Theorem 6.2. Assume that the problem (6.1_0), (1.2_0) has a nontrivial solution u.

First suppose that u does not change its sign in $[\tau_0,\tau_1]$. Without loss of generality we can assume that (6.30) is fulfilled. Define numbers M and m by (6.31) and choose $t_M, t_m\in[\tau_0,\tau_1]$ such that (6.32) holds. Furthermore, define numbers α_0, α_1 and A_{2i}, B_{2i} $(i=1,2,3)$ by (6.33) and (6.34), respectively. It is clear that (6.35) is satisfied, since if $M=0$, then, in view of (6.1_0), (6.30), and (6.31), we obtain $u(\tau_0)=0$ and $u'(t)=0$ for $t\in[a,b]$, i.e., $u\equiv 0$. It is also evident that either (6.36) or (6.37) is fulfilled.

First suppose that (6.36) holds. The integrations of (6.1_0) from t_M to t_m, from a to τ_0, and from τ_0 to b, in view of (6.29)–(6.32), result in

$$M-m=\int_{t_M}^{t_m}[p(s)]_-u(\tau(s))ds-\int_{t_M}^{t_m}[p(s)]_+u(\tau(s))ds\leq MB_2, \tag{6.62}$$

$$\begin{aligned}u(a)-u(\tau_0)=\int_a^{\tau_0}[p(s)]_-u(\tau(s))ds-\int_a^{\tau_0}[p(s)]_+u(\tau(s))ds\leq\\ \leq MB_1-mA_1,\end{aligned} \tag{6.63}$$

$$\begin{aligned}u(\tau_0)-u(b)=\int_{\tau_0}^{b}[p(s)]_-u(\tau(s))ds-\int_{\tau_0}^{b}[p(s)]_+u(\tau(s))ds\leq\\ \leq M\big(B_2+B_3\big)-m\big(A_2+A_3\big).\end{aligned} \tag{6.64}$$

Multiplying both sides of (6.64) by $\left|\frac{\mu}{\lambda}\right|$, summing with (6.63), and taking into account (1.2_0), (2.1), (6.31), (6.35), and the assumption $\left|\frac{\mu}{\lambda}\right|\in\,]0,1]$, we get

$$M\left(\left|\frac{\mu}{\lambda}\right|-1\right) \le u(\tau_0)\left(\left|\frac{\mu}{\lambda}\right|-1\right) \le$$

$$M\left(B_1+\left|\frac{\mu}{\lambda}\right|B_2+\left|\frac{\mu}{\lambda}\right|B_3\right)-m\left(A_1+\left|\frac{\mu}{\lambda}\right|A_2+\left|\frac{\mu}{\lambda}\right|A_3\right),$$

i.e.,

$$\begin{aligned}0 \le m\Big(A_1+\left|\frac{\mu}{\lambda}\right|A_2+\left|\frac{\mu}{\lambda}\right|A_3\Big) &\le \\ \le M\Big(1-\left|\frac{\mu}{\lambda}\right|+B_1+\left|\frac{\mu}{\lambda}\right|B_2+\left|\frac{\mu}{\lambda}\right|B_3\Big).\end{aligned} \tag{6.65}$$

On the other hand, with respect to (6.10), (6.11), (6.29), (6.35), and the assumption $\left|\frac{\mu}{\lambda}\right| \in\,]0,1]$, (6.62) yields

$$0 < M\big(1-B_2\big) \le m. \tag{6.66}$$

Thus, it follows from (6.65) and (6.66) that

$$\Big(A_1+\left|\frac{\mu}{\lambda}\right|A_2+\left|\frac{\mu}{\lambda}\right|A_3\Big)\big(1-B_2\big) \le 1-\left|\frac{\mu}{\lambda}\right|+B_1+\left|\frac{\mu}{\lambda}\right|B_2+\left|\frac{\mu}{\lambda}\right|B_3,$$

which, on account of (6.29), contradicts (6.13).

Now suppose that (6.37) is fulfilled. The integrations of (6.1_0) from a to t_m, from t_m to t_M, and from t_M to b, on account of (6.29) and (6.31)–(6.34), yield

$$\begin{aligned}u(a)-m = \int_a^{t_m}[p(s)]_-u(\tau(s))ds-\int_a^{t_m}[p(s)]_+u(\tau(s))ds &\le \\ \le M\big(B_1+B_{21}\big)-m\big(A_1+A_{21}\big),\end{aligned} \tag{6.67}$$

$$\begin{aligned}m-M = \int_{t_m}^{t_M}[p(s)]_-u(\tau(s))ds-\int_{t_m}^{t_M}[p(s)]_+u(\tau(s))ds &\le \\ \le MB_{22}-mA_{22},\end{aligned} \tag{6.68}$$

$$\begin{aligned}M-u(b) = \int_{t_M}^{b}[p(s)]_-u(\tau(s))ds-\int_{t_M}^{b}[p(s)]_+u(\tau(s))ds &\le \\ \le M\big(B_{23}+B_3\big)-m\big(A_{23}+A_3\big).\end{aligned} \tag{6.69}$$

Multiplying both sides of (6.69) by $\left|\frac{\mu}{\lambda}\right|$, summing with (6.67), and taking into account (1.2_0) and (2.1), we get

$$\left|\frac{\mu}{\lambda}\right| M - m \leq M\left(B_1 + B_{21} + \left|\frac{\mu}{\lambda}\right| B_{23} + \left|\frac{\mu}{\lambda}\right| B_3\right) - \\ -m\left(A_1 + A_{21} + \left|\frac{\mu}{\lambda}\right| A_{23} + \left|\frac{\mu}{\lambda}\right| A_3\right).$$

Hence, by virtue of (6.10), (6.11), (6.29), (6.34), (6.35), and the assumption $\left|\frac{\mu}{\lambda}\right| \in \,]0,1]$, the last inequality results in

$$\begin{aligned} 0 < M\bigg(\left|\frac{\mu}{\lambda}\right| - B_1 - B_{21} - \left|\frac{\mu}{\lambda}\right| B_{23} - \left|\frac{\mu}{\lambda}\right| B_3\bigg) \leq \\ \leq m\bigg(1 - A_1 - A_{21} - \left|\frac{\mu}{\lambda}\right| A_{23} - \left|\frac{\mu}{\lambda}\right| A_3\bigg). \end{aligned} \tag{6.70}$$

On the other hand, with respect to (6.34) and (6.35), (6.68) implies

$$0 \leq m\big(1 + A_{22}\big) \leq M\big(1 + B_{22}\big). \tag{6.71}$$

Thus, it follows from (6.70) and (6.71) that

$$\begin{aligned} \bigg(\left|\frac{\mu}{\lambda}\right| - B_1 - B_{21} - \left|\frac{\mu}{\lambda}\right| B_{23} - \left|\frac{\mu}{\lambda}\right| B_3\bigg)\big(1 + A_{22}\big) \leq \\ \leq \bigg(1 - A_1 - A_{21} - \left|\frac{\mu}{\lambda}\right| A_{23} - \left|\frac{\mu}{\lambda}\right| A_3\bigg)\big(1 + B_{22}\big). \end{aligned} \tag{6.72}$$

Obviously, on account of (6.10), (6.29), (6.34), and the assumption $\left|\frac{\mu}{\lambda}\right| \in \,]0,1]$, we obtain

$$\left(\left|\frac{\mu}{\lambda}\right| - B_1 - B_{21} - \left|\frac{\mu}{\lambda}\right| B_{23} - \left|\frac{\mu}{\lambda}\right| B_3\right)\left(1 + A_{22}\right) = \\ = \left(\left|\frac{\mu}{\lambda}\right| - B_1 - B_{21} - B_{22} - \left|\frac{\mu}{\lambda}\right| B_{23} - \left|\frac{\mu}{\lambda}\right| B_3\right)\left(1 + A_2\right) + B_{22}\big(1 + A_2\big) - \\ - \left(\left|\frac{\mu}{\lambda}\right| - B_1 - B_{21} - \left|\frac{\mu}{\lambda}\right| B_{23} - \left|\frac{\mu}{\lambda}\right| B_3\right)\left(A_{21} + A_{23}\right) \geq \\ \geq \left(\left|\frac{\mu}{\lambda}\right| - B\right)\left(1 + A_2\right) + B_{22} - \left|\frac{\mu}{\lambda}\right|\left(A_{21} + A_{23}\right)$$

and

$$\Big(1 - A_1 - A_{21} - \Big|\frac{\mu}{\lambda}\Big| A_{23} - \Big|\frac{\mu}{\lambda}\Big| A_3\Big)\,\big(1 + B_{22}\big) = 1 - A_1 -$$

$$- \Big|\frac{\mu}{\lambda}\Big| A_3 - \Big(A_{21} + \Big|\frac{\mu}{\lambda}\Big| A_{23}\Big) + \Big(1 - A_1 - A_{21} - \Big|\frac{\mu}{\lambda}\Big| A_{23} - \Big|\frac{\mu}{\lambda}\Big| A_3\Big) B_{22} \leq$$

$$\leq 1 - A_1 - \Big|\frac{\mu}{\lambda}\Big| A_3 - \Big|\frac{\mu}{\lambda}\Big| \big(A_{21} + A_{23}\big) + B_{22}.$$

By virtue of the last two inequalities, (6.72) yields

$$\Big(\Big|\frac{\mu}{\lambda}\Big| - B\Big)\,\big(1 + A_2\big) \leq 1 - A_1 - \Big|\frac{\mu}{\lambda}\Big| A_3,$$

which, in view of (6.29), contradicts (6.12).

Now suppose that u changes its sign in $[\tau_0, \tau_1]$. Define numbers m_0 and M_0 by (6.50) and choose $\alpha_0, \alpha_1 \in [\tau_0, \tau_1]$ such that (6.51) holds. It is clear that (6.52) is satisfied and without loss of generality we can assume that $\alpha_0 < \alpha_1$. Moreover, define numbers A_{2i}, B_{2i} $(i = 1, 2, 3)$ by (6.34) and put

$$g(x) \overset{\text{def}}{=} \frac{\left|\frac{\mu}{\lambda}\right| - B}{x + A_1 + \left|\frac{\mu}{\lambda}\right| A_3 - 1} + x \quad \text{for} \quad x > 1 - A_1 - \Big|\frac{\mu}{\lambda}\Big| A_3, \tag{6.73}$$

where B is given by (6.10).

In a similar manner as in the second part of the proof of Theorem 6.1, it can be shown that the inequalities (6.55) and (6.57) hold. Due to (6.10), (6.11), (6.29), (6.34), and the assumption $\left|\frac{\mu}{\lambda}\right| \in \,]0, 1]$, we have

$$B_1 + B_{21} + B_{23} + \Big|\frac{\mu}{\lambda}\Big| B_3 < \Big|\frac{\mu}{\lambda}\Big|, \qquad B_{22} < 1.$$

Thus, by virtue of (6.52), it follows from (6.55) and (6.57) that

$$A_{22} > 1, \qquad A_1 + A_{21} + A_{23} + \Big|\frac{\mu}{\lambda}\Big| A_3 > 1, \tag{6.74}$$

and

$$A_{22} \geq 1 + \frac{m_0}{M_0}\big(1 - B_{22}\big)\,, \tag{6.75}$$

$$\frac{m_0}{M_0} \geq \frac{\left|\frac{\mu}{\lambda}\right| - B_1 - B_{21} - B_{23} - \left|\frac{\mu}{\lambda}\right| B_3}{A_1 + A_{21} + A_{23} + \left|\frac{\mu}{\lambda}\right| A_3 - 1}\,. \tag{6.76}$$

According to (6.74), the assumption $\left|\frac{\mu}{\lambda}\right| \in]0,1]$, and the fact that

$$\big(1-B_{22}\big)\left(\left|\frac{\mu}{\lambda}\right| - B_1 - B_{21} - B_{23} - \left|\frac{\mu}{\lambda}\right| B_3\right) \geq$$
$$\geq \left|\frac{\mu}{\lambda}\right| - B_1 - B_{21} - \left|\frac{\mu}{\lambda}\right| B_{22} - B_{23} - \left|\frac{\mu}{\lambda}\right| B_3 \geq \left|\frac{\mu}{\lambda}\right| - B,$$

from (6.75) and (6.76) we get

$$A_{22} \geq 1 + \frac{\left|\frac{\mu}{\lambda}\right| - B}{A_1 + A_{21} + A_{23} + \left|\frac{\mu}{\lambda}\right| A_3 - 1}\,. \tag{6.77}$$

First suppose that (6.14) and (6.15) are satisfied. By virtue of (6.74), from (6.77) we have

$$\left|\frac{\mu}{\lambda}\right| - B \leq \big(A_{22}-1\big)\left(A_1 + A_{21} + A_{23} + \left|\frac{\mu}{\lambda}\right| A_3 - 1\right) \leq$$
$$\leq \frac{1}{4}\left(A_1 + A_{21} + A_{22} + A_{23} + \left|\frac{\mu}{\lambda}\right| A_3 - 2\right)^2 =$$
$$= \frac{1}{4}\left(A_1 + A_2 + \left|\frac{\mu}{\lambda}\right| A_3 - 2\right)^2,$$

which, in view of (6.11), (6.29), (6.34), and (6.74), contradicts (6.15).

Now suppose that (6.16) and (6.17) are fulfilled. It is not difficult to verify that, on account of (6.16) and (6.29), the function g defined by (6.73) is nondecreasing in $[0,+\infty[$. Therefore, from (6.77) we obtain

$$A_{21} + A_{22} + A_{23} \geq 1 + \frac{\left|\frac{\mu}{\lambda}\right| - B}{A_1 + A_{21} + A_{23} + \left|\frac{\mu}{\lambda}\right| A_3 - 1} + A_{21} + A_{23} =$$
$$= 1 + g\big(A_{21} + A_{23}\big) \geq 1 + g(0) = 1 + \frac{\left|\frac{\mu}{\lambda}\right| - B}{A_1 + \left|\frac{\mu}{\lambda}\right| A_3 - 1}\,,$$

which, in view of (6.29) and (6.34), contradicts (6.17). □

Proof of Theorem 6.3. Assume that the problem (6.1_0), (1.2_0) has a nontrivial solution u.

First suppose that u has a zero in $[\tau_0,\tau_1]$. Define numbers m_0 and M_0 by (6.50) and choose $\alpha_0, \alpha_1 \in [\tau_0,\tau_1]$ such that (6.51) holds. Obviously,

$$m_0 \geq 0, \qquad M_0 \geq 0, \qquad m_0 + M_0 > 0, \tag{6.78}$$

since if $m_0 = 0$ and $M_0 = 0$, then, in view of (6.1_0) and (6.50), we obtain $u(\tau_0) = 0$ and $u'(t) = 0$ for $t \in [a, b]$, i.e., $u \equiv 0$. It is also evident that without loss of generality we can assume that $\alpha_0 < \alpha_1$.

The integration of (6.1_0) from α_0 to α_1, on account of (6.50), (6.51), and (6.78), yields

$$\begin{aligned} M_0 + m_0 &= \int_{\alpha_0}^{\alpha_1} [p(s)]_+ u(\tau(s))ds - \int_{\alpha_0}^{\alpha_1} [p(s)]_- u(\tau(s))ds \leq \\ &\leq M_0 \int_{\tau_0}^{\tau_1} [p(s)]_+ ds + m_0 \int_{\tau_0}^{\tau_1} [p(s)]_- ds, \end{aligned} \tag{6.79}$$

which, by virtue of (6.18) and (6.78), results in $M_0 + m_0 < M_0 + m_0$, a contradiction.

Now suppose that u has no zero in $[\tau_0, \tau_1]$. Without loss of generality we can assume that $u(t) > 0$ for $t \in [\tau_0, \tau_1]$. Define numbers M and m by (6.31) and choose $t_M, t_m \in [\tau_0, \tau_1]$ such that (6.32) holds. Furthermore, denote

$$\begin{aligned} f_+(t) &\stackrel{\text{def}}{=} \int_a^t [p(s)]_+ ds + \left|\frac{\mu}{\lambda}\right| \int_t^b [p(s)]_+ ds \quad \text{for} \quad t \in [a, b], \\ f_-(t) &\stackrel{\text{def}}{=} \int_a^t [p(s)]_- ds + \left|\frac{\mu}{\lambda}\right| \int_t^b [p(s)]_- ds \quad \text{for} \quad t \in [a, b]. \end{aligned} \tag{6.80}$$

It is obvious that

$$M > 0, \qquad m > 0, \tag{6.81}$$

and either (6.36) or (6.37) is satisfied.

If (6.36) holds, then the integration of (6.1_0) from t_M to t_m, on account of (6.31), (6.32), and (6.81), results in

$$M - m = \int_{t_M}^{t_m} [p(s)]_- u(\tau(s))ds - \int_{t_M}^{t_m} [p(s)]_+ u(\tau(s))ds \leq M \int_{\tau_0}^{\tau_1} [p(s)]_- ds.$$

If (6.37) holds, then the integration of (6.1_0) from t_m to t_M, in view of

(6.31), (6.32), and (6.81), results in

$$M - m = \int_{t_m}^{t_M} [p(s)]_+ u(\tau(s))ds - \int_{t_m}^{t_M} [p(s)]_- u(\tau(s))ds \le M \int_{\tau_0}^{\tau_1} [p(s)]_+ ds.$$

Therefore, with respect to (6.18) and (6.81), in both cases (6.36) and (6.37) we have

$$0 < M\big(1 - T\big) \le m, \tag{6.82}$$

where T is defined by (6.21).

First suppose that (6.19) holds with T given by (6.21). The integrations of (6.1_0) from a to t_M and from t_M to b, on account of (6.31) and (6.32), imply

$$\begin{aligned} M - u(a) = \int_a^{t_M} [p(s)]_+ u(\tau(s))ds - \int_a^{t_M} [p(s)]_- u(\tau(s))ds \le \\ \le M \int_a^{t_M} [p(s)]_+ ds - m \int_a^{t_M} [p(s)]_- ds, \end{aligned} \tag{6.83}$$

$$\begin{aligned} u(b) - M = \int_{t_M}^{b} [p(s)]_+ u(\tau(s))ds - \int_{t_M}^{b} [p(s)]_- u(\tau(s))ds \le \\ \le M \int_{t_M}^{b} [p(s)]_+ ds - m \int_{t_M}^{b} [p(s)]_- ds. \end{aligned} \tag{6.84}$$

Multiplying both sides of (6.84) by $\left|\frac{\mu}{\lambda}\right|$, summing with (6.83), and taking into account (1.2_0), (2.1), and (6.80), we get

$$\begin{aligned} M\left(1 - \left|\frac{\mu}{\lambda}\right|\right) \le M\left(\int_a^{t_M} [p(s)]_+ ds + \left|\frac{\mu}{\lambda}\right| \int_{t_M}^{b} [p(s)]_+ ds\right) - \\ -m\left(\int_a^{t_M} [p(s)]_- ds + \left|\frac{\mu}{\lambda}\right| \int_{t_M}^{b} [p(s)]_- ds\right) = M f_+(t_M) - m f_-(t_M). \end{aligned} \tag{6.85}$$

It is easy to verify that, in view of the assumption $\left|\frac{\mu}{\lambda}\right| \in [0,1]$, the functions f_+ and f_- defined by (6.80) are nondecreasing in $[a,b]$ and thus, with respect to (6.29) and (6.80), it follows from (6.85) that

$$M\left(1-\left|\frac{\mu}{\lambda}\right|\right)\leq Mf_+(t_M)-mf_-(t_M)\leq Mf_+(\tau_1)-mf_-(\tau_0)=$$
$$=M\left(A_1+A_2+\left|\frac{\mu}{\lambda}\right|A_3\right)-m\left(B_1+\left|\frac{\mu}{\lambda}\right|B_2+\left|\frac{\mu}{\lambda}\right|B_3\right). \tag{6.86}$$

By virtue of (6.82), (6.86) yields

$$M\left(1-\left|\frac{\mu}{\lambda}\right|\right)\leq M\left(A_1+A_2+\left|\frac{\mu}{\lambda}\right|A_3\right)-$$
$$-M\left(B_1+\left|\frac{\mu}{\lambda}\right|B_2+\left|\frac{\mu}{\lambda}\right|B_3\right)(1-T),$$

which, in view of (6.29) and (6.81), contradicts (6.19).

Now suppose that (6.20) holds with T given by (6.21). The integrations of (6.1_0) from a to t_m and from t_m to b, on account of (6.31) and (6.32), imply

$$m-u(a)=\int_a^{t_m}[p(s)]_+u(\tau(s))ds-\int_a^{t_m}[p(s)]_-u(\tau(s))ds\geq$$
$$\geq m\int_a^{t_m}[p(s)]_+ds-M\int_a^{t_m}[p(s)]_-ds, \tag{6.87}$$

$$u(b)-m=\int_{t_m}^{b}[p(s)]_+u(\tau(s))ds-\int_{t_m}^{b}[p(s)]_-u(\tau(s))ds\geq$$
$$\geq m\int_{t_m}^{b}[p(s)]_+ds-M\int_{t_m}^{b}[p(s)]_-ds. \tag{6.88}$$

Multiplying both sides of (6.88) by $\left|\frac{\mu}{\lambda}\right|$, summing with (6.87), and taking into account (1.2_0), (2.1), and (6.80), we obtain

$$m\left(1-\left|\frac{\mu}{\lambda}\right|\right)\geq m\left(\int_a^{t_m}[p(s)]_+ds+\left|\frac{\mu}{\lambda}\right|\int_{t_m}^{b}[p(s)]_+ds\right)-$$
$$-M\left(\int_a^{t_m}[p(s)]_-ds+\left|\frac{\mu}{\lambda}\right|\int_{t_m}^{b}[p(s)]_-ds\right)=mf_+(t_m)-Mf_-(t_m). \tag{6.89}$$

As above, in view of the assumption $\left|\frac{\mu}{\lambda}\right| \in [0,1]$, the functions f_+ and f_- defined by (6.80) are nondecreasing in $[a,b]$ and thus, with respect to (6.29) and (6.80), it follows from (6.89) that

$$\begin{aligned} m\left(1-\left|\frac{\mu}{\lambda}\right|\right) &\geq mf_+(t_m) - Mf_-(t_m) \geq mf_+(\tau_0) - Mf_-(\tau_1) = \\ &= m\left(A_1 + \left|\frac{\mu}{\lambda}\right| A_2 + \left|\frac{\mu}{\lambda}\right| A_3\right) - M\left(B_1 + B_2 + \left|\frac{\mu}{\lambda}\right| B_3\right). \end{aligned} \tag{6.90}$$

By virtue of (6.18), (6.21), and (6.82), (6.90) implies

$$\begin{aligned} m\left(1-\left|\frac{\mu}{\lambda}\right|\right)(1-T) \geq m\left(A_1 + \left|\frac{\mu}{\lambda}\right| A_2 + \left|\frac{\mu}{\lambda}\right| A_3\right)(1-T) - \\ - m\left(B_1 + B_2 + \left|\frac{\mu}{\lambda}\right| B_3\right), \end{aligned}$$

which, in view of (6.29) and (6.81), contradicts (6.20). □

Proof of Theorem 6.4. Assume that the problem (6.1_0), (1.2_0) has a nontrivial solution u.

According to Theorem 2.1 (see p. 17) and the assumptions (6.22) and (6.23), it is clear that $G \in V_{ab}^+(\lambda,\mu)$, where

$$G(v)(t) \stackrel{\text{def}}{=} [p(t)]_+ v(\tau(t)) \quad \text{for} \quad t \in [a,b].$$

Now it follows easily from Definition 2.1 (see p. 15) that u changes its sign in $[\tau_0,\tau_1]$. Define numbers m_0 and M_0 by (6.50) and choose $\alpha_0, \alpha_1 \in [\tau_0,\tau_1]$ such that (6.51) holds. Obviously, (6.52) is satisfied and without loss of generality we can assume that $\alpha_1 < \alpha_0$. From (6.1_0), (1.2_0), (6.22), and (6.23), with respect to (2.1), (6.50), and (6.52), we obtain

$$\begin{aligned} &\big(M_0\gamma(t) + u(t)\big)' \geq \\ &\geq [p(t)]_+\big(M_0\gamma(\tau(t)) + u(\tau(t))\big) + [p(t)]_-\big(M_0 - u(\tau(t))\big) \geq \\ &\geq G(M_0\gamma + u)(t) \quad \text{for} \quad t \in [a,b], \end{aligned} \tag{6.91}$$

$$|\lambda|\big(M_0\gamma(a) + u(a)\big) - |\mu|\big(M_0\gamma(b) + u(b)\big) > 0,$$

and

$$\begin{gathered}\big(m_0\gamma(t)-u(t)\big)'\geq\\ \geq [p(t)]_+\big(m_0\gamma(\tau(t))-u(\tau(t))\big)+[p(t)]_-\big(m_0+u(\tau(t))\big)\geq\\ \geq G(m_0\gamma-u)(t)\quad\text{for}\quad t\in[a,b],\end{gathered}\tag{6.92}$$

$$|\lambda|\big(m_0\gamma(a)-u(a)\big)-|\mu|\big(m_0\gamma(b)-u(b)\big)>0.$$

Hence, according to the condition $G\in V_{ab}^+(\lambda,\mu)$ and Remark 2.3 (see p. 16), we get

$$M_0\gamma(t)+u(t)\geq 0,\qquad m_0\gamma(t)-u(t)\geq 0\quad\text{for}\quad t\in[a,b].$$

By virtue of the last inequalities, it follows from (6.91) and (6.92) that

$$\big(M_0\gamma(t)+u(t)\big)'\geq 0,\qquad \big(m_0\gamma(t)-u(t)\big)'\geq 0\quad\text{for}\quad t\in[a,b].\tag{6.93}$$

The integration of the first inequality in (6.93) from α_1 to α_0, in view of (6.51) and (6.52), yields

$$M_0\gamma(\alpha_0)-m_0-M_0\gamma(\alpha_1)-M_0\geq 0,$$

i.e.,

$$\gamma(\alpha_0)-\gamma(\alpha_1)\geq 1+\frac{m_0}{M_0}\,.\tag{6.94}$$

On the other hand, the integrations of the second inequality in (6.93) from a to α_1 and from α_0 to b, on account of (6.51), imply

$$m_0\gamma(\alpha_1)-M_0-m_0\gamma(a)+u(a)\geq 0,\tag{6.95}$$

$$m_0\gamma(b)-u(b)-m_0\gamma(\alpha_0)-m_0\geq 0.\tag{6.96}$$

Multiplying both sides of (6.96) by $\left|\frac{\mu}{\lambda}\right|$, summing with (6.95), and taking into account (1.2_0), (2.1), and (6.52), we get

$$\gamma(\alpha_1)-\gamma(a)+\left|\frac{\mu}{\lambda}\right|\big(\gamma(b)-\gamma(\alpha_0)\big)\geq\left|\frac{\mu}{\lambda}\right|+\frac{M_0}{m_0}\,.\tag{6.97}$$

First suppose that (6.24) and (6.25) are fulfilled. Summing (6.94) and (6.97) and taking into account (6.52), we obtain

$$\begin{gathered}\gamma(\alpha_0)-\gamma(a)+\left|\frac{\mu}{\lambda}\right|\big(\gamma(b)-\gamma(\alpha_0)\big)\geq\\ \geq 1+\left|\frac{\mu}{\lambda}\right|+\frac{M_0}{m_0}+\frac{m_0}{M_0}\geq 3+\left|\frac{\mu}{\lambda}\right|.\end{gathered}\tag{6.98}$$

On the other hand, by virtue of the fact that the function γ is nondecreasing in $[a,b]$, and the assumption $\left|\frac{\mu}{\lambda}\right| \in [0,1[$, we get

$$\left|\frac{\mu}{\lambda}\right| \gamma(b) + \Big(1 - \left|\frac{\mu}{\lambda}\right|\Big)\, \gamma(\tau_1) - \gamma(a) \geq \left|\frac{\mu}{\lambda}\right| \gamma(b) + \Big(1 - \left|\frac{\mu}{\lambda}\right|\Big)\, \gamma(\alpha_0) - \gamma(a),$$

which, together with (6.98), contradicts (6.25).

Now suppose that (6.26) and (6.27) are satisfied. According to (6.26), (6.52), and the fact that the function γ is nondecreasing in $[a,b]$, it follows from (6.97) that

$$\frac{m_0}{M_0} \geq \frac{|\lambda|}{|\lambda|\big(\gamma(\alpha_1) - \gamma(a)\big) + |\mu|\big(\gamma(b) - \gamma(\alpha_0)\big) - |\mu|}$$

and thus, (6.94) implies

$$\gamma(\alpha_0) - \gamma(\alpha_1) \geq 1 + \frac{|\lambda|}{|\lambda|\big(\gamma(\alpha_1) - \gamma(a)\big) + |\mu|\big(\gamma(b) - \gamma(\alpha_0)\big) - |\mu|}\,. \tag{6.99}$$

Let

$$g(x) \overset{\text{def}}{=} \frac{|\lambda|}{x - |\mu|} + \frac{x}{|\lambda|} \quad \text{for} \quad x > |\mu|. \tag{6.100}$$

By virtue of (6.99), (6.100), the assumption $\left|\frac{\mu}{\lambda}\right| \in [0,1[$, and the fact that the function γ is nondecreasing in $[a,b]$, we get

$$\begin{aligned} &\gamma(\tau_1) - \gamma(\tau_0) = \\ &= \gamma(\alpha_0) - \gamma(\alpha_1) + \gamma(\tau_1) - \gamma(\alpha_0) + \gamma(\alpha_1) - \gamma(\tau_0) \geq \\ &\geq 1 + \frac{|\lambda|}{|\lambda|\big(\gamma(\alpha_1) - \gamma(a)\big) + |\mu|\big(\gamma(b) - \gamma(\alpha_0)\big) - |\mu|} + \\ &\quad + \left|\frac{\mu}{\lambda}\right| \big(\gamma(\tau_1) - \gamma(\alpha_0)\big) + \gamma(\alpha_1) - \gamma(\tau_0) = \\ &= 1 + g\big(|\lambda|\big(\gamma(\alpha_1) - \gamma(a)\big) + |\mu|\big(\gamma(b) - \gamma(\alpha_0)\big)\big) + \\ &\quad + \gamma(a) - \gamma(\tau_0) + \left|\frac{\mu}{\lambda}\right| \big(\gamma(\tau_1) - \gamma(b)\big). \end{aligned} \tag{6.101}$$

It is easy to verify that the function g is nondecreasing in $[|\lambda|+|\mu|, +\infty[$ and thus, according to (6.26) and the fact that the function γ is nondecreasing in $[a,b]$, we find

$$\begin{aligned} &g\big(|\lambda|\big(\gamma(\alpha_1) - \gamma(a)\big) + |\mu|\big(\gamma(b) - \gamma(\alpha_0)\big)\big) \geq \\ &\qquad \geq g\big(|\lambda|\big(\gamma(\tau_0) - \gamma(a)\big) + |\mu|\big(\gamma(b) - \gamma(\tau_1)\big)\big). \end{aligned}$$

Therefore, (6.101) yields

$$\gamma(\tau_1)-\gamma(\tau_0)\geq 1+g\big(|\lambda|\big(\gamma(\tau_0)-\gamma(a)\big)+|\mu|\big(\gamma(b)-\gamma(\tau_1)\big)\big)+$$
$$+\gamma(a)-\gamma(\tau_0)+\left|\frac{\mu}{\lambda}\right|\big(\gamma(\tau_1)-\gamma(b)\big)=$$
$$=1+\frac{|\lambda|}{|\lambda|\big(\gamma(\tau_0)-\gamma(a)\big)+|\mu|\big(\gamma(b)-\gamma(\tau_1)\big)-|\mu|}\,,$$

which contradicts (6.27). □

6.3. Comments and Examples

Example 6.1. Let $|\mu|\leq|\lambda|$ and let $x_i, y_i\in R_+$ $(i=1,2,3)$ be such that

$$x_1+x_2+\left|\frac{\mu}{\lambda}\right|x_3<1 \tag{6.102}$$

and

$$\left(y_1+\left|\frac{\mu}{\lambda}\right|y_2+\left|\frac{\mu}{\lambda}\right|y_3\right)(1-x_2)=\left|\frac{\mu}{\lambda}\right|-1+x_1+x_2+\left|\frac{\mu}{\lambda}\right|x_3.$$

Let, moreover, $a=0$, $b=7$,

$$p(t)=\begin{cases}-y_1 & \text{for} \quad t\in[0,1[\\ x_1 & \text{for} \quad t\in[1,2[\\ x_2 & \text{for} \quad t\in[2,3[\\ -y_2 & \text{for} \quad t\in[3,4[\\ 0 & \text{for} \quad t\in[4,5[\\ x_3 & \text{for} \quad t\in[5,6[\\ -y_3 & \text{for} \quad t\in[6,7]\end{cases}, \tag{6.103}$$

and

$$\tau(t)=\begin{cases}2 & \text{for} \quad t\in[0,1[\,\cup\,[3,4[\,\cup\,[6,7]\\ 3 & \text{for} \quad t\in[1,3[\,\cup\,[5,6[\\ 4 & \text{for} \quad t\in[4,5[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 4$, and

$$\begin{aligned} &\int_a^{\tau_0} [p(s)]_+ ds = x_1, \quad \int_{\tau_0}^{\tau_1} [p(s)]_+ ds = x_2, \quad \int_{\tau_1}^{b} [p(s)]_+ ds = x_3, \\ &\int_a^{\tau_0} [p(s)]_- ds = y_1, \quad \int_{\tau_0}^{\tau_1} [p(s)]_- ds = y_2, \quad \int_{\tau_1}^{b} [p(s)]_- ds = y_3. \end{aligned} \tag{6.104}$$

On the other hand, the function

$$u(t) = \begin{cases} y_1(1-x_2)(1-t) + 1 - x_1 - x_2 & \text{for} \quad t \in [0,1[\\ x_1(t-2) + 1 - x_2 & \text{for} \quad t \in [1,2[\\ x_2(t-3) + 1 & \text{for} \quad t \in [2,3[\\ y_2(1-x_2)(3-t) + 1 & \text{for} \quad t \in [3,4[\\ 1 - y_2(1-x_2) & \text{for} \quad t \in [4,5[\\ x_3(t-5) + 1 - y_2(1-x_2) & \text{for} \quad t \in [5,6[\\ y_3(1-x_2)(7-t) + 1 + x_3 - (y_2+y_3)(1-x_2) & \text{for} \quad t \in [6,7] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

This example shows that in Theorem 6.1 the strict inequality (6.4) cannot be replaced by the nonstrict one.

Example 6.2. Let $|\mu| \leq |\lambda|$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that (6.102) holds and

$$\left(1 - x_1 - x_2 - \left|\frac{\mu}{\lambda}\right| x_3\right)\left(1 + y_2\right) = \left|\frac{\mu}{\lambda}\right| - y_1 - \left|\frac{\mu}{\lambda}\right| y_3.$$

Let, moreover, $a = 0$, $b = 7$, $p \in L\big([a,b];R\big)$ be defined by (6.103), and

$$\tau(t) = \begin{cases} 2 & \text{for} \quad t \in [4,5[\\ 3 & \text{for} \quad t \in [1,3[\cup [5,6[\\ 4 & \text{for} \quad t \in [0,1[\cup [3,4[\cup [6,7] \end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 4$, and (6.104) is fulfilled.

On the other hand, the function

$$u(t)=\begin{cases} y_1(1-t)+1+y_2-(x_1+x_2)(1+y_2) & \text{for} \quad t\in[0,1[\\ x_1(1+y_2)(t-2)+1+y_2-x_2(1+y_2) & \text{for} \quad t\in[1,2[\\ x_2(1+y_2)(t-3)+1+y_2 & \text{for} \quad t\in[2,3[\\ y_2(4-t)+1 & \text{for} \quad t\in[3,4[\\ 1 & \text{for} \quad t\in[4,5[\\ x_3(1+y_2)(t-5)+1 & \text{for} \quad t\in[5,6[\\ y_3(7-t)+1-y_3+x_3(1+y_2) & \text{for} \quad t\in[6,7] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q\in L\big([a,b];R\big)$ and $c\in R$ such that the problem (6.1), (1.2) has no solution.

This example shows that in Theorem 6.1 the strict inequality (6.5) cannot be replaced by the nonstrict one.

Example 6.3. Let $|\mu|\le|\lambda|$ and let $x_i,y_i\in R_+$ $(i=1,2,3)$ be such that (6.102) holds and

$$y_1+\left|\frac{\mu}{\lambda}\right|y_3<\left|\frac{\mu}{\lambda}\right|+\sqrt{1-x_1-x_2-\left|\frac{\mu}{\lambda}\right|x_3}\,,$$

$$y_1+y_2+\left|\frac{\mu}{\lambda}\right|y_3\ge 1+\left|\frac{\mu}{\lambda}\right|+2\sqrt{1-x_1-x_2-\left|\frac{\mu}{\lambda}\right|x_3}\,.$$

Put $\alpha=\sqrt{1-x_1-x_2-\left|\frac{\mu}{\lambda}\right|x_3}$ and $k=\left|\frac{\mu}{\lambda}\right|+\alpha-y_1-\left|\frac{\mu}{\lambda}\right|y_3$. Obviously, $k>0$ and $y_2\ge 1+\alpha+k$. Let, moreover, $a=0$, $b=10$,

$$p(t)=\begin{cases} -y_1 & \text{for} \quad t\in[0,1[\\ x_1 & \text{for} \quad t\in[1,2[\\ x_2 & \text{for} \quad t\in[2,3[\\ -k & \text{for} \quad t\in[3,4[\\ -1 & \text{for} \quad t\in[4,5[\\ -\big(y_2-1-\alpha-k\big) & \text{for} \quad t\in[5,6[\\ -\alpha & \text{for} \quad t\in[6,7[\\ 0 & \text{for} \quad t\in[7,8[\\ x_3 & \text{for} \quad t\in[8,9[\\ -y_3 & \text{for} \quad t\in[9,10] \end{cases}, \tag{6.105}$$

and

$$\tau(t) = \begin{cases} 7 & \text{for} \quad t \in [0,1[\cup [3,4[\cup [9,10] \\ 4 & \text{for} \quad t \in [1,3[\cup [4,5[\cup [6,7[\cup [8,9[\\ 5 & \text{for} \quad t \in [5,6[\\ 2 & \text{for} \quad t \in [7,8[\end{cases}. \tag{6.106}$$

Obviously, $\tau_0 = 2$, $\tau_1 = 7$, and (6.104) is satisfied.

On the other hand, the function

$$u(t) = \begin{cases} \alpha y_1(1-t) + \alpha k + x_1 + x_2 - 1 & \text{for} \quad t \in [0,1[\\ x_1(2-t) + \alpha k + x_2 - 1 & \text{for} \quad t \in [1,2[\\ x_2(3-t) + \alpha k - 1 & \text{for} \quad t \in [2,3[\\ \alpha k(4-t) - 1 & \text{for} \quad t \in [3,4[\\ t - 5 & \text{for} \quad t \in [4,5[\\ 0 & \text{for} \quad t \in [5,6[\\ \alpha(t-6) & \text{for} \quad t \in [6,7[\\ \alpha & \text{for} \quad t \in [7,8[\\ x_3(8-t) + \alpha & \text{for} \quad t \in [8,9[\\ \alpha y_3(10-t) + \alpha - x_3 - \alpha y_3 & \text{for} \quad t \in [9,10] \end{cases} \tag{6.107}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L([a,b];R)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

This example shows that in Theorem 6.1 the strict inequality (6.7) cannot be replaced by the nonstrict one.

Further, in addition, let $|\mu| < |\lambda|$ and $x_i = 0$ $(i = 1,2,3)$. Put

$$\gamma(t) = \delta + \int_a^t [p(s)]_- ds \quad \text{for} \quad t \in [a,b], \tag{6.108}$$

where $\delta > \frac{|\mu|}{|\lambda|-|\mu|}(y_1 + y_2 + y_3)$ and $p \in L([a,b];R)$ is defined by (6.105). Obviously, γ satisfies (6.22) with $\tau \in \mathcal{M}_{ab}$ given by (6.106), (6.23), and

$$\gamma(\tau_0) - \gamma(a) = y_1, \quad \gamma(\tau_1) - \gamma(\tau_0) = y_2, \quad \gamma(b) - \gamma(\tau_1) = y_3. \tag{6.109}$$

Thus, (6.24) is fulfilled and

$$\left|\frac{\mu}{\lambda}\right| \big(\gamma(b) - \gamma(\tau_1)\big) + \gamma(\tau_1) - \gamma(a) = y_1 + y_2 + \left|\frac{\mu}{\lambda}\right| y_3 \geq 3 + \left|\frac{\mu}{\lambda}\right|.$$

On the other hand, as we have shown, the problem (6.1_0), (1.2_0) has a nontrivial solution u given by (6.107). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L([a,b];R)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

Consequently, this example also shows that in Theorem 6.4 the strict inequality (6.25) cannot be replaced by the nonstrict one.

Example 6.4. Let $|\mu| \leq |\lambda|$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that (6.102) holds and

$$y_1 + \left|\frac{\mu}{\lambda}\right| y_3 \geq \left|\frac{\mu}{\lambda}\right| + \sqrt{1 - x_1 - x_2 - \left|\frac{\mu}{\lambda}\right| x_3}\,,$$

$$y_2 \geq 1 + \frac{1 - x_1 - x_2 - \left|\frac{\mu}{\lambda}\right| x_3}{y_1 + \left|\frac{\mu}{\lambda}\right| y_3 - \left|\frac{\mu}{\lambda}\right|}\,.$$

Put $\alpha = 1 - x_1 - x_2 - \left|\frac{\mu}{\lambda}\right| x_3$ and $\beta = y_1 + \left|\frac{\mu}{\lambda}\right| y_3 - \left|\frac{\mu}{\lambda}\right|$. Obviously, $\alpha > 0$, $\beta > 0$, and $y_2 \geq 1 + \frac{\alpha}{\beta}$. Let, moreover, $a = 0$, $b = 9$,

$$p(t) = \begin{cases} x_1 & \text{for} \quad t \in [0,1[\\ -y_1 & \text{for} \quad t \in [1,2[\\ x_2 & \text{for} \quad t \in [2,3[\\ -1 & \text{for} \quad t \in [3,4[\\ -\left(y_2 - 1 - \frac{\alpha}{\beta}\right) & \text{for} \quad t \in [4,5[\\ -\frac{\alpha}{\beta} & \text{for} \quad t \in [5,6[\\ 0 & \text{for} \quad t \in [6,7[\\ -y_3 & \text{for} \quad t \in [7,8[\\ x_3 & \text{for} \quad t \in [8,9] \end{cases}, \tag{6.110}$$

and

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [0,1[\cup [2,4[\cup [5,6[\cup [8,9] \\ 6 & \text{for} \quad t \in [1,2[\cup [7,8[\\ 4 & \text{for} \quad t \in [4,5[\\ 2 & \text{for} \quad t \in [6,7[\end{cases}. \tag{6.111}$$

Obviously, $\tau_0 = 2$, $\tau_1 = 6$, and (6.104) is satisfied.

On the other hand, the function

$$u(t)=\begin{cases}\beta x_1(1-t)+y_1\alpha-(1-x_2)\beta & \text{for} \quad t\in[0,1[\\ \alpha y_1(2-t)-(1-x_2)\beta & \text{for} \quad t\in[1,2[\\ \beta x_2(3-t)-\beta & \text{for} \quad t\in[2,3[\\ \beta(t-4) & \text{for} \quad t\in[3,4[\\ 0 & \text{for} \quad t\in[4,5[\\ \alpha(t-5) & \text{for} \quad t\in[5,6[\\ \alpha & \text{for} \quad t\in[6,7[\\ \alpha y_3(7-t)+\alpha & \text{for} \quad t\in[7,8[\\ \beta x_3(9-t)+\alpha(1-y_3)-\beta x_3 & \text{for} \quad t\in[8,9]\end{cases} \tag{6.112}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q\in L([a,b];R)$ and $c\in R$ such that the problem (6.1), (1.2) has no solution.

This example shows that in Theorem 6.1 the strict inequality (6.9) cannot be replaced by the nonstrict one.

Further, in addition, let $|\mu|<|\lambda|$ and $x_i=0$ $(i=1,2,3)$. Define the function $\gamma\in\widetilde{C}([a,b];]0,+\infty[)$ by (6.108), where $\delta>\frac{|\mu|}{|\lambda|-|\mu|}(y_1+y_2+y_3)$ and $p\in L([a,b];R)$ is given by (6.110). Obviously, γ satisfies (6.22) with $\tau\in\mathcal{M}_{ab}$ given by (6.111), (6.23), and (6.109). Thus, (6.26) is fulfilled and

$$\gamma(\tau_1)-\gamma(\tau_0)=y_2\geq 1+\frac{\alpha}{\beta}=1+\frac{1}{\gamma(\tau_0)-\gamma(a)+\left|\frac{\mu}{\lambda}\right|\big(\gamma(b)-\gamma(\tau_1)\big)-\left|\frac{\mu}{\lambda}\right|}\,.$$

On the other hand, as we have shown, the problem (6.1_0), (1.2_0) has a nontrivial solution u given by (6.112). Therefore, according to Remark 1.1 (see p. 14), there exist $q\in L([a,b];R)$ and $c\in R$ such that the problem (6.1), (1.2) has no solution.

Consequently, this example also shows that in Theorem 6.4 the strict inequality (6.27) cannot be replaced by the nonstrict one.

Example 6.5. Let $0\neq|\mu|\leq|\lambda|$ and let $x_i,y_i\in R_+$ $(i=1,2,3)$ be such that

$$y_1+y_2+\left|\frac{\mu}{\lambda}\right|y_3<\left|\frac{\mu}{\lambda}\right| \tag{6.113}$$

and

$$\left(\left|\frac{\mu}{\lambda}\right|-y_1-y_2-\left|\frac{\mu}{\lambda}\right|y_3\right)\big(1+x_2\big)=1-x_1-\left|\frac{\mu}{\lambda}\right|x_3.$$

Let, moreover, $a = 0$, $b = 7$,

$$p(t) = \begin{cases} -y_1 & \text{for} \quad t \in [0,1[\\ x_1 & \text{for} \quad t \in [1,2[\\ 0 & \text{for} \quad t \in [2,3[\\ -y_2 & \text{for} \quad t \in [3,4[\\ x_2 & \text{for} \quad t \in [4,5[\\ x_3 & \text{for} \quad t \in [5,6[\\ -y_3 & \text{for} \quad t \in [6,7] \end{cases}, \tag{6.114}$$

and

$$\tau(t) = \begin{cases} 5 & \text{for} \quad t \in [0,1[\cup [3,4[\cup [6,7] \\ 4 & \text{for} \quad t \in [1,2[\cup [4,6[\\ 3 & \text{for} \quad t \in [2,3[\end{cases}.$$

Obviously, $\tau_0 = 3$, $\tau_1 = 5$, and (6.104) is fulfilled.

On the other hand, the function

$$u(t) = \begin{cases} y_1(1+x_2)(1-t) + 1 - x_1 + y_2(1+x_2) & \text{for} \quad t \in [0,1[\\ x_1(t-2) + 1 + y_2(1+x_2) & \text{for} \quad t \in [1,2[\\ 1 + y_2(1+x_2) & \text{for} \quad t \in [2,3[\\ y_2(1+x_2)(4-t) + 1 & \text{for} \quad t \in [3,4[\\ x_2(t-5) + 1 + x_2 & \text{for} \quad t \in [4,5[\\ x_3(t-5) + 1 + x_2 & \text{for} \quad t \in [5,6[\\ y_3(1+x_2)(7-t) + 1 + x_2 + x_3 - y_3(1+x_2) & \text{for} \quad t \in [6,7] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b]; R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

This example shows that in Theorem 6.2 the strict inequality (6.12) cannot be replaced by the nonstrict one.

Example 6.6. Let $0 \neq |\mu| \leq |\lambda|$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that (6.113) holds and

$$\left(x_1 + \left|\frac{\mu}{\lambda}\right| x_2 + \left|\frac{\mu}{\lambda}\right| x_3\right)\big(1 - y_2\big) = 1 - \left|\frac{\mu}{\lambda}\right| + y_1 + \left|\frac{\mu}{\lambda}\right| \big(y_2 + y_3\big).$$

Let, moreover, $a = 0$, $b = 7$, $p \in L\big([a,b];R\big)$ be defined by (6.114), and

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [0,1[\, \cup\, [3,4[\, \cup\, [6,7] \\ 4 & \text{for} \quad t \in [1,2[\, \cup\, [4,6[\\ 5 & \text{for} \quad t \in [2,3[\end{cases}.$$

Obviously, $\tau_0 = 3$, $\tau_1 = 5$, and (6.104) is fulfilled.

On the other hand, the function

$$u(t) = \begin{cases} y_1(1-t) + 1 - x_1(1-y_2) & \text{for} \quad t \in [0,1[\\ x_1(1-y_2)(t-2) + 1 & \text{for} \quad t \in [1,2[\\ 1 & \text{for} \quad t \in [2,3[\\ y_2(3-t) + 1 & \text{for} \quad t \in [3,4[\\ x_2(1-y_2)(t-4) + 1 - y_2 & \text{for} \quad t \in [4,5[\\ x_3(1-y_2)(t-5) + 1 - y_2 + x_2(1-y_2) & \text{for} \quad t \in [5,6[\\ y_3(7-t) + 1 - y_2 - y_3 + (x_2+x_3)(1-y_2) & \text{for} \quad t \in [6,7] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

This example shows that in Theorem 6.2 the strict inequality (6.13) cannot be replaced by the nonstrict one.

Example 6.7. Let $0 \neq |\mu| \leq |\lambda|$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that (6.113) holds and

$$x_1 + \left|\frac{\mu}{\lambda}\right| x_3 < 1 + \sqrt{\left|\frac{\mu}{\lambda}\right| - y_1 - y_2 - \left|\frac{\mu}{\lambda}\right| y_3}\,,$$

$$x_1 + x_2 + \left|\frac{\mu}{\lambda}\right| x_3 \geq 2 + 2\sqrt{\left|\frac{\mu}{\lambda}\right| - y_1 - y_2 - \left|\frac{\mu}{\lambda}\right| y_3}\,.$$

Put $\alpha = \sqrt{\left|\frac{\mu}{\lambda}\right| - y_1 - y_2 - \left|\frac{\mu}{\lambda}\right| y_3}$ and $k = 1 + \alpha - x_1 - \left|\frac{\mu}{\lambda}\right| x_3$. Obviously,

$k > 0$ and $x_2 \geq 1 + \alpha + k$. Let, moreover, $a = 0$, $b = 10$,

$$p(t) = \begin{cases} x_1 & \text{for} \quad t \in [0,1[\\ -y_1 & \text{for} \quad t \in [1,2[\\ -y_2 & \text{for} \quad t \in [2,3[\\ k & \text{for} \quad t \in [3,4[\\ \alpha & \text{for} \quad t \in [4,5[\\ x_2 - 1 - \alpha - k & \text{for} \quad t \in [5,6[\\ 1 & \text{for} \quad t \in [6,7[\\ 0 & \text{for} \quad t \in [7,8[\\ x_3 & \text{for} \quad t \in [8,9[\\ -y_3 & \text{for} \quad t \in [9,10] \end{cases},$$

and

$$\tau(t) = \begin{cases} 4 & \text{for} \quad t \in [0,1[\cup [3,4[\cup [8,9[\\ 7 & \text{for} \quad t \in [1,3[\cup [4,5[\cup [6,7[\cup [9,10] \\ 5 & \text{for} \quad t \in [5,6[\\ 2 & \text{for} \quad t \in [7,8[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 7$, and (6.104) is satisfied.

On the other hand, the function

$$u(t) = \begin{cases} \alpha x_1(1-t) + y_1 + y_2 + \alpha(k-1) & \text{for} \quad t \in [0,1[\\ y_1(2-t) + y_2 + \alpha(k-1) & \text{for} \quad t \in [1,2[\\ y_2(3-t) + \alpha(k-1) & \text{for} \quad t \in [2,3[\\ \alpha k(4-t) - \alpha & \text{for} \quad t \in [3,4[\\ \alpha(t-5) & \text{for} \quad t \in [4,5[\\ 0 & \text{for} \quad t \in [5,6[\\ t-6 & \text{for} \quad t \in [6,7[\\ 1 & \text{for} \quad t \in [7,8[\\ \alpha x_3(8-t) + 1 & \text{for} \quad t \in [8,9[\\ y_3(10-t) + 1 - y_3 - \alpha x_3 & \text{for} \quad t \in [9,10] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

This example shows that in Theorem 6.2 the strict inequality (6.15) cannot be replaced by the nonstrict one.

Example 6.8. Let $0 \neq |\mu| \leq |\lambda|$ and let $x_i, y_i \in R_+$ $(i = 1, 2, 3)$ be such that (6.113) holds and

$$x_1 + \left|\frac{\mu}{\lambda}\right| x_3 \geq 1 + \sqrt{\left|\frac{\mu}{\lambda}\right| - y_1 - y_2 - \left|\frac{\mu}{\lambda}\right| y_3}\,,$$

$$x_2 \geq 1 + \frac{\left|\frac{\mu}{\lambda}\right| - y_1 - y_2 - \left|\frac{\mu}{\lambda}\right| y_3}{x_1 + \left|\frac{\mu}{\lambda}\right| x_3 - 1}\,.$$

Put $\alpha = \left|\frac{\mu}{\lambda}\right| - y_1 - y_2 - \left|\frac{\mu}{\lambda}\right| y_3$ and $\beta = x_1 + \left|\frac{\mu}{\lambda}\right| x_3 - 1$. Obviously, $\alpha > 0$, $\beta > 0$, and $x_2 \geq 1 + \frac{\alpha}{\beta}$. Let, moreover, $a = 0$, $b = 9$,

$$p(t) = \begin{cases} x_1 & \text{for} \quad t \in [0, 1[\\ -y_1 & \text{for} \quad t \in [1, 2[\\ -y_2 & \text{for} \quad t \in [2, 3[\\ \frac{\alpha}{\beta} & \text{for} \quad t \in [3, 4[\\ x_2 - 1 - \frac{\alpha}{\beta} & \text{for} \quad t \in [4, 5[\\ 1 & \text{for} \quad t \in [5, 6[\\ 0 & \text{for} \quad t \in [6, 7[\\ -y_3 & \text{for} \quad t \in [7, 8[\\ x_3 & \text{for} \quad t \in [8, 9] \end{cases},$$

and

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [0, 1[\cup [8, 9] \\ 6 & \text{for} \quad t \in [1, 4[\cup [5, 6[\cup [7, 8[\\ 4 & \text{for} \quad t \in [4, 5[\\ 2 & \text{for} \quad t \in [6, 7[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 6$, and (6.104) is satisfied.

On the other hand, the function

$$u(t)=\begin{cases}\alpha x_1(1-t)+\beta(y_1+y_2)-\alpha & \text{for}\quad t\in[0,1[\\ \beta y_1(2-t)+\beta y_2-\alpha & \text{for}\quad t\in[1,2[\\ \beta y_2(3-t)-\alpha & \text{for}\quad t\in[2,3[\\ \alpha(t-4) & \text{for}\quad t\in[3,4[\\ 0 & \text{for}\quad t\in[4,5[\\ \beta(t-5) & \text{for}\quad t\in[5,6[\\ \beta & \text{for}\quad t\in[6,7[\\ \beta y_3(7-t)+\beta & \text{for}\quad t\in[7,8[\\ \alpha x_3(9-t)+\beta(1-y_3)-\alpha x_3 & \text{for}\quad t\in[8,9]\end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q\in L\big([a,b];R\big)$ and $c\in R$ such that the problem (6.1), (1.2) has no solution.

This example shows that in Theorem 6.2 the strict inequality (6.17) cannot be replaced by the nonstrict one.

Example 6.9. Let $0\neq|\mu|\leq|\lambda|$ (for the case $\mu=0$ see Example 6.10), $k\in\,]0,1[$, and $\varepsilon\geq 0$. Choose $m>0$ such that

$$m\leq\min\left\{\left|\frac{\mu}{\lambda}\right|,\ \frac{|\mu|(1-k)k}{|\mu|(1-k)+\varepsilon k}\right\}$$

and put $a=0$, $b=3$, and

$$p(t)=\begin{cases}-\frac{|\mu|-|\lambda|m}{|\lambda|m} & \text{for}\quad t\in[0,1[\\ \frac{k-m}{k} & \text{for}\quad t\in[1,2[\\ \frac{|\mu|(1-k)+\varepsilon k}{|\mu|k} & \text{for}\quad t\in[2,3]\end{cases},\qquad \tau(t)=\begin{cases}1 & \text{for}\quad t\in[0,1[\\ 2 & \text{for}\quad t\in[1,2[\\ t^* & \text{for}\quad t\in[2,3]\end{cases},$$

where

$$t^*=\begin{cases}1+\frac{1}{k-m}\left(\frac{|\mu|(1-k)k}{|\mu|(1-k)+\varepsilon k}-m\right) & \text{if}\quad m\neq k\\ 2 & \text{if}\quad m=k\end{cases}.$$

It is not difficult to verify that $\tau_0 = 1$, $\tau_1 = 2$, and

$$\int_a^{\tau_1} [p(s)]_+ ds = \int_{\tau_0}^{\tau_1} [p(s)]_+ ds = \frac{k-m}{k}\,, \qquad \int_{\tau_1}^{b} [p(s)]_+ ds = \frac{|\mu|(1-k)+\varepsilon k}{|\mu| k}\,,$$

$$\int_a^{\tau_0} [p(s)]_- ds = \frac{|\mu| - |\lambda| m}{|\lambda| m}\,, \qquad \int_{\tau_0}^{b} [p(s)]_- ds = \int_{\tau_0}^{\tau_1} [p(s)]_- ds = 0\,.$$

Thus, the conditon (6.18) holds, $T = \frac{k-m}{k}$, and instead of (6.19) we have

$$|\lambda| \int_a^{\tau_1} [p(s)]_+ ds + |\mu| \int_{\tau_1}^{b} [p(s)]_+ ds -$$

$$- \left(|\lambda| \int_a^{\tau_0} [p(s)]_- ds + |\mu| \int_{\tau_0}^{b} [p(s)]_- ds \right) \big(1 - T\big) = |\lambda| - |\mu| + \varepsilon\,.$$

On the other hand, the function

$$u(t) = \begin{cases} |\mu| - (|\mu| - |\lambda| m)t & \text{for} \quad t \in [0,1[\\ |\lambda|(k-m)(t-1) + |\lambda| m & \text{for} \quad t \in [1,2[\\ |\lambda|(1-k)(t-3) + |\lambda| & \text{for} \quad t \in [2,3] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

Example 6.10. Let $\mu = 0$, $k > 1$, and $\varepsilon \geq 0$. Choose $m > 0$ such that

$$m \leq \frac{k|\lambda|}{|\lambda| + \varepsilon}$$

and put $a = 0$, $b = 2$, and

$$p(t) = \begin{cases} \frac{|\lambda|+\varepsilon}{|\lambda|} & \text{for} \quad t \in [0,1[\\ -\frac{k-m}{k} & \text{for} \quad t \in [1,2] \end{cases}, \qquad \tau(t) = \begin{cases} t^* & \text{for} \quad t \in [0,1[\\ 1 & \text{for} \quad t \in [1,2] \end{cases},$$

where

$$t^* = \begin{cases} 2 - \frac{1}{k-m}\left(\frac{k|\lambda|}{|\lambda|+\varepsilon} - m\right) & \text{if} \quad m \neq k \\ 1 & \text{if} \quad m = k \end{cases}.$$

It is not difficult to verify that $\tau_0 = 1$, $\tau_1 = t^*$, and

$$\int_a^{\tau_1} [p(s)]_+ ds = \frac{|\lambda| + \varepsilon}{|\lambda|}\,, \qquad \int_{\tau_0}^{\tau_1} [p(s)]_+ ds = 0\,,$$

$$\int_a^{\tau_0} [p(s)]_- ds = 0\,, \qquad \int_{\tau_0}^{\tau_1} [p(s)]_- ds = \frac{(k-m)(t^*-1)}{k}.$$

Thus, the conditon (6.18) holds, $T = \frac{(k-m)(t^*-1)}{k}$, and instead of (6.19) we have

$$|\lambda| \int_a^{\tau_1} [p(s)]_+ ds - |\lambda|\big(1 - T\big) \int_a^{\tau_0} [p(s)]_- ds = |\lambda| + \varepsilon\,.$$

On the other hand, the function

$$u(t) = \begin{cases} |\lambda| k t & \text{for} \quad t \in [0, 1[\\ |\lambda|(k-m)(2-t) + |\lambda| m & \text{for} \quad t \in [1, 2] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a, b]; R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

Example 6.11. Let $|\mu| \le |\lambda|$, $k > 1$, and $\varepsilon \in [0, |\lambda|[\,$. Choose $M \ge \frac{|\mu| + |\lambda| k}{|\lambda| - \varepsilon}$ and put $a = 0$, $b = 4$,

$$p(t) = \begin{cases} \frac{|\lambda| M - |\mu| - \varepsilon M}{|\lambda| k} & \text{for} \quad t \in [0, 1[\\ -\frac{M-k}{M} & \text{for} \quad t \in [1, 2[\\ -\frac{k-1}{M} & \text{for} \quad t \in [2, 3[\\ 0 & \text{for} \quad t \in [3, 4] \end{cases}, \qquad \tau(t) = \begin{cases} t^* & \text{for} \quad t \in [0, 1[\\ 1 & \text{for} \quad t \in [1, 3[\\ 2 & \text{for} \quad t \in [3, 4] \end{cases},$$

where

$$t^* = \begin{cases} 2 - \frac{\varepsilon M k}{(M-k)(|\lambda| M - |\mu| - \varepsilon M)} & \text{if} \quad M \ne k \\ 2 & \text{if} \quad M = k \end{cases}.$$

It is not difficult to verify that $\tau_0 = 1$, $\tau_1 = 2$, and

$$\int_a^{\tau_0} [p(s)]_+ ds = \frac{|\lambda| M - |\mu| - \varepsilon M}{|\lambda| k}, \qquad \int_{\tau_0}^{b} [p(s)]_+ ds = \int_{\tau_0}^{\tau_1} [p(s)]_+ ds = 0,$$

$$\int_a^{\tau_1} [p(s)]_- ds = \int_{\tau_0}^{\tau_1} [p(s)]_- ds = \frac{M-k}{M}, \qquad \int_{\tau_1}^{b} [p(s)]_- ds = \frac{k-1}{M}.$$

Thus, the conditon (6.18) holds, $T = \frac{M-k}{M}$, and instead of (6.20) we have

$$\left(|\lambda| \int_a^{\tau_0} [p(s)]_+ ds + |\mu| \int_{\tau_0}^{b} [p(s)]_+ ds \right) \big(1 - T\big) -$$

$$- |\lambda| \int_a^{\tau_1} [p(s)]_- ds - |\mu| \int_{\tau_1}^{b} [p(s)]_- ds = \big(|\lambda| - |\mu|\big)\big(1 - T\big) - \varepsilon.$$

On the other hand, the function

$$u(t) = \begin{cases} |\mu| + (|\lambda| M - |\mu|)t & \text{for} \quad t \in [0, 1[\\ |\lambda|(M - k)(2 - t) + |\lambda| k & \text{for} \quad t \in [1, 2[\\ |\lambda|(k - 1)(3 - t) + |\lambda| & \text{for} \quad t \in [2, 3[\\ |\lambda| & \text{for} \quad t \in [3, 4] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a, b]; R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

§7. Antiperiodic Type BVP

In this section, we will establish nonimprovable, in a certain sense, sufficient conditions for unique solvability of the problem (1.1), (1.2), where the boundary condition (1.2) is of an antiperiodic type, i.e., when the inequality

$$\lambda\mu > 0 \tag{7.1}$$

holds. In Subsection 7.1, the main results are formulated. Theorem 7.1 deals with the case $|\mu| \leq |\lambda|$, while the case $|\mu| \geq |\lambda|$ is considered in Theorem 7.2. The proof of Theorem 7.1 can be found in Subsection 7.2. Subsection 7.3 is devoted to the examples verifying the optimality of the main results.

7.1. Existence and Uniqueness Theorems

In the case, where $|\mu| \leq |\lambda|$, the following assertion holds.

Theorem 7.1. *Let $|\mu| \leq |\lambda|$, the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and let either*

$$\|\ell_0(1)\|_L < 1 - \left(\frac{\mu}{\lambda}\right)^2, \tag{7.2}$$

$$\|\ell_1(1)\|_L < 1 - \frac{\mu}{\lambda} + 2\sqrt{1 - \|\ell_0(1)\|_L} \tag{7.3}$$

or

$$1 - \left(\frac{\mu}{\lambda}\right)^2 \leq \|\ell_0(1)\|_L, \tag{7.4}$$

$$\|\ell_0(1)\|_L + \frac{\mu}{\lambda}\|\ell_1(1)\|_L < 1 + \frac{\mu}{\lambda}\,. \tag{7.5}$$

Then the problem (1.1), (1.2) *has a unique solution.*

Remark 7.1. Let $|\mu| \leq |\lambda|$. Denote by G the set of pairs $(x, y) \in R_+ \times R_+$ satisfying either

$$x < 1 - \left(\frac{\mu}{\lambda}\right)^2, \qquad y < 1 - \frac{\mu}{\lambda} + 2\sqrt{1 - x}$$

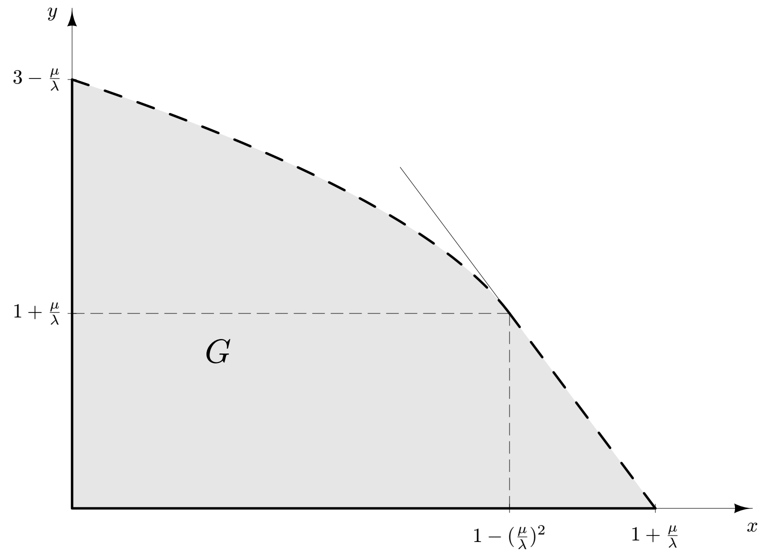

Fig. 7.1.

or

$$1-\left(\frac{\mu}{\lambda}\right)^2 \leq x, \qquad \frac{\mu}{\lambda}y < 1+\frac{\mu}{\lambda}-x$$

(see Fig. 7.1).

According to Theorem 7.1, if $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and

$$\Big(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L\Big) \in G,$$

then the problem (1.1), (1.2) has a unique solution. Below we will show (see On Remark 7.1, p. 163) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin G$ there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $q \in L\big([a,b]; R\big)$, and $c \in R$ such that (2.30) holds, and the problem (1.1), (1.2) with $\ell = \ell_0 - \ell_1$ has no solution. In particular, neither one of the strict inequalities (7.3) and (7.5) can be replaced by the nonstrict one.

In the case, where $|\mu| \geq |\lambda|$, the following statement holds.

Theorem 7.2. *Let $|\mu| \geq |\lambda|$, the operator ℓ admit the representation $\ell =$*

$\ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and let either

$$\|\ell_1(1)\|_L < 1 - \left(\frac{\lambda}{\mu}\right)^2, \qquad \|\ell_0(1)\|_L < 1 - \frac{\lambda}{\mu} + 2\sqrt{1 - \|\ell_1(1)\|_L}$$

or

$$1 - \left(\frac{\lambda}{\mu}\right)^2 \leq \|\ell_1(1)\|_L, \qquad \|\ell_1(1)\|_L + \frac{\lambda}{\mu}\|\ell_0(1)\|_L < 1 + \frac{\lambda}{\mu}\,.$$

Then the problem (1.1), (1.2) *has a unique solution.*

Remark 7.2. According to Remark 2.16 (see p. 28), Theorem 7.2 immediately follows from Theorem 7.1. Moreover, by virtue of Remark 7.1, Theorem 7.2 is nonimprovable in an appropriate sense.

7.2. Proofs

According to Theorem 1.1 (see p. 14), it is sufficient to show that the homogeneous problem (1.1_0), (1.2_0) has no nontrivial solution.

Proof of Theorem 7.1. Assume that the problem (1.1_0), (1.2_0) has a nontrivial solution u. It follows from (1.2_0) and (7.1) that u has a zero. Define numbers M and m by (2.94) and choose $t_M, t_m \in [a, b]$ such that (2.95) is fulfilled. Obviously,

$$M \geq 0, \qquad m \geq 0, \qquad M + m > 0. \tag{7.6}$$

Without loss of generality we can assume that $t_M < t_m$.

The integration of (1.1_0) from a to t_M and from t_m to b, in view of (2.94), (2.95), and the assumptions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, results in

$$M - u(a) = \int_a^{t_M} [\ell_0(u)(s) - \ell_1(u)(s)]ds \leq M \int_a^{t_M} \ell_0(1)(s)ds + m \int_a^{t_M} \ell_1(1)(s)ds,$$

$$u(b) + m = \int_{t_m}^{b} [\ell_0(u)(s) - \ell_1(u)(s)]ds \leq M \int_{t_m}^{b} \ell_0(1)(s)ds + m \int_{t_m}^{b} \ell_1(1)(s)ds.$$

Summing the last two inequalities and taking into account (1.2_0), (7.1), and (2.94), we obtain

$$M + m - m\left(1 + \frac{\mu}{\lambda}\right) \leq M + m + u(b)\left(1 + \frac{\mu}{\lambda}\right) \leq$$
$$\leq M\int_J \ell_0(1)(s)ds + m\int_J \ell_1(1)(s)ds$$

and

$$M + m - M\left(1 + \frac{\lambda}{\mu}\right) \leq M + m - u(a)\left(1 + \frac{\lambda}{\mu}\right) \leq$$
$$\leq M\int_J \ell_0(1)(s)ds + m\int_J \ell_1(1)(s)ds,$$

where $J = [a, t_M] \cup [t_m, b]$. Thus,

$$M - \frac{\mu}{\lambda}\, m \leq MC + mA \tag{7.7}$$

and

$$m - \frac{\lambda}{\mu}\, M \leq MC + mA, \tag{7.8}$$

where

$$A = \int_J \ell_1(1)(s)ds, \qquad C = \int_J \ell_0(1)(s)ds. \tag{7.9}$$

On the other hand, the integration of (1.1_0) from t_M to t_m, on account of (2.94), (2.95), and the assumptions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, implies

$$M + m = \int_{t_M}^{t_m} [\ell_1(u)(s) - \ell_0(u)(s)]ds \leq M\int_{t_M}^{t_m} \ell_1(1)(s)ds + m\int_{t_M}^{t_m} \ell_0(1)(s)ds.$$

Hence,

$$M + m \leq MB + mD, \tag{7.10}$$

where

$$B = \int_{t_M}^{t_m} \ell_1(1)(s)ds, \qquad D = \int_{t_M}^{t_m} \ell_0(1)(s)ds. \tag{7.11}$$

First suppose that $\|\ell_0(1)\|_L \geq 1$ holds, i.e., the conditions (7.4) and (7.5) are fulfilled. According to (7.5), $\|\ell_1(1)\|_L < 1$ and thus, $A < 1$ and $B < 1$. Therefore, it follows from (7.6), (7.8), and (7.10) that

$$0 \leq m(1-A) \leq M\left(C + \frac{\lambda}{\mu}\right), \qquad 0 \leq M(1-B) \leq m(D-1).$$

Consequently, $M > 0$, $m > 0$, $D > 1$, and

$$0 < (1-A)(1-B) \leq \left(C + \frac{\lambda}{\mu}\right)(D-1). \tag{7.12}$$

Obviously,

$$(1-A)(1-B) \geq 1 - (A+B) = 1 - \|\ell_1(1)\|_L. \tag{7.13}$$

According to (7.5) and the condition $\frac{\mu}{\lambda} \in \,]0,1]$, we have $\|\ell_0(1)\|_L < 1 + \frac{\lambda}{\mu}$. Hence, $D - 1 < \frac{\lambda}{\mu}$ and thus

$$\left(C + \frac{\lambda}{\mu}\right)(D-1) = \frac{\lambda}{\mu}D - \frac{\lambda}{\mu} + C(D-1) \leq \frac{\lambda}{\mu}(C+D) - \frac{\lambda}{\mu} = \frac{\lambda}{\mu}\|\ell_0(1)\|_L - \frac{\lambda}{\mu}\,.$$

By the last inequality and (7.13), it follows from (7.12) that

$$1 - \|\ell_1(1)\|_L \leq \frac{\lambda}{\mu}\|\ell_0(1)\|_L - \frac{\lambda}{\mu},$$

which contradicts the inequality (7.5).

Now suppose that $\|\ell_0(1)\|_L < 1$. Obviously, $C < 1$, $D < 1$, and by (7.6), (7.7), and (7.10) we get

$$0 \leq M(1-C) \leq m\left(A + \frac{\mu}{\lambda}\right), \qquad 0 \leq m(1-D) \leq M(B-1).$$

Consequently, $M > 0$, $m > 0$, $B > 1$, and

$$0 < (1-C)(1-D) \leq \left(A + \frac{\mu}{\lambda}\right)(B-1). \tag{7.14}$$

It is clear that

$$(1-C)(1-D) \geq 1 - (C+D) = 1 - \|\ell_0(1)\|_L. \tag{7.15}$$

First assume that (7.4) and (7.5) hold. Then we have $\|\ell_1(1)\|_L < 1 + \frac{\mu}{\lambda}$. Hence, $B - 1 < \frac{\mu}{\lambda}$ and

$$\left(A + \frac{\mu}{\lambda}\right)(B-1) = \frac{\mu}{\lambda}B - \frac{\mu}{\lambda} + A(B-1) \leq \frac{\mu}{\lambda}(A+B) - \frac{\mu}{\lambda} = \frac{\mu}{\lambda}\|\ell_1(1)\|_L - \frac{\mu}{\lambda}\,.$$

By the last inequality and (7.15), it follows from (7.14) that

$$1 - \|\ell_0(1)\|_L \leq \frac{\mu}{\lambda}\|\ell_1(1)\|_L - \frac{\mu}{\lambda},$$

which contradicts the inequality (7.5).

Now assume that (7.2) and (7.3) are satisfied. According to (7.15) and the fact that

$$4\left(A + \frac{\mu}{\lambda}\right)(B-1) \leq \left(A + B - 1 + \frac{\mu}{\lambda}\right)^2 = \left(\|\ell_1(1)\|_L - 1 + \frac{\mu}{\lambda}\right)^2,$$

the inequality (7.14) implies

$$0 < 4(1 - \|\ell_0(1)\|_L) \leq \left(\|\ell_1(1)\|_L - \left(1 - \frac{\mu}{\lambda}\right)\right)^2. \tag{7.16}$$

On the other hand, since $B > 1$, we have

$$\|\ell_1(1)\|_L > 1,$$

which, together with (7.16), contradicts the inequality (7.3). □

7.3. Comments and Examples

On Remark 7.1. Let $|\mu| \leq |\lambda|$. Below, for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin G$ the functions $p \in L\big([a,b]; R\big)$ and $\tau \in \mathcal{M}_{ab}$ are constructed such that (2.130) holds, and the problem (4.58) has a nontrivial solution. Then, by Remark 1.1 (see p. 14), there exist $q \in L\big([a,b]; R\big)$ and $c \in R$ such that the problem (1.1), (1.2), where $\ell = \ell_0 - \ell_1$, ℓ_0, ℓ_1 are defined by (2.132), has no solution.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin G$, then (x_0, y_0) belongs at least to one of the following sets:

$$G_1 = \left\{(x,y) \in R_+ \times R_+ \ : \ y < 1, \ \frac{\mu}{\lambda}(1-y) + 1 \leq x\right\},$$

$$G_2 = \{(x,y) \in R_+ \times R_+ \ : \ 1 \leq x, \ 1 \leq y\},$$

$$G_3 = \left\{(x,y) \in R_+ \times R_+ \ : \ 1 - \left(\frac{\mu}{\lambda}\right)^2 \leq x < 1, \ 1 - x + \frac{\mu}{\lambda} \leq \frac{\mu}{\lambda}y\right\},$$

$$G_4 = \left\{(x,y) \in R_+ \times R_+ \ : \ x \leq 1 - \left(\frac{\mu}{\lambda}\right)^2, \ 1 - \frac{\mu}{\lambda} + 2\sqrt{1-x} \leq y\right\}.$$

Let $(x_0, y_0) \in G_1$. Put $a = 0$, $b = 2$, $\alpha = \frac{\mu(1-y_0)+\lambda}{1-y_0}$, $\beta = \frac{\lambda y_0}{1-y_0}$, $t_0 = \frac{\mu}{\alpha} + \frac{1}{x_0}$,

$$p(t) = \begin{cases} x_0 & \text{for } t \in [0,1[\\ -y_0 & \text{for } t \in [1,2] \end{cases}, \quad \tau(t) = \begin{cases} t_0 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,2] \end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t) = \begin{cases} -\alpha t + \mu & \text{for } t \in [0,1[\\ \beta(t-2) - \lambda & \text{for } t \in [1,2] \end{cases}.$$

Let $(x_0, y_0) \in G_2$. Put $a = 0$, $b = 4$,

$$p(t) = \begin{cases} x_0 - 1 & \text{for } t \in [0,1[\\ 1 - y_0 & \text{for } t \in [1,2[\\ 1 & \text{for } t \in [2,3[\\ -1 & \text{for } t \in [3,4] \end{cases}, \quad \tau(t) = \begin{cases} 0 & \text{for } t \in [0,2[\\ 3 & \text{for } t \in [2,4] \end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t) = \begin{cases} 0 & \text{for } t \in [0,2[\\ t-2 & \text{for } t \in [2,3[\\ 4-t & \text{for } t \in [3,4] \end{cases}.$$

Let $(x_0, y_0) \in G_3$. Put $a = 0$, $b = 2$, $\alpha = \frac{\mu x_0}{1-x_0}$, $\beta = \frac{\lambda(1-x_0)+\mu}{1-x_0}$, $t_0 = 2 - \frac{1}{y_0} - \frac{\lambda}{\beta}$,

$$p(t) = \begin{cases} x_0 & \text{for } t \in [0,1[\\ -y_0 & \text{for } t \in [1,2] \end{cases}, \quad \tau(t) = \begin{cases} 1 & \text{for } t \in [0,1[\\ t_0 & \text{for } t \in [1,2] \end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t) = \begin{cases} \alpha t + \mu & \text{for } t \in [0,1[\\ \beta(2-t) - \lambda & \text{for } t \in [1,2] \end{cases}.$$

Let $(x_0, y_0) \in G_4$. Put $a = 0$, $b = 5$, $\alpha = \sqrt{1 - x_0}$, $\beta = 1 - y_0 + 2\alpha - \frac{\mu}{\lambda}$, $t_0 = 3 - \alpha$,

$$p(t) = \begin{cases} \frac{\mu}{\lambda} - \alpha & \text{for } t \in [0,1[\\ -\alpha & \text{for } t \in [1,2[\\ -1 & \text{for } t \in [2,3[\\ \beta & \text{for } t \in [3,4[\\ x_0 & \text{for } t \in [4,5] \end{cases}, \quad \tau(t) = \begin{cases} 5 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,3[\\ t_0 & \text{for } t \in [3,4[\\ 5 & \text{for } t \in [4,5] \end{cases}.$$

It is not difficult to verify that (2.130) holds, and the problem (4.58) has the nontrivial solution

$$u(t) = \begin{cases} (\alpha - \frac{\mu}{\lambda})t + \frac{\mu}{\lambda} & \text{for } t \in [0,1[\\ \alpha^2(1-t) + \alpha & \text{for } t \in [1,2[\\ \alpha(3-t) - \alpha^2 & \text{for } t \in [2,3[\\ -\alpha^2 & \text{for } t \in [3,4[\\ x_0(5-t) - 1 & \text{for } t \in [4,5] \end{cases}.$$

§8. Antiperiodic Type BVP for EDA

In this section, we will establish consequences of Theorems 7.1 and 7.2 from §7 for the equation with deviating arguments $(1.1')$.

In what follows we will use the notation

$$p_0(t) = \sum_{j=1}^{m} p_j(t), \qquad g_0(t) = \sum_{j=1}^{m} g_j(t) \quad \text{for} \quad t \in [a,b].$$

and we will suppose that the inequality (7.1) is fulfilled.

From Theorems 7.1 and 7.2 immediately follows the following statements. The first of them deals with the case $|\mu| \le |\lambda|$ and the second one with the case $|\mu| \ge |\lambda|$.

Theorem 8.1. *Let $|\mu| \le |\lambda|$, $p_k, g_k \in L\big([a,b]; R_+\big)$ $(k = 1, \dots, m)$, and let either*

$$\int_a^b p_0(s)ds < 1 - \Big(\frac{\mu}{\lambda}\Big)^2,$$

$$\int_a^b g_0(s)ds < 1 - \frac{\mu}{\lambda} + 2\sqrt{1 - \int_a^b p_0(s)ds} \tag{8.1}$$

or

$$1 - \Big(\frac{\mu}{\lambda}\Big)^2 \le \int_a^b p_0(s)ds,$$

$$\int_a^b p_0(s)ds + \frac{\mu}{\lambda} \int_a^b g_0(s)ds < 1 + \frac{\mu}{\lambda}\,. \tag{8.2}$$

Then the problem $(1.1')$, (1.2) *has a unique solution.*

Remark 8.1. The examples constructed in Subsection 7.3 (see On Remark 7.1, p. 163) also show that the strict inequalities (8.1) and (8.2) cannot be replaced by the nonstrict ones.

Theorem 8.2. *Let* $|\mu| \ge |\lambda|$, $p_k, g_k \in L\big([a,b]; R_+\big)$ $(k = 1, \dots, m)$, *and let either*

$$\int_a^b g_0(s)ds < 1 - \left(\frac{\lambda}{\mu}\right)^2,$$

$$\int_a^b p_0(s)ds < 1 - \frac{\lambda}{\mu} + 2\sqrt{1 - \int_a^b g_0(s)ds}$$

or

$$1 - \left(\frac{\lambda}{\mu}\right)^2 \le \int_a^b g_0(s)ds,$$

$$\int_a^b g_0(s)ds + \frac{\lambda}{\mu}\int_a^b p_0(s)ds < 1 + \frac{\lambda}{\mu}.$$

Then the problem (1.1′), (1.2) *has a unique solution.*

Remark 8.2. Similarly as in the case $|\mu| \le |\lambda|$ one can show that Theorem 8.2 is also nonimprovable in a certain sense.

§9. Antiperiodic Type BVP for Two Terms EDA

This section deals with the special case of the equation (1.1′) with $m = 1$ and $\tau_1 \equiv \nu_1$. In that case the equation (1.1′) can be rewritten in the form (6.1). Throughout the section we will also suppose that the inequality (7.1) is satisfied.

In §8, there were established effective sufficient conditions for unique solvability of the problem (6.1), (1.2). Although those results are, in general, nonimprovable, in the special case, where τ maps the segment $[a, b]$ into some subsegment $[\tau_0, \tau_1] \subseteq [a, b]$, those results can be improved in a certain way.

Therefore, in the sequel we will assume that there exist $\tau_0, \tau_1 \in [a, b]$, $\tau_0 \le \tau_1$ such that $\tau(t) \in [\tau_0, \tau_1]$ for almost all $t \in [a, b]$. Thus, it will be supposed that

$$\tau_0 = \text{ess inf}\{\tau(t) : t \in [a, b]\}, \qquad \tau_1 = \text{ess sup}\{\tau(t) : t \in [a, b]\}.$$

Note also that if $\tau_0 = a$ and $\tau_1 = b$, then obtained results coincide with the appropriate ones from §8.

In Subsection 9.1, the main results are formulated, Subsection 9.2 is devoted to their proofs, and the examples verifying the optimality of the main results can be found in Subsection 9.3.

9.1. Existence and Uniqueness Theorems

Theorem 9.1. *Let the condition* (6.18) *be fulfilled and let either*

$$\begin{aligned} &|\lambda| \int_a^{\tau_1} [p(s)]_+ ds + |\mu| \int_{\tau_0}^{b} [p(s)]_- ds - \\ &- \left(|\lambda| \int_a^{\tau_0} [p(s)]_- ds + |\mu| \int_{\tau_1}^{b} [p(s)]_+ ds \right) \big(1 - T\big) < |\lambda| + |\mu| \end{aligned} \tag{9.1}$$

or

$$\left(|\lambda| \int_a^{\tau_0} [p(s)]_+ ds + |\mu| \int_{\tau_1}^b [p(s)]_- ds\right) (1 - T) - \\ -|\lambda| \int_a^{\tau_1} [p(s)]_- ds - |\mu| \int_{\tau_0}^b [p(s)]_+ ds > (|\lambda| + |\mu|)(1 - T), \tag{9.2}$$

where T is defined by (6.21)*. Then the problem* (6.1)*,* (1.2) *has a unique solution.*

Remark 9.1. Theorem 9.1 is nonimprovable in the sense that the strict inequalities (9.1) and (9.2) cannot be replaced by the nonstrict ones (see Examples 9.1 and 9.2, p. 181).

Note also that if the segment $[\tau_0, \tau_1]$ is degenerated to a point $c \in [a, b]$, i.e., $\tau(t) = c$ for $t \in [a, b]$, then $T = 0$ and the inequalities (9.1) and (9.2) can be rewritten as

$$\lambda \int_a^c p(s)ds - \mu \int_c^b p(s)ds \neq \lambda + \mu,$$

which is sufficient and necessary for the unique solvability of the problem (6.1), (1.2) with $\tau(t) = c$ for $t \in [a, b]$.

The following theorems can be understood as a supplement of the previous one for the case $T \geq 1$, where T is given by (6.21). The first of them deals with the case $|\mu| \leq |\lambda|$ and the second one with the case $|\mu| \geq |\lambda|$.

Theorem 9.2. *Let $|\mu| \leq |\lambda|$,*

$$H = \int_a^{\tau_1} [p(s)]_+ ds + \frac{\mu}{\lambda} \int_{\tau_1}^b [p(s)]_- ds, \tag{9.3}$$

and let one of the following items be fulfilled:

a)

$$\int_{\tau_0}^{\tau_1} [p(s)]_+ ds \geq 1, \tag{9.4}$$

$$\left(|\lambda|\int_a^{\tau_0}[p(s)]_-ds+|\mu|\int_{\tau_1}^{b}[p(s)]_+ds\right)\left(\int_{\tau_0}^{\tau_1}[p(s)]_+ds-1\right)+ \\ +|\lambda|\int_a^{\tau_1}[p(s)]_+ds+|\mu|\int_{\tau_0}^{b}[p(s)]_-ds<|\lambda|+|\mu|\,; \tag{9.5}$$

b)

$$\int_{\tau_0}^{\tau_1}[p(s)]_-ds\ge 1\,, \tag{9.6}$$

$$H\ge 1-\left(\frac{\mu}{\lambda}\right)^2\,, \tag{9.7}$$

$$\left(|\lambda|\int_a^{\tau_0}[p(s)]_-ds+|\mu|\int_{\tau_1}^{b}[p(s)]_+ds\right)\left(\int_{\tau_0}^{\tau_1}[p(s)]_-ds-1\right)+ \\ +|\lambda|\int_a^{\tau_1}[p(s)]_+ds+|\mu|\int_{\tau_0}^{b}[p(s)]_-ds<|\lambda|+|\mu|\,; \tag{9.8}$$

c) the condition (9.6) *holds,*

$$H<1-\left(\frac{\mu}{\lambda}\right)^2\,, \tag{9.9}$$

and either

$$\int_a^{\tau_0}[p(s)]_-ds+\frac{\mu}{\lambda}\int_{\tau_1}^{b}[p(s)]_+ds<-\frac{\mu}{\lambda}+\sqrt{1-H}\,, \tag{9.10}$$

$$\int_a^{\tau_1}[p(s)]_-ds+\frac{\mu}{\lambda}\int_{\tau_1}^{b}[p(s)]_+ds<1-\frac{\mu}{\lambda}+2\sqrt{1-H} \tag{9.11}$$

or

$$\int_a^{\tau_0}[p(s)]_-ds+\frac{\mu}{\lambda}\int_{\tau_1}^{b}[p(s)]_+ds\ge-\frac{\mu}{\lambda}+\sqrt{1-H} \tag{9.12}$$

and the condition (9.8) *holds.*

Then the problem (6.1), (1.2) *has a unique solution.*

Remark 9.2. Theorem 9.2 is nonimprovable in the sense that neither one of the strict inequalities (9.5), (9.8), and (9.11) can be replaced by the nonstrict one (see Examples 9.3–9.6, pp. 183–186).

Note also that if $\tau_0 = a$ and $\tau_1 = b$, then from Theorems 9.1 and 9.2 we obtain Theorem 8.1 (see p. 166).

Theorem 9.3. *Let* $|\mu| \geq |\lambda|$,

$$\widetilde{H} = \frac{\lambda}{\mu} \int_a^{\tau_0} [p(s)]_+ ds + \int_{\tau_0}^{b} [p(s)]_- ds \,,$$

and let one of the following items be fulfilled:

a)

$$\int_{\tau_0}^{\tau_1} [p(s)]_- ds \geq 1$$

and the condition (9.8) *holds;*

b)

$$\int_{\tau_0}^{\tau_1} [p(s)]_+ ds \geq 1 \,, \qquad \widetilde{H} \geq 1 - \left(\frac{\lambda}{\mu}\right)^2 \,,$$

and the condition (9.5) *holds;*

c)

$$\int_{\tau_0}^{\tau_1} [p(s)]_+ ds \geq 1 \,, \qquad \widetilde{H} < 1 - \left(\frac{\lambda}{\mu}\right)^2 \,,$$

and either

$$\frac{\lambda}{\mu} \int_a^{\tau_0} [p(s)]_- ds + \int_{\tau_1}^{b} [p(s)]_+ ds < -\frac{\lambda}{\mu} + \sqrt{1 - \widetilde{H}} \,,$$

$$\frac{\lambda}{\mu} \int_a^{\tau_0} [p(s)]_- ds + \int_{\tau_0}^{b} [p(s)]_+ ds < 1 - \frac{\lambda}{\mu} + 2\sqrt{1 - \widetilde{H}}$$

or

$$\frac{\lambda}{\mu}\int\limits_a^{\tau_0}[p(s)]_-ds+\int\limits_{\tau_1}^{b}[p(s)]_+ds\geq-\frac{\lambda}{\mu}+\sqrt{1-\widetilde{H}}$$

and the condition (9.5) *holds.*

Then the problem (6.1)*,* (1.2) *has a unique solution.*

Remark 9.3. According to Remark 6.5 (see p. 126), Theorem 9.3 can be immediately derived from Theorem 9.2. Moreover, by virtue of Remark 9.2, Theorem 9.3 is nonimprovable in an appropriate sense.

9.2. Proofs

According to Theorem 1.1 (see p. 14), to prove Theorems 9.1 and 9.2 it is sufficient to show that the homogeneous problem (6.1_0), (1.2_0) (see p. 126) has only the trivial solution.

In the sequel, numbers A_i, B_i $(i = 1, 2, 3)$ are defined by (6.29).

Proof of Theorem 9.1. Assume that the problem (6.1_0), (1.2_0) possesses a nontrivial solution u.

First suppose that u has a zero in $[\tau_0, \tau_1]$. Define numbers m_0 and M_0 by (6.50) and choose $\alpha_0, \alpha_1 \in [\tau_0, \tau_1]$ such that (6.51) holds. Obviously, (6.78) is satisfied, since if $m_0 = 0$ and $M_0 = 0$, then, in view of (6.1_0) and (6.50), we obtain $u(\tau_0) = 0$ and $u'(t) = 0$ for $t \in [a, b]$, i.e., $u \equiv 0$. It is also evident that without loss of generality we can assume that $\alpha_0 < \alpha_1$.

The integration of (6.1_0) from α_0 to α_1, by virtue of (6.50), (6.51), and (6.78), yields the inequality (6.79), which, on account of (6.18) and (6.78), results in $M_0 + m_0 < M_0 + m_0$, a contradiction.

Now suppose that u has no zero in $[\tau_0, \tau_1]$. Without loss of generality we can assume that $u(t) > 0$ for $t \in [\tau_0, \tau_1]$. Define numbers M and m by (6.31) and choose $t_M, t_m \in [\tau_0, \tau_1]$ such that (6.32) holds. It is obvious that (6.81) is fulfilled and either (6.36) or (6.37) is satisfied. Analogously as in the proof of Theorem 6.3 one can show that, in both cases (6.36) and (6.37), the inequality (6.82) holds, where T is defined by (6.21).

On the other hand, the integrations of (6.1_0) from a to t_M, from t_M to

b, from a to t_m, and from t_m to b, in view of (6.31) and (6.32), yield

$$\begin{aligned} M-u(a) &= \int_a^{t_M}[p(s)]_+u(\tau(s))ds - \int_a^{t_M}[p(s)]_-u(\tau(s))ds \le \\ &\le M\int_a^{t_M}[p(s)]_+ds - m\int_a^{t_M}[p(s)]_-ds, \end{aligned} \tag{9.13}$$

$$\begin{aligned} M-u(b) &= \int_{t_M}^{b}[p(s)]_-u(\tau(s))ds - \int_{t_M}^{b}[p(s)]_+u(\tau(s))ds \le \\ &\le M\int_{t_M}^{b}[p(s)]_-ds - m\int_{t_M}^{b}[p(s)]_+ds, \end{aligned} \tag{9.14}$$

$$\begin{aligned} m-u(a) &= \int_a^{t_m}[p(s)]_+u(\tau(s))ds - \int_a^{t_m}[p(s)]_-u(\tau(s))ds \ge \\ &\ge m\int_a^{t_m}[p(s)]_+ds - M\int_a^{t_m}[p(s)]_-ds, \end{aligned} \tag{9.15}$$

$$\begin{aligned} m-u(b) &= \int_{t_m}^{b}[p(s)]_-u(\tau(s))ds - \int_{t_m}^{b}[p(s)]_+u(\tau(s))ds \ge \\ &\ge m\int_{t_m}^{b}[p(s)]_-ds - M\int_{t_m}^{b}[p(s)]_+ds. \end{aligned} \tag{9.16}$$

Put

$$\begin{aligned} f_1(t) &\overset{\text{def}}{=} M\int_a^{t}[p(s)]_+ds - \frac{\mu}{\lambda}m\int_t^{b}[p(s)]_+ds \quad \text{for} \quad t\in[a,b], \\ f_2(t) &\overset{\text{def}}{=} m\int_a^{t}[p(s)]_-ds - \frac{\mu}{\lambda}M\int_t^{b}[p(s)]_-ds \quad \text{for} \quad t\in[a,b], \end{aligned} \tag{9.17}$$

$$
\begin{aligned}
f_3(t) &\stackrel{\text{def}}{=} m\int_a^t [p(s)]_+ ds - \frac{\mu}{\lambda} M \int_t^b [p(s)]_+ ds \quad \text{for} \quad t\in[a,b],\\
f_4(t) &\stackrel{\text{def}}{=} M\int_a^t [p(s)]_- ds - \frac{\mu}{\lambda} m \int_t^b [p(s)]_- ds \quad \text{for} \quad t\in[a,b].
\end{aligned} \tag{9.18}
$$

First suppose that (9.1) holds, where T is defined by (6.21). Multiplying both sides of (9.14) by $\frac{\mu}{\lambda}$, summing with (9.13), and taking into account (1.2_0), (7.1), and (9.17), we get

$$
\begin{aligned}
M\left(1+\frac{\mu}{\lambda}\right) &\le M\int_a^{t_M}[p(s)]_+ds - \frac{\mu}{\lambda} m\int_{t_M}^b [p(s)]_+ds-\\
-\left(m\int_a^{t_M}[p(s)]_-ds - \frac{\mu}{\lambda} M\int_{t_M}^b [p(s)]_-ds\right) &= f_1(t_M)-f_2(t_M).
\end{aligned} \tag{9.19}
$$

It is easy to verify that the functions f_1 and f_2 defined by (9.17) are nondecreasing in $[a,b]$ and therefore, with respect to (6.29) and (9.17), it follows from (9.19) that

$$
\begin{aligned}
M\left(1+\frac{\mu}{\lambda}\right) &\le f_1(t_M)-f_2(t_M) \le f_1(\tau_1)-f_2(\tau_0) =\\
&= M\left(A_1+A_2+\frac{\mu}{\lambda}B_2+\frac{\mu}{\lambda}B_3\right) - m\left(B_1+\frac{\mu}{\lambda}A_3\right).
\end{aligned} \tag{9.20}
$$

Thus, (6.82) and (9.20) imply

$$
M\left(1+\frac{\mu}{\lambda}\right) \le M\left(A_1+A_2+\frac{\mu}{\lambda}B_2+\frac{\mu}{\lambda}B_3\right) - M\left(B_1+\frac{\mu}{\lambda}A_3\right)(1-T),
$$

which, in view of (6.29) and (6.81), contradicts (9.1).

Now suppose that (9.2) is satisfied, where T is defined by (6.21). Multiplying both sides of (9.16) by $\frac{\mu}{\lambda}$, summing with (9.15), and taking into account (1.2_0), (7.1), and (9.18), we obtain

$$
\begin{aligned}
m\left(1+\frac{\mu}{\lambda}\right) &\ge m\int_a^{t_m}[p(s)]_+ds - \frac{\mu}{\lambda} M\int_{t_m}^b [p(s)]_+ds-\\
-\left(M\int_a^{t_m}[p(s)]_-ds - \frac{\mu}{\lambda} m\int_{t_m}^b [p(s)]_-ds\right) &= f_3(t_m)-f_4(t_m).
\end{aligned} \tag{9.21}
$$

It is easy to verify that the functions f_3 and f_4 defined by (9.18) are nondecreasing in $[a, b]$ and thus, with respect to (6.29) and (9.18), it follows from (9.21) that

$$\begin{aligned} m\left(1+\frac{\mu}{\lambda}\right) &\geq f_3(t_m) - f_4(t_m) \geq f_3(\tau_0) - f_4(\tau_1) = \\ &= m\left(A_1 + \frac{\mu}{\lambda}\, B_3\right) - M\left(B_1 + B_2 + \frac{\mu}{\lambda}\, A_2 + \frac{\mu}{\lambda}\, A_3\right). \end{aligned} \tag{9.22}$$

By virtue of (6.18) and (6.21), (6.82) and (9.22) yield

$$\begin{aligned} m\left(1+\frac{\mu}{\lambda}\right)(1-T) \geq m\left(A_1 + \frac{\mu}{\lambda}\, B_3\right)(1-T) - \\ -m\left(B_1 + B_2 + \frac{\mu}{\lambda}\, A_2 + \frac{\mu}{\lambda}\, A_3\right), \end{aligned}$$

which, in view of (6.29) and (6.81), contradicts (9.2). □

Proof of Theorem 9.2. Assume that the problem (6.1_0), (1.2_0) possesses a nontrivial solution u.

First suppose that u changes its sign in $[\tau_0, \tau_1]$. Define numbers m_0 and M_0 by (6.50) and choose $\alpha_0, \alpha_1 \in [\tau_0, \tau_1]$ such that (6.51) holds. It is clear that (6.52) is satisfied and without loss of generality we can assume that $\alpha_0 < \alpha_1$. Furthermore, define numbers A_{2i}, B_{2i} $(i = 1, 2, 3)$ by (6.34).

The integrations of (6.1_0) from a to α_0, from α_0 to α_1, from α_1 to b, and from τ_1 to b, in view of (6.29), (6.34), (6.50), and (6.51), result in

$$\begin{aligned} -m_0 - u(a) = \int_a^{\alpha_0} [p(s)]_+ u(\tau(s))ds - \int_a^{\alpha_0} [p(s)]_- u(\tau(s))ds \leq \\ \leq M_0\big(A_1 + A_{21}\big) + m_0\big(B_1 + B_{21}\big), \end{aligned} \tag{9.23}$$

$$\begin{aligned} u(a) + m_0 = \int_a^{\alpha_0} [p(s)]_- u(\tau(s))ds - \int_a^{\alpha_0} [p(s)]_+ u(\tau(s))ds \leq \\ \leq M_0\big(B_1 + B_{21}\big) + m_0\big(A_1 + A_{21}\big), \end{aligned} \tag{9.24}$$

$$\begin{aligned} M_0 + m_0 = \int_{\alpha_0}^{\alpha_1} [p(s)]_+ u(\tau(s))ds - \int_{\alpha_0}^{\alpha_1} [p(s)]_- u(\tau(s))ds \leq \\ \leq M_0 A_{22} + m_0 B_{22}, \end{aligned} \tag{9.25}$$

$$M_0 - u(b) = \int_{\alpha_1}^{b} [p(s)]_- u(\tau(s))ds - \int_{\alpha_1}^{b} [p(s)]_+ u(\tau(s))ds \leq \\ \leq M_0\big(B_{23} + B_3\big) + m_0\big(A_{23} + A_3\big), \tag{9.26}$$

$$u(b) - M_0 \leq u(b) - u(\tau_1) = \int_{\tau_1}^{b} [p(s)]_+ u(\tau(s))ds - \\ - \int_{\tau_1}^{b} [p(s)]_- u(\tau(s))ds \leq M_0 A_3 + m_0 B_3. \tag{9.27}$$

Multiplying both sides of (9.26) by $\frac{\mu}{\lambda}$, summing with (9.23), and taking into account (1.2_0) and (7.1), we get

$$\frac{\mu}{\lambda}\, M_0 - m_0 \leq M_0 \left(A_1 + A_{21} + \frac{\mu}{\lambda}\, B_{23} + \frac{\mu}{\lambda}\, B_3\right) + \\ + m_0 \left(B_1 + B_{21} + \frac{\mu}{\lambda}\, A_{23} + \frac{\mu}{\lambda}\, A_3\right). \tag{9.28}$$

Analogously, (9.24) and (9.27) imply

$$m_0 - \frac{\mu}{\lambda}\, M_0 \leq M_0 \left(B_1 + B_{21} + \frac{\mu}{\lambda}\, A_3\right) + m_0 \left(A_1 + A_{21} + \frac{\mu}{\lambda}\, B_3\right). \tag{9.29}$$

First suppose that the assumption a) holds. According to (6.34), (9.4), and (9.5), we have $B_{22} < 1$. Consequently, in view of (6.52), (9.25) yields

$$0 < m_0\big(1 - B_{22}\big) \leq M_0\big(A_{22} - 1\big). \tag{9.30}$$

Moreover, by virtue of (7.1) and (9.4), it follows from (9.5) that

$$A_{22} < 1 + \frac{\mu}{\lambda}\,. \tag{9.31}$$

From (9.28) we get

$$M_0\left(\frac{\mu}{\lambda} - A_1 - A_{21} - \frac{\mu}{\lambda}\, B_{23} - \frac{\mu}{\lambda}\, B_3\right) \leq \\ \leq m_0\left(B_1 + B_{21} + \frac{\mu}{\lambda}\, A_{23} + \frac{\mu}{\lambda}\, A_3 + 1\right),$$

which, together with (9.30), implies

$$\begin{aligned}\Big(\frac{\mu}{\lambda}-A_1-A_{21}-\frac{\mu}{\lambda}\,B_{23}-\frac{\mu}{\lambda}\,B_3\Big)\big(1-B_{22}\big)\leq\\ \leq\Big(B_1+B_{21}+\frac{\mu}{\lambda}\,A_{23}+\frac{\mu}{\lambda}\,A_3+1\Big)\big(A_{22}-1\big).\end{aligned}\tag{9.32}$$

Obviously,

$$\begin{aligned}\Big(\frac{\mu}{\lambda}-A_1-A_{21}-\frac{\mu}{\lambda}\,B_{23}-\frac{\mu}{\lambda}\,B_3\Big)\big(1-B_{22}\big)\geq\\ \geq\frac{\mu}{\lambda}-A_1-A_{21}-\frac{\mu}{\lambda}\big(B_{22}+B_{23}+B_3\big).\end{aligned}\tag{9.33}$$

On the other hand, by virtue of (9.31) and the assumption $\frac{\mu}{\lambda}\in\,]0,1]$, we obtain

$$\begin{aligned}\Big(B_1+B_{21}+\frac{\mu}{\lambda}\,A_{23}+\frac{\mu}{\lambda}\,A_3+1\Big)\,\big(A_{22}-1\big)=\\ =\Big(B_1+\frac{\mu}{\lambda}\,A_3\Big)\,\big(A_{22}-1\big)+B_{21}\big(A_{22}-1\big)+\frac{\mu}{\lambda}\,A_{23}\big(A_{22}-1\big)+\\ +A_{22}-1\leq\Big(B_1+\frac{\mu}{\lambda}A_3\Big)\,\big(A_2-1\big)+\frac{\mu}{\lambda}\,B_{21}+A_{22}+A_{23}-1.\end{aligned}\tag{9.34}$$

Using (6.29), (6.34), (9.33), and (9.34), (9.32) results in

$$A_1+A_2+\frac{\mu}{\lambda}\big(B_2+B_3\big)+\Big(B_1+\frac{\mu}{\lambda}\,A_3\Big)\,\big(A_2-1\big)\geq 1+\frac{\mu}{\lambda}\,,$$

which, in view of (6.29) and (7.1), contradicts (9.5).

Now suppose that the assumption b) holds. According to (6.34), (9.6), and (9.8), we have $A_{22}<1$. Consequently, on account of (6.52), (9.25) implies

$$0<M_0\big(1-A_{22}\big)\leq m_0\big(B_{22}-1\big).\tag{9.35}$$

Moreover, it follows from (7.1), (9.3), and (9.6)–(9.8) that

$$B_{22}<1+\frac{\mu}{\lambda}\,.\tag{9.36}$$

From (9.29) we obtain

$$m_0\Big(1-A_1-A_{21}-\frac{\mu}{\lambda}\,B_3\Big)\leq M_0\Big(B_1+B_{21}+\frac{\mu}{\lambda}\,A_3+\frac{\mu}{\lambda}\Big)\,,$$

which, together with (9.35), yields

$$\begin{aligned}\Big(1-A_1-A_{21}-\frac{\mu}{\lambda}\,B_3\Big)\big(1-A_{22}\big)\leq\\ \leq\Big(B_1+B_{21}+\frac{\mu}{\lambda}\,A_3+\frac{\mu}{\lambda}\Big)\,\big(B_{22}-1\big).\end{aligned}\tag{9.37}$$

Clearly,

$$\begin{aligned}\Big(1-A_1-A_{21}-\frac{\mu}{\lambda}\,B_3\Big)\big(1-A_{22}\big)\geq\\ \geq 1-A_1-A_{21}-A_{22}-\frac{\mu}{\lambda}\,B_3\,.\end{aligned}\tag{9.38}$$

On the other hand, by virtue of (9.36) and the assumption $\frac{\mu}{\lambda}\in\,]0,1]$, we get

$$\begin{aligned}\Big(B_1+B_{21}+\frac{\mu}{\lambda}\,A_3+\frac{\mu}{\lambda}\Big)\,\big(B_{22}-1\big)=\\ =\Big(B_1+\frac{\mu}{\lambda}\,A_3\Big)\,\big(B_{22}-1\big)+B_{21}\big(B_{22}-1\big)+\frac{\mu}{\lambda}\,B_{22}-\frac{\mu}{\lambda}\leq\\ \leq\Big(B_1+\frac{\mu}{\lambda}\,A_3\Big)\,\big(B_2-1\big)+\frac{\mu}{\lambda}\big(B_{21}+B_{22}\big)-\frac{\mu}{\lambda}\,.\end{aligned}\tag{9.39}$$

Using (6.29), (6.34), (9.38), and (9.39), (9.37) implies

$$A_1+A_2+\frac{\mu}{\lambda}\big(B_2+B_3\big)+\Big(B_1+\frac{\mu}{\lambda}\,A_3\Big)\,\big(B_2-1\big)\geq 1+\frac{\mu}{\lambda}\,,$$

which, in view of (6.29) and (7.1), contradicts (9.8).

Finally suppose that the assumption c) holds. According to (6.29), (6.34), (9.3), and (9.9), we have

$$A_{22}<1,\qquad A_1+A_{21}+\frac{\mu}{\lambda}\,B_3<1.$$

Thus, it follows from (6.52), (9.25), and (9.29) that

$$B_{22}>1,\qquad B_1+B_{21}+\frac{\mu}{\lambda}\,A_3+\frac{\mu}{\lambda}>0,\tag{9.40}$$

and

$$\begin{aligned}\Big(1-A_1-A_{21}-\frac{\mu}{\lambda}\,B_3\Big)\big(1-A_{22}\big)\leq\\ \leq\Big(B_1+B_{21}+\frac{\mu}{\lambda}\,A_3+\frac{\mu}{\lambda}\Big)\big(B_{22}-1\big).\end{aligned}\tag{9.41}$$

According to (9.3), (9.40), and the fact that

$$\Big(1 - A_1 - A_{21} - \frac{\mu}{\lambda} B_3\Big)\big(1 - A_{22}\big) \geq$$

$$\geq 1 - A_1 - A_{21} - A_{22} - \frac{\mu}{\lambda} B_3 \geq 1 - H,$$

from (9.41) we get

$$B_{22} \geq 1 + \frac{1 - H}{B_1 + B_{21} + \frac{\mu}{\lambda} A_3 + \frac{\mu}{\lambda}}\,. \tag{9.42}$$

First suppose that (9.10) and (9.11) are satisfied. By virtue of (9.40), from (9.42) we obtain

$$1 - H \leq \Big(B_1 + B_{21} + \frac{\mu}{\lambda} A_3 + \frac{\mu}{\lambda}\Big)\big(B_{22} - 1\big) \leq$$

$$\leq \frac{1}{4}\Big(B_1 + B_{21} + B_{22} + \frac{\mu}{\lambda} A_3 - 1 + \frac{\mu}{\lambda}\Big)^2 \leq$$

$$\leq \frac{1}{4}\Big(B_1 + B_2 + \frac{\mu}{\lambda} A_3 - 1 + \frac{\mu}{\lambda}\Big)^2,$$

which, in view of (6.29), (9.9), and (9.40), contradicts (9.11).

Now suppose that (9.8) and (9.12) are fulfilled. Let

$$g(x) \overset{\text{def}}{=} \frac{1 - H}{x + B_1 + \frac{\mu}{\lambda} A_3 + \frac{\mu}{\lambda}} + x \quad \text{for} \quad x > -B_1 - \frac{\mu}{\lambda} A_3 - \frac{\mu}{\lambda}\,,$$

where H is given by (9.3). It is not difficult to verify that, on account of (6.29) and (9.12), the function g is nondecreasing in $[0, +\infty[$. Therefore, from (9.42) we obtain

$$B_{21} + B_{22} + B_{23} \geq 1 + \frac{1 - H}{B_1 + B_{21} + \frac{\mu}{\lambda} A_3 + \frac{\mu}{\lambda}} + B_{21} =$$

$$= 1 + g\big(B_{21}\big) \geq 1 + g(0) = 1 + \frac{1 - H}{B_1 + \frac{\mu}{\lambda} A_3 + \frac{\mu}{\lambda}}\,,$$

which, in view of (6.29), (6.34), (7.1), and (9.3), contradicts (9.8).

Now suppose that u does not change its sign in $[\tau_0, \tau_1]$. Without loss of generality we can assume that (6.30) is satisfied. Define numbers M and m by (6.31) and choose $t_M, t_m \in [\tau_0, \tau_1]$ such that (6.32) holds. It is clear that (6.35) is satisfied, since if $M = 0$, then, in view of (6.1_0), (6.30), and (6.31), we obtain $u(\tau_0) = 0$ and $u'(t) = 0$ for $t \in [a, b]$, i.e., $u \equiv 0$.

The integrations of (6.1_0) from a to t_M, from t_M to b, from a to τ_0, and from τ_1 to b, in view of (6.29)–(6.32), result in (9.13), (9.14),

$$\begin{aligned} -u(a) &\leq u(\tau_0) - u(a) = \\ &= \int_a^{\tau_0} [p(s)]_+ u(\tau(s))ds - \int_a^{\tau_0} [p(s)]_- u(\tau(s))ds \leq MA_1, \end{aligned} \tag{9.43}$$

$$\begin{aligned} -u(b) &\leq u(\tau_1) - u(b) = \\ &= \int_{\tau_1}^{b} [p(s)]_- u(\tau(s))ds - \int_{\tau_1}^{b} [p(s)]_+ u(\tau(s))ds \leq MB_3. \end{aligned} \tag{9.44}$$

Moreover, from (9.13) and (9.14), in view of (6.29) and (6.35), we find

$$M - u(a) \leq M\big(A_1 + A_2\big), \tag{9.45}$$

$$M - u(b) \leq M\big(B_2 + B_3\big). \tag{9.46}$$

Multiplying both sides of (9.44) by $\frac{\mu}{\lambda}$, summing with (9.45), and taking into account (1.2_0), (7.1), and (6.35), we get

$$A_1 + A_2 + \frac{\mu}{\lambda} B_3 \geq 1. \tag{9.47}$$

Analogously, (9.43) and (9.46) yield

$$A_1 + \frac{\mu}{\lambda} B_2 + \frac{\mu}{\lambda} B_3 \geq \frac{\mu}{\lambda}\,. \tag{9.48}$$

First suppose that the assumption a) holds. By virtue of (6.29), (7.1), (9.4), and (9.48), (9.5) results in

$$1 + \frac{\mu}{\lambda} > A_1 + A_2 + \frac{\mu}{\lambda}\big(B_2 + B_3\big) \geq 1 + \frac{\mu}{\lambda}\,, \tag{9.49}$$

a contradiction.

Now suppose that the assumption b) holds. With respect to (6.29), (7.1), (9.6), and (9.47), (9.8) implies (9.49), a contradiction.

Finally suppose that the assumption c) holds. On account of (6.29) and (9.3), (9.47) contradicts (9.9). □

9.3. Comments and Examples

Example 9.1. Let $k > 1$, and $\varepsilon \geq 0$. Choose $m > 0$ such that

$$m \leq \min\left\{1,\ \frac{k(|\lambda|k + |\mu|)}{(|\lambda| + \varepsilon)k + |\mu|}\right\}$$

and put $a = 0$, $b = 3$, and

$$p(t) = \begin{cases} \frac{(|\lambda|+\varepsilon)k+|\mu|}{|\lambda|k} & \text{for} \quad t \in [0, 1[\\ -\frac{k-m}{k} & \text{for} \quad t \in [1, 2[\\ \frac{1-m}{m} & \text{for} \quad t \in [2, 3] \end{cases}, \qquad \tau(t) = \begin{cases} t^* & \text{for} \quad t \in [0, 1[\\ 1 & \text{for} \quad t \in [1, 2[\\ 2 & \text{for} \quad t \in [2, 3] \end{cases},$$

where

$$t^* = \begin{cases} 2 - \frac{1}{k-m}\left(\frac{k(|\lambda|k+|\mu|)}{(|\lambda|+\varepsilon)k+|\mu|} - m\right) & \text{if} \quad m \neq k \\ 1 & \text{if} \quad m = k \end{cases}.$$

It is not difficult to verify that $\tau_0 = 1$, $\tau_1 = 2$, and

$$\int_a^{\tau_1} [p(s)]_+ ds = \frac{(|\lambda| + \varepsilon)k + |\mu|}{|\lambda|k}, \qquad \int_{\tau_1}^{b} [p(s)]_+ ds = \frac{1-m}{m}, \qquad \int_{\tau_0}^{\tau_1} [p(s)]_+ ds = 0,$$

$$\int_a^{\tau_0} [p(s)]_- ds = 0, \qquad \int_{\tau_0}^{b} [p(s)]_- ds = \int_{\tau_0}^{\tau_1} [p(s)]_- ds = \frac{k-m}{k}.$$

Thus, the conditon (6.18) holds, $T = \frac{k-m}{k}$, and

$$|\lambda| \int_a^{\tau_1} [p(s)]_+ ds + |\mu| \int_{\tau_0}^{b} [p(s)]_- ds -$$

$$- \left(|\lambda| \int_a^{\tau_0} [p(s)]_- ds + |\mu| \int_{\tau_1}^{b} [p(s)]_+ ds \right) \big(1 - T\big) = |\lambda| + |\mu| + \varepsilon.$$

On the other hand, the function

$$u(t) = \begin{cases} (|\lambda|k + |\mu|)t - |\mu| & \text{for} \quad t \in [0, 1[\\ |\lambda|(k-m)(2-t) + |\lambda|m & \text{for} \quad t \in [1, 2[\\ |\lambda|(1-m)(t-3) + |\lambda| & \text{for} \quad t \in [2, 3] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

Example 9.2. Let $k \in \left]0, \frac{\mu}{\lambda}\right[$, and $\varepsilon \in \left[0, \frac{|\mu|-|\lambda|k}{k}\right[$. Choose

$$M \geq \frac{k(|\mu|-|\lambda|k)}{|\mu|-|\lambda|k-\varepsilon k}$$

and put $a=0$, $b=3$, and

$$p(t)=\begin{cases} -\frac{|\mu|-|\lambda|k+\varepsilon M}{|\lambda|M} & \text{for} \quad t \in [0,1[\\ \frac{M-k}{M} & \text{for} \quad t \in [1,2[\\ -\frac{M+1}{k} & \text{for} \quad t \in [2,3] \end{cases}, \qquad \tau(t)=\begin{cases} t^* & \text{for} \quad t \in [0,1[\\ 2 & \text{for} \quad t \in [1,2[\\ 1 & \text{for} \quad t \in [2,3] \end{cases},$$

where

$$t^*=\begin{cases} 1+\frac{1}{M-k}\left(\frac{M(|\mu|-|\lambda|k)}{|\mu|-|\lambda|k+\varepsilon M}-k\right) & \text{if} \quad M \neq k \\ 2 & \text{if} \quad M = k \end{cases}.$$

It is not difficult to verify that $\tau_0=1$, $\tau_1=2$, and

$$\int_a^{\tau_0}[p(s)]_+ds=0\,, \qquad \int_{\tau_0}^{b}[p(s)]_+ds=\int_{\tau_0}^{\tau_1}[p(s)]_+ds=\frac{M-k}{M}\,,$$

$$\int_a^{\tau_1}[p(s)]_-ds=\frac{|\mu|-|\lambda|k+\varepsilon M}{|\lambda|M}\,, \quad \int_{\tau_0}^{\tau_1}[p(s)]_-ds=0\,, \quad \int_{\tau_1}^{b}[p(s)]_-ds=\frac{M+1}{k}\,.$$

Thus, the conditon (6.18) holds, $T=\frac{M-k}{M}$, and

$$\left(|\lambda|\int_a^{\tau_0}[p(s)]_+ds+|\mu|\int_{\tau_1}^{b}[p(s)]_-ds\right)\big(1-T\big)-$$

$$-|\lambda|\int_a^{\tau_1}[p(s)]_-ds-|\mu|\int_{\tau_0}^{b}[p(s)]_+ds=\big(|\lambda|+|\mu|\big)\big(1-T\big)-\varepsilon\,.$$

On the other hand, the function

$$u(t)=\begin{cases} |\mu|-(|\mu|-|\lambda|k)t & \text{for} \quad t \in [0,1[\\ |\lambda|(M-k)(t-1)+|\lambda|k & \text{for} \quad t \in [1,2[\\ |\lambda|(M+1)(3-t)-|\lambda| & \text{for} \quad t \in [2,3] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

Example 9.3. Let $|\mu| \le |\lambda|$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that

$$x_2 \ge 1$$

and

$$|\lambda|\big(x_1 + x_2\big) + |\mu|\big(y_2 + y_3\big) + \big(|\lambda|y_1 + |\mu|x_3\big)\big(x_2 - 1\big) = |\lambda| + |\mu|.$$

Let, moreover, $a = 0$, $b = 8$,

$$p(t) = \begin{cases} -y_1 & \text{for} \quad t \in [0,1[\\ x_1 & \text{for} \quad t \in [1,2[\\ x_2 - 1 & \text{for} \quad t \in [2,3[\\ 0 & \text{for} \quad t \in [3,4[\\ 1 & \text{for} \quad t \in [4,5[\\ -y_2 & \text{for} \quad t \in [5,6[\\ -y_3 & \text{for} \quad t \in [6,7[\\ x_3 & \text{for} \quad t \in [7,8] \end{cases},$$

and

$$\tau(t) = \begin{cases} 2 & \text{for} \quad t \in [0,1[\, \cup [7,8] \\ 5 & \text{for} \quad t \in [1,3[\, \cup [4,7[\\ 6 & \text{for} \quad t \in [3,4[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 6$, and (6.104) is satisfied.

On the other hand, the function

$$u(t) = \begin{cases} y_1(x_2-1)(t-1) + 1 - x_2 - x_1 & \text{for} \quad t \in [0,1[\\ x_1(t-2) + 1 - x_2 & \text{for} \quad t \in [1,2[\\ (x_2-1)(t-3) & \text{for} \quad t \in [2,3[\\ 0 & \text{for} \quad t \in [3,4[\\ t-4 & \text{for} \quad t \in [4,5[\\ y_2(5-t) + 1 & \text{for} \quad t \in [5,6[\\ y_3(6-t) + 1 - y_2 & \text{for} \quad t \in [6,7[\\ x_3(x_2-1)(7-t) + 1 - y_2 - y_3 & \text{for} \quad t \in [7,8] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

Example 9.4. Let $|\mu| \le |\lambda|$ and let $x_i, y_i \in R_+$ $(i = 1,2,3)$ be such that

$$y_2 \ge 1\,, \qquad x_1 + x_2 + \frac{\mu}{\lambda}\, y_3 \ge 1 - \Big(\frac{\mu}{\lambda}\Big)^2\,,$$

and

$$|\lambda|\big(x_1 + x_2\big) + |\mu|\big(y_2 + y_3\big) + \big(|\lambda| y_1 + |\mu| x_3\big)\big(y_2 - 1\big) = |\lambda| + |\mu|.$$

Let, moreover, $a = 0$, $b = 8$,

$$p(t) = \begin{cases} x_1 & \text{for} \quad t \in [0,1[\\ -y_1 & \text{for} \quad t \in [1,2[\\ x_2 & \text{for} \quad t \in [2,3[\\ -1 & \text{for} \quad t \in [3,4[\\ 0 & \text{for} \quad t \in [4,5[\\ -(y_2 - 1) & \text{for} \quad t \in [5,6[\\ x_3 & \text{for} \quad t \in [6,7[\\ -y_3 & \text{for} \quad t \in [7,8] \end{cases},$$

and

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [0,1[\,\cup\, [2,4[\,\cup\, [5,6[\,\cup\, [7,8] \\ 6 & \text{for} \quad t \in [1,2[\,\cup\, [6,7[\\ 2 & \text{for} \quad t \in [4,5[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 6$, and (6.104) is satisfied.

On the other hand, the function

$$u(t) = \begin{cases} x_1(t-1) + 1 - x_2 - y_1(y_2 - 1) & \text{for} \quad t \in [0,1[\\ y_1(y_2 - 1)(t-2) + 1 - x_2 & \text{for} \quad t \in [1,2[\\ x_2(t-3) + 1 & \text{for} \quad t \in [2,3[\\ 4 - t & \text{for} \quad t \in [3,4[\\ 0 & \text{for} \quad t \in [4,5[\\ (y_2 - 1)(6-t) + 1 - y_2 & \text{for} \quad t \in [5,6[\\ x_3(y_2 - 1)(7-t) + 1 - y_2 - x_3(y_2 - 1) & \text{for} \quad t \in [6,7[\\ y_3(8-t) + 1 - y_2 - x_3(y_2 - 1) - y_3 & \text{for} \quad t \in [7,8] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

Example 9.5. Let $|\mu| \le |\lambda|$ and let $x_i, y_i \in R_+$ $(i = 1, 2, 3)$ be such that

$$y_2 \ge 1\,, \qquad x_1 + x_2 + \frac{\mu}{\lambda}\, y_3 < 1 - \Big(\frac{\mu}{\lambda}\Big)^2, \tag{9.50}$$

$$y_1 + \frac{\mu}{\lambda}\, x_3 < -\frac{\mu}{\lambda} + \sqrt{1 - x_1 - x_2 - \frac{\mu}{\lambda}\, y_3}\,,$$

$$y_1 + y_2 + \frac{\mu}{\lambda}\, x_3 \ge 1 - \frac{\mu}{\lambda} + 2\sqrt{1 - x_1 - x_2 - \frac{\mu}{\lambda}\, y_3}\,.$$

Put $\alpha = \sqrt{1 - x_1 - x_2 - \frac{\mu}{\lambda}\, y_3}$ and $k = \alpha - \frac{\mu}{\lambda} - y_1 - \frac{\mu}{\lambda}\, x_3$. Obviously, $k > 0$ and $y_2 \ge 1 + \alpha + k$. Let, moreover, $a = 0$, $b = 10$, $p \in L\big([a,b];R\big)$ be defined by (6.105), and

$$\tau(t) = \begin{cases} 8 & \text{for} \quad t \in [0,1[\, \cup [3,4[\, \cup [8,9[\\ 4 & \text{for} \quad t \in [1,3[\, \cup [4,5[\, \cup [6,7[\, \cup [9,10] \\ 5 & \text{for} \quad t \in [5,6[\\ 2 & \text{for} \quad t \in [7,8[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 8$, and (6.104) is satisfied.

On the other hand, the function

$$u(t) = \begin{cases} \alpha y_1(1-t) + x_1 + x_2 + k\alpha - 1 & \text{for} \quad t \in [0,1[\\ x_1(2-t) + x_2 + k\alpha - 1 & \text{for} \quad t \in [1,2[\\ x_2(3-t) + k\alpha - 1 & \text{for} \quad t \in [2,3[\\ k\alpha(4-t) - 1 & \text{for} \quad t \in [3,4[\\ t - 5 & \text{for} \quad t \in [4,5[\\ 0 & \text{for} \quad t \in [5,6[\\ \alpha(t-6) & \text{for} \quad t \in [6,7[\\ \alpha & \text{for} \quad t \in [7,8[\\ \alpha x_3(t-8) + \alpha & \text{for} \quad t \in [8,9[\\ y_3(t-9) + \alpha + \alpha x_3 & \text{for} \quad t \in [9,10] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a,b];R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

Example 9.6. Let $|\mu| \le |\lambda|$ and let $x_i, y_i \in R_+$ $(i = 1, 2, 3)$ be such that (9.50) holds and

$$y_1 + \frac{\mu}{\lambda} x_3 \ge -\frac{\mu}{\lambda} + \sqrt{1 - x_1 - x_2 - \frac{\mu}{\lambda} y_3}\,,$$

$$|\lambda|\big(x_1 + x_2\big) + |\mu|\big(y_2 + y_3\big) + \big(|\lambda| y_1 + |\mu| x_3\big)\big(y_2 - 1\big) \ge |\lambda| + |\mu|.$$

Put $\alpha = 1 - x_1 - x_2 - \frac{\mu}{\lambda} y_3$ and $\beta = y_1 + \frac{\mu}{\lambda} x_3 + \frac{\mu}{\lambda}$. Obviously, $\alpha > 0$, $\beta > 0$, and $y_2 \ge 1 + \frac{\alpha}{\beta}$. Let, moreover, $a = 0$, $b = 9$, $p \in L\big([a, b]; R\big)$ be defined by (6.110), and

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [0,1[\cup [2,4[\cup [5,6[\cup [7,8[\\ 7 & \text{for} \quad t \in [1,2[\cup [8,9] \\ 4 & \text{for} \quad t \in [4,5[\\ 2 & \text{for} \quad t \in [6,7[\end{cases}.$$

Obviously, $\tau_0 = 2$, $\tau_1 = 7$, and (6.104) is satisfied.

On the other hand, the function

$$u(t) = \begin{cases} x_1\beta(1 - t) + y_1\alpha + x_2\beta - \beta & \text{for} \quad t \in [0,1[\\ y_1\alpha(2 - t) + x_2\beta - \beta & \text{for} \quad t \in [1,2[\\ x_2\beta(3 - t) - \beta & \text{for} \quad t \in [2,3[\\ \beta(t - 4) & \text{for} \quad t \in [3,4[\\ 0 & \text{for} \quad t \in [4,5[\\ \alpha(t - 5) & \text{for} \quad t \in [5,6[\\ \alpha & \text{for} \quad t \in [6,7[\\ y_3\beta(t - 7) + \alpha & \text{for} \quad t \in [7,8[\\ x_3\alpha(t - 8) + \alpha + y_3\beta & \text{for} \quad t \in [8,9] \end{cases}$$

is a nontrivial solution of the problem (6.1_0), (1.2_0). Therefore, according to Remark 1.1 (see p. 14), there exist $q \in L\big([a, b]; R\big)$ and $c \in R$ such that the problem (6.1), (1.2) has no solution.

Suplementary Remarks

The main ideas of the results presented in Chapter I can be found in [22,24, 26–29], where the special case of the boundary condition (1.2) with $\lambda = 1$ is considered.

Theorems 2.1–2.3 and 2.5 are proved in [27], Theorems 2.9 and 2.10 are proved in [26], Theorems 2.4, 2.11, and 4.1 are proved in [24], Theorems 4.3 and 4.6 are proved in [29], Theorem 4.4 is proved in [29], and Theorem 7.1 one can find in [28].

CHAPTER II

Nonlinear Problem

§10. Statement of the Problem

In this chapter, we will consider the problem on the existence and uniqueness of a solution of the equation

$$u'(t) = F(u)(t) \tag{10.1}$$

satisfying the boundary condition

$$\lambda u(a) + \mu u(b) = h(u), \tag{10.2}$$

where $F \in K_{ab}$, $\lambda, \mu \in R$, $|\lambda| + |\mu| \neq 0$, and $h : C\big([a,b]; R\big) \to R$ is a continuous functional satisfying that for every $r > 0$ there exists $M_r \in R_+$ such that

$$|h(v)| \leq M_r \quad \text{for} \quad \|v\|_C \leq r.$$

By a solution of the equation (10.1) is understood a function $u \in \widetilde{C}\big([a,b]; R\big)$ satisfying this equation almost everywhere in $[a,b]$. Note also that as in Chapter I, the equalities and inequalities with integrable functions are understood almost everywhere.

The following result is well–known from the general theory of boundary value problems for functional differential equations (see, e.g., [39]).

Proposition 10.1. *Let there exist $\ell \in \mathcal{L}_{ab}$ such that the problem*

$$u'(t) = \ell(u)(t), \qquad \lambda u(a) + \mu u(b) = 0$$

has only the trivial solution and on the set $C\big([a,b]; R\big)$ the inequalities

$$|F(v)(t) - \ell(v)(t)| \leq q(t, \|v\|_C) \quad \textit{for} \quad t \in [a,b], \tag{10.3}$$

$$|h(v)| \leq c \tag{10.4}$$

hold, where $c \in R_+$ and $q \in K\big([a,b] \times R_+; R_+\big)$ is nondecreasing in the second argument and satisfies

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)\,ds = 0. \tag{10.5}$$

Then the problem (10.1), (10.2) *has at least one solution.*

According to Proposition 10.1 and the results from Chapter I, effective sufficient conditions for solvability of the problem (10.1), (10.2) can be immediately established. Although these results are nonimprovable (since they are nonimprovable for the special case of the problem (10.1), (10.2), for the linear problem (1.1), (1.2)), we will show that, in some cases, the assumptions (10.3) and (10.4) can be weakened to the one–side restrictions.

All results will be concretized for the differential equation with deviating arguments (EDA) of the form

$$\begin{aligned} u'(t) = \sum_{k=1}^{m} \Big(p_k(t)u(\tau_k(t)) - g_k(t)u(\nu_k(t)) \Big) + \\ + f(t, u(t), u(\zeta_1(t)), \dots, u(\zeta_n(t))), \end{aligned} \tag{10.1$'$}$$

where $f \in K([a,b] \times R^{n+1}; R)$, $p_k, g_k \in L\big([a,b]; R_+\big)$, $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, $\zeta_j \in \mathcal{M}_{ab}$ $(j = 1, \dots, n)$, $m, n \in N$.

§11. Auxiliary Propositions

In this section, we will establish some auxiliary results for solvability and unique solvability of the problem (10.1), (10.2).

Lemma 11.1. *Let $\ell_0 \in \mathcal{L}_{ab}$ and let the homogeneous problem*

$$v'(t) = \ell_0(v)(t), \qquad \lambda v(a) + \mu v(b) = 0$$

have only the trivial solution. Then there exists a positive number r_0 such that for any $\overline{q} \in L\big([a,b];R\big)$ and $\overline{c} \in R$ the solution v of the problem

$$v'(t) = \ell_0(v)(t) + \overline{q}(t), \qquad \lambda v(a) + \mu v(b) = \overline{c} \tag{11.1}$$

admits the estimate

$$\|v\|_C \leq r_0\big(|\overline{c}| + \|\overline{q}\|_L\big). \tag{11.2}$$

Proof. Let

$$R \times L\big([a,b];R\big) = \big\{(\overline{c},\overline{q}) : \overline{c} \in R,\ \overline{q} \in L\big([a,b];R\big)\big\}$$

be the Banach space with the norm

$$\|(\overline{c},\overline{q})\|_{R\times L} = |\overline{c}| + \|\overline{q}\|_L,$$

and let Ω be an operator, which assigns to every $(\overline{c},\overline{q}) \in R \times L\big([a,b];R\big)$ the solution v of the problem (11.1). According to Theorem 1.4 in [42], $\Omega : R \times L\big([a,b];R\big) \to C\big([a,b];R\big)$ is a linear bounded operator. Denote by r_0 the norm of Ω. Then, clearly, for any $(\overline{c},\overline{q}) \in R \times L\big([a,b];R\big)$ the inequality

$$\|\Omega(\overline{c},\overline{q})\|_C \leq r_0\big(|\overline{c}| + \|\overline{q}\|_L\big)$$

holds. Consequently, the solution $v = \Omega(\overline{c},\overline{q})$ of the problem (11.1) admits the estimate (11.2). □

Now let us formulate the result from [41, Theorem 1] in a suitable for us form.

Lemma 11.2. *Let there exist a positive number ρ and an operator $\ell \in \mathcal{L}_{ab}$ such that the homogeneous problem* (1.1_0), (1.2_0) *has only the trivial* *solution, and let for every* $\delta \in \,]0,1[$ *and for an arbitrary function* $u \in \widetilde{C}\big([a,b];R\big)$ *satisfying*

$$u'(t) = \ell(u)(t) + \delta\big[F(u)(t) - \ell(u)(t)\big] \quad \textit{for} \quad t \in [a,b], \tag{11.3}$$

$$\lambda u(a) + \mu u(b) = \delta h(u), \tag{11.4}$$

the estimate

$$\|u\|_C \le \rho \tag{11.5}$$

hold. Then the problem (10.1), (10.2) *has at least one solution.*

Proof. Since $\ell \in \mathcal{L}_{ab}$ and $F \in K_{ab}$, there exist $\eta, \omega \in L\big([a,b];R_+\big)$ such that

$$|\ell(v)(t)| \le \eta(t)\|v\|_C \quad \text{for} \quad t \in [a,b], \quad v \in C\big([a,b];R\big),$$

$$|F(v)(t)| \le \omega(t) \quad \text{for} \quad t \in [a,b], \quad \|v\|_C \le 2\rho.$$

Moreover, there exists $\alpha \in R_+$ such that

$$|h(v)| \le \alpha \quad \text{for} \quad \|v\|_C \le 2\rho$$

(see §10). Put

$$\gamma(t) \overset{\text{def}}{=} \omega(t) + 2\rho\eta(t) \quad \text{for} \quad t \in [a,b],$$

$$\sigma(s) \overset{\text{def}}{=} \begin{cases} 1 & \text{for} \quad 0 \le s \le \rho \\ 2 - \frac{s}{\rho} & \text{for} \quad \rho < s < 2\rho \\ 0 & \text{for} \quad s \ge 2\rho \end{cases}, \tag{11.6}$$

$$\begin{gathered} q_0(v)(t) \overset{\text{def}}{=} \sigma\big(\|v\|_C\big)\big[F(v)(t) - \ell(v)(t)\big] \quad \text{for} \quad t \in [a,b], \\ c_0(v) \overset{\text{def}}{=} \sigma\big(\|v\|_C\big)h(v). \end{gathered} \tag{11.7}$$

Then for every $v \in C\big([a,b];R\big)$ and almost all $t \in [a,b]$, the inequalities

$$|q_0(v)(t)| \le \gamma(t), \qquad |c_0(v)| \le \alpha$$

hold.

For arbitrarily fixed $u \in C\big([a,b];R\big)$, let us consider the problem

$$v'(t) = \ell(v)(t) + q_0(u)(t), \qquad \lambda v(a) + \mu v(b) = c_0(u). \tag{11.8}$$

According to Theorem 1.1 (see p. 14), the problem (11.8) has a unique solution v and, moreover, by virtue of Lemma 11.1, there exists $\beta > 0$ such that

$$\|v\|_C \le \beta\big(|c_0(u)| + \|q_0(u)\|_L\big).$$

Therefore, for arbirarily fixed $u \in C\big([a,b];R\big)$, the solution v of the problem (11.8) admits the estimates

$$\|v\|_C \le \rho_0, \qquad |v'(t)| \le \gamma^*(t) \quad \text{for} \quad t \in [a,b], \tag{11.9}$$

where $\rho_0 = \beta\big(\|\gamma\|_L + \alpha\big)$ and $\gamma^*(t) = \rho_0\eta(t) + \gamma(t)$ for $t \in [a,b]$.

Let $\Omega : C\big([a,b];R\big) \to C\big([a,b];R\big)$ be an operator which to every $u \in C\big([a,b];R\big)$ assigns the solution v of the problem (11.8). Due to Theorem 1.4 from [42], the operator Ω is continuous. On the other hand, by virtue of (11.9), for every $u \in C\big([a,b];R\big)$ we have

$$\|\Omega(u)\|_C \le \rho_0, \qquad \big|\Omega(u)(t) - \Omega(u)(s)\big| \le \left|\int_s^t \gamma^*(\xi)d\xi\right| \quad \text{for} \quad s,t \in [a,b].$$

Thus the operator Ω continuously maps the Banach space $C\big([a,b];R\big)$ into its relatively compact subset. Therefore, using the Schauder's principle, there exists $u \in C\big([a,b];R\big)$ such that

$$\Omega(u)(t) = u(t) \quad \text{for} \quad t \in [a,b].$$

By the equalities (11.7), u is obviously a solution of the problem (11.3), (11.4) with

$$\delta = \sigma\big(\|u\|_C\big). \tag{11.10}$$

Now we will show that u admits the estimate (11.5). Suppose the contrary. Then either

$$\rho < \|u\|_C < 2\rho \tag{11.11}$$

or

$$\|u\|_C \ge 2\rho. \tag{11.12}$$

If we assume that the inequalities (11.11) are fulfilled, then, on account of (11.6) and (11.10), we have $\delta \in \,]0,1[$. However, by the conditions of the lemma, in this case we have the estimate (11.5), which contradicts (11.11).

Suppose now that (11.12) is satisfied. Then by (11.6) and (11.10), we have $\delta = 0$. Hence u is a solution of the problem (1.1_0), (1.2_0). But this is imposible because the problem (1.1_0), (1.2_0) has only the trivial solution. Thus, above–obtained contradiction proves the validity of the estimate (11.5).

By virtue of (11.5), (11.6), and (11.10), it is clear that $\delta = 1$ and thus, u is a solution of the problem (10.1), (10.2). $\square$

Definition 11.1. We will say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\mathcal{A}^i\big(\lambda,\mu\big)$, $i \in \{1,2\}$, if there exists a positive number r such that for any $q^* \in L\big([a,b];R_+\big)$ and $c \in R_+$ every function $u \in \widetilde{C}\big([a,b];R\big)$ satisfying the inequalities

$$\big[\lambda u(a) + \mu u(b)\big]\operatorname{sgn}\big((2-i)\lambda u(a) + (i-1)\mu u(b)\big) \le c, \tag{11.13}$$

$$(-1)^{i+1}\big[u'(t) - \ell(u)(t)\big]\operatorname{sgn} u(t) \le q^*(t) \quad \text{for} \quad t \in [a,b] \tag{11.14}$$

admits the estimate

$$\|u\|_C \le r\big(c + \|q^*\|_L\big). \tag{11.15}$$

Lemma 11.3. *Let $i \in \{1,2\}$, $c \in R_+$,*

$$h(v)\operatorname{sgn}\big((2-i)\lambda v(a) + (i-1)\mu v(b)\big) \le c \quad \textit{for} \quad v \in C\big([a,b];R\big), \tag{11.16}$$

and let there exist $\ell \in \mathcal{A}^i\big(\lambda,\mu\big)$ such that on the set $B^i_{\lambda\mu c}\big([a,b];R\big)$ the inequality

$$(-1)^{i+1}\big[F(v)(t) - \ell(v)(t)\big]\operatorname{sgn} v(t) \le q(t,\|v\|_C) \quad \textit{for} \quad t \in [a,b] \tag{11.17}$$

is fulfilled, where $q \in K\big([a,b]\times R_+;R_+\big)$ is nondecreasing in the second argument and satisfies (10.5)*. Then the problem* (10.1)*,* (10.2) *has at least one solution.*

Proof. First note that due to the condition $\ell \in \mathcal{A}^i\big(\lambda,\mu\big)$ the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution.

Let r be the number appearing in Definition 11.1. According to (10.5), there exists $\rho > 2rc$ such that

$$\frac{1}{x}\int_a^b q(s,x)ds < \frac{1}{2r} \quad \text{for} \quad x > \rho. \tag{11.18}$$

Now assume that a function $u \in \widetilde{C}\big([a,b];R\big)$ satisfies (11.3) and (11.4) for some $\delta \in \,]0,1[\,$. Then, according to (11.16), u satisfies the inequality (11.13), i.e., $u \in B^i_{\lambda\mu c}\big([a,b];R\big)$. By (11.17) we obtain that the inequality

(11.14) is fulfilled with $q^*(t) = q(t, \|u\|_C)$ for $t \in [a, b]$. Hence, by the condition $\ell \in \mathcal{A}^i(\lambda, \mu)$ and the definition of the number ρ we get the estimate (11.5).

Since ρ depends neither on u nor on δ, it follows from Lemma 11.2 that the problem (10.1), (10.2) has at least one solution. □

Lemma 11.4. *Let* $i \in \{1, 2\}$,

$$\begin{gathered}\big[h(v) - h(w)\big] \operatorname{sgn}\big((2-i)\lambda(v(a) - w(a)) + \\ (i-1)\mu(v(b) - w(b))\big) \le 0 \quad \textit{for} \quad v, w \in C\big([a, b]; R\big),\end{gathered} \tag{11.19}$$

and let there exist $\ell \in \mathcal{A}^i(\lambda, \mu)$ *such that on the set* $B^i_{\lambda\mu c}\big([a, b]; R\big)$, *where* $c = |h(0)|$, *the inequality*

$$\begin{gathered}(-1)^{i+1}\big[F(v)(t) - F(w)(t) - \\ -\ell(v - w)(t)\big] \operatorname{sgn}\big(v(t) - w(t)\big) \le 0 \quad \textit{for} \quad t \in [a, b]\end{gathered} \tag{11.20}$$

holds. Then the problem (10.1), (10.2) *is uniquely solvable.*

Proof. It follows from (11.19) that the condition (11.16) is fulfilled, where $c = |h(0)|$. By (11.20) we see that on the set $B^i_{\lambda\mu c}\big([a, b]; R\big)$ the inequality (11.17) holds, where $q \equiv |F(0)|$. Consequently, the assumptions of Lemma 11.3 are fulfilled and so the problem (10.1), (10.2) has at least one solution. It remains to show that the problem (10.1), (10.2) has at most one solution.

Let u_1, u_2 be arbitrary solutions of the problem (10.1), (10.2). Put

$$u(t) = u_1(t) - u_2(t) \quad \text{for} \quad t \in [a, b].$$

Then, by (11.19) and (11.20) we get

$$\big[\lambda u(a) + \mu u(b)\big] \operatorname{sgn}\big((2-i)\lambda u(a) + (i-1)\mu u(b)\big) \le 0,$$

$$(-1)^{i+1}[u'(t) - \ell(u)(t)] \operatorname{sgn} u(t) \le 0 \quad \text{for} \quad t \in [a, b].$$

This, together with the condition $\ell \in \mathcal{A}^i(\lambda, \mu)$, results in $u \equiv 0$. Consequently, $u_1 \equiv u_2$. □

Definition 11.2. We will say that a pair $(\ell_0, \ell_1) \in \mathcal{P}_{ab} \times \mathcal{L}_{ab}$ belongs to the set $\mathcal{B}(\lambda, \mu)$ if there exists a positive number r such that for any

$q^* \in L([a,b];R_+)$ and $c \in R_+$ every function $u \in \widetilde{C}([a,b];R)$ satisfying the inequalities

$$[\lambda u(a) + \mu u(b)] \operatorname{sgn}(\lambda u(a)) \leq c, \tag{11.21}$$

$$[u'(t) + \ell_1(u)(t)] \operatorname{sgn} u(t) \leq \ell_0(|u|)(t) + q^*(t) \quad \text{for} \quad t \in [a,b] \tag{11.22}$$

admits the estimate (11.15).

Lemma 11.5. *Let $c \in R_+$,*

$$h(v) \operatorname{sgn}(\lambda v(a)) \leq c \quad \textit{for} \quad v \in C([a,b];R), \tag{11.23}$$

and let there exist $(\ell_0, \ell_1) \in \mathcal{B}(\lambda, \mu)$ such that on the set $B^1_{\lambda\mu c}([a,b];R)$ the inequality

$$\begin{gathered} [F(v)(t) + \ell_1(v)(t)] \operatorname{sgn} v(t) \leq \ell_0(|v|)(t) + \\ + q(t, \|v\|_C) \quad \textit{for} \quad t \in [a,b] \end{gathered} \tag{11.24}$$

holds, where $q \in K([a,b] \times R_+; R_+)$ is nondecreasing in the second argument and satisfies (10.5)*. Then the problem* (10.1), (10.2) *has at least one solution.*

Proof. First note that due to the condition $(\ell_0, \ell_1) \in \mathcal{B}(\lambda, \mu)$ the homogeneous problem (1.1_0), (1.2_0) with $\ell \equiv -\ell_1$ has only the trivial solution.

Let r be the number appearing in Definition 11.2. According to (10.5), there exists $\rho > 2rc$ such that (11.18) holds.

Now assume that a function $u \in \widetilde{C}([a,b];R)$ satisfies (11.3) and (11.4) for some $\delta \in\,]0,1[$ with $\ell \equiv -\ell_1$, i.e.,

$$\begin{gathered} u'(t) + \ell_1(u)(t) = \delta[F(u)(t) + \ell_1(u)(t)] \quad \text{for} \quad t \in [a,b], \\ \lambda u(a) + \mu u(b) = \delta h(u). \end{gathered} \tag{11.25}$$

According to (11.23), the function u satisfies the inequality (11.21), i.e., $u \in B^1_{\lambda\mu c}([a,b];R)$. By virtue of (11.24) and (11.25), we obtain that the inequality (11.22) is fulfilled with $q^*(t) = q(t, \|u\|_C)$ for $t \in [a,b]$. Hence, by the condition $(\ell_0, \ell_1) \in \mathcal{B}(\lambda, \mu)$ and the definition of the number ρ we get the estimate (11.5).

Since ρ depends neither on u nor on δ, it follows from Lemma 11.2 that the problem (10.1), (10.2) has at least one solution. $\square$

Lemma 11.6. *Let*

$$\big[h(v) - h(w)\big] \operatorname{sgn}\big(\lambda(v(a) - w(a))\big) \le 0 \quad \textit{for} \quad v, w \in C\big([a,b];R\big) \tag{11.26}$$

and let there exist $(\ell_0, \ell_1) \in \mathcal{B}(\lambda, \mu)$ *such that on the set* $B^1_{\lambda\mu c}\big([a,b];R\big)$, *where* $c = |h(0)|$, *the inequality*

$$\begin{gathered}\big[F(v)(t) - F(w)(t) + \ell_1(v-w)(t)\big] \operatorname{sgn}\big(v(t) - w(t)\big) \le \\ \le \ell_0(|v-w|)(t) \quad \textit{for} \quad t \in [a,b]\end{gathered} \tag{11.27}$$

holds. Then the problem (10.1), (10.2) *is uniquely solvable.*

Proof. It follows from (11.26) that the condition (11.23) is fulfilled, where $c = |h(0)|$. By (11.27) we see that on the set $B^1_{\lambda\mu c}\big([a,b];R\big)$ the inequality (11.24) holds, where $q \equiv |F(0)|$. Consequently, the assumptions of Lemma 11.5 are fulfilled and so the problem (10.1), (10.2) has at least one solution. It remains to show that the problem (10.1), (10.2) has at most one solution.

Let u_1, u_2 be arbitrary solutions of the problem (10.1), (10.2). Put

$$u(t) = u_1(t) - u_2(t) \quad \text{for} \quad t \in [a,b].$$

Then, by (11.26) and (11.27) we get

$$\big[\lambda u(a) + \mu u(b)\big] \operatorname{sgn}\big(\lambda u(a)\big) \le 0,$$
$$[u'(t) + \ell_1(u)(t)] \operatorname{sgn} u(t) \le \ell_0(|u|)(t) \quad \text{for} \quad t \in [a,b].$$

This, together with the condition $(\ell_0, \ell_1) \in \mathcal{B}(\lambda, \mu)$, results in $u \equiv 0$. Consequently, $u_1 \equiv u_2$. $\square$

§12. Periodic Type BVP

In this section, we will establish nonimprovable, in a certain sense, sufficient conditions for solvability and unique solvability of the problem (10.1), (10.2), where the boundary condition (10.2) is of a periodic type, i.e., when the inequality (2.1) is satisfied. In Subsection 12.1, the main results are formulated. Theorems 12.1–12.12 deal with the case $|\mu| \leq |\lambda|$, while the case $|\mu| \geq |\lambda|$ is considered in Theorems 12.13–12.24. The proofs of the main results can be found in Subsection 12.2. Subsection 12.3 is devoted to the examples verifying the optimality of the main results.

In the sequel, we will assume that the function $q \in K\big([a,b] \times R_+; R_+\big)$ is nondecreasing in the second argument and satisfies (10.5), i.e.,

$$\lim_{x \to +\infty} \frac{1}{x} \int\limits_a^b q(s,x)ds = 0.$$

12.1. Existence and Uniqueness Theorems

In the case, where $|\mu| \leq |\lambda|$, the following statements hold.

Theorem 12.1. *Let* $0 \neq |\mu| \leq |\lambda|$, $c \in R_+$,

$$h(v)\operatorname{sgn}\big(\lambda v(a)\big) \leq c \quad \textit{for} \quad v \in C\big([a,b];R\big), \tag{12.1}$$

and let there exist

$$\ell_0, \ell_1 \in \mathcal{P}_{ab} \tag{12.2}$$

such that on the set $B^1_{\lambda\mu c}\big([a,b];R\big)$ *the inequality*

$$\big[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big]\operatorname{sgn} v(t) \leq q(t, \|v\|_C) \quad \textit{for} \quad t \in [a,b] \tag{12.3}$$

holds. If, moreover,

$$\|\ell_0(1)\|_L < 1 \tag{12.4}$$

and

$$\frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{|\lambda| - |\mu|}{|\mu|} < \|\ell_1(1)\|_L < 2\sqrt{1 - \|\ell_0(1)\|_L}\ , \tag{12.5}$$

then the problem (10.1), (10.2) *has at least one solution.*

Remark 12.1. Let $0 \neq |\mu| \leq |\lambda|$. Denote by D the set of pairs $(x, y) \in R_+ \times R_+$ such that

$$x < 1, \qquad \frac{x}{1-x} - \frac{|\lambda| - |\mu|}{|\mu|} < y < 2\sqrt{1-x}$$

(see Fig. 12.1, p. 201).

According to Theorem 12.1, if (12.1) holds, there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that the inequality (12.3) is satisfied on the set $B^1_{\lambda\mu c}\big([a,b]; R\big)$, and

$$\Big(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L\Big) \in D,$$

then the problem (10.1), (10.2) is solvable. Below we will show (see On Remark 12.1, p. 240) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin D$ there exist $F \in K_{ab}$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and $c_0 \in R$ such that (12.1) (with $h \equiv c_0$, $c = |c_0|$) and (12.3) hold,

$$x_0 = \|\ell_0(1)\|_L, \qquad y_0 = \|\ell_1(1)\|_L,$$

and the problem (10.1), (10.2) with $h \equiv c_0$ has no solution. In particular, neither one of the strict inequalities in (12.4) and (12.5) can be replaced by the nonstrict one.

The next theorem can be understood as a supplement of the previous one for the case $\mu = 0$.

Theorem 12.2. *Let $\mu = 0$, $c \in R_+$, the condition* (12.1) *be fulfilled, and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that on the set $B^1_{\lambda\mu c}\big([a,b]; R\big)$ the inequality* (12.3) *holds. If, moreover,*

$$\|\ell_0(1)\|_L < 1 \tag{12.6}$$

and

$$\|\ell_1(1)\|_L < 2\sqrt{1 - \|\ell_0(1)\|_L}\,, \tag{12.7}$$

then the problem (10.1), (10.2) *has at least one solution.*

Remark 12.2. Let $\mu = 0$. Denote by E the set of pairs $(x, y) \in R_+ \times R_+$ such that

$$x < 1, \qquad y < 2\sqrt{1-x}$$

(see Fig. 12.2).

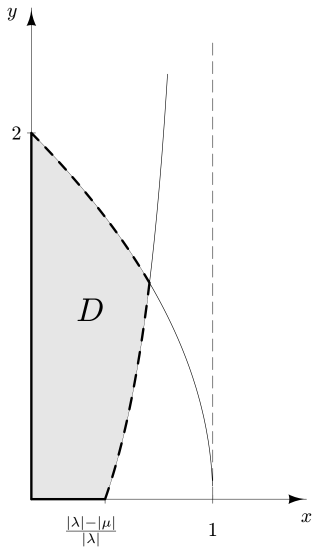

Fig. 12.1.

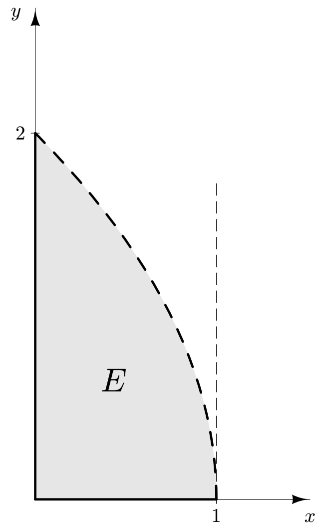

Fig. 12.2.

According to Theorem 12.2, if (12.1) holds, there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that the inequality (12.3) is satisfied on the set $B^1_{\lambda\mu c}\big([a,b];R\big)$, and

$$\Big(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L\Big) \in E,$$

then the problem (10.1), (10.2) is solvable. Below we will show (see On Remark 12.2, p. 243) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin E$ there exist $F \in K_{ab}$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and $c_0 \in R$ such that (12.1) (with $h \equiv c_0$, $c = |c_0|$) and (12.3) hold,

$$x_0 = \|\ell_0(1)\|_L, \qquad y_0 = \|\ell_1(1)\|_L,$$

and the problem (10.1), (10.2) with $h \equiv c_0$ has no solution. In particular, the strict inequalities (12.6) and (12.7) cannot be replaced by the nonstrict ones.

Theorem 12.3. *Let* $0 \neq |\mu| \leq |\lambda|$, $c \in R_+$,

$$h(v)\operatorname{sgn}\big(\mu v(b)\big) \leq c \quad \textit{for} \quad v \in C\big([a,b];R\big), \tag{12.8}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}\big([a,b];R\big)$ *the in-*

equality

$$\big[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big]\operatorname{sgn} v(t) \geq \\ \geq -q(t, \|v\|_C) \quad \textit{for} \quad t \in [a,b] \tag{12.9}$$

holds. If, moreover,

$$\|\ell_1(1)\|_L < \left|\frac{\mu}{\lambda}\right| \tag{12.10}$$

and

$$\frac{|\lambda|}{|\mu| - |\lambda|\|\ell_1(1)\|_L} - 1 < \|\ell_0(1)\|_L < 2\sqrt{\left|\frac{\mu}{\lambda}\right| - \|\ell_1(1)\|_L}\ , \tag{12.11}$$

then the problem (10.1), (10.2) *has at least one solution.*

Remark 12.3. Let $0 \neq |\mu| \leq |\lambda|$. Denote by W the set of pairs $(x, y) \in R_+ \times R_+$ such that

$$y < \left|\frac{\mu}{\lambda}\right|, \qquad \frac{|\lambda|}{|\mu| - |\lambda| y} - 1 < x < 2\sqrt{\left|\frac{\mu}{\lambda}\right| - y}$$

(see Fig. 12.3).

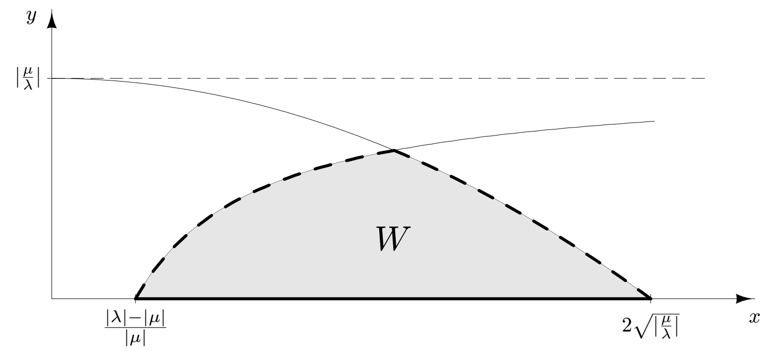

Fig. 12.3.

According to Theorem 12.3, if (12.8) holds, there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that the inequality (12.9) is satisfied on the set $B^2_{\lambda\mu c}\big([a,b]; R\big)$, and

$$\Big(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L\Big) \in W,$$

then the problem (10.1), (10.2) is solvable. Below we will show (see On Remark 12.3, p. 245) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin W$ there exist $F \in K_{ab}$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and $c_0 \in R$ such that (12.8) (with $h \equiv c_0$, $c = |c_0|$) and (12.9) hold,

$$x_0 = \|\ell_0(1)\|_L, \qquad y_0 = \|\ell_1(1)\|_L,$$

and the problem (10.1), (10.2) with $h \equiv c_0$ has no solution. In particular, neither one of the strict inequalities in (12.10) and (12.11) can be replaced by the nonstrict one.

Theorem 12.4. *Let* $|\mu| < |\lambda|$, $c \in R_+$, *the inequality* (12.1) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}\big([a,b];R\big)$ *the inequality*

$$\begin{gathered}\big[F(v)(t) + \ell_1(v)(t)\big] \operatorname{sgn} v(t) \le \ell_0(|v|)(t) + \\ + q(t, \|v\|_C) \quad \text{for} \quad t \in [a,b]\end{gathered} \tag{12.12}$$

holds. If, moreover,

$$\ell_0 \in V^+_{ab}(\lambda,\mu), \qquad -\ell_1 \in V^+_{ab}(\lambda,\mu), \tag{12.13}$$

then the problem (10.1), (10.2) *has at least one solution.*

Remark 12.4. Theorem 12.4 is nonimprovable. More precisely, the inequality (12.12) cannot be replaced by the inequality

$$\big[F(v)(t) + \ell_1(v)(t)\big] \operatorname{sgn} v(t) \le (1+\varepsilon)\ell_0(|v|)(t) + q(t, \|v\|_C), \tag{12.14}$$

no matter how small $\varepsilon > 0$ would be. Moreover, the assumption (12.13) can be replaced neither by the assumption

$$(1-\varepsilon)\ell_0 \in V^+_{ab}(\lambda,\mu), \qquad -\ell_1 \in V^+_{ab}(\lambda,\mu), \tag{12.15}$$

nor by the assumption

$$\ell_0 \in V^+_{ab}(\lambda,\mu), \qquad -(1-\varepsilon)\ell_1 \in V^+_{ab}(\lambda,\mu), \tag{12.16}$$

no matter how small $\varepsilon > 0$ would be (see On Remark 12.4 and Example 12.1, p. 247).

Remark 12.5. By the last theorem and the results from §2, one can obtain several effective sufficient solvability conditions for the problem (10.1), (10.2).

Theorem 12.5. *Let* $|\mu| < |\lambda|$, $c \in R_+$, *the inequality* (12.1) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}([a,b];R)$ *the inequality* (12.3) *holds. If, moreover, there exists* $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *satisfying*

$$\gamma'(t) \geq \ell_0(\gamma)(t) + \ell_1(1)(t) \quad \textit{for} \quad t \in [a,b], \tag{12.17}$$

$$|\lambda|\gamma(a) > |\mu|\gamma(b), \tag{12.18}$$

$$\gamma(b) - \gamma(a) < 2, \tag{12.19}$$

then the problem (10.1), (10.2) *has at least one solution.*

Remark 12.6. Theorem 12.5 is nonimprovable in the sense that the inequality (12.19) cannot be replaced by the nonstrict one (see Example 12.2, p. 249).

In the next theorem if $|\mu| = |\lambda|$, then the operator $\ell_0 \in \mathcal{P}_{ab}$ is supposed to be nontrivial.

Theorem 12.6. *Let* $0 \neq |\mu| \leq |\lambda|$, $c \in R_+$, *the inequality* (12.8) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}([a,b];R)$ *the inequality*

$$\begin{gathered}\big[F(v)(t) - \ell_0(v)(t) - \ell_1(v)(t)\big] \operatorname{sgn} v(t) \geq \\ \geq -q(t, \|v\|_C) \quad \textit{for} \quad t \in [a,b]\end{gathered} \tag{12.20}$$

holds. If, moreover,

$$\ell_0 \in V^-_{ab}(\lambda,\mu) \tag{12.21}$$

and

$$\int_a^b \big(\ell_0(1)(s) + \ell_1(1)(s)\big)ds < 2\sqrt{\left|\frac{\mu}{\lambda}\right|}, \tag{12.22}$$

then the problem (10.1), (10.2) *has at least one solution.*

Remark 12.7. Theorem 12.6 is nonimprovable in the sense that, in general, the strict inequality (12.22) cannot be replaced by the nonstrict one (see Example 12.3, p. 251).

Note also, that if $|\lambda| = |\mu|$ and the conditions (12.20) and (12.22) are fulfilled for some $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, then, without loss of generality, the operator ℓ_0 can be chosen such that the condition (12.21) is satisfied. Indeed, in this case the operator ℓ_0 is supposed to be nontrivial and thus, it can be chosen such that

$$0 < \int_a^b \ell_0(1)(s)ds \le 1,$$

which guarantees that the condition (12.21) is fulfilled (see Theorem 2.11 with $\ell_1 \equiv 0$, p. 26).

Nevertheless, if $0 \ne |\mu| < |\lambda|$, then, in general, the assumption (12.21) cannot be replaced by the assumption

$$(1+\varepsilon)\ell_0 \in V_{ab}^-(\lambda,\mu)\,, \tag{12.23}$$

no matter how small $\varepsilon > 0$ would be (see On Remark 12.7, p. 253).

In Theorems 12.7–12.12, the conditions guaranteeing the unique solvability of the problem (10.1), (10.2) are established.

Theorem 12.7. *Let* $0 \ne |\mu| \le |\lambda|$,

$$\big[h(v)-h(w)\big]\operatorname{sgn}\big(\lambda\big(v(a)-w(a)\big)\big) \le 0 \quad for \quad v,w \in C\big([a,b];R\big), \tag{12.24}$$

and let there exist $\ell_0,\ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}\big([a,b];R\big)$, *where* $c = |h(0)|$, *the inequality*

$$\begin{gathered}\big[F(v)(t)-F(w)(t)-\ell_0(v-w)(t)+\\ +\ell_1(v-w)(t)\big]\operatorname{sgn}\big(v(t)-w(t)\big) \le 0 \quad for \quad t\in[a,b]\end{gathered} \tag{12.25}$$

holds. If, moreover, the conditions (12.4) *and* (12.5) *are fulfilled, then the problem* (10.1), (10.2) *is uniquely solvable.*

Remark 12.8. The examples constructed in Subsection 12.3 (see On Remark 12.1, p. 240) also show that neither one of the strict inequalities in (12.4) and (12.5) can be replaced by the nonstrict one.

The next theorem can be understood as a supplement of the previous one for the case $\mu = 0$.

Theorem 12.8. *Let* $\mu = 0$, *the condition* (12.24) *be fulfilled, and let there exist* $\ell_0,\ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}\big([a,b];R\big)$, *where* $c = |h(0)|$, *the inequality* (12.25) *holds. If, moreover, the conditions* (12.6) *and* (12.7) *are fulfilled, then the problem* (10.1), (10.2) *is uniquely solvable.*

Remark 12.9. The examples constructed in Subsection 12.3 (see On Remark 12.2, p. 243) also show that the strict inequalities (12.6) and (12.7) cannot be replaced by the nonstrict ones.

Theorem 12.9. *Let* $0 \neq |\mu| \leq |\lambda|$,

$$[h(v) - h(w)] \operatorname{sgn}\big(\mu\big(v(b) - w(b)\big)\big) \leq 0 \quad \textit{for} \quad v, w \in C\big([a,b]; R\big), \tag{12.26}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}\big([a,b]; R\big)$, *where* $c = |h(0)|$, *the inequality*

$$\begin{aligned} \big[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \\ + \ell_1(v-w)(t)\big] \operatorname{sgn}\big(v(t) - w(t)\big) \geq 0 \quad \textit{for} \quad t \in [a,b] \end{aligned} \tag{12.27}$$

holds. If, moreover, the conditions (12.10) *and* (12.11) *are fulfilled, then the problem* (10.1), (10.2) *is uniquely solvable.*

Remark 12.10. The examples constructed in Subsection 12.3 (see On Remark 12.3, p. 245) also show that neither one of the strict inequalities in (12.10) and (12.11) can be replaced by the nonstrict one.

Theorem 12.10. *Let* $|\mu| < |\lambda|$, *the condition* (12.24) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}\big([a,b]; R\big)$, *where* $c = |h(0)|$, *the inequality*

$$\begin{aligned} \big[F(v)(t) - F(w)(t) + \ell_1(v-w)(t)\big] \operatorname{sgn}\big(v(t) - w(t)\big) \leq \\ \leq \ell_0(|v-w|)(t) \quad \textit{for} \quad t \in [a,b] \end{aligned} \tag{12.28}$$

holds. If, moreover, the condition (12.13) *is satisfied, then the problem* (10.1), (10.2) *has a unique solution.*

Remark 12.11. The examples constructed in Subsection 12.3 (see On Remark 12.4 and Example 12.1, p. 247) also show that the the inequality (12.28) cannot be replaced by the inequality

$$\begin{aligned} \big[F(v)(t) - F(w)(t) + \ell_1(v-w)(t)\big] \operatorname{sgn}\big(v(t) - w(t)\big) \leq \\ \leq (1+\varepsilon)\ell_0(|v-w|)(t) \quad \text{for} \quad t \in [a,b] \end{aligned}$$

and the assumption (12.13) can be replaced neither by the assumption (12.15) nor by the assumption (12.16), no matter how small $\varepsilon > 0$ would be.

Theorem 12.11. *Let* $|\mu| < |\lambda|$, *the inequality* (12.24) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}([a,b];R)$, *where* $c = |h(0)|$, *the inequality* (12.25) *holds. If, moreover, there exists a function* $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *satisfying the inequalities* (12.17)–(12.19), *then the problem* (10.1), (10.2) *is uniquely solvable.*

Remark 12.12. The examples constructed in Subsection 12.3 (see Example 12.2, p. 249) also show that the strict inequality (12.19) cannot be replaced by the nonstrict one.

In the next theorem if $|\mu| = |\lambda|$, then the operator $\ell_0 \in \mathcal{P}_{ab}$ is supposed to be nontrivial.

Theorem 12.12. *Let* $0 \neq |\mu| \leq |\lambda|$, *the inequality* (12.26) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}([a,b];R)$, *where* $c = |h(0)|$, *the inequality*

$$\begin{gathered}\big[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) - \\ -\ell_1(v-w)(t)\big]\operatorname{sgn}\big(v(t) - w(t)\big) \geq 0 \quad \text{for} \quad t \in [a,b]\end{gathered} \tag{12.29}$$

holds. If, moreover, the conditions (12.21) *and* (12.22) *are satisfied, then the problem* (10.1), (10.2) *has a unique solution.*

Remark 12.13. The examples constructed in Subsection 12.3 (see Example 12.3 and On Remark 12.7, pp. 251 and 253) also show that, in general, the assumption (12.21) in Theorem 12.12 cannot be replaced by the assumption (12.23), no matter how small $\varepsilon > 0$ would be, and the strict inequality (12.22) cannot be replaced by the nonstrict one.

In the case, where $|\mu| \geq |\lambda|$, the following assertions hold.

Theorem 12.13. *Let* $|\mu| \geq |\lambda| \neq 0$, $c \in R_+$, *the inequality* (12.8) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}([a,b];R)$ *the inequality* (12.9) *holds. If, moreover,*

$$\|\ell_1(1)\|_L < 1 \tag{12.30}$$

and

$$\frac{\|\ell_1(1)\|_L}{1 - \|\ell_1(1)\|_L} - \frac{|\mu| - |\lambda|}{|\lambda|} < \|\ell_0(1)\|_L < 2\sqrt{1 - \|\ell_1(1)\|_L}\,, \tag{12.31}$$

then the problem (10.1), (10.2) *has at least one solution.*

The next theorem can be understood as a supplement of the previous one for the case $\lambda = 0$.

Theorem 12.14. *Let* $\lambda = 0$, $c \in R_+$, *the inequality* (12.8) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}\big([a,b];R\big)$ *the inequality* (12.9) *holds. If, moreover,*

$$\|\ell_1(1)\|_L < 1 \tag{12.32}$$

and

$$\|\ell_0(1)\|_L < 2\sqrt{1 - \|\ell_1(1)\|_L}\ , \tag{12.33}$$

then the problem (10.1), (10.2) *has at least one solution.*

Theorem 12.15. *Let* $|\mu| \geq |\lambda| \neq 0$, $c \in R_+$, *the inequality* (12.1) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}\big([a,b];R\big)$ *the inequality* (12.3) *holds. If, moreover,*

$$\|\ell_0(1)\|_L < \left|\frac{\lambda}{\mu}\right| \tag{12.34}$$

and

$$\frac{|\mu|}{|\lambda| - |\mu|\|\ell_0(1)\|_L} - 1 < \|\ell_1(1)\|_L < 2\sqrt{\left|\frac{\lambda}{\mu}\right| - \|\ell_0(1)\|_L}\ , \tag{12.35}$$

then the problem (10.1), (10.2) *has at least one solution.*

Theorem 12.16. *Let* $|\mu| > |\lambda|$, $c \in R_+$, *the inequality* (12.8) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}\big([a,b];R\big)$ *the inequality*

$$\big[F(v)(t) - \ell_1(v)(t)\big]\operatorname{sgn} v(t) \geq -\ell_0(|v|)(t) -$$
$$-q(t, \|v\|_C) \quad \textit{for} \quad t \in [a,b]$$

holds. If, moreover,

$$-\ell_0 \in V^-_{ab}(\lambda,\mu), \qquad \ell_1 \in V^-_{ab}(\lambda,\mu), \tag{12.36}$$

then the problem (10.1), (10.2) *has at least one solution.*

Theorem 12.17. *Let* $|\mu| > |\lambda|$, $c \in R_+$, *the inequality* (12.8) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}\big([a,b];R\big)$ *the inequality* (12.9) *holds. If, moreover, there exists* $\gamma \in \widetilde{C}\big([a,b];]0,+\infty[\big)$ *satisfying*

$$-\gamma'(t) \geq \ell_1(\gamma)(t) + \ell_0(1)(t) \quad \text{for} \quad t \in [a,b], \tag{12.37}$$

$$|\lambda|\gamma(a) < |\mu|\gamma(b), \tag{12.38}$$

$$\gamma(a) - \gamma(b) < 2, \tag{12.39}$$

then the problem (10.1), (10.2) *has at least one solution.*

In the next theorem if $|\mu| = |\lambda|$, then the operator $\ell_0 \in \mathcal{P}_{ab}$ is supposed to be nontrivial.

Theorem 12.18. *Let* $|\mu| \geq |\lambda| \neq 0$, $c \in R_+$, *the inequality* (12.1) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}\big([a,b];R\big)$ *the inequality*

$$\begin{gathered}\big[F(v)(t) + \ell_0(v)(t) + \ell_1(v)(t)\big]\operatorname{sgn} v(t) \leq \\ \leq q(t, \|v\|_C) \quad \text{for} \quad t \in [a,b]\end{gathered} \tag{12.40}$$

holds. If, moreover,

$$-\ell_0 \in V^+_{ab}(\lambda,\mu) \tag{12.41}$$

and

$$\int_a^b \big(\ell_0(1)(s) + \ell_1(1)(s)\big)ds < 2\sqrt{\left|\frac{\lambda}{\mu}\right|}, \tag{12.42}$$

then the problem (10.1), (10.2) *has at least one solution.*

In Theorems 12.19–12.24, the conditions guaranteeing the unique solvability of the problem (10.1), (10.2) are established.

Theorem 12.19. *Let* $|\mu| \geq |\lambda| \neq 0$, *the condition* (12.26) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}\big([a,b];R\big)$, *where* $c = |h(0)|$, *the inequality* (12.27) *holds. If, moreover, the conditions* (12.30) *and* (12.31) *are satisfied, then the problem* (10.1), (10.2) *is uniquely solvable.*

The next theorem can be understood as a supplement of the previous one for the case $\lambda = 0$.

Theorem 12.20. *Let $\lambda = 0$, the condition* (12.26) *be fulfilled, and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that on the set $B^2_{\lambda\mu c}\big([a,b];R\big)$, where $c = |h(0)|$, the inequality* (12.27) *holds. If, moreover, the conditions* (12.32) *and* (12.33) *are satisfied, then the problem* (10.1), (10.2) *is uniquely solvable.*

Theorem 12.21. *Let $|\mu| \geq |\lambda| \neq 0$, the condition* (12.24) *be fulfilled, and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that on the set $B^1_{\lambda\mu c}\big([a,b];R\big)$, where $c = |h(0)|$, the inequality* (12.25) *holds. If, moreover, the conditions* (12.34) *and* (12.35) *are satisfied, then the problem* (10.1), (10.2) *is uniquely solvable.*

Theorem 12.22. *Let $|\mu| > |\lambda|$, the inequality* (12.26) *be fulfilled, and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that on the set $B^2_{\lambda\mu c}\big([a,b];R\big)$, where $c = |h(0)|$, the inequality*

$$\big[F(v)(t) - F(w)(t) - \ell_1(v-w)(t)\big]\operatorname{sgn}\big(v(t)-w(t)\big) \geq$$
$$\geq -\ell_0(|v-w|)(t) \quad \text{for} \quad t \in [a,b]$$

holds. If, moreover, the condition (12.36) *is satisfied, then the problem* (10.1), (10.2) *has a unique solution.*

Theorem 12.23. *Let $|\mu| > |\lambda|$, the inequality* (12.26) *be fulfilled, and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that on the set $B^2_{\lambda\mu c}\big([a,b];R\big)$, where $c = |h(0)|$, the inequality* (12.27) *holds. If, moreover, there exists a function $\gamma \in \widetilde{C}\big([a,b];]0,+\infty[\big)$ satisfying* (12.37)–(12.39), *then the problem* (10.1), (10.2) *is uniquely solvable.*

In the next theorem if $|\mu| = |\lambda|$, then the operator $\ell_0 \in \mathcal{P}_{ab}$ is supposed to be nontrivial.

Theorem 12.24. *Let $|\mu| \geq |\lambda| \neq 0$, the inequality* (12.24) *be fulfilled, and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that on the set $B^1_{\lambda\mu c}\big([a,b];R\big)$, where $c = |h(0)|$, the inequality*

$$\begin{aligned}\big[F(v)(t) - F(w)(t) + \ell_0(v-w)(t) +\\ +\ell_1(v-w)(t)\big]\operatorname{sgn}\big(v(t)-w(t)\big) \leq 0 \quad \text{for} \quad t \in [a,b]\end{aligned} \tag{12.43}$$

holds. If, moreover, the conditions (12.41) *and* (12.42) *are satisfied, then the problem* (10.1), (10.2) *has a unique solution.*

Remark 12.14. Let ψ and φ be operators defined in Remark 2.16 (see p. 28). Put

$$\widehat{F}(w)(t) \overset{\text{def}}{=} -\psi(F(\varphi(w)))(t) \quad \text{for} \quad t \in [a,b], \qquad \widehat{h}(w) \overset{\text{def}}{=} h(\varphi(w)).$$

It is clear that if u is a solution of the problem (10.1), (10.2), then the function $v \overset{\text{def}}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{F}(v)(t), \qquad \mu v(a) + \lambda v(b) = \widehat{h}(v), \tag{12.44}$$

and vice versa, if v is a solution of the problem (12.44), then the function $u \overset{\text{def}}{=} \varphi(v)$ is a solution of the problem (10.1), (10.2).

Remark 12.15. According to Remark 12.14 and Remark 2.16 (see p. 28), Theorems 12.13–12.24 can be immediately derived from Theorems 12.1–12.12. Moreover, by virtue of Remarks 12.1–12.4 and 12.6–12.13, Theorems 12.13–12.24 are nonimprovable in an appropriate sense.

12.2. Proofs

First we will prove several lemmas.

Lemma 12.1. *Let $0 \neq |\mu| \leq |\lambda|$ and let the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. If, moreover, the conditions* (12.4) *and* (12.5) *hold, then $\ell \in \mathcal{A}^1\big(\lambda,\mu\big)$.*

Proof. Let $q^* \in L\big([a,b];R_+\big)$, $c \in R_+$, and $u \in \widetilde{C}\big([a,b];R\big)$ satisfy (11.13) and (11.14) for $i = 1$. Put

$$\mu_0 = \max\{1, |\mu|\}, \qquad \lambda_0 = \max\left\{1, \frac{1}{|\lambda|}\right\}. \tag{12.45}$$

We will show that (11.15) holds with

$$r = \frac{\mu_0 + \lambda_0(|\mu|\|\ell_1(1)\|_L + |\lambda| - |\mu|)}{(1 - \|\ell_0(1)\|_L)(|\mu|\|\ell_1(1)\|_L + |\lambda| - |\mu|) - |\mu|\|\ell_0(1)\|_L} + \\ + \frac{\lambda_0(\|\ell_1(1)\|_L + 1)}{1 - \|\ell_0(1)\|_L - \frac{1}{4}\|\ell_1(1)\|_L^2}\,. \tag{12.46}$$

It is clear that

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \widetilde{q}(t) \quad \text{for} \quad t \in [a,b], \tag{12.47}$$

where
$$\widetilde{q}(t) = u'(t) - \ell(u)(t) \quad \text{for} \quad t \in [a, b]. \tag{12.48}$$
Obviously,
$$\widetilde{q}(t) \operatorname{sgn} u(t) \le q^*(t) \quad \text{for} \quad t \in [a, b], \tag{12.49}$$
and
$$[\lambda u(a) + \mu u(b)] \operatorname{sgn} \big(\lambda u(a)\big) \le c. \tag{12.50}$$

First suppose that u does not change its sign. According to (2.1), (12.50), and the assumption $\left|\frac{\mu}{\lambda}\right| \in \,]0, 1]$, we obtain
$$|u(a)| - |u(b)| \le \frac{c}{|\lambda|} \tag{12.51}$$
and
$$|u(a)| - |u(b)| \le |u(a)| \, \frac{|\mu| - |\lambda|}{|\mu|} + \frac{c}{|\mu|} \, . \tag{12.52}$$
Put
$$\overline{M} = \max\{|u(t)| : t \in [a, b]\}, \qquad \overline{m} = \min\{|u(t)| : t \in [a, b]\} \tag{12.53}$$
and choose $t_1, t_2 \in [a, b]$ such that $t_1 \ne t_2$ and
$$|u(t_1)| = \overline{M}, \qquad |u(t_2)| = \overline{m}. \tag{12.54}$$
Obviously, $\overline{M} \ge 0$, $\overline{m} \ge 0$, and either
$$t_1 < t_2 \tag{12.55}$$
or
$$t_1 > t_2. \tag{12.56}$$
Due to (12.2), (12.49), and (12.53), (12.47) implies
$$|u(t)|' \le \overline{M} \ell_0(1)(t) - \overline{m} \, \ell_1(1)(t) + q^*(t) \quad \text{for} \quad t \in [a, b]. \tag{12.57}$$
If (12.55) holds, then the integration of (12.57) from a to t_1 and from t_2 to b, in view of (12.2), $\overline{m} \ge 0$, and (12.54), results in
$$\overline{M} - |u(a)| \le \overline{M} \int\limits_a^{t_1} \ell_0(1)(s) ds + \int\limits_a^{t_1} q^*(s) ds,$$
$$|u(b)| - \overline{m} \le \overline{M} \int\limits_{t_2}^{b} \ell_0(1)(s) ds + \int\limits_{t_2}^{b} q^*(s) ds.$$

Summing the last two inequalities and taking into account (12.2) and (12.51), we obtain

$$\overline{M} - \overline{m} - \frac{c}{|\lambda|} \leq \overline{M} - \overline{m} + |u(b)| - |u(a)| \leq \overline{M}\|\ell_0(1)\|_L + \|q^*\|_L.$$

If (12.56) is fulfilled, then the integration of (12.57) from t_2 to t_1, on account of (12.2), $\overline{m} \geq 0$, and (12.54), yields

$$\overline{M} - \overline{m} - \frac{c}{|\lambda|} \leq \overline{M} - \overline{m} \leq \overline{M} \int_{t_2}^{t_1} \ell_0(1)(s)ds + \int_{t_2}^{t_1} q^*(s)ds \leq \overline{M}\|\ell_0(1)\|_L + \|q^*\|_L.$$

Therefore, in both cases (12.55) and (12.56), the inequality

$$\overline{M} - \overline{m} - \frac{c}{|\lambda|} \leq \overline{M}\|\ell_0(1)\|_L + \|q^*\|_L \tag{12.58}$$

holds.

On the other hand, the integration of (12.57) from a to b yields

$$|u(b)| - |u(a)| \leq \overline{M}\|\ell_0(1)\|_L - \overline{m}\|\ell_1(1)\|_L + \|q^*\|_L.$$

Hence, by (12.52), (12.53), and the assumption $|\mu| \leq |\lambda|$, we get

$$\begin{aligned} \overline{m}\|\ell_1(1)\|_L &\leq \overline{M}\|\ell_0(1)\|_L + |u(a)|\, \frac{|\mu| - |\lambda|}{|\mu|} + \frac{c}{|\mu|} + \|q^*\|_L \leq \\ &\leq \overline{M}\|\ell_0(1)\|_L + \overline{m}\, \frac{|\mu| - |\lambda|}{|\mu|} + \|q^*\|_L + \frac{c}{|\mu|}. \end{aligned}$$

From the last inequality and (12.58), in view of (12.4) and the assumption $\left|\frac{\mu}{\lambda}\right| \in\,]0, 1]$, it follows that

$$0 \leq \overline{m}\left(\|\ell_1(1)\|_L + \frac{|\lambda| - |\mu|}{|\mu|}\right) \leq \overline{M}\|\ell_0(1)\|_L + \|q^*\|_L + \frac{c}{|\mu|},$$

$$0 \leq \overline{M}(1 - \|\ell_0(1)\|_L) \leq \overline{m} + \|q^*\|_L + \frac{c}{|\lambda|}.$$

Thus, on account of (12.5) and (12.45),

$$\|u\|_C = \overline{M} \leq r_0\big(\mu_0 + \lambda_0(|\mu|\|\ell_1(1)\|_L + |\lambda| - |\mu|)\big)(c + \|q^*\|_L),$$

where $r_0 = \big[(1 - \|\ell_0(1)\|_L)(|\mu|\|\ell_1(1)\|_L + |\lambda| - |\mu|) - |\mu|\|\ell_0(1)\|_L\big]^{-1}$. Therefore, the estimate (11.15) holds.

Now suppose that u changes its sign. Put

$$M = \max\{u(t) : t \in [a,b]\}, \quad m = -\min\{u(t) : t \in [a,b]\}, \tag{12.59}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \quad u(t_m) = -m. \tag{12.60}$$

Obviously, $M > 0$, $m > 0$, and either

$$t_m < t_M \tag{12.61}$$

or

$$t_m > t_M. \tag{12.62}$$

First suppose that (12.61) is fulfilled. It is clear that there exists $\alpha_2 \in]t_m, t_M[$ such that

$$u(t) > 0 \quad \text{for} \quad \alpha_2 < t \leq t_M, \qquad u(\alpha_2) = 0. \tag{12.63}$$

Let

$$\alpha_1 = \inf\{t \in [a, t_m] : u(s) < 0 \text{ for } t \leq s \leq t_m\}. \tag{12.64}$$

Obviously,

$$u(t) < 0 \quad \text{for} \quad \alpha_1 < t \leq t_m \tag{12.65}$$

and

$$\text{if} \quad \alpha_1 > a, \quad \text{then} \quad u(\alpha_1) = 0. \tag{12.66}$$

Put

$$\alpha_3 = \begin{cases} b & \text{if } \ u(b) \geq 0 \\ \inf\{t \in]t_M, b] : u(s) < 0 \text{ for } t \leq s \leq b\} & \text{if } \ u(b) < 0 \end{cases}. \tag{12.67}$$

It is clear that

$$\text{if} \quad \alpha_3 < b, \quad \text{then} \quad u(t) < 0 \quad \text{for} \quad \alpha_3 < t \leq b, \quad u(\alpha_3) = 0. \tag{12.68}$$

The integration of (12.47) from α_1 to t_m, from α_2 to t_M, and from α_3 to b, in view of (12.2), (12.49), (12.59), (12.60), (12.63), (12.65), and (12.68), yields

$$u(\alpha_1) + m \leq M \int\limits_{\alpha_1}^{t_m} \ell_1(1)(s)ds + m \int\limits_{\alpha_1}^{t_m} \ell_0(1)(s)ds + \int\limits_{\alpha_1}^{t_m} q^*(s)ds, \quad (12.69)$$

$$M \leq M \int\limits_{\alpha_2}^{t_M} \ell_0(1)(s)ds + m \int\limits_{\alpha_2}^{t_M} \ell_1(1)(s)ds + \int\limits_{\alpha_2}^{t_M} q^*(s)ds, \quad (12.70)$$

$$u(\alpha_3) - u(b) \leq M \int\limits_{\alpha_3}^{b} \ell_1(1)(s)ds + m \int\limits_{\alpha_3}^{b} \ell_0(1)(s)ds + \int\limits_{\alpha_3}^{b} q^*(s)ds. \quad (12.71)$$

Evidently, either

$$u(b) \geq 0 \quad (12.72)$$

or

$$u(b) < 0. \quad (12.73)$$

If (12.72) holds, then, according to (2.1), (12.50), (12.64), and (12.66), we obtain $u(\alpha_1) \geq -\frac{c}{|\lambda|}$. Thus, it follows from (12.69) that

$$-\frac{c}{|\lambda|} + m \leq M \int\limits_{I} \ell_1(1)(s)ds + m \int\limits_{I} \ell_0(1)(s)ds + \int\limits_{I} q^*(s)ds, \quad (12.74)$$

where $I = [\alpha_1, t_m]$.

Now let (12.73) hold. According to (2.1) and (12.50), it is clear that

$$u(a) - \left|\frac{\mu}{\lambda}\right| u(b) \geq -\frac{1}{|\lambda|}\left[\lambda u(a) + \mu u(b)\right] \operatorname{sgn}\big(\lambda u(a)\big) \geq -\frac{c}{|\lambda|}\,.$$

By virtue of (12.64) and (12.66), we find

$$u(\alpha_1) - \left|\frac{\mu}{\lambda}\right| u(b) \geq -\frac{c}{|\lambda|}\,. \quad (12.75)$$

Multiplying both sides of (12.71) by $\left|\frac{\mu}{\lambda}\right|$ and taking into account (12.67), (12.68), and the assumption $\left|\frac{\mu}{\lambda}\right| \in\,]0,1]$, we get

$$-\left|\frac{\mu}{\lambda}\right| u(b) \leq M \int\limits_{\alpha_3}^{b} \ell_1(1)(s)ds + m \int\limits_{\alpha_3}^{b} \ell_0(1)(s)ds + \int\limits_{\alpha_3}^{b} q^*(s)ds.$$

Summing the last inequality and (12.69), by (12.75) we obtain that the inequality (12.74) holds, where $I = [\alpha_1, t_m] \cup [\alpha_3, b]$.

Thus, in both cases (12.72) and (12.73), the inequality (12.74) is fulfilled, where $I = [\alpha_1, t_m] \cup [\alpha_3, b]$.

It follows from (12.70) and (12.74) that

$$\begin{aligned} M(1 - C_1) &\le mA_1 + \|q^*\|_L + \frac{c}{|\lambda|}, \\ m(1 - D_1) &\le MB_1 + \|q^*\|_L + \frac{c}{|\lambda|}, \end{aligned} \tag{12.76}$$

where

$$A_1 = \int_{\alpha_2}^{t_M} \ell_1(1)(s)ds, \qquad B_1 = \int_I \ell_1(1)(s)ds,$$
$$C_1 = \int_{\alpha_2}^{t_M} \ell_0(1)(s)ds, \qquad D_1 = \int_I \ell_0(1)(s)ds.$$

Due to (12.4), $C_1 < 1$, $D_1 < 1$. Consequently, (12.45) and (12.76) imply

$$\begin{aligned} 0 &< M(1 - C_1)(1 - D_1) \le A_1 \left(MB_1 + \|q^*\|_L + \frac{c}{|\lambda|} \right) + \\ &+ \|q^*\|_L + \frac{c}{|\lambda|} \le MA_1B_1 + \lambda_0 \left(\|q^*\|_L + c \right) \left(\|\ell_1(1)\|_L + 1 \right), \\ 0 &< m(1 - C_1)(1 - D_1) \le B_1 \left(mA_1 + \|q^*\|_L + \frac{c}{|\lambda|} \right) + \\ &+ \|q^*\|_L + \frac{c}{|\lambda|} \le mA_1B_1 + \lambda_0 \left(\|q^*\|_L + c \right) \left(\|\ell_1(1)\|_L + 1 \right). \end{aligned} \tag{12.77}$$

Obviously,

$$(1 - C_1)(1 - D_1) \ge 1 - (C_1 + D_1) \ge 1 - \|\ell_0(1)\|_L > 0,$$
$$4A_1B_1 \le (A_1 + B_1)^2 \le \|\ell_1(1)\|_L^2.$$

By the last inequalities and (12.5), from (12.77) we get

$$\begin{aligned} M &\le r_1\lambda_0(\|\ell_1(1)\|_L + 1)(c + \|q^*\|_L), \\ m &\le r_1\lambda_0(\|\ell_1(1)\|_L + 1)(c + \|q^*\|_L), \end{aligned} \tag{12.78}$$

where

$$r_1 = \left(1 - \|\ell_0(1)\|_L - \frac{1}{4}\|\ell_1(1)\|_L^2\right)^{-1}. \tag{12.79}$$

The inequalities (12.78), on account of (12.46), (12.59), and (12.79), imply that the estimate (11.15) holds.

Now suppose that (12.62) is fulfilled. It is clear that there exists $\alpha_5 \in]t_M, t_m[$ such that

$$u(t) < 0 \quad \text{for} \quad \alpha_5 < t \le t_m, \qquad u(\alpha_5) = 0. \tag{12.80}$$

Let

$$\alpha_4 = \inf\{t \in [a, t_M] : u(s) > 0 \text{ for } t \le s \le t_M\}. \tag{12.81}$$

Obviously,

$$u(t) > 0 \quad \text{for} \quad \alpha_4 < t \le t_M, \tag{12.82}$$

and

$$\text{if} \quad \alpha_4 > a, \quad \text{then} \quad u(\alpha_4) = 0. \tag{12.83}$$

Put

$$\alpha_6 = \begin{cases} b & \text{if } \ u(b) \le 0 \\ \inf\{t \in]t_m, b] : u(s) > 0 \text{ for } t \le s \le b\} & \text{if } \ u(b) > 0 \end{cases}. \tag{12.84}$$

It is clear that

$$\text{if} \quad \alpha_6 < b, \quad \text{then} \quad u(t) > 0 \quad \text{for} \quad \alpha_6 < t \le b, \qquad u(\alpha_6) = 0. \tag{12.85}$$

The integration of (12.47) from α_4 to t_M, from α_5 to t_m, and from α_6 to b, in view of (12.2), (12.49), (12.59), (12.60), (12.80), (12.82), and (12.85), results in

$$M - u(\alpha_4) \le M \int_{\alpha_4}^{t_M} \ell_0(1)(s)ds + m \int_{\alpha_4}^{t_M} \ell_1(1)(s)ds + \int_{\alpha_4}^{t_M} q^*(s)ds, \tag{12.86}$$

$$m \le M \int_{\alpha_5}^{t_m} \ell_1(1)(s)ds + m \int_{\alpha_5}^{t_m} \ell_0(1)(s)ds + \int_{\alpha_5}^{t_m} q^*(s)ds, \tag{12.87}$$

$$u(b) - u(\alpha_6) \le M \int_{\alpha_6}^{b} \ell_0(1)(s)ds + m \int_{\alpha_6}^{b} \ell_1(1)(s)ds + \int_{\alpha_6}^{b} q^*(s)ds. \tag{12.88}$$

Evidently, either

$$u(b) \le 0 \tag{12.89}$$

or

$$u(b) > 0. \tag{12.90}$$

If (12.89) holds, then, according to (2.1), (12.50), (12.81), and (12.83), we obtain $u(\alpha_4) \le \frac{c}{|\lambda|}$. Thus, it follows from (12.86) that

$$-\frac{c}{|\lambda|} + M \le M \int\limits_J \ell_0(1)(s)ds + m \int\limits_J \ell_1(1)(s)ds + \int\limits_J q^*(s)ds, \tag{12.91}$$

where $J = [\alpha_4, t_M]$.

Now let (12.90) hold. According to (2.1) and (12.50), it is clear that

$$\left|\frac{\mu}{\lambda}\right| u(b) - u(a) \ge -\frac{1}{|\lambda|}\big[\lambda u(a) + \mu u(b)\big] \operatorname{sgn}\big(\lambda u(a)\big) \ge -\frac{c}{|\lambda|}\,.$$

By virtue of (12.81) and (12.83), we find

$$\left|\frac{\mu}{\lambda}\right| u(b) - u(\alpha_4) \ge -\frac{c}{|\lambda|}\,. \tag{12.92}$$

Multiplying both sides of (12.88) by $\left|\frac{\mu}{\lambda}\right|$ and taking into account (12.84), (12.85) and the assumption $\left|\frac{\mu}{\lambda}\right| \in \,]0, 1]$, we get

$$\left|\frac{\mu}{\lambda}\right| u(b) \le M \int\limits_{\alpha_6}^{b} \ell_0(1)(s)ds + m \int\limits_{\alpha_6}^{b} \ell_1(1)(s)ds + \int\limits_{\alpha_6}^{b} q^*(s)ds.$$

Summing the last inequality and (12.86), by (12.92), we obtain that the inequality (12.91) holds, where $J = [\alpha_4, t_M] \cup [\alpha_6, b]$.

Thus, in both cases (12.89) and (12.90), the inequality (12.91) is fulfilled, where $J = [\alpha_4, t_M] \cup [\alpha_6, b]$.

It follows from (12.87) and (12.91) that

$$\begin{aligned} m(1 - C_2) &\le MA_2 + \|q^*\|_L + \frac{c}{|\lambda|}, \\ M(1 - D_2) &\le mB_2 + \|q^*\|_L + \frac{c}{|\lambda|}, \end{aligned} \tag{12.93}$$

where

$$A_2 = \int_{\alpha_5}^{t_m} \ell_1(1)(s)ds, \qquad B_2 = \int_J \ell_1(1)(s)ds,$$

$$C_2 = \int_{\alpha_5}^{t_m} \ell_0(1)(s)ds, \qquad D_2 = \int_J \ell_0(1)(s)ds.$$

Due to (12.4), $C_2 < 1$, $D_2 < 1$. Consequently, (12.45) and (12.93) imply

$$\begin{aligned} 0 < m(1-C_2)(1-D_2) &\le A_2\left(mB_2 + \|q^*\|_L + \frac{c}{|\lambda|}\right) + \\ + \|q^*\|_L + \frac{c}{|\lambda|} &\le mA_2B_2 + \lambda_0(\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1), \\ 0 < M(1-C_2)(1-D_2) &\le B_2\left(MA_2 + \|q^*\|_L + \frac{c}{|\lambda|}\right) + \\ + \|q^*\|_L + \frac{c}{|\lambda|} &\le MA_2B_2 + \lambda_0(\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1). \end{aligned} \tag{12.94}$$

Obviously,

$$(1-C_2)(1-D_2) \ge 1 - (C_2 + D_2) \ge 1 - \|\ell_0(1)\|_L > 0,$$

$$4A_2B_2 \le (A_2 + B_2)^2 \le \|\ell_1(1)\|_L^2.$$

By the last inequalities and (12.5), (12.94) implies (12.78), where r_1 is defined by (12.79). The inequalities (12.78), on account of (12.46), (12.59) and (12.79), imply that the estimate (11.15) holds. $\square$

Lemma 12.2. *Let $0 \ne |\mu| \le |\lambda|$ and let the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. If, moreover, the conditions* (12.10) *and* (12.11) *hold, then $\ell \in \mathcal{A}^2(\lambda, \mu)$.*

Proof. Let $q^* \in L([a,b]; R_+)$, $c \in R_+$, and $u \in \widetilde{C}([a,b]; R)$ satisfy (11.13) and (11.14) for $i = 2$. Define the number λ_0 by (12.45). We will show that (11.15) holds with

$$r = \frac{\lambda_0|\lambda|(\|\ell_0(1)\|_L + 1)}{(|\mu| - |\lambda|\|\ell_1(1)\|_L)\|\ell_0(1)\|_L - |\lambda|\|\ell_1(1)\|_L - |\lambda| + |\mu|} + \\ + \frac{\lambda_0(\|\ell_0(1)\|_L + 1)}{\left|\frac{\mu}{\lambda}\right| - \|\ell_1(1)\|_L - \frac{1}{4}\|\ell_0(1)\|_L^2}. \tag{12.95}$$

Obviously, u satisfies (12.47), where $\widetilde{q}$ is defined by (12.48). Clearly,

$$-\widetilde{q}(t)\operatorname{sgn} u(t) \le q^*(t) \quad \text{for} \quad t \in [a,b] \tag{12.96}$$

and

$$[\lambda u(a) + \mu u(b)] \operatorname{sgn}\big(\mu u(b)\big) \le c. \tag{12.97}$$

First suppose that u does not change its sign. According to (2.1), (12.97), and the assumption $\left|\frac{\mu}{\lambda}\right| \in\,]0,1]$, we obtain

$$\left|\frac{\mu}{\lambda}\right| |u(b)| - |u(a)| \le \frac{c}{|\lambda|} \tag{12.98}$$

and

$$|u(b)| - |u(a)| \le |u(b)| \frac{|\lambda| - |\mu|}{|\lambda|} + \frac{c}{|\lambda|}\,. \tag{12.99}$$

Define the numbers $\overline{M}$ and $\overline{m}$ by (12.53) and choose $t_1, t_2 \in [a,b]$ such that $t_1 \neq t_2$ and (12.54) is fulfilled. Obviously, $\overline{M} \ge 0$, $\overline{m} \ge 0$, and either (12.55) or (12.56) holds. Due to (12.2), (12.53), and (12.96), (12.47) implies

$$-|u(t)|' \le \overline{M}\ell_1(1)(t) - \overline{m}\ell_0(1)(t) + q^*(t) \quad \text{for} \quad t \in [a,b]. \tag{12.100}$$

If (12.56) holds, then the integration of (12.100) from a to t_2 and from t_1 to b, in view of (12.2), $\overline{m} \ge 0$, and (12.54), results in

$$|u(a)| - \overline{m} \le \overline{M} \int_a^{t_2} \ell_1(1)(s)ds + \int_a^{t_2} q^*(s)ds, \tag{12.101}$$

$$\overline{M} - |u(b)| \le \overline{M} \int_{t_1}^{b} \ell_1(1)(s)ds + \int_{t_1}^{b} q^*(s)ds. \tag{12.102}$$

Multiplying both sides of (12.102) by $\left|\frac{\mu}{\lambda}\right|$ and taking into accout (12.2) and the condition $\left|\frac{\mu}{\lambda}\right| \in\,]0,1]$, we obtain

$$\left|\frac{\mu}{\lambda}\right| \overline{M} - \left|\frac{\mu}{\lambda}\right| |u(b)| \le \overline{M} \int_{t_1}^{b} \ell_1(1)(s)ds + \int_{t_1}^{b} q^*(s)ds.$$

Summing the last inequality and (12.101), in view of (12.98), we get

$$\left|\frac{\mu}{\lambda}\right| \overline{M} - \overline{m} - \frac{c}{|\lambda|} \le \left|\frac{\mu}{\lambda}\right| \overline{M} - \overline{m} + |u(a)| - \left|\frac{\mu}{\lambda}\right| |u(b)| \le \overline{M}\|\ell_1(1)\|_L + \|q^*\|_L.$$

If (12.55) is fulfilled, then the integration of (12.100) from t_1 to t_2, on account of (12.2), $\overline{m} \geq 0$, (12.54), and the assumption $\left|\frac{\mu}{\lambda}\right| \in \,]0,1]$, yields

$$\left|\frac{\mu}{\lambda}\right| \overline{M} - \overline{m} - \frac{c}{|\lambda|} \leq \overline{M} - \overline{m} \leq \overline{M} \int_{t_1}^{t_2} \ell_1(1)(s)ds + \int_{t_1}^{t_2} q^*(s)ds \leq$$

$$\leq \overline{M} \|\ell_1(1)\|_L + \|q^*\|_L.$$

Therefore, in both cases (12.55) and (12.56), in view of (12.10), the inequality

$$0 \leq \overline{M} \left(\left|\frac{\mu}{\lambda}\right| - \|\ell_1(1)\|_L \right) \leq \overline{m} + \|q^*\|_L + \frac{c}{|\lambda|} \tag{12.103}$$

holds.

On the other hand, the integration of (12.100) from a to b implies

$$|u(a)| - |u(b)| \leq \overline{M} \|\ell_1(1)\|_L - \overline{m} \|\ell_0(1)\|_L + \|q^*\|_L.$$

Hence, by (12.99), (12.53), and the assumption $|\mu| \leq |\lambda|$, we get

$$\begin{aligned} \overline{m} \|\ell_0(1)\|_L &\leq \overline{M} \|\ell_1(1)\|_L + |u(b)| \frac{|\lambda| - |\mu|}{|\lambda|} + \frac{c}{|\lambda|} + \|q^*\|_L \leq \\ &\leq \overline{M} \|\ell_1(1)\|_L + \overline{M} \frac{|\lambda| - |\mu|}{|\lambda|} + \|q^*\|_L + \frac{c}{|\lambda|} = \\ &= \overline{M} \left(\|\ell_1(1)\|_L + \frac{|\lambda| - |\mu|}{|\lambda|} \right) + \|q^*\|_L + \frac{c}{|\lambda|}. \end{aligned}$$

The last inequality and (12.103), in view of (12.11), (12.45), and (12.53), result in

$$\|u\|_C = \overline{M} \leq r_0 \lambda_0 |\lambda| (\|\ell_0(1)\|_L + 1)(c + \|q^*\|_L),$$

where $r_0 = \left[(|\mu| - |\lambda| \|\ell_1(1)\|_L) \|\ell_0(1)\|_L - |\lambda| \|\ell_1(1)\|_L - |\lambda| + |\mu| \right]^{-1}$. Therefore, the estimate (11.15) holds.

Now suppose that u changes its sign. Define the numbers M and m by (12.59) and choose $t_M, t_m \in [a, b]$ such that (12.60) is fulfilled. Obviously, $M > 0$, $m > 0$, and either (12.61) or (12.62) holds.

First suppose that (12.61) is fulfilled. It is clear that there exists $\alpha_1 \in \,]t_m, t_M[$ such that

$$u(t) < 0 \quad \text{for} \quad t_m \leq t < \alpha_1, \qquad u(\alpha_1) = 0. \tag{12.104}$$

Let

$$\alpha_2 = \sup\{t \in [t_M, b] : u(s) > 0 \text{ for } t_M \le s \le t\}. \tag{12.105}$$

Obviously,

$$u(t) > 0 \quad \text{for} \quad t_M \le t < \alpha_2 \tag{12.106}$$

and

$$\text{if} \quad \alpha_2 < b, \quad \text{then} \quad u(\alpha_2) = 0. \tag{12.107}$$

Put

$$\alpha_3 = \begin{cases} a & \text{if } \ u(a) \le 0 \\ \sup\{t \in [a, t_m[\, : u(s) > 0 \text{ for } a \le s \le t\} & \text{if } \ u(a) > 0 \end{cases}. \tag{12.108}$$

It is clear that

$$\text{if} \quad \alpha_3 > a, \quad \text{then} \quad u(t) > 0 \quad \text{for} \quad a \le t < \alpha_3, \quad u(\alpha_3) = 0. \tag{12.109}$$

The integration of (12.47) from t_m to α_1, from t_M to α_2, and from a to α_3, in view of (12.2), (12.59), (12.60), (12.96), (12.104), (12.106), and (12.109), yields

$$m \le M \int_{t_m}^{\alpha_1} \ell_0(1)(s)ds + m \int_{t_m}^{\alpha_1} \ell_1(1)(s)ds + \int_{t_m}^{\alpha_1} q^*(s)ds, \tag{12.110}$$

$$M - u(\alpha_2) \le M \int_{t_M}^{\alpha_2} \ell_1(1)(s)ds + m \int_{t_M}^{\alpha_2} \ell_0(1)(s)ds + \int_{t_M}^{\alpha_2} q^*(s)ds, \tag{12.111}$$

$$u(a) - u(\alpha_3) \le M \int_a^{\alpha_3} \ell_1(1)(s)ds + m \int_a^{\alpha_3} \ell_0(1)(s)ds + \int_a^{\alpha_3} q^*(s)ds. \tag{12.112}$$

Evidently, either

$$u(a) \le 0 \tag{12.113}$$

or

$$u(a) > 0. \tag{12.114}$$

If (12.113) holds, then, in view of (2.1), (12.97), (12.105), and (12.107), we obtain $u(\alpha_2) \le \frac{c}{|\mu|}$. Thus, from (12.111), on account of $\left|\frac{\mu}{\lambda}\right| \in \,]0,1]$, it follows that

$$\left|\frac{\mu}{\lambda}\right| M - \frac{c}{|\lambda|} \le M \int_I \ell_1(1)(s)ds + m \int_I \ell_0(1)(s)ds + \int_I q^*(s)ds, \tag{12.115}$$

where $I = [t_M, \alpha_2]$.

Now let (12.114) hold. According to (2.1) and (12.97), it is clear that

$$u(a) - \left|\frac{\mu}{\lambda}\right| u(b) \ge -\frac{1}{|\lambda|}[\lambda u(a) + \mu u(b)] \operatorname{sgn}\big(\mu u(b)\big) \ge -\frac{c}{|\lambda|}\,.$$

By virtue of (12.105) and (12.107), we find

$$u(a) - \left|\frac{\mu}{\lambda}\right| u(\alpha_2) \ge -\frac{c}{|\lambda|}\,. \tag{12.116}$$

Multiplying both sides of (12.111) by $\left|\frac{\mu}{\lambda}\right|$ and taking into account the assumption $\left|\frac{\mu}{\lambda}\right| \in \,]0,1]$, we get

$$\left|\frac{\mu}{\lambda}\right| M - \left|\frac{\mu}{\lambda}\right| u(\alpha_2) \le M \int_{t_M}^{\alpha_2} \ell_1(1)(s)ds + m \int_{t_M}^{\alpha_2} \ell_0(1)(s)ds + \int_{t_M}^{\alpha_2} q^*(s)ds.$$

Summing the last inequality and (12.112), according to (12.108), (12.109), and (12.116), we obtain that the inequality (12.115) holds, where $I = [a, \alpha_3] \cup [t_M, \alpha_2]$.

Thus, in both cases (12.113) and (12.114), the inequality (12.115) is fulfilled, where $I = [a, \alpha_3] \cup [t_M, \alpha_2]$.

It follows from (12.110) and (12.115) that

$$\begin{gathered} m(1 - A_1) \le MC_1 + \|q^*\|_L + \frac{c}{|\lambda|}, \\ M\left(\left|\frac{\mu}{\lambda}\right| - B_1\right) \le mD_1 + \|q^*\|_L + \frac{c}{|\lambda|}, \end{gathered} \tag{12.117}$$

where

$$A_1 = \int_{t_m}^{\alpha_1} \ell_1(1)(s)ds, \qquad B_1 = \int_I \ell_1(1)(s)ds,$$
$$C_1 = \int_{t_m}^{\alpha_1} \ell_0(1)(s)ds, \qquad D_1 = \int_I \ell_0(1)(s)ds.$$

Due to (12.10), $A_1 < \left|\frac{\mu}{\lambda}\right|$, $B_1 < \left|\frac{\mu}{\lambda}\right|$. Consequently, (12.45) and (12.117) imply

$$
\begin{gathered}
0 < m(1-A_1)\left(\left|\frac{\mu}{\lambda}\right| - B_1\right) \leq C_1\left(mD_1 + \|q^*\|_L + \frac{c}{|\lambda|}\right) + \\
+ \|q^*\|_L + \frac{c}{|\lambda|} \leq mC_1D_1 + \lambda_0(\|q^*\|_L + c)(\|\ell_0(1)\|_L + 1), \\
0 < M(1-A_1)\left(\left|\frac{\mu}{\lambda}\right| - B_1\right) \leq D_1\left(MC_1 + \|q^*\|_L + \frac{c}{|\lambda|}\right) + \\
+ \|q^*\|_L + \frac{c}{|\lambda|} \leq MC_1D_1 + \lambda_0(\|q^*\|_L + c)(\|\ell_0(1)\|_L + 1).
\end{gathered}
\tag{12.118}
$$

Obviously, in view of the assumption $\left|\frac{\mu}{\lambda}\right| \in \,]0,1]$,

$$
(1-A_1)\left(\left|\frac{\mu}{\lambda}\right| - B_1\right) \geq \left|\frac{\mu}{\lambda}\right| - \left|\frac{\mu}{\lambda}\right| A_1 - B_1 \geq \left|\frac{\mu}{\lambda}\right| - \|\ell_1(1)\|_L > 0,
$$

$$
4C_1D_1 \leq (C_1 + D_1)^2 \leq \|\ell_0(1)\|_L^2.
$$

By the last inequalities and (12.11), from (12.118) we get

$$
\begin{gathered}
M \leq r_1\lambda_0(\|\ell_0(1)\|_L + 1)(c + \|q^*\|_L), \\
m \leq r_1\lambda_0(\|\ell_0(1)\|_L + 1)(c + \|q^*\|_L),
\end{gathered}
\tag{12.119}
$$

where

$$
r_1 = \left(\left|\frac{\mu}{\lambda}\right| - \|\ell_1(1)\|_L - \frac{1}{4}\|\ell_0(1)\|_L^2\right)^{-1}. \tag{12.120}
$$

The inequalities (12.119), on account of (12.59), (12.95), and (12.120), imply that the estimate (11.15) holds.

Now suppose that (12.62) is fulfilled. It is clear that there exists $\alpha_4 \in \,]t_M, t_m[$ such that

$$
u(t) > 0 \quad \text{for} \quad t_M \leq t < \alpha_4, \qquad u(\alpha_4) = 0. \tag{12.121}
$$

Let

$$
\alpha_5 = \sup\{t \in [t_m, b] : u(s) < 0 \text{ for } t_m \leq s \leq t\}. \tag{12.122}
$$

Obviously,

$$
u(t) < 0 \quad \text{for} \quad t_m \leq t < \alpha_5 \tag{12.123}
$$

and

$$
\text{if} \quad \alpha_5 < b, \quad \text{then} \quad u(\alpha_5) = 0. \tag{12.124}
$$

Put

$$\alpha_6 = \begin{cases} a & \text{if } \ u(a) \geq 0 \\ \sup\{t \in [a, t_M[\, : u(s) < 0 \text{ for } a \leq s \leq t\} & \text{if } \ u(a) < 0 \end{cases} . \quad (12.125)$$

It is clear that

$$\text{if} \quad \alpha_6 > a, \quad \text{then} \quad u(t) < 0 \quad \text{for} \quad a \leq t < \alpha_6, \quad u(\alpha_6) = 0. \quad (12.126)$$

The integration of (12.47) from t_M to α_4, from t_m to α_5, and from a to α_6, in view of (12.2), (12.59), (12.60), (12.96), (12.121), (12.123), and (12.126), yields

$$M \leq M \int_{t_M}^{\alpha_4} \ell_1(1)(s)ds + m \int_{t_M}^{\alpha_4} \ell_0(1)(s)ds + \int_{t_M}^{\alpha_4} q^*(s)ds, \quad (12.127)$$

$$u(\alpha_5) + m \leq M \int_{t_m}^{\alpha_5} \ell_0(1)(s)ds + m \int_{t_m}^{\alpha_5} \ell_1(1)(s)ds + \int_{t_m}^{\alpha_5} q^*(s)ds, \quad (12.128)$$

$$u(\alpha_6) - u(a) \leq M \int_a^{\alpha_6} \ell_0(1)(s)ds + m \int_a^{\alpha_6} \ell_1(1)(s)ds + \int_a^{\alpha_6} q^*(s)ds. \quad (12.129)$$

Evidently, either

$$u(a) \geq 0 \quad (12.130)$$

or

$$u(a) < 0. \quad (12.131)$$

If (12.130) holds, then, in view of (2.1), (12.97), (12.122), and (12.124), we obtain $u(\alpha_5) \geq -\frac{c}{|\mu|}$. Thus, from (12.128), on account of $\left|\frac{\mu}{\lambda}\right| \in \,]0, 1]$, it follows that

$$\left|\frac{\mu}{\lambda}\right| m - \frac{c}{|\lambda|} \leq M \int_J \ell_0(1)(s)ds + m \int_J \ell_1(1)(s)ds + \int_J q^*(s)ds, \quad (12.132)$$

where $J = [t_m, \alpha_5]$.

Now let (12.131) be satisfied. According to (2.1) and (12.97), it is clear that

$$\left|\frac{\mu}{\lambda}\right| u(b) - u(a) \geq -\frac{1}{|\lambda|}[\lambda u(a) + \mu u(b)] \operatorname{sgn}\big(\mu u(b)\big) \geq -\frac{c}{|\lambda|} \, .$$

By virtue of (12.122) and (12.124), we find

$$\left|\frac{\mu}{\lambda}\right| u(\alpha_5) - u(a) \geq -\frac{c}{|\lambda|} \,. \tag{12.133}$$

Multiplying both sides of (12.128) by $\left|\frac{\mu}{\lambda}\right|$ and taking into account the assumption $\left|\frac{\mu}{\lambda}\right| \in]0,1]$, we get

$$\left|\frac{\mu}{\lambda}\right| u(\alpha_5) + \left|\frac{\mu}{\lambda}\right| m \leq M \int\limits_{t_m}^{\alpha_5} \ell_0(1)(s)ds + m \int\limits_{t_m}^{\alpha_5} \ell_1(1)(s)ds + \int\limits_{t_m}^{\alpha_5} q^*(s)ds.$$

Summing the last inequality and (12.129), according to (12.125), (12.126), and (12.133), we obtain that the inequality (12.132) holds, where $J = [a,\alpha_6] \cup [t_m,\alpha_5]$.

Thus, in both cases (12.130) and (12.131), the inequality (12.132) is fulfilled, where $J = [a,\alpha_6] \cup [t_m,\alpha_5]$

It follows from (12.127) and (12.132) that

$$\begin{gathered} M(1-A_2) \leq mC_2 + \|q^*\|_L + \frac{c}{|\lambda|}, \\ m\left(\left|\frac{\mu}{\lambda}\right| - B_2\right) \leq MD_2 + \|q^*\|_L + \frac{c}{|\lambda|}, \end{gathered} \tag{12.134}$$

where

$$A_2 = \int\limits_{t_M}^{\alpha_4} \ell_1(1)(s)ds, \qquad B_2 = \int\limits_{J} \ell_1(1)(s)ds,$$
$$C_2 = \int\limits_{t_M}^{\alpha_4} \ell_0(1)(s)ds, \qquad D_2 = \int\limits_{J} \ell_0(1)(s)ds.$$

Due to (12.10), $A_2 < \left|\frac{\mu}{\lambda}\right|$, $B_2 < \left|\frac{\mu}{\lambda}\right|$. Consequently, (12.45) and (12.134) yield

$$\begin{aligned} &0 < M(1-A_2)\left(\left|\frac{\mu}{\lambda}\right| - B_2\right) \leq C_2\left(MD_2 + \|q^*\|_L + \frac{c}{|\lambda|}\right) + \\ &\quad + \|q^*\|_L + \frac{c}{|\lambda|} \leq MC_2D_2 + \lambda_0(\|q^*\|_L + c)(\|\ell_0(1)\|_L + 1), \\ &0 < m(1-A_2)\left(\left|\frac{\mu}{\lambda}\right| - B_2\right) \leq D_2\left(mC_2 + \|q^*\|_L + \frac{c}{|\lambda|}\right) + \\ &\quad + \|q^*\|_L + \frac{c}{|\lambda|} \leq mC_2D_2 + \lambda_0(\|q^*\|_L + c)(\|\ell_0(1)\|_L + 1). \end{aligned} \tag{12.135}$$

Obviously, in view of the assumption $\left|\frac{\mu}{\lambda}\right| \in]0,1]$,

$$(1-A_2)\left(\left|\frac{\mu}{\lambda}\right| - B_2\right) \geq \left|\frac{\mu}{\lambda}\right| - \left|\frac{\mu}{\lambda}\right| A_2 - B_2 \geq \left|\frac{\mu}{\lambda}\right| - \|\ell_1(1)\|_L > 0,$$

$$4C_2D_2 \leq (C_2+D_2)^2 \leq \|\ell_0(1)\|_L^2.$$

By the last inequalities and (12.11), (12.135) implies (12.119), where r_1 is defined by (12.120). The inequalities (12.119), on account of (12.59), (12.95), and (12.120), imply that the estimate (11.15) holds. □

Lemma 12.3. *Let* $|\mu| < |\lambda|$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, *and*

$$\ell_0 \in V_{ab}^{+}(\lambda,\mu), \qquad -\ell_1 \in V_{ab}^{+}(\lambda,\mu).$$

Then $(\ell_0,\ell_1) \in \mathcal{B}(\lambda,\mu)$.

Proof. Let $q^* \in L\big([a,b];R_+\big)$, $c \in R_+$, and $u \in \widetilde{C}\big([a,b];R\big)$ satisfy the inequalities (11.21) and (11.22). In view of the condition $\ell_0 \in V_{ab}^{+}(\lambda,\mu)$, the assumptions of Lemma 11.1 (see p. 192) are satisfied. We will show that (11.15) holds, where $r = r_0$ is the number appearing in Lemma 11.1 (see p. 192).

It is clear that

$$u'(t) = -\ell_1(u)(t) + \widetilde{q}(t) \quad \text{for} \quad t \in [a,b], \tag{12.136}$$

where

$$\widetilde{q}(t) = u'(t) + \ell_1(u)(t) \quad \text{for} \quad t \in [a,b].$$

According to (11.22), evidently

$$\widetilde{q}(t)\operatorname{sgn} u(t) \leq \ell_0(|u|)(t) + q^*(t) \quad \text{for} \quad t \in [a,b]. \tag{12.137}$$

From (12.136), in view of the assumption $\ell_1 \in \mathcal{P}_{ab}$ and the inequality (12.137), we get

$$\begin{aligned}[u(t)]'_+ &\leq \ell_1([u]_-)(t) + \ell_0(|u|)(t) + q^*(t) = \\ &= -\ell_1([u]_+)(t) + \ell_1(|u|)(t) + \ell_0(|u|)(t) + q^*(t) \quad \text{for} \quad t \in [a,b],\end{aligned} \tag{12.138}$$

and

$$\begin{aligned}[u(t)]'_- &\leq \ell_1([u]_+)(t) + \ell_0(|u|)(t) + q^*(t) = \\ &= -\ell_1([u]_-)(t) + \ell_1(|u|)(t) + \ell_0(|u|)(t) + q^*(t) \quad \text{for} \quad t \in [a,b].\end{aligned} \tag{12.139}$$

Since $-\ell_1 \in V_{ab}^+(\lambda,\mu)$, by virtue of Theorem 1.1 (see p. 14), the problem

$$\alpha'(t) = -\ell_1(\alpha)(t) + \ell_1(|u|)(t) + \ell_0(|u|)(t) + q^*(t), \qquad \lambda\alpha(a) + \mu\alpha(b) = c \operatorname{sgn} \lambda \tag{12.140}$$

has a unique solution α. According to (2.1) and (11.21), we find

$$|\lambda|[u(a)]_+ - |\mu|[u(b)]_+ \le c, \qquad |\lambda|[u(a)]_- - |\mu|[u(b)]_- \le c. \tag{12.141}$$

From (2.1), (12.138)–(12.141), on account of the condition $-\ell_1 \in V_{ab}^+(\lambda,\mu)$ and Remark 2.3 (see p. 16), it follows that

$$[u(t)]_+ \le \alpha(t), \qquad [u(t)]_- \le \alpha(t) \quad \text{for} \quad t \in [a,b],$$

consequently,

$$|u(t)| \le \alpha(t) \quad \text{for} \quad t \in [a,b]. \tag{12.142}$$

By (12.142) and the conditions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, (12.140) results in

$$\alpha'(t) \le \ell_0(\alpha)(t) + q^*(t) \quad \text{for} \quad t \in [a,b].$$

By virtue of $\ell_0 \in V_{ab}^+(\lambda,\mu)$ and the boundary condition in (12.140), the latter inequality yields

$$\alpha(t) \le v(t) \quad \text{for} \quad t \in [a,b], \tag{12.143}$$

where v is a solution of the problem (11.1) with $\overline{q} \equiv q^*$ and $\overline{c} = c \operatorname{sgn} \lambda$. Now it follows from (12.142) and (12.143), according to Lemma 11.1 (see p. 192), that the estimate (11.15) holds with $r = r_0$. $\square$

Lemma 12.4. *Let* $|\mu| < |\lambda|$ *and the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$*, where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$*. If, moreover, there exists a function* $\gamma \in \widetilde{C}([a,b];\,]0,+\infty[)$ *satisfying the inequalities* (12.17)–(12.19)*, then* $\ell \in \mathcal{A}^1(\lambda,\mu)$*.*

Proof. Let $q^* \in L([a,b];R_+)$, $c \in R_+$, and $u \in \widetilde{C}([a,b];R)$ satisfy the inequalities (11.13) and (11.14) for $i = 1$. Obviously, u satisfies (12.47), where $\widetilde{q}$ is defined by (12.48). It is also evident that the inequalities (12.49) and (12.50) hold.

According to Theorem 2.1 (see p. 17), the conditions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and the inequalities (12.17) and (12.18) yield $\ell_0 \in V_{ab}^+(\lambda,\mu)$. Therefore, the assumptions of Lemma 11.1 (see p. 192) are fulfilled. Let r_0 be the number appearing in Lemma 11.1 and put

$$r = r_0 \Big(1 + 4\big(1 + \gamma(b) - \gamma(a)\big)\big(4 - (\gamma(b) - \gamma(a))^2\big)^{-1} \Big) . \tag{12.144}$$

We will show that (11.15) holds, where r is defined by (12.144).

First suppose that u does not change its sign. Then from (12.47), in view of (12.49) and the assumptions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, we have

$$|u(t)|' \leq \ell_0(|u|)(t) + q^*(t) \quad \text{for} \quad t \in [a,b],$$

and from (2.1) and (12.50) we get

$$|\lambda u(a)| - |\mu u(b)| \leq c.$$

Therefore, by Remark 2.3 (see p. 16), in view of the condition $\ell_0 \in V_{ab}^+(\lambda, \mu)$, we have

$$|u(t)| \leq v(t) \quad \text{for} \quad t \in [a,b],$$

where v is a solution of the problem (11.1) with $\overline{q} \equiv q^*$ and $\overline{c} = c \operatorname{sgn} \lambda$. Due to Lemma 11.1 (see p. 192), the function v admits the estimate (11.2), and thus, in view of (12.144), the estimate (11.15) holds.

Now assume that u changes its sign. Define the numbers M and m by (12.59) and choose $t_M, t_m \in [a,b]$ such that (12.60) holds. It is clear that

$$M > 0, \qquad m > 0,$$

and either (12.61) or (12.62) is fulfilled.

Let v be a solution of the problem (11.1) with $\overline{q} \equiv q^*$ and $\overline{c} = c \operatorname{sgn} \lambda$. According to (2.1), (12.17), (12.18), (12.59), (11.1), and the assumptions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, we have

$$\begin{gathered} (M\gamma(t) + v(t))' \geq \ell_0(M\gamma + v)(t) + M\ell_1(1)(t) + q^*(t) \geq \\ \geq \ell_0(M\gamma + v)(t) + \ell_1([u]_+)(t) + q^*(t) \quad \text{for} \quad t \in [a,b], \\ |\lambda|(M\gamma(a) + v(a)) - |\mu|(M\gamma(b) + v(b)) \geq c, \end{gathered} \tag{12.145}$$

and

$$\begin{gathered} (m\gamma(t) + v(t))' \geq \ell_0(m\gamma + v)(t) + m\ell_1(1)(t) + q^*(t) \geq \\ \geq \ell_0(m\gamma + v)(t) + \ell_1([u]_-)(t) + q^*(t) \quad \text{for} \quad t \in [a,b], \\ |\lambda|(m\gamma(a) + v(a)) - |\mu|(m\gamma(b) + v(b)) \geq c. \end{gathered} \tag{12.146}$$

On the other hand, from (12.47) and (12.50), on account of (2.1) and (12.49), we obtain

$$\begin{gathered}[u(t)]'_+ \leq \ell_0([u]_+)(t) + \ell_1([u]_-)(t) + q^*(t) \quad \text{for} \quad t \in [a,b], \\ |\lambda|[u(a)]_+ - |\mu|[u(b)]_+ \leq c,\end{gathered} \tag{12.147}$$

and

$$\begin{gathered}[u(t)]'_- \leq \ell_0([u]_-)(t) + \ell_1([u]_+)(t) + q^*(t) \quad \text{for} \quad t \in [a,b], \\ |\lambda|[u(a)]_- - |\mu|[u(b)]_- \leq c.\end{gathered} \tag{12.148}$$

Since $\ell_0 \in V_{ab}^+(\lambda,\mu)$, from (12.145), (12.148) and from (12.146), (12.147), on account of Remark 2.3 (see p. 16), we get

$$\begin{gathered}M\gamma(t) + v(t) \geq [u(t)]_- \quad \text{for} \quad t \in [a,b], \\ m\gamma(t) + v(t) \geq [u(t)]_+ \quad \text{for} \quad t \in [a,b].\end{gathered} \tag{12.149}$$

Inequalities (12.145)–(12.148), by virtue of (12.149) and the assumption $\ell_0 \in \mathcal{P}_{ab}$, imply

$$[u(t)]'_- \leq \big(M\gamma(t) + v(t)\big)' \quad \text{for} \quad t \in [a,b], \tag{12.150}$$

$$[u(t)]'_+ \leq \big(m\gamma(t) + v(t)\big)' \quad \text{for} \quad t \in [a,b]. \tag{12.151}$$

Note also that, in view of the condition $\ell_0 \in V_{ab}^+(\lambda,\mu)$,

$$v(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{12.152}$$

First suppose that (12.62) is fulfilled. The integration of (12.150) from t_M to t_m, on account of (12.59), (12.60), and (12.152), results in

$$\begin{gathered}m \leq M\gamma(t_m) + v(t_m) - M\gamma(t_M) - v(t_M) \leq \\ \leq M(\gamma(t_m) - \gamma(t_M)) + \|v\|_C.\end{gathered} \tag{12.153}$$

On the other hand, the integration of (12.151) from a to t_M and from t_m to b, in view of (12.59), (12.60), and (12.152), yields

$$\begin{gathered}M - [u(a)]_+ \leq m\gamma(t_M) + v(t_M) - m\gamma(a) - v(a) \leq \\ \leq m\big(\gamma(t_M) - \gamma(a)\big) - v(a) + \|v\|_C,\end{gathered} \tag{12.154}$$

$$\begin{gathered}[u(b)]_+ \leq m\gamma(b) + v(b) - m\gamma(t_m) - v(t_m) \leq \\ \leq m\big(\gamma(b) - \gamma(t_m)\big) + v(b).\end{gathered} \tag{12.155}$$

Multiplying both sides of (12.155) by $\left|\frac{\mu}{\lambda}\right|$ and taking into account the facts that $m > 0$, γ is a nondecreasing function, and $\left|\frac{\mu}{\lambda}\right| \in [0, 1[$, we obtain

$$\left|\frac{\mu}{\lambda}\right| [u(b)]_+ \le m(\gamma(b) - \gamma(t_m)) + \left|\frac{\mu}{\lambda}\right| v(b).$$

Summing the last inequality and (12.154) and taking into account (2.1), and the boundary conditions in (11.1) and (12.147), we get

$$M \le m(\gamma(t_M) - \gamma(t_m) + \gamma(b) - \gamma(a)) + \|v\|_C. \tag{12.156}$$

From (12.153) and (12.156), with respect to (12.59), (12.62), and the condition $\gamma'(t) \ge 0$ for $t \in [a, b]$, it follows that

$$\|u\|_C \le \|u\|_C\big(\gamma(t_m) - \gamma(t_M)\big)\big(\gamma(t_M) - \gamma(t_m) + \gamma(b) - \gamma(a)\big) +$$
$$+ \big(1 + \gamma(b) - \gamma(a)\big)\|v\|_C.$$

Consequently, by virtue of the inequality

$$AB \le \frac{1}{4}(A + B)^2, \tag{12.157}$$

the inequality

$$\|u\|_C \le \frac{\|u\|_C}{4}\big(\gamma(b) - \gamma(a)\big)^2 + \big(1 + \gamma(b) - \gamma(a)\big)\|v\|_C$$

holds. Hence, by virtue of (12.19) we find

$$\|u\|_C \le 4\big(1 + \gamma(b) - \gamma(a)\big)\big(4 - (\gamma(b) - \gamma(a))^2\big)^{-1}\|v\|_C. \tag{12.158}$$

Therefore, according to (11.2) and (12.144), the estimate (11.15) holds.

Now suppose that (12.61) is fulfilled. The integration of (12.151) from t_m to t_M, on account of (12.59), (12.60), and (12.152), results in

$$\begin{aligned} M &\le m\gamma(t_M) + v(t_M) - m\gamma(t_m) - v(t_m) \le \\ &\le m(\gamma(t_M) - \gamma(t_m)) + \|v\|_C. \end{aligned} \tag{12.159}$$

On the other hand, the integration of (12.150) from a to t_m and from t_M to b, in view of (12.59), (12.60), and (12.152), yields

$$\begin{aligned} m - [u(a)]_- &\le M\gamma(t_m) + v(t_m) - M\gamma(a) - v(a) \le \\ &\le M\big(\gamma(t_m) - \gamma(a)\big) - v(a) + \|v\|_C, \end{aligned} \tag{12.160}$$

$$\begin{aligned} [u(b)]_- &\le M\gamma(b) + v(b) - M\gamma(t_M) - v(t_M) \le \\ &\le M\big(\gamma(b) - \gamma(t_M)\big) + v(b). \end{aligned} \tag{12.161}$$

Multiplying both sides of (12.161) by $\left|\frac{\mu}{\lambda}\right|$ and taking into account the facts that $M > 0$, γ is a nondecreasing function, and $\left|\frac{\mu}{\lambda}\right| \in [0,1[$, we obtain

$$\left|\frac{\mu}{\lambda}\right| [u(b)]_- \leq M\big(\gamma(b) - \gamma(t_M)\big) + \left|\frac{\mu}{\lambda}\right| v(b).$$

Summing the last inequality and (12.160) and taking into account (2.1) and the boundary conditions in (11.1) and (12.148), we get

$$m \leq M\big(\gamma(t_m) - \gamma(t_M) + \gamma(b) - \gamma(a)\big) + \|v\|_C. \tag{12.162}$$

From (12.159) and (12.162), with respect to (12.59), (12.61), and the condition $\gamma'(t) \geq 0$ for $t \in [a,b]$, it follows that

$$\|u\|_C \leq \|u\|_C\big(\gamma(t_M) - \gamma(t_m)\big)\big(\gamma(t_m) - \gamma(t_M) + \gamma(b) - \gamma(a)\big)+$$
$$+\big(1 + \gamma(b) - \gamma(a)\big)\|v\|_C.$$

Consequently, by virtue of (12.19) and (12.157), the inequality (12.158) is fulfilled. Therefore, according to (11.2) and (12.144), the estimate (11.15) holds. □

Lemma 12.5. *Let* $0 < |\mu| \leq |\lambda|$ *and* $\ell = \ell_0 + \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. *If, moreover, the conditions* (12.21) *and* (12.22) *are fulfilled, then* $\ell \in \mathcal{A}^2\big(\lambda, \mu\big)$.

Proof. Let $q^* \in L\big([a,b]; R_+\big)$, $c \in R_+$, and $u \in \widetilde{C}\big([a,b]; R\big)$ satisfy the inequalities (11.13) and (11.14) for $i = 2$. Obviously, u satisfies

$$u'(t) = \ell(u)(t) + \widetilde{q}(t) \quad \text{for} \quad t \in [a,b], \tag{12.163}$$

where $\widetilde{q}$ is defined by (12.48). It is also evident that the inequalities (12.96) and (12.97) hold.

Since the inclusion $\ell_0 \in V_{ab}^-(\lambda,\mu)$ holds, the assumptions of Lemma 11.1 (see p. 192) are fulfilled. Let r_0 be the number appearing in Lemma 11.1 and put

$$r = r_0 + \frac{\lambda_0\big(\|\ell(1)\|_L + 1\big)}{\left|\frac{\mu}{\lambda}\right| - \frac{1}{4}\|\ell(1)\|_L^2}\ , \tag{12.164}$$

where λ_0 is given by (12.45). We will show that (11.15) holds, where r is defined by (12.164).

First suppose that u does not change its sign. It is evident that either there exists $t_0 \in [a,b]$ such that $u(t_0) = 0$ or $|u(t)| > 0$ for $t \in [a,b]$.

Let there exists $t_0 \in [a, b]$ such that

$$u(t_0) = 0. \tag{12.165}$$

According to (12.163), (12.96), and the assumption $\ell \in \mathcal{P}_{ab}$, we have

$$|u(t)|' \geq -q^*(t) \quad \text{for} \quad t \in [a, b] \tag{12.166}$$

and from (2.1) and (12.97) we get

$$|\lambda u(a)| - |\mu u(b)| \geq -c. \tag{12.167}$$

Put

$$\overline{M} = \max\{|u(t)| : t \in [a, b]\} \tag{12.168}$$

and choose $t_1 \in [a, b]$ such that

$$|u(t_1)| = \overline{M}. \tag{12.169}$$

If $t_1 < t_0$, then the integration of (12.166) from t_1 to t_0, in view of of (12.165) and (12.169), results in

$$\overline{M} \leq \int\limits_{t_1}^{t_0} q^*(s)ds \leq \|q^*\|_L + c.$$

Thus, on account of (12.45), (12.164), and (12.168), we find that the estimate (11.15) holds.

If $t_1 \geq t_0$, then the integration of (12.166) from a to t_0 and from t_1 to b, in view of (12.165) and (12.169), yields

$$|u(a)| \leq \int\limits_{a}^{t_0} q^*(s)ds,$$

$$\overline{M} - |u(b)| \leq \int\limits_{t_1}^{b} q^*(s)ds.$$

From the last two inequalities, using (12.45), (12.167), and the assumption $\left|\frac{\mu}{\lambda}\right| \in]0, 1]$, we get

$$\overline{M} \leq \int\limits_{t_1}^{b} q^*(s)ds + \left|\frac{\lambda}{\mu}\right| |u(a)| + \frac{c}{|\mu|} \leq \left|\frac{\lambda}{\mu}\right| \lambda_0\big(\|q^*\|_L + c\big).$$

Thus, on account of (12.164) and (12.168), we find that the estimate (11.15) holds.

Now let

$$|u(t)| > 0 \quad \text{for} \quad t \in [a, b]. \tag{12.170}$$

According to (12.163), (12.96), (12.170), and the assumption $\ell = \ell_0 + \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, we have

$$|u(t)|' \ge \ell_0(|u|)(t) + \ell_1(|u|)(t) - q^*(t) \ge \ell_0(|u|)(t) - q^*(t) \quad \text{for} \quad t \in [a, b].$$

Moreover, (2.1) and (12.97) yield (12.167). Therefore, by Remark 2.3 (see p. 16), the condition $\ell_0 \in V_{ab}^-(\lambda, \mu)$ implies

$$|u(t)| \le v(t) \quad \text{for} \quad t \in [a, b], \tag{12.171}$$

where v is a solution of the problem (11.1) with $\overline{q} \equiv -q^*$ and $\overline{c} = -c \operatorname{sgn} \lambda$. According to Lemma 11.1 (see p. 192), the function v admits the estimate (11.2) and thus, on account of (12.164) and (12.171), the estimate (11.15) holds.

Now suppose that u changes its sign. Define numbers M and m by (12.59) and choose $t_M, t_m \in [a, b]$ such that (12.60) holds. Obviously, $M > 0$, $m > 0$, and either (12.61) or (12.62) is satisfied.

First suppose that (12.61) is fulfilled. It is clear that there exists $\alpha_1 \in]t_m, t_M[$ such that

$$u(t) < 0 \quad \text{for} \quad t_m \le t < \alpha_1, \qquad u(\alpha_1) = 0. \tag{12.172}$$

Let

$$\alpha_2 = \sup \{t \in [t_M, b] : u(s) > 0 \text{ for } t_M \le s \le t\}. \tag{12.173}$$

Obviously,

$$u(t) > 0 \quad \text{for} \quad t_M \le t < \alpha_2 \tag{12.174}$$

and

$$\text{if} \quad \alpha_2 < b \quad \text{then} \quad u(\alpha_2) = 0. \tag{12.175}$$

Put

$$\alpha_3 = \begin{cases} a & \text{if } u(a) \le 0 \\ \sup \{t \in [a, t_m[: u(s) > 0 \text{ for } a \le s \le t\} & \text{if } u(a) > 0 \end{cases}. \tag{12.176}$$

It is clear that

$$\text{if} \quad \alpha_3 > a, \quad \text{then} \quad u(t) > 0 \quad \text{for} \quad a \le t < \alpha_3, \; u(\alpha_3) = 0. \tag{12.177}$$

The integration of (12.163) from t_m to α_1, from t_M to α_2, and from a to α_3, in view of (12.59), (12.60), (12.96), (12.172), (12.174), (12.177), and the assumption $\ell \in \mathcal{P}_{ab}$, yields

$$m \le M \int_{t_m}^{\alpha_1} \ell(1)(s)ds + \int_{t_m}^{\alpha_1} q^*(s)ds, \tag{12.178}$$

$$M - u(\alpha_2) \le m \int_{t_M}^{\alpha_2} \ell(1)(s)ds + \int_{t_M}^{\alpha_2} q^*(s)ds, \tag{12.179}$$

$$u(a) - u(\alpha_3) \le m \int_{a}^{\alpha_3} \ell(1)(s)ds + \int_{a}^{\alpha_3} q^*(s)ds. \tag{12.180}$$

Evidently, either

$$u(a) \le 0 \tag{12.181}$$

or

$$u(a) > 0. \tag{12.182}$$

If (12.181) holds, then, in view of (2.1), (12.97), (12.173), and (12.175), we obtain $u(\alpha_2) \le \frac{c}{|\mu|}$. Thus, from (12.179), on account of the assumption $\left|\frac{\mu}{\lambda}\right| \in]0,1]$, it follows that

$$\left|\frac{\mu}{\lambda}\right| M - \frac{c}{|\lambda|} \le m \int_I \ell(1)(s)ds + \int_I q^*(s)ds, \tag{12.183}$$

where $I = [t_M, \alpha_2]$.

Now let (12.182) hold. According to (2.1) and (12.97), it is clear that

$$u(a) - \left|\frac{\mu}{\lambda}\right| u(b) \ge -\frac{1}{|\lambda|}\big[\lambda u(a) + \mu u(b)\big] \operatorname{sgn}\big(\mu u(b)\big) \ge -\frac{c}{|\lambda|}.$$

Hence, by virtue of (12.173), (12.175), and (12.182), we find

$$u(a) - \left|\frac{\mu}{\lambda}\right| u(\alpha_2) \ge -\frac{c}{|\lambda|}. \tag{12.184}$$

Multiplying both sides of (12.179) by $\left|\frac{\mu}{\lambda}\right|$ and taking into account the assumption $\left|\frac{\mu}{\lambda}\right| \in]0,1]$, we get

$$\left|\frac{\mu}{\lambda}\right| M - \left|\frac{\mu}{\lambda}\right| u(\alpha_2) \le m \int_{t_M}^{\alpha_2} \ell(1)(s)ds + \int_{t_M}^{\alpha_2} q^*(s)ds.$$

Summing the last inequality and (12.180), according to (12.176), (12.177), and (12.184), we obtain that the inequality (12.183) holds, where $I = [a, \alpha_3] \cup [t_M, \alpha_2]$.

Thus, in both cases (12.181) and (12.182), the inequality (12.183) is fulfilled, where $I = [a, \alpha_3] \cup [t_M, \alpha_2]$.

It follows from (12.178) and (12.183) that

$$\begin{aligned} m &\le M A_1 + \|q^*\|_L, \\ \left|\frac{\mu}{\lambda}\right| M &\le m B_1 + \|q^*\|_L + \frac{c}{|\lambda|}, \end{aligned} \tag{12.185}$$

where

$$A_1 = \int\limits_{t_m}^{\alpha_1} \ell(1)(s)ds, \qquad B_1 = \int\limits_I \ell(1)(s)ds.$$

Consequently, on account of (12.45) and the assumption $|\mu| \le |\lambda|$, the inequalities (12.185) imply

$$\begin{aligned} \left|\frac{\mu}{\lambda}\right| m &\le A_1 \left(m B_1 + \|q^*\|_L + \frac{c}{|\lambda|} \right) + \|q^*\|_L \le \\ &\le m A_1 B_1 + \lambda_0 (\|q^*\|_L + c)(\|\ell(1)\|_L + 1), \\ \left|\frac{\mu}{\lambda}\right| M &\le B_1 \big(M A_1 + \|q^*\|_L \big) + \|q^*\|_L + \frac{c}{|\lambda|} \le \\ &\le M A_1 B_1 + \lambda_0 (\|q^*\|_L + c)(\|\ell(1)\|_L + 1). \end{aligned} \tag{12.186}$$

Obviously,

$$4 A_1 B_1 \le (A_1 + B_1)^2 \le \|\ell(1)\|_L^2.$$

By the last inequality and (12.22), from (12.186) we get

$$\begin{aligned} m &\le r_1 \lambda_0 (\|\ell(1)\|_L + 1)(c + \|q^*\|_L), \\ M &\le r_1 \lambda_0 (\|\ell(1)\|_L + 1)(c + \|q^*\|_L), \end{aligned} \tag{12.187}$$

where

$$r_1 = \left(\left|\frac{\mu}{\lambda}\right| - \frac{1}{4} \|\ell(1)\|_L^2 \right)^{-1}. \tag{12.188}$$

Inequalities (12.187), on account of (12.59), (12.164), and (12.188), imply that the estimate (11.15) holds.

Now suppose that (12.62) is fulfilled. It is clear that there exists $\alpha_4 \in]t_M, t_m[$ such that

$$u(t) > 0 \quad \text{for} \quad t_M \le t < \alpha_4, \qquad u(\alpha_4) = 0. \tag{12.189}$$

Let

$$\alpha_5 = \sup \{ t \in [t_m, b] : u(s) < 0 \text{ for } t_m \le s \le t \}. \tag{12.190}$$

Obviously,

$$u(t) < 0 \quad \text{for} \quad t_m \le t < \alpha_5 \tag{12.191}$$

and

$$\text{if} \quad \alpha_5 < b \quad \text{then} \quad u(\alpha_5) = 0. \tag{12.192}$$

Put

$$\alpha_6 = \begin{cases} a & \text{if } u(a) \ge 0 \\ \sup \{ t \in [a, t_M[: u(s) < 0 \text{ for } a \le s \le t \} & \text{if } u(a) < 0 \end{cases}. \tag{12.193}$$

It is clear that

$$\text{if} \quad \alpha_6 > a, \quad \text{then} \quad u(t) < 0 \quad \text{for} \quad a \le t < \alpha_6, \quad u(\alpha_6) = 0. \tag{12.194}$$

The integration of (12.163) from t_M to α_4, from t_m to α_5, and from a to α_6, in view of (12.59), (12.60), (12.96), (12.189), (12.191), (12.194), and the assumption $\ell \in \mathcal{P}_{ab}$, yields

$$M \le m \int_{t_M}^{\alpha_4} \ell(1)(s)ds + \int_{t_M}^{\alpha_4} q^*(s)ds, \tag{12.195}$$

$$u(\alpha_5) + m \le M \int_{t_m}^{\alpha_5} \ell(1)(s)ds + \int_{t_m}^{\alpha_5} q^*(s)ds, \tag{12.196}$$

$$u(\alpha_6) - u(a) \le M \int_{a}^{\alpha_6} \ell(1)(s)ds + \int_{a}^{\alpha_6} q^*(s)ds. \tag{12.197}$$

Evidently, either

$$u(a) \ge 0 \tag{12.198}$$

or

$$u(a) < 0. \tag{12.199}$$

If (12.198) holds, then, in view of (2.1), (12.97), (12.190), and (12.192), we obtain $u(\alpha_5) \geq -\frac{c}{|\mu|}$. Thus, from (12.196), on account $\left|\frac{\mu}{\lambda}\right| \in \,]0,1]$, it follows that

$$\left|\frac{\mu}{\lambda}\right| m - \frac{c}{|\lambda|} \leq M \int\limits_J \ell(1)(s)ds + \int\limits_J q^*(s)ds, \tag{12.200}$$

where $J = [t_m, \alpha_5]$.

Now let (12.199) hold. According to (2.1) and (12.97), it is clear that

$$\left|\frac{\mu}{\lambda}\right| u(b) - u(a) \geq -\frac{1}{|\lambda|}\big[\lambda u(a) + \mu u(b)\big] \operatorname{sgn}\big(\mu u(b)\big) \geq -\frac{c}{|\lambda|}.$$

Hence, by virtue of (12.190), (12.192), and (12.199), we find

$$\left|\frac{\mu}{\lambda}\right| u(\alpha_5) - u(a) \geq -\frac{c}{|\lambda|}. \tag{12.201}$$

Multiplying both sides of (12.196) by $\left|\frac{\mu}{\lambda}\right|$ and taking into account the assumption $\left|\frac{\mu}{\lambda}\right| \in \,]0,1]$, we get

$$\left|\frac{\mu}{\lambda}\right| u(\alpha_5) + \left|\frac{\mu}{\lambda}\right| m \leq M \int\limits_{t_m}^{\alpha_5} \ell(1)(s)ds + \int\limits_{t_m}^{\alpha_5} q^*(s)ds.$$

Summing the last inequality and (12.197), according to (12.193), (12.194), and (12.201), we obtain that the inequality (12.200) holds, where $J = [a, \alpha_6] \cup [t_m, \alpha_5]$.

Thus, in both cases (12.198) and (12.199), the inequality (12.200) is fulfilled, where $J = [a, \alpha_6] \cup [t_m, \alpha_5]$.

It follows from (12.195) and (12.200) that

$$\begin{gathered} M \leq mA_2 + \|q^*\|_L, \\ \left|\frac{\mu}{\lambda}\right| m \leq MB_2 + \|q^*\|_L + \frac{c}{|\lambda|}, \end{gathered} \tag{12.202}$$

where

$$A_2 = \int\limits_{t_M}^{\alpha_4} \ell(1)(s)ds, \qquad B_2 = \int\limits_J \ell(1)(s)ds.$$

Consequently, in view of (12.45) and the assumption $|\mu| \leq |\lambda|$, the inequalities (12.202) imply

$$\begin{aligned} \left|\frac{\mu}{\lambda}\right| M &\leq A_2\left(MB_2 + \|q^*\|_L + \frac{c}{|\lambda|}\right) + \|q^*\|_L \leq \\ &\leq MA_2B_2 + \lambda_0(\|q^*\|_L + c)(\|\ell(1)\|_L + 1), \\ \left|\frac{\mu}{\lambda}\right| m &\leq B_2\big(mA_2 + \|q^*\|_L\big) + \|q^*\|_L + \frac{c}{|\lambda|} \leq \\ &\leq mA_2B_2 + \lambda_0(\|q^*\|_L + c)(\|\ell(1)\|_L + 1). \end{aligned} \tag{12.203}$$

Obviously,

$$4A_2B_2 \leq (A_2 + B_2)^2 \leq \|\ell(1)\|_L^2.$$

By the last inequality and (12.22), (12.203) implies (12.187), where r_1 is defined by (12.188). Inequalities (12.187), on account of (12.59), (12.164), and (12.188), imply that the estimate (11.15) holds. □

Theorem 12.1 follows from Lemma 11.3 (see p. 195) and Lemma 12.1 (see p. 211).

Proof of Theorem 12.2. It can be proved in a similar manner as Theorem 12.1. Moreover, the proof of Theorem 12.2 can be found in [4]. □

Theorem 12.3 follows from Lemma 11.3 (see p. 195) and Lemma 12.2 (see p. 219). Theorem 12.4 follows from Lemma 11.5 (see p. 197) and Lemma 12.3 (see p. 227). Theorem 12.5 follows from Lemma 11.3 (see p. 195) and Lemma 12.4 (see p. 228). Theorem 12.6 follows from Lemma 11.3 (see p. 195) and Lemma 12.5 (see p. 232). Theorem 12.7 follows from Lemma 11.4 (see p. 196) and Lemma 12.1 (see p. 211). Theorem 12.9 follows from Lemma 11.4 (see p. 196) and Lemma 12.2 (see p. 219). Theorem 12.10 follows from Lemma 11.6 (see p. 198) and Lemma 12.3 (see p. 227). Theorem 12.11 follows from Lemma 11.4 (see p. 196) and Lemma 12.4 (see p. 228). Theorem 12.12 follows from Lemma 11.4 (see p. 196) and Lemma 12.5 (see p. 232).

12.3. Comments and Examples

On Remark 12.1. Let $0 \neq |\mu| \leq |\lambda|$. It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin D$, then (x_0, y_0) belongs at least to one of the following sets:

$$D_1 = \{(x,y) \in R_+ \times R_+ \ : \ 1 \leq x\},$$
$$D_2 = \left\{(x,y) \in R_+ \times R_+ \ : \ \frac{|\lambda| - |\mu|}{|\lambda|} \leq x < 1, \ y \leq \frac{x}{1-x} - \frac{|\lambda| - |\mu|}{|\mu|}\right\},$$
$$D_3 = \left\{(x,y) \in R_+ \times R_+ \ : \ x < 1, \ 2\sqrt{1-x} \leq y\right\}.$$

Let $(x_0, y_0) \in D_1$. Put $a = 0$, $b = 3$, $\varepsilon = \frac{|\mu|}{|\lambda|(1+y_0)}$,

$$p(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[\\ x_0 & \text{for } t \in [1,2[\\ 0 & \text{for } t \in [2,3] \end{cases}, \quad z(t) = \begin{cases} 0 & \text{for } t \in [0,2[\\ -\frac{x_0+\varepsilon-1}{1-(x_0+\varepsilon-1)(t-3)} & \text{for } t \in [2,3] \end{cases},$$

$$\tau(t) = \begin{cases} 1 & \text{for } t \in [0,1[\\ 3 & \text{for } t \in [1,3] \end{cases}.$$

It is not difficult to verify that

$$x_0 = \int_a^b [p(s)]_+ ds, \qquad y_0 = \int_a^b [p(s)]_- ds, \tag{12.204}$$

and the problem

$$u'(t) = p(t)u(\tau(t)) + z(t)u(t), \quad \lambda u(a) + \mu u(b) = 0 \tag{12.205}$$

has the nontrivial solution

$$u(t) = \begin{cases} (|\mu| - |\lambda|\varepsilon)t - |\mu| & \text{for } t \in [0,1[\\ -x_0|\lambda|(t-1) - |\lambda|\varepsilon & \text{for } t \in [1,2[\\ |\lambda|(x_0+\varepsilon-1)(t-3) - |\lambda| & \text{for } t \in [2,3] \end{cases}.$$

Then, by Remark 1.1 (see p. 14), there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (10.1), (10.2) with

$$F(v)(t) \stackrel{\text{def}}{=} p(t)v(\tau(t)) + z(t)v(t) + q_0(t), \qquad h(v) \stackrel{\text{def}}{=} c_0 \tag{12.206}$$

has no solution, while the conditions (12.1) and (12.3) are fulfilled, where

$$\ell_0(v)(t) \stackrel{\text{def}}{=} [p(t)]_+ v(\tau(t)), \qquad \ell_1(v)(t) \stackrel{\text{def}}{=} [p(t)]_- v(\tau(t)),$$
$$q \equiv |q_0|, \qquad c = |c_0|. \tag{12.207}$$

Let $(x_0, y_0) \in D_2$. Put $a = 0$, $b = 3$,

$$p(t) = \begin{cases} x_0 & \text{for } t \in [0,1[\\ -y_0 & \text{for } t \in [1,2[\\ 0 & \text{for } t \in [2,3] \end{cases}, \quad \tau(t) = \begin{cases} 1 & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,3] \end{cases},$$

$$z(t) = \begin{cases} 0 & \text{for } t \in [0,2[\\ -\frac{|\mu|-|\lambda|(1-x_0)-|\mu|y_0(1-x_0)}{|\lambda|(1-x_0)-(|\mu|-|\lambda|(1-x_0)-|\mu|y_0(1-x_0))(t-3)} & \text{for } t \in [2,3] \end{cases}.$$

It is not difficult to verify that (12.204) holds, and the problem (12.205) has the nontrivial solution

$$u(t) = \begin{cases} -\frac{|\mu|x_0}{1-x_0}t - |\mu| & \text{for } t \in [0,1[\\ |\mu|y_0(t-1) - \frac{|\mu|}{1-x_0} & \text{for } t \in [1,2[\\ \left(\frac{|\mu|-|\lambda|(1-x_0)}{1-x_0} - |\mu|y_0\right)(t-3) - |\lambda| & \text{for } t \in [2,3] \end{cases}.$$

Then, by Remark 1.1 (see p. 14), there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (10.1), (10.2) with F and h given by (12.206) has no solution, while the conditions (12.1) and (12.3) are fulfilled, where ℓ_0, ℓ_1, q, and c are defined by (12.207).

Let $(x_0, y_0) \in D_3$. Put $a = 0$, $b = 6$,

$$p(t) = \begin{cases} 0 & \text{for } t \in [0,1[\cup [2,3[\\ -\sqrt{1-x_0} & \text{for } t \in [1,2[\cup [3,4[\\ x_0 & \text{for } t \in [4,5[\\ 2\sqrt{1-x_0} - y_0 & \text{for } t \in [5,6] \end{cases},$$

$$\tau(t) = \begin{cases} 6 & \text{for } t \in [0,3[\cup [4,5[\\ 2 & \text{for } t \in [3,4[\\ 1 & \text{for } t \in [5,6] \end{cases}.$$

Obviously, (12.204) holds. Furthermore, define the operator $G \in K_{ab}$ by

$$G(v)(t) = \begin{cases} -v(t)|v(t)| & \text{for } t \in [0,1[\cup [2,3[\\ 0 & \text{for } t \in [1,2[\cup [3,5[\\ q_0(t) & \text{for } t \in [5,6] \end{cases},$$

where $q_0 \in L([a,b];R)$ is such that

$$\int_5^6 q_0(s)ds \geq 1 + y_0 - \sqrt{1-x_0}\,. \tag{12.208}$$

We will show that the problem (10.1), (10.2) with

$$F(v)(t) \stackrel{\text{def}}{=} p(t)v(\tau(t)) + G(v)(t), \qquad h(v) \stackrel{\text{def}}{=} 0 \tag{12.209}$$

has no solution, while the conditions (12.1) and (12.3) are fulfilled, where

$$\begin{gathered} \ell_0(v)(t) \stackrel{\text{def}}{=} [p(t)]_+ v(\tau(t)), \qquad \ell_1(v)(t) \stackrel{\text{def}}{=} [p(t)]_- v(\tau(t)), \\ q \equiv |q_0|, \qquad c = 0. \end{gathered} \tag{12.210}$$

Indeed, suppose on the contrary that u is a solution of the problem (10.1), (10.2) with F and h given by (12.209), i.e., the equality (1.2_0) holds and

$$u'(t) = p(t)u(\tau(t)) + G(u)(t) \quad \text{for} \quad t \in [a,b]. \tag{12.211}$$

From (12.211) we get

$$u(1) = \frac{u(0)}{1+|u(0)|}\,, \tag{12.212}$$

$$u(2) = u(1) - u(6)\sqrt{1-x_0}\,, \tag{12.213}$$

$$u(3) = \frac{u(2)}{1+|u(2)|}\,, \tag{12.214}$$

$$u(4) = u(3) - u(2)\sqrt{1-x_0}\,, \tag{12.215}$$

$$u(5) = u(4) + u(6)x_0\,, \tag{12.216}$$

$$u(6) = u(5) - \left(y_0 - 2\sqrt{1-x_0}\right)u(1) + \int_5^6 q_0(s)ds. \tag{12.217}$$

The equalities (12.213) and (12.215)–(12.217) imply

$$u(3) = \left(y_0 - \sqrt{1-x_0}\right)u(1) - \int_5^6 q_0(s)ds. \tag{12.218}$$

Hence, by virtue of (12.212), (12.214) and (12.218) yield

$$\int_5^6 q_0(s)ds = \left(y_0 - \sqrt{1-x_0}\right)\frac{u(0)}{1+|u(0)|} - \frac{u(2)}{1+|u(2)|} \leq$$

$$\leq \left(y_0 - \sqrt{1-x_0}\right)\frac{|u(0)|}{1+|u(0)|} + \frac{|u(2)|}{1+|u(2)|} < 1 + y_0 - \sqrt{1-x_0}\,,$$

which contradicts (12.208).

On Remark 12.2. Let $\mu = 0$. It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin E$, then (x_0, y_0) belongs at least to one of the following sets:

$$E_1 = \{(x,y) \in R_+ \times R_+ \ : \ 1 \leq x\}\,,$$
$$E_2 = \left\{(x,y) \in R_+ \times R_+ \ : \ x < 1,\ 2\sqrt{1-x} \leq y\right\}.$$

Let $(x_0, y_0) \in E_1$. In the example appearing in On Remark 4.2 (see the case $(x_0, y_0) \in \widetilde{H}_1$, p. 97), the functions p and τ are constructed such that (12.204) holds, and the problem

$$u'(t) = p(t)u(\tau(t)), \qquad \lambda u(a) + \mu u(b) = 0$$

has a nontrivial solution. Then, by Remark 1.1 (see p. 14), there exist $q_0 \in L\big([a,b];R\big)$ and $c_0 \in R$ such that the problem (10.1), (10.2) with

$$F(v)(t) \stackrel{\text{def}}{=} p(t)v(\tau(t)) + q_0(t), \qquad h(v) \stackrel{\text{def}}{=} c_0$$

has no solution, while the conditions (12.1) and (12.3) are fulfilled, where ℓ_0, ℓ_1, q, and c are defined by (12.207).

Let $(x_0, y_0) \in E_2$. Put $a = 0$, $b = 5$,

$$p(t) = \begin{cases} -\sqrt{1-x_0} & \text{for } t \in [0,1[\, \cup [2,3[\\ 0 & \text{for } t \in [1,2[\\ x_0 & \text{for } t \in [3,4[\\ 2\sqrt{1-x_0} - y_0 & \text{for } t \in [4,5] \end{cases},$$

$$\tau(t) = \begin{cases} 4 & \text{for } t \in [0,2[\, \cup [3,5] \\ 1 & \text{for } t \in [2,3[\end{cases}.$$

Obviously, (12.204) holds. Furthermore, define the operator $G \in K_{ab}$ by

$$G(v)(t) = \begin{cases} 0 & \text{for } t \in [0,1[\cup [2,3[\cup [4,5] \\ -v(t)|v(t)| & \text{for } t \in [1,2[\\ q_0(t) & \text{for } t \in [3,4[\end{cases},$$

where $q_0 \in L\big([a,b];R\big)$ is such that

$$\int_3^4 q_0(s)ds \geq 1\,. \tag{12.219}$$

We will show that the problem (10.1), (10.2) with F and h given by (12.209) has no solution, while the conditions (12.1) and (12.3) are fulfilled, where ℓ_0, ℓ_1, q, and c are defined by (12.210).

Indeed, suppose on the contrary that u is a solution of the problem (10.1), (10.2) with F and h given by (12.209), i.e., the equalities (1.2_0) and (12.211) hold. From (12.211), in view of (1.2_0), we get

$$u(1) = -u(4)\sqrt{1-x_0}\,, \tag{12.220}$$

$$u(2) = \frac{u(1)}{1+|u(1)|}\,, \tag{12.221}$$

$$u(3) = u(2) - u(1)\sqrt{1-x_0}\,, \tag{12.222}$$

$$u(4) = u(3) + u(4)x_0 + \int_3^4 q_0(s)ds\,. \tag{12.223}$$

The equalities (12.220), (12.222), and (12.223) imply

$$\int_3^4 q_0(s)ds = -u(2).$$

Hence, the last equality, by virtue of (12.221), yields

$$\int_3^4 q_0(s)ds = -\frac{u(1)}{1+|u(1)|} \leq \frac{|u(1)|}{1+|u(1)|} < 1,$$

which contradicts (12.219).

On Remark 12.3. Let $0 \neq |\mu| \leq |\lambda|$. It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin W$, then (x_0, y_0) belongs at least to one of the following sets:

$$W_1 = \left\{(x, y) \in R \times R \ : \ \left|\frac{\mu}{\lambda}\right| \leq y\right\},$$
$$W_2 = \left\{(x, y) \in R \times R \ : \ y < \left|\frac{\mu}{\lambda}\right|, \ x \leq \frac{|\lambda|}{|\mu| - |\lambda| y} - 1\right\},$$
$$W_3 = \left\{(x, y) \in R \times R \ : \ y < \left|\frac{\mu}{\lambda}\right|, \ 2\sqrt{\left|\frac{\mu}{\lambda}\right| - y} \leq x\right\}.$$

Let $(x_0, y_0) \in W_1$. Put $a = 0$, $b = 3$, $\varepsilon = \frac{1}{1+x_0}$,

$$p(t) = \begin{cases} 0 & \text{for } t \in [0, 1[\\ -y_0 & \text{for } t \in [1, 2[\\ x_0 & \text{for } t \in [2, 3] \end{cases}, \quad \tau(t) = \begin{cases} 3 & \text{for } t \in [0, 2[\\ 2 & \text{for } t \in [2, 3] \end{cases},$$

$$z(t) = \begin{cases} \frac{|\lambda|(y_0+\varepsilon)-|\mu|}{(|\lambda|(y_0+\varepsilon)-|\mu|)t+|\mu|} & \text{for } t \in [0, 1[\\ 0 & \text{for } t \in [1, 3] \end{cases}.$$

It is not difficult to verify that (12.204) holds, and the problem (12.205) has the nontrivial solution

$$u(t) = \begin{cases} (|\lambda|(y_0 + \varepsilon) - |\mu|)t + |\mu| & \text{for } t \in [0, 1[\\ y_0|\lambda|(2 - t) + |\lambda|\varepsilon & \text{for } t \in [1, 2[\\ |\lambda|(1 - \varepsilon)(t - 3) + |\lambda| & \text{for } t \in [2, 3] \end{cases}.$$

Then, by Remark 1.1 (see p. 14), there exist $q_0 \in L([a, b]; R)$ and $c_0 \in R$ such that the problem (10.1), (10.2) with F and h given by (12.206) has no solution, while the conditions (12.8) and (12.9) are fulfilled, where ℓ_0, ℓ_1, q, and c are defined by (12.207).

Let $(x_0, y_0) \in W_2$. Put $a = 0$, $b = 3$,

$$p(t) = \begin{cases} -y_0 & \text{for } t \in [0, 1[\\ x_0 & \text{for } t \in [1, 2[\\ 0 & \text{for } t \in [2, 3] \end{cases}, \quad \tau(t) = \begin{cases} 3 & \text{for } t \in [0, 1[\\ 1 & \text{for } t \in [1, 3] \end{cases},$$

$$z(t) = \begin{cases} 0 & \text{for } t \in [0, 2[\\ \frac{|\lambda|-(|\mu|-|\lambda|y_0)(1+x_0)}{(|\lambda|-(|\mu|-|\lambda|y_0)(1+x_0))(t-3)+|\lambda|} & \text{for } t \in [2, 3] \end{cases}.$$

It is not difficult to verify that (12.204) holds, and the problem (12.205) has the nontrivial solution

$$u(t) = \begin{cases} -y_0|\lambda|t + |\mu| & \text{for } t \in [0,1[\\ x_0(|\mu| - |\lambda|y_0)(t-1) + |\mu| - |\lambda|y_0 & \text{for } t \in [1,2[\\ (|\lambda| - (|\mu| - |\lambda|y_0)(1+x_0))(t-3) + |\lambda| & \text{for } t \in [2,3] \end{cases}.$$

Then, by Remark 1.1 (see p. 14), there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (10.1), (10.2) with F and h given by (12.206) has no solution, while the conditions (12.8) and (12.9) are fulfilled, where ℓ_0, ℓ_1, q, and c are defined by (12.207).

Let $(x_0, y_0) \in W_3$. Put $a = 0$, $b = 6$,

$$p(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[\\ \sqrt{\left|\frac{\mu}{\lambda}\right| - y_0} & \text{for } t \in [1,2[\cup [3,4[\\ 0 & \text{for } t \in [2,3[\cup [4,5[\\ x_0 - 2\sqrt{\left|\frac{\mu}{\lambda}\right| - y_0} & \text{for } t \in [5,6] \end{cases},$$

and

$$\tau(t) = \begin{cases} 6 & \text{for } t \in [0,1[\cup [3,6] \\ 3 & \text{for } t \in [1,3[\end{cases}.$$

Obviously, (12.204) holds. Furthermore, define the operator $G \in K_{ab}$ by

$$G(v)(t) = \begin{cases} 0 & \text{for } t \in [0,1[\cup [3,4[\cup [5,6] \\ q_0(t) & \text{for } t \in [1,2[\\ v(t)|v(t)| & \text{for } t \in [2,3[\cup [4,5[\end{cases},$$

where $q_0 \in L([a,b];R)$ is such that

$$\int_1^2 q_0(s)ds \ge 1 + \sqrt{\left|\frac{\mu}{\lambda}\right| - y_0}\,. \tag{12.224}$$

We will show that the problem (10.1), (10.2) with F and h given by (12.209) has no solution, while the conditions (12.8) and (12.9) are fulfilled, where ℓ_0, ℓ_1, q, and c are defined by (12.210).

Indeed, suppose on the contrary that u is a solution of the problem (10.1), (10.2) with F and h given by (12.209), i.e., the equalities (1.2_0) and (12.211) hold. From (12.211) we get

$$u(1) = u(0) - u(6)y_0\,, \tag{12.225}$$

$$u(2) = u(1) + u(3)\sqrt{\left|\frac{\mu}{\lambda}\right| - y_0} + \int_1^2 q_0(s)ds\,, \tag{12.226}$$

$$u(2) = \frac{u(3)}{1 + |u(3)|}\,, \tag{12.227}$$

$$u(4) = u(3) + u(6)\sqrt{\left|\frac{\mu}{\lambda}\right| - y_0}\,, \tag{12.228}$$

$$u(4) = \frac{u(5)}{1 + |u(5)|}\,. \tag{12.229}$$

The equalities (12.225), (12.226), and (12.228), in view of (1.2_0) and (2.1), result in

$$u(2) = u(4)\sqrt{\left|\frac{\mu}{\lambda}\right| - y_0} + \int_1^2 q_0(s)ds.$$

Hence, by virtue of (12.227) and (12.229), we get

$$\int_1^2 q_0(s)ds = \frac{u(3)}{1 + |u(3)|} - \frac{u(5)}{1 + |u(5)|}\sqrt{\left|\frac{\mu}{\lambda}\right| - y_0} \leq$$

$$\leq \frac{|u(3)|}{1 + |u(3)|} + \frac{|u(5)|}{1 + |u(5)|}\sqrt{\left|\frac{\mu}{\lambda}\right| - y_0} < 1 + \sqrt{\left|\frac{\mu}{\lambda}\right| - y_0}\,,$$

which contradicts (12.224).

On Remark 12.4. Let $|\mu| < |\lambda|$, $\varepsilon > 0$, $\ell, \ell_0 \in \mathcal{L}_{ab}$ be defined by (4.59), where $p \in L\big([a,b]; R_+\big)$ satisfies (4.60). According to Example 4.1 (see p. 98), the problem (1.1_0), (1.2_0) has a nontrivial solution. By Remark 1.1 (see p. 14), there exist $q_0 \in L\big([a,b]; R\big)$ and $c_0 \in R$ such that the problem (10.1), (10.2) with

$$F(v)(t) \stackrel{\text{def}}{=} \ell(v)(t) + q_0(t) \quad \text{for} \quad t \in [a,b], \qquad h(v) \stackrel{\text{def}}{=} c_0 \tag{12.230}$$

has no solution, while the conditions (12.1), (12.13), and (12.14) are fulfilled with $\ell_1 \equiv 0$, $c = |c_0|$, and $q \equiv |q_0|$. Thus, the condition (12.12) in Theorem 12.4 cannot be replaced by the condition (12.14), no matter how small $\varepsilon > 0$ would be.

Let $|\mu| < |\lambda|$, $\varepsilon \in \,]0,1[\,$, $\ell \in \mathcal{L}_{ab}$ be defined by (4.61), where $p \in L\big([a,b];R_+\big)$ satisfies (4.62). According to Example 4.2 (see p. 98), the problem (1.1_0), (1.2_0) has a nontrivial solution. By Remark 1.1 (see p. 14), there exist $q_0 \in L\big([a,b];R\big)$ and $c_0 \in R$ such that the problem (10.1), (10.2) with F and h given by (12.230) has no solution, while the conditions (12.1), (12.12), and (12.15) are fulfilled with $\ell_0 \equiv \ell$, $\ell_1 \equiv 0$, $c = |c_0|$, and $q \equiv |q_0|$. Thus, the condition (12.13) in Theorem 12.4 cannot be replaced by the condition (12.15), no matter how small $\varepsilon > 0$ would be.

Example 12.1. Let $|\mu| < |\lambda|$, $a = 0$, $b = 5$, and $\varepsilon \in \,]0,1[\,$. Choose $\delta \in \,]0,\varepsilon[$ and $\vartheta > 0$ such that

$$\vartheta \le \min\left\{\frac{\varepsilon-\delta}{1-\varepsilon}, 1-\delta\right\}.$$

Let, moreover, $\ell \in \mathcal{L}_{ab}$ be an operator defined by (4.63), where

$$p(t) = \begin{cases} \frac{|\lambda|-|\mu|}{|\lambda|} - \delta & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,2[\,\cup\,[3,4[\\ -\frac{1+\vartheta}{1-\delta} & \text{for } t \in [2,3[\\ -(1+\vartheta) & \text{for } t \in [4,5] \end{cases}, \quad \tau(t) = \begin{cases} 5 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,3[\\ 3 & \text{for } t \in [3,5] \end{cases},$$

and let $\ell_0, \ell_1 \in \mathcal{L}_{ab}$ be defined by (4.64), where $p_0 \equiv [p]_+$, $p_1 \equiv [p]_-$, $\tau_0 \equiv 5$, and

$$\tau_1(t) = \begin{cases} 0 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,3[\\ 3 & \text{for } t \in [3,5] \end{cases}.$$

Put

$$z(t) = \begin{cases} 0 & \text{for } t \in [0,1[\,\cup\,[2,3[\,\cup\,[4,5[\\ -\frac{1-\delta-\vartheta}{(1-\delta-\vartheta)(1-t)+1-\delta} & \text{for } t \in [1,2[\\ -\frac{1-\vartheta}{1-(1-\vartheta)(t-3)} & \text{for } t \in [3,4[\end{cases}.$$

It is clear that $z \in L\big([0,5];R_-\big)$, $\ell_0, \ell_1 \in \mathcal{P}_{05}$, and

$$\int_0^5 \ell_0(1)(s)ds = \int_0^1 p_0(s)ds = \frac{|\lambda|-|\mu|}{|\lambda|} - \delta < \frac{|\lambda|-|\mu|}{|\lambda|}.$$

Consequently, according to Remark 2.5 (see p. 19), we have $\ell_0 \in V_{ab}^+(\lambda,\mu)$. Furthermore, it is not difficult to verify that the homogeneous problem

$$u'(t) = -(1-\varepsilon)\ell_1(u)(t), \qquad \lambda u(0) + \mu u(5) = 0$$

has only the trivial solution and, for arbitrary $q_0 \in L\big([0,5];R_+\big)$ and $c_0 \in R$ satisfying $c_0 \operatorname{sgn}\lambda \geq 0$, the solution of the problem

$$u'(t) = -(1-\varepsilon)\ell_1(u)(t) + q_0(t), \qquad \lambda u(0) + \mu u(5) = c_0$$

is nonnegative. Therefore, by Definition 2.1 (see p. 15), we obtain

$$-(1-\varepsilon)\ell_1 \in V_{ab}^+(\lambda,\mu).$$

On the other hand, the function

$$u(t) = \begin{cases} \left(\frac{|\lambda|-|\mu|}{|\lambda|} - \delta\right) t + \left|\frac{\mu}{\lambda}\right| & \text{for } t \in [0,1[\\ (1-\delta-\vartheta)(1-t) + 1 - \delta & \text{for } t \in [1,2[\\ (1+\vartheta)(2-t) + \vartheta & \text{for } t \in [2,3[\\ (1-\vartheta)(t-3) - 1 & \text{for } t \in [3,4[\\ (1+\vartheta)(t-4) - \vartheta & \text{for } t \in [4,5] \end{cases}$$

is a nontrivial solution of the problem (12.205). Therefore, according to Remark 1.1 (see p. 14), there exist $q_0 \in L\big([a,b];R\big)$ and $c_0 \in R$ such that the problem (10.1), (10.2) with F and h given by (12.206) has no solution, while the conditions (12.1), (12.12), and (12.16) are fulfilled with $c = |c_0|$ and $q \equiv |q_0|$.

Example 12.2. Let $|\mu| < |\lambda|$ and $\varepsilon \geq 0$. Put $a = 0$, $b = 5$,

$$g(t) = \begin{cases} 0 & \text{for } t \in [0,1[\, \cup\, [2,3[\\ 1 & \text{for } t \in [1,2[\, \cup\, [3,4[\\ \varepsilon & \text{for } t \in [4,5] \end{cases}, \qquad \nu(t) = \begin{cases} 5 & \text{for } t \in [0,3[\\ 2 & \text{for } t \in [3,4[\\ 1 & \text{for } t \in [4,5] \end{cases}.$$

Furthermore, define the operator $G \in K_{ab}$ by

$$G(v)(t) = \begin{cases} -v(t)|v(t)| & \text{for } t \in [0,1[\, \cup\, [2,3[\\ 0 & \text{for } t \in [1,2[\, \cup\, [3,4[\\ q_0(t) & \text{for } t \in [4,5] \end{cases},$$

where $q_0 \in L\big([a,b];R\big)$ is such that

$$\int_4^5 q_0(s)ds \ge 2+\varepsilon\,. \tag{12.231}$$

Let, moreover, the function γ be defined by (4.66), where $\delta > \frac{|\mu|}{|\lambda|-|\mu|}(2+\varepsilon)$. It is clear that $\gamma \in \widetilde{C}\big([a,b];]0,+\infty[\big)$,

$$\gamma(b)-\gamma(a) = 2+\varepsilon,$$

and the conditions (12.17) and (12.18) hold with ℓ_0 and ℓ_1 given by (4.65).

We will show that the problem (10.1), (10.2) with

$$F(v)(t) \stackrel{\text{def}}{=} -g(t)v(\nu(t)) + G(v)(t), \qquad h(v) \stackrel{\text{def}}{=} 0 \tag{12.232}$$

has no solution, while the conditions (12.1) and (12.3) are fulfilled, whith $q \equiv |q_0|$ and $c=0$.

Indeed, suppose on the contrary that u is a solution of the problem (10.1), (10.2) with F and h given by (12.232), i.e., the equality (1.2_0) holds and

$$u'(t) = -g(t)u(\nu(t)) + G(u)(t) \quad \text{for} \quad t\in[a,b]. \tag{12.233}$$

From (12.233) we get

$$u(1) = \frac{u(0)}{1+|u(0)|}\,, \tag{12.234}$$

$$u(2) = u(1)-u(5)\,, \tag{12.235}$$

$$u(3) = \frac{u(2)}{1+|u(2)|}\,, \tag{12.236}$$

$$u(4) = u(3)-u(2)\,, \tag{12.237}$$

$$u(5) = u(4) - \varepsilon u(1) + \int_4^5 q_0(s)ds\,. \tag{12.238}$$

The equalities (12.235), (12.237), and (12.238) imply

$$u(3) = \big(1+\varepsilon\big)u(1) - \int_4^5 q_0(s)ds.$$

Hence, by virtue of (12.234) and (12.236), we get

$$\int_4^5 q_0(s)ds = \big(1+\varepsilon\big)\frac{u(0)}{1+|u(0)|} - \frac{u(2)}{1+|u(2)|} \leq$$

$$\leq \big(1+\varepsilon\big)\frac{|u(0)|}{1+|u(0)|} + \frac{|u(2)|}{1+|u(2)|} < 2+\varepsilon\,,$$

which contradicts (12.231).

Example 12.3. Let $\delta_0 \in \,]0,1[$ be a number satisfying

$$\frac{1-\delta_0}{\delta_0} = \sqrt{\delta_0}\,,$$

$\left|\frac{\mu}{\lambda}\right| \in \,]\delta_0, 1]$, $\varepsilon \geq 0$, $a = 0$, $b = 5$, and let $\ell_0, \ell_1 \in \mathcal{L}_{ab}$ be operators defined by

$$\ell_0(v)(t) \overset{\text{def}}{=} p_0(t)v(\tau(t)), \qquad \ell_1(v)(t) \overset{\text{def}}{=} p_1(t)v(\tau(t)) \quad \text{for} \quad t \in [a,b],$$

where

$$p_0(t) = \begin{cases} \sqrt{\left|\frac{\mu}{\lambda}\right|} & \text{for} \quad t \in [0,1[\\ 0 & \text{for} \quad t \in [1,5] \end{cases}, \quad p_1(t) = \begin{cases} 0 & \text{for} \quad t \in [0,2[\, \cup\, [3,4[\\ \sqrt{\left|\frac{\mu}{\lambda}\right|} & \text{for} \quad t \in [2,3[\\ \varepsilon & \text{for} \quad t \in [4,5] \end{cases},$$

$$\tau(t) = \begin{cases} 2 & \text{for} \quad t \in [0,2[\\ 5 & \text{for} \quad t \in [2,5] \end{cases}.$$

Put

$$G(v)(t) = \begin{cases} q_0(t) & \text{for } t \in [0,1[\\ v(t)|v(t)| & \text{for } t \in [1,2[\, \cup\, [3,4[\\ 0 & \text{for } t \in [2,3[\, \cup\, [4,5] \end{cases},$$

where $q_0 \in L\big([a,b];R\big)$ is such that

$$\int_0^1 q_0(s)ds \geq 1 + \sqrt{\left|\frac{\mu}{\lambda}\right|}. \tag{12.239}$$

It is clear that $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and

$$\int\limits_a^b \big(\ell_0(1)(s) + \ell_1(1)(s)ds\big) = \int\limits_0^5 \big(p_0(s) + p_1(s)\big)ds = 2\sqrt{\left|\frac{\mu}{\lambda}\right|} + \varepsilon.$$

Moreover, according to the condition

$$\int\limits_a^b p_0(s)ds = \sqrt{\left|\frac{\mu}{\lambda}\right|}$$

and Theorem 2.11 (with $\ell_1 \equiv 0$, see p. 26), we find $\ell_0 \in V_{ab}^-(\lambda, \mu)$.

We will show that the problem (10.1), (10.2) with

$$F(v)(t) \overset{\text{def}}{=} \big(p_0(t) + p_1(t)\big)v(\tau(t)) + G(v)(t), \qquad h(v) \overset{\text{def}}{=} 0 \qquad (12.240)$$

has no solution, while the conditions (12.8), (12.20), and (12.21) are fulfilled with $q \equiv |q_0|$ and $c = 0$.

Indeed, suppose on the contrary that u is a solution of the problem (10.1), (10.2) with F and h given by (12.240), i.e., the equality (1.2_0) holds and

$$u'(t) = \big(p_0(t) + p_1(t)\big)u(\tau(t)) + G(u)(t) \quad \text{for} \quad t \in [a, b]. \qquad (12.241)$$

From (12.241) we get

$$u(1) = u(0) + u(2)\sqrt{\left|\frac{\mu}{\lambda}\right|} + \int\limits_0^1 q_0(s)ds\,, \qquad (12.242)$$

$$u(1) = \frac{u(2)}{1 + |u(2)|}\,, \qquad (12.243)$$

$$u(3) = u(2) + u(5)\sqrt{\left|\frac{\mu}{\lambda}\right|}\,, \qquad (12.244)$$

$$u(3) = \frac{u(4)}{1 + |u(4)|}\,. \qquad (12.245)$$

The equalities (12.242) and (12.244), in view of (1.2_0) and (2.1), result in

$$u(1) = u(3)\sqrt{\left|\frac{\mu}{\lambda}\right|} + \int\limits_0^1 q_0(s)ds.$$

Hence, by virtue of (12.243) and (12.245), we get

$$\int_0^1 q_0(s)ds = \frac{u(2)}{1+|u(2)|} - \frac{u(4)}{1+|u(4)|}\sqrt{\left|\frac{\mu}{\lambda}\right|} \le$$

$$\le \frac{|u(2)|}{1+|u(2)|} + \frac{|u(4)|}{1+|u(4)|}\sqrt{\left|\frac{\mu}{\lambda}\right|} < 1 + \sqrt{\left|\frac{\mu}{\lambda}\right|},$$

which contradicts (12.239).

On Remark 12.7. Let $\delta_0 \in \,]0,1[$ be a number satisfying

$$\ln\frac{1}{\delta_0} = 2\sqrt{\delta_0}\,,$$

$\left|\frac{\mu}{\lambda}\right| \in \,]\delta_0, 1[$, $\varepsilon > 0$, and $\ell_0 \in \mathcal{L}_{ab}$ be defined by

$$\ell_0(v)(t) \overset{\text{def}}{=} p(t)v(t) \quad \text{for} \quad t \in [a,b],$$

where $p \in L\big([a,b];R_+\big)$ is such that

$$\int_a^b p(s)ds = \ln\left|\frac{\lambda}{\mu}\right| .$$

Put $\ell_1 \equiv 0$. Then the condition (12.22) holds and according to Corollary 3.5 (see p. 70), we have $(1+\varepsilon)\ell_0 \in V_{ab}^-(\lambda,\mu)$.

On the other hand, the problem

$$u'(t) = \ell_0(u)(t), \qquad \lambda u(a) + \mu u(b) = 0$$

has a nontrivial solution

$$u(t) = |\mu| \exp\left(\int_a^t p(s)ds\right) \quad \text{for} \quad t \in [a,b].$$

Therefore, by Remark 1.1 (see p. 14), there exist $q_0 \in L\big([a,b];R\big)$ and $c_0 \in R$ such that the problem (10.1), (10.2) with

$$F(v)(t) \overset{\text{def}}{=} \ell_0(v)(t) + \ell_1(v)(t) + q_0(t) \quad \text{for} \quad t \in [a,b], \qquad h(v) \overset{\text{def}}{=} c_0$$

has no solution, while the conditions (12.8), (12.20), (12.22), and (12.23) are fulfilled with $q \equiv |q_0|$ and $c = |c_0|$.

§13. Periodic Type BVP for EDA

In this section we will establish some consequences of the main results from §12 for the equation with deviating arguments (10.1′). Here we will also suppose that the inequality (2.1) is fulfilled.

In what follows we will use the notation

$$p_0(t) = \sum_{j=1}^{m} p_j(t), \qquad g_0(t) = \sum_{j=1}^{m} g_j(t) \quad \text{for} \quad t \in [a,b]$$

and we will suppose that the function $q \in K\big([a,b] \times R_+; R_+\big)$ is nondecreasing in the second argument and satisfies (10.5), i.e.,

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)ds = 0.$$

13.1. Existence and Uniqueness Theorems

In the case, where $|\mu| \le |\lambda|$, the following assertions hold.

Theorem 13.1. *Let* $0 \ne |\mu| \le |\lambda|$, $p_k, g_k \in L\big([a,b]; R_+\big)$ $(k = 1, \dots, m)$, $c \in R_+$, *the condition* (12.1) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality*

$$f(t, x, x_1, \dots, x_n) \operatorname{sgn} x \le q(t, |x|) \tag{13.1}$$

hold. If, moreover,

$$\|p_0\|_L < 1, \tag{13.2}$$

$$\frac{\|p_0\|_L}{1 - \|p_0\|_L} - \frac{|\lambda| - |\mu|}{|\mu|} < \|g_0\|_L < 2\sqrt{1 - \|p_0\|_L}\ , \tag{13.3}$$

then the problem (10.1′), (10.2) *has at least one solution.*

Remark 13.1. The examples constructed in Subsection 12.3 (see On Remark 12.1, p. 240) also show that neither one of the strict inequalities in (13.2) and (13.3) can be replaced by the nonstrict one.

The next theorem can be understood as a supplement of the previous one for the case $\mu = 0$.

Theorem 13.2. *Let* $\mu = 0$, $p_k, g_k \in L([a,b]; R_+)$ $(k = 1, \dots, m)$, $c \in R_+$, *the condition* (12.1) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.1) *hold. If, moreover,*

$$\|p_0\|_L < 1, \tag{13.4}$$

$$\|g_0\|_L < 2\sqrt{1 - \|p_0\|_L}\ , \tag{13.5}$$

then the problem (10.1′), (10.2) *has at least one solution.*

Remark 13.2. The examples constructed in Subsection 12.3 (see On Remark 12.2, p. 243) also show that the strict inequalities (13.4) and (13.5) cannot be replaced by the nonstrict ones.

Theorem 13.3. *Let* $0 \neq |\mu| \leq |\lambda|$, $p_k, g_k \in L([a,b]; R_+)$ $(k = 1, \dots, m)$, $c \in R_+$, *the condition* (12.8) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality*

$$f(t, x, x_1, \dots, x_n) \operatorname{sgn} x \geq -q(t, |x|) \tag{13.6}$$

hold. If, moreover,

$$\|g_0\|_L < \left|\frac{\mu}{\lambda}\right|, \tag{13.7}$$

$$\frac{|\lambda|}{|\mu| - |\lambda| \|g_0\|_L} - 1 < \|p_0\|_L < 2\sqrt{\left|\frac{\mu}{\lambda}\right| - \|g_0\|_L}\ , \tag{13.8}$$

then the problem (10.1′), (10.2) *has at least one solution.*

Remark 13.3. The examples constructed in Subsection 12.3 (see On Remark 12.3, p. 245) also show that neither one of the strict inequalities in (13.7) and (13.8) can be replaced by the nonstrict one.

Theorem 13.4. *Let* $|\mu| < |\lambda|$, $p_k, g_k \in L([a,b]; R_+)$, $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, $c \in R_+$, *the condition* (12.1) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.1) *hold. Let, moreover, the functions* p_k, τ_k $(k = 1, \dots, m)$ *satisfy at least one of the conditions a), b) or c) in Theorem 3.1 (see p. 63) or the assumptions of Theorem 3.2 (see p. 64), while the functions* g_k, ν_k $(k = 1, \dots, m)$ *satisfy* $\nu_k(t) \leq t$ *for* $t \in [a,b]$ $(k = 1, \dots, m)$ *and at least one of the conditions a), b) or c) in Theorem 3.3 (see p. 65). Then the problem* (10.1′), (10.2) *has at least one solution.*

Theorem 13.5. *Let* $|\mu| < |\lambda|$, $p_k, g_k \in L([a,b]; R_+)$, $\tau_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, $c \in R_+$, *the conndition* (12.1) *be fulfilled and let on the set*

$[a,b] \times R^{n+1}$ *the inequality* (13.1) *hold. If, moreover,*

$$|\mu| \exp\left(\int_a^b p_0(s)ds\right) < |\lambda|, \tag{13.9}$$

$$\tau_k(t) \le t \quad for \quad t \in [a,b] \quad (k=1,\dots,m), \tag{13.10}$$

and

$$\frac{|\lambda| - |\mu|}{|\lambda| - |\mu| \exp\left(\int_a^b p_0(s)ds\right)} \int_a^b g_0(s) \exp\left(\int_s^b p_0(\xi)d\xi\right) ds < 2, \tag{13.11}$$

then the problem (10.1′), (10.2) *has at least one solution.*

Theorem 13.6. *Let* $|\mu| < |\lambda|$, $p_k, g_k \in L([a,b];R_+)$, $\tau_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$, $c \in R_+$, *the condition* (12.1) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.1) *hold. If, moreover,*

$$\frac{|\lambda| - |\mu|}{|\lambda|} \left(\int_a^b g_0(s)ds + \alpha_1\right) + 2\beta_1 < 2, \tag{13.12}$$

where

$$\alpha_1 = \int_a^b \sum_{k=1}^m p_k(s) \left(\int_a^{\tau_k(s)} g_0(\xi)d\xi\right) \exp\left(\int_s^b p_0(\xi)d\xi\right) ds, \tag{13.13}$$

$$\begin{aligned} \beta_1 = \left|\frac{\mu}{\lambda}\right| \exp\left(\int_a^b p_0(s)ds\right) + \\ + \int_a^b \sum_{k=1}^m p_k(s)\sigma_k(s) \left(\int_s^{\tau_k(s)} p_0(\xi)d\xi\right) \exp\left(\int_s^b p_0(\xi)d\xi\right) ds, \end{aligned} \tag{13.14}$$

$$\sigma_k(t) = \frac{1}{2}\left(1 + \operatorname{sgn}(\tau_k(t) - t)\right) \quad for \quad t \in [a,b] \quad (k=1,\dots,m), \tag{13.15}$$

then the problem (10.1′), (10.2) *has at least one solution.*

Remark 13.4. Example 12.2 (see p. 249) also shows that the strict inequalities (13.11) in Theorem 13.5 and (13.12) in Theorem 13.6 cannot be replaced by the nonstrict ones.

The next theorem concerns the equation with deviating arguments of the form

$$u'(t) = \sum_{k=1}^{m} \Big(p_k(t)u(\tau_k(t)) + g_k(t)u(\nu_k(t)) \Big) + \\ + f(t, u(t), u(\zeta_1(t)), \dots, u(\zeta_n(t))), \tag{13.16}$$

where $f \in K([a,b] \times R^{n+1}; R)$, $p_k, g_k \in L\big([a,b]; R_+\big)$, $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, $\zeta_j \in \mathcal{M}_{ab}$ $(j = 1, \dots, n)$, and $m, n \in N$.

Theorem 13.7. *Let* $0 \neq |\mu| < |\lambda|$, $p_k, g_k \in L\big([a,b]; R_+\big)$, $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, $c \in R_+$, *the condition* (12.8) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.6) *hold. Let, moreover,*

$$\int_a^b \big(p_0(s) + g_0(s)\big) ds < 2\sqrt{\left|\frac{\mu}{\lambda}\right|} \tag{13.17}$$

and the functions p_k, τ_k $(k = 1, \dots, m)$ *satisfy the assumptions of Theorem 3.9 (see p. 69) or Theorem 3.10 (see p. 69). Then the problem* (13.16), (10.2) *has at least one solution.*

In Theorems 13.8–13.14, the conditions guaranteeing the unique solvability of the problem (10.1′), (10.2) are established.

Theorem 13.8. *Let* $0 \neq |\mu| \leq |\lambda|$, $p_k, g_k \in L\big([a,b]; R_+\big)$ $(k = 1, \dots, m)$, *the condition* (12.24) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality*

$$[f(t, x, x_1, \dots, x_n) - f(t, y, y_1, \dots, y_n)] \operatorname{sgn}(x - y) \leq 0 \tag{13.18}$$

hold. If, moreover, the inequalities (13.2) *and* (13.3) *are fulfilled, then the problem* (10.1′), (10.2) *is uniquely solvable.*

Remark 13.5. The examples constructed in Subsection 12.3 (see On Remark 12.1, p. 240) also show that neither one of the strict inequalities in (13.2) and (13.3) can be replaced by the nonstrict one.

The next theorem can be understood as a supplement of the previous one for the case $\mu = 0$.

Theorem 13.9. *Let* $\mu = 0$, $p_k, g_k \in L([a,b]; R_+)$ $(k = 1, \dots, m)$, *the condition* (12.24) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.18) *hold. If, moreover, the inequalities* (13.4) *and* (13.5) *are fulfilled, then the problem* (10.1′), (10.2) *is uniquely solvable.*

Remark 13.6. The examples constructed in Subsection 12.3 (see On Remark 12.2, p. 243) also show that the strict inequalities (13.4) and (13.5) cannot be replaced by the nonstrict ones.

Theorem 13.10. *Let* $0 \neq |\mu| \leq |\lambda|$, $p_k, g_k \in L([a,b]; R_+)$ $(k = 1, \dots, m)$, *the condition* (12.26) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality*

$$[f(t, x, x_1, \dots, x_n) - f(t, y, y_1, \dots, y_n)] \operatorname{sgn}(x - y) \geq 0 \tag{13.19}$$

hold. If, moreover, the inequalities (13.7) *and* (13.8) *are fulfilled, then the problem* (10.1′), (10.2) *is uniquely solvable.*

Remark 13.7. The examples constructed in Subsection 12.3 (see On Remark 12.3, p. 245) also show that neither one of the strict inequalities in (13.7) and (13.8) can be replaced by the nonstrict one.

Theorem 13.11. *Let* $|\mu| < |\lambda|$, $p_k, g_k \in L([a,b]; R_+)$, $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, *the condition* (12.24) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.18) *hold. Let, moreover, the functions* p_k, τ_k $(k = 1, \dots, m)$ *satisfy at least one of the conditions a), b) or c) in Theorem 3.1 (see p. 63) or the assumptions of Theorem 3.2 (see p. 64), while the functions* g_k, ν_k $(k = 1, \dots, m)$ *satisfy* $\nu_k(t) \leq t$ *for* $t \in [a,b]$ $(k = 1, \dots, m)$ *and at least one of the conditions a), b) or c) in Theorem 3.3 (see p. 65). Then the problem* (10.1′), (10.2) *is uniquely solvable.*

Theorem 13.12. *Let* $|\mu| < |\lambda|$, $p_k, g_k \in L([a,b]; R_+)$, $\tau_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, *the condition* (12.24) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.18) *hold. If, moreover, the inequalities* (13.9)–(13.11) *are satisfied, then the problem* (10.1′), (10.2) *is uniquely solvable.*

Theorem 13.13. *Let* $|\mu| < |\lambda|$, $p_k, g_k \in L([a,b]; R_+)$, $\tau_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, *the condition* (12.24) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.18) *hold. If, moreover, the inequality* (13.12) *holds, where* α_1 *and* β_1 *are defined by* (13.13) *and* (13.14) *with* σ_k $(k = 1, \dots, m)$ *given by* (13.15), *then the problem* (10.1′), (10.2) *is uniquely solvable.*

Remark 13.8. Example 12.2 (see p. 249) also shows that the strict inequalities (13.11) in Theorem 13.12 and (13.12) in Theorem 13.13 cannot be replaced by the nonstrict ones.

The next theorem deals with the equation with deviating arguments (13.16).

Theorem 13.14. *Let* $0 \neq |\mu| < |\lambda|$, $p_k, g_k \in L\big([a,b]; R_+\big)$, $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, *the condition* (12.26) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.19) *hold. Let, moreover, the inequality* (13.17) *be fulfilled and the functions* p_k, τ_k $(k = 1, \dots, m)$ *satisfy the assumptions of Theorem 3.9 (see p. 69) or Theorem 3.10 (see p. 69). Then the problem* (13.16), (10.2) *has a unique solution.*

In the case, where $|\mu| \geq |\lambda|$, the following statements hold.

Theorem 13.15. *Let* $|\mu| \geq |\lambda| \neq 0$, $p_k, g_k \in L\big([a,b]; R_+\big)$ $(k = 1, \dots, m)$, $c \in R_+$, *the conndition* (12.8) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.6) *hold. If, moreover,*

$$\|g_0\|_L < 1, \tag{13.20}$$

$$\frac{\|g_0\|_L}{1 - \|g_0\|_L} - \frac{|\mu| - |\lambda|}{|\lambda|} < \|p_0\|_L < 2\sqrt{1 - \|g_0\|_L}\ , \tag{13.21}$$

then the problem (10.1′), (10.2) *has at least one solution.*

The next theorem can be understood as a supplement of the previous one for the case $\lambda = 0$.

Theorem 13.16. *Let* $\lambda = 0$, $p_k, g_k \in L\big([a,b]; R_+\big)$ $(k = 1, \dots, m)$, $c \in R_+$, *the conndition* (12.8) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.6) *hold. If, moreover,*

$$\|g_0\|_L < 1, \tag{13.22}$$

$$\|p_0\|_L < 2\sqrt{1 - \|g_0\|_L}\ , \tag{13.23}$$

then the problem (10.1′), (10.2) *has at least one solution.*

Theorem 13.17. *Let* $|\mu| \geq |\lambda| \neq 0$*,* $p_k, g_k \in L\big([a,b]; R_+\big)$ $(k = 1, \dots, m)$*,* $c \in R_+$*, the condition* (12.1) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.1) *hold. If, moreover,*

$$\|p_0\|_L < \left|\frac{\lambda}{\mu}\right|, \tag{13.24}$$

$$\frac{|\mu|}{|\lambda| - |\mu|\|p_0\|_L} - 1 < \|g_0\|_L < 2\sqrt{\left|\frac{\lambda}{\mu}\right| - \|p_0\|_L}\ , \tag{13.25}$$

then the problem (10.1′), (10.2) *has at least one solution.*

Theorem 13.18. *Let* $|\mu| > |\lambda|$*,* $p_k, g_k \in L\big([a,b]; R_+\big)$*,* $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$*,* $c \in R_+$*, the condition* (12.8) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.6) *hold. Let, moreover, the functions* p_k, τ_k $(k = 1, \dots, m)$ *satisfy* $\tau_k(t) \geq t$ *for* $t \in [a,b]$ $(k = 1, \dots, m)$ *and at least one of the conditions a), b) or c) in Theorem 3.14 (see p. 72), while the functions* g_k, ν_k $(k = 1, \dots, m)$ *satisfy at least one of the conditions a), b) or c) in Theorem 3.12 (see p. 70) or the assumptions of Theorem 3.13 (see p. 71). Then the problem* (10.1′), (10.2) *has at least one solution.*

Theorem 13.19. *Let* $|\mu| > |\lambda|$*,* $p_k, g_k \in L\big([a,b]; R_+\big)$*,* $\nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$*,* $c \in R_+$*, the conndition* (12.8) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.6) *hold. If, moreover,*

$$|\lambda| \exp\left(\int_a^b g_0(s)ds\right) < |\mu|, \tag{13.26}$$

$$\nu_k(t) \geq t \quad \textit{for} \quad t \in [a,b] \quad (k = 1, \dots, m), \tag{13.27}$$

and

$$\frac{|\mu| - |\lambda|}{|\mu| - |\lambda| \exp\left(\int_a^b g_0(s)ds\right)} \int_a^b p_0(s) \exp\left(\int_a^s g_0(\xi)d\xi\right) ds < 2, \tag{13.28}$$

then the problem (10.1′), (10.2) *has at least one solution.*

Theorem 13.20. *Let* $|\mu| > |\lambda|$, $p_k, g_k \in L\big([a,b];R_+\big)$, $\nu_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$, $c \in R_+$, *the condition* (12.8) *be fulfilled, and let on the set* $[a,b]\times R^{n+1}$ *the inequality* (13.6) *hold. If, moreover,*

$$\frac{|\mu|-|\lambda|}{|\mu|}\left(\int_a^b p_0(s)ds + \alpha_2\right) + 2\beta_2 < 2, \tag{13.29}$$

where

$$\alpha_2 = \int_a^b \sum_{k=1}^m g_k(s)\left(\int_{\nu_k(s)}^b p_0(\xi)d\xi\right)\exp\left(\int_a^s g_0(\xi)d\xi\right)ds, \tag{13.30}$$

$$\begin{gathered}\beta_2 = \left|\frac{\lambda}{\mu}\right|\exp\left(\int_a^b g_0(s)ds\right) + \\ + \int_a^b \sum_{k=1}^m g_k(s)\sigma_k(s)\left(\int_{\nu_k(s)}^s g_0(\xi)d\xi\right)\exp\left(\int_a^s g_0(\xi)d\xi\right)ds,\end{gathered} \tag{13.31}$$

$$\sigma_k(t) = \frac{1}{2}\left(1+\operatorname{sgn}(t-\nu_k(t))\right) \quad \textit{for} \quad t\in[a,b] \quad (k=1,\dots,m), \tag{13.32}$$

then the problem $(10.1')$, (10.2) *has at least one solution.*

The next theorem concerns the equation with deviating arguments of the form

$$\begin{aligned} u'(t) = -\sum_{k=1}^m \Big(p_k(t)u(\tau_k(t)) + g_k(t)u(\nu_k(t))\Big) + \\ + f(t,u(t),u(\zeta_1(t)),\dots,u(\zeta_n(t))),\end{aligned} \tag{13.33}$$

where $f \in K([a,b]\times R^{n+1};R)$, $p_k, g_k \in L\big([a,b];R_+\big)$, $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$, $\zeta_j \in \mathcal{M}_{ab}$ $(j = 1,\dots,n)$, and $m,n \in N$.

Theorem 13.21. *Let* $|\mu| > |\lambda| \neq 0$, $p_k, g_k \in L\big([a,b];R_+\big)$, $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1,\dots,m)$, $c \in R_+$, *the condition* (12.1) *be fulfilled, and let on the set* $[a,b]\times R^{n+1}$ *the inequality* (13.1) *hold. Let, moreover,*

$$\int_a^b \big(p_0(s)+g_0(s)\big)ds < 2\sqrt{\left|\frac{\lambda}{\mu}\right|} \tag{13.34}$$

and the functions g_k, ν_k $(k = 1, \dots, m)$ *satisfy the assumptions of Theorem 3.6 (see p. 67) or Theorem 3.7 (see p. 68). Then the problem* (13.33), (10.2) *has at least one solution.*

In Theorems 13.22–13.28, the conditions guaranteeing the unique solvability of the problem (10.1′), (10.2) are established.

Theorem 13.22. *Let* $|\mu| \geq |\lambda| \neq 0$, $p_k, g_k \in L([a,b]; R_+)$ $(k = 1, \dots, m)$, *the condition* (12.26) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.19) *hold. If, moreover, the inequalities* (13.20) *and* (13.21) *are fulfilled, then the problem* (10.1′), (10.2) *is uniquely solvable.*

The next theorem can be understood as a supplement of the previous one for the case $\lambda = 0$.

Theorem 13.23. *Let* $\lambda = 0$, $p_k, g_k \in L([a,b]; R_+)$ $(k = 1, \dots, m)$, *the condition* (12.26) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.19) *hold. If, moreover, the inequalities* (13.22) *and* (13.23) *are fulfilled, then the problem* (10.1′), (10.2) *is uniquely solvable.*

Theorem 13.24. *Let* $|\mu| \geq |\lambda| \neq 0$, $p_k, g_k \in L([a,b]; R_+)$ $(k = 1, \dots, m)$, *the condition* (12.24) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.18) *hold. If, moreover, the inequalities* (13.24) *and* (13.25) *are fulfilled, then the problem* (10.1′), (10.2) *is uniquely solvable.*

Theorem 13.25. *Let* $|\mu| > |\lambda|$, $p_k, g_k \in L([a,b]; R_+)$, $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, *the condition* (12.26) *be fulfilled and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.19) *hold. Let, furthermore, the functions* p_k, τ_k $(k = 1, \dots, m)$ *satisfy* $\tau_k(t) \geq t$ *for* $t \in [a,b]$ $(k = 1, \dots, m)$ *and at least one of the conditions a), b) or c) in Theorem 3.14 (see p. 72), while the functions* g_k, ν_k $(k = 1, \dots, m)$ *satisfy at least one of the conditions a), b) or c) in Theorem 3.12 (see p. 70) or the assumptions of Theorem 3.13 (see p. 71). Then the problem* (10.1′), (10.2) *is uniquely solvable.*

Theorem 13.26. *Let* $|\mu| > |\lambda|$, $p_k, g_k \in L([a,b]; R_+)$, $\nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, *the condition* (12.26) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.19) *hold. If, moreover, the inequalities* (13.26)–(13.28) *are satisfied, then the problem* (10.1′), (10.2) *is uniquely solvable.*

Theorem 13.27. *Let* $|\mu| > |\lambda|$, $p_k, g_k \in L([a,b]; R_+)$, $\nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$, *the condition* (12.26) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$

the inequality (13.19) *hold. If, moreover, the inequality* (13.29) *holds, where* α_2 *and* β_2 *are defined by* (13.30) *and* (13.31) *with* σ_k $(k = 1, \dots, m)$ *given by* (13.32)*, then the problem* (10.1′)*,* (10.2) *is uniquely solvable.*

The next theorem deals with the equation with deviating arguments (13.33).

Theorem 13.28. *Let* $|\mu| > |\lambda| \neq 0$*,* $p_k, g_k \in L\big([a,b]; R_+\big)$*,* $\tau_k, \nu_k \in \mathcal{M}_{ab}$ $(k = 1, \dots, m)$*, the condition* (12.24) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.18) *hold. Let, moreover, the inequality* (13.34) *be fulfilled and the functions* g_k, ν_k $(k = 1, \dots, m)$ *satisfy the assumptions of Theorem 3.6 (see p. 67) or Theorem 3.7 (see p. 68). Then the problem* (13.33)*,* (10.2) *has a unique solution.*

Remark 13.9. According to Remark 12.14 (see p. 211), Theorems 13.15–13.28 can be derived from Theorems 13.1–13.14. Moreover, by virtue of Remarks 13.1–13.8, Theorems 13.15–13.17, 13.19, 13.20, 13.22–13.24, 13.26, and 13.27 are nonimprovable in an appropriate sense.

13.2. Proofs

Proof of Theorem 13.1. Obviously, the conditions (13.1)–(13.3) yield the conditions (12.3)–(12.5), where

$$F(v)(t) \overset{\text{def}}{=} \sum_{k=1}^{m} \Big(p_k(t) v(\tau_k(t)) - g_k(t) v(\nu_k(t)) \Big) + \\ + f(t, u(t), u(\zeta_1(t)), \dots, u(\zeta_n(t))), \tag{13.35}$$

$$\ell_0(v)(t) \overset{\text{def}}{=} \sum_{k=1}^{m} p_k(t) v(\tau_k(t)), \qquad \ell_1(v)(t) \overset{\text{def}}{=} \sum_{k=1}^{m} g_k(t) v(\nu_k(t)).$$

Consequently, the assumptions of Theorem 12.1 (see p. 199) are fulfilled. □

Proof of Theorem 13.2. Similarly to the proof of Theorem 13.1 one can show that the assumptions of Theorem 12.2 (see p. 200) are satisfied. □

Proof of Theorem 13.3. Similarly to the proof of Theorem 13.1 one can show that the assumptions of Theorem 12.3 (see p. 201) are satisfied. □

Proof of Theorem 13.4. Clearly, the condition (13.1) yields the condition (12.12), where F, ℓ_0, and ℓ_1 are defined by (13.35). Moreover, according to Theorems 3.1–3.3 (see pp. 63–65), the condition (12.13) holds. Therefore, the assumptions of Theorem 12.4 (see p. 203) are satisfied. $\square$

Proof of Theorem 13.5. Obviously, the condition (13.1) yields the condition (12.3), where F, ℓ_0, and ℓ_1 are defined by (13.35). Moreover, similarly to the proof of Theorem 5.4 (see p. 112), one can show that there exists a function $\gamma \in \widetilde{C}\big([a,b];\,]0,+\infty[\big)$ satisfying the inequalities (12.17)–(12.19). Therefore, the assumptions of Theorem 12.5 (see p. 204) are fulfilled. $\square$

Proof of Theorem 13.6. Obviously, the condition (13.1) yields the condition (12.3), where F, ℓ_0, and ℓ_1 are defined by (13.35). Moreover, similarly to the proof of Theorem 5.5 (see p. 113), one can show that there exists a function $\gamma \in \widetilde{C}\big([a,b];\,]0,+\infty[\big)$ satisfying the inequalities (12.17)–(12.19). Therefore, the assumptions of Theorem 12.5 (see p. 204) are fulfilled. $\square$

Proof of Theorem 13.7. Obviously, the conditions (13.6) and (13.17) yield the conditions (12.20) and (12.22), where

$$F(v)(t) \overset{\text{def}}{=} \sum_{k=1}^{m} \Big(p_k(t)v(\tau_k(t)) + g_k(t)v(\nu_k(t)) \Big) + \\ + f(t, u(t), u(\zeta_1(t)), \dots, u(\zeta_n(t))), \tag{13.36}$$

$$\ell_0(v)(t) \overset{\text{def}}{=} \sum_{k=1}^{m} p_k(t)v(\tau_k(t)), \qquad \ell_1(v)(t) \overset{\text{def}}{=} \sum_{k=1}^{m} g_k(t)v(\nu_k(t)).$$

Consequently, the assumptions of Theorem 12.6 (see p. 204) are fulfilled. $\square$

Proof of Theorem 13.8. Obviously, the conditions (13.2), (13.3), and (13.18) yield the conditions (12.4), (12.5), and (12.25), where F, ℓ_0, and ℓ_1 are defined by (13.35). Consequently, the assumptions of Theorem 12.7 (see p. 205) are fulfilled. $\square$

Proof of Theorem 13.9. Similarly to the proof of Theorem 13.8, one can show that the assumptions of Theorem 12.8 (see p. 205) are satisfied. $\square$

Proof of Theorem 13.10. Similarly to the proof of Theorem 13.8, one can show that the assumptions of Theorem 12.9 (see p. 206) are satisfied. $\square$

Proof of Theorem 13.11. Clearly, the condition (13.18) yields the condition (12.28), where F, ℓ_0, and ℓ_1 are defined by (13.35). Moreover, according to Theorems 3.1–3.3 (see pp. 63–65), the condition (12.13) holds. Therefore, the assumptions of Theorem 12.10 (see p. 206) are satisfied. □

Proof of Theorem 13.12. Similarly to the proof of Theorem 13.5 one can show that the assumptions of Theorem 12.11 (see p. 207) are satisfied. □

Proof of Theorem 13.13. Similarly to the proof of Theorem 13.6 one can show that the assumptions of Theorem 12.11 (see p. 207) are satisfied. □

Proof of Theorem 13.14. Obviously, the conditions (13.17) and (13.19) yield the conditions (12.22) and (12.29), where F, ℓ_0, and ℓ_1 are defined by (13.36). Consequently, the assumptions of Theorem 12.12 (see p. 207) are fulfilled. □

§14. Antiperiodic Type BVP

In this section, we will establish nonimprovable, in a certain sense, sufficient conditions for solvability and unique solvability of the problem (10.1), (10.2), where the boundary condition (10.2) is of an antiperiodic type, i.e., when the inequality (7.1) is satisfied. In Subsection 14.1, the main results are formulated. Theorems 14.1–14.4 deal with the case $|\mu| \leq |\lambda|$, while the case $|\mu| \geq |\lambda|$ is considered in Theorems 14.5–14.8. The proofs of the main results can be found in Subsection 14.2. Subsection 14.3 is devoted to the examples verifying the optimality of the main results.

In the sequel, we will assume that the function $q \in K\big([a,b] \times R_+; R_+\big)$ is nondecreasing in the second argument and satisfies (10.5), i.e.,

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)ds = 0.$$

14.1. Existence and Uniqueness Theorems

In the case, where $|\mu| \leq |\lambda|$, the following statements hold.

Theorem 14.1. *Let* $|\mu| \leq |\lambda|$, $c \in R_+$, *the inequality* (12.1) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}\big([a,b]; R\big)$ *the inequality* (12.3) *holds. If, moreover,*

$$\|\ell_1(1)\|_L < \alpha(\lambda, \mu), \tag{14.1}$$

where

$$\alpha(\lambda,\mu) = \begin{cases} -\frac{\mu}{\lambda} + 2\sqrt{1 - \|\ell_0(1)\|_L} & \text{if} \quad \|\ell_0(1)\|_L < 1 - \left(\frac{\mu}{\lambda}\right)^2 \\ \frac{\lambda}{\mu}\left(1 - \|\ell_0(1)\|_L\right) & \text{if} \quad \|\ell_0(1)\|_L \geq 1 - \left(\frac{\mu}{\lambda}\right)^2 \end{cases}, \tag{14.2}$$

then the problem (10.1), (10.2) *has at least one solution.*

Remark 14.1. Note that the condition $\|\ell_0(1)\|_L < 1$ is necessary for operators ℓ_0, ℓ_1 to satisfy the condition (14.1) with α given by (14.2).

Let $|\mu| \leq |\lambda|$. Denote by U the set of pairs $(x,y) \in R_+ \times R_+$ satisfying either

$$x < 1 - \left(\frac{\mu}{\lambda}\right)^2, \qquad y < -\frac{\mu}{\lambda} + 2\sqrt{1-x}$$

or

$$1 - \left(\frac{\mu}{\lambda}\right)^2 \le x, \qquad y < \frac{\lambda}{\mu}(1-x)$$

(see Fig. 14.1).

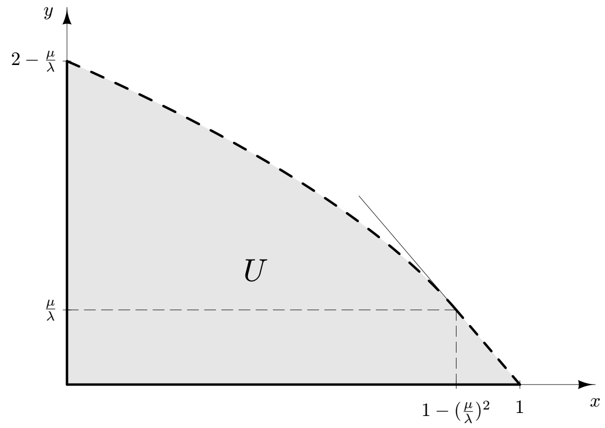

Fig. 14.1.

According to Theorem 14.1, if (12.1) holds, there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that the inequality (12.3) is satisfied on the set $B^1_{\lambda\mu c}\big([a,b];R\big)$, and

$$\Big(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L\Big) \in U,$$

then the problem (10.1), (10.2) is solvable. Below we will show (see On Remark 14.1, see p. 280) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin U$ there exist $F \in K_{ab}$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and $c_0 \in R$ such that the conditions (12.1) (with $h \equiv c_0$, $c = |c_0|$) and (12.3) hold,

$$x_0 = \|\ell_0(1)\|_L, \qquad y_0 = \|\ell_1(1)\|_L,$$

and the problem (10.1), (10.2) with $h \equiv c_0$ has no solution. In particular, the strict inequality (14.1) cannot be replaced by the nonstrict one.

Theorem 14.2. *Let $|\mu| \le |\lambda|$, $c \in R_+$, the condition* (12.8) *be fulfilled, and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that on the set $B^2_{\lambda\mu c}\big([a,b];R\big)$ the inequality* (12.9) *holds. If, moreover,*

$$\|\ell_0(1)\|_L + \frac{\mu}{\lambda}\|\ell_1(1)\|_L < \frac{\mu}{\lambda}\,, \tag{14.3}$$

then the problem (10.1), (10.2) *has at least one solution.*

Remark 14.2. Let $|\mu| \le |\lambda|$ and

$$S = \Big\{(x, y) \in R_+ \times R_+ :\ x + \frac{\mu}{\lambda} y < \frac{\mu}{\lambda}\Big\}$$

(see Fig. 14.2).

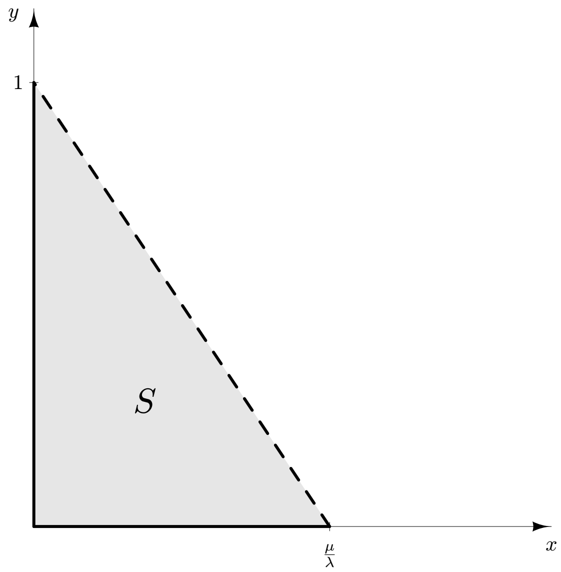

Fig. 14.2.

According to Theorem 14.2, if (12.8) holds, there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that the inequality (12.9) is satisfied on the set $B^2_{\lambda\mu c}\big([a, b]; R\big)$, and

$$\Big(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L\Big) \in S,$$

then the problem (10.1), (10.2) is solvable. Below we will show (see On Remark 14.2, see p. 284) that for every $x_0, y_0 \in R_+$ such that $(x_0, y_0) \notin S$ there exist $F \in K_{ab}$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and $c_0 \in R$ such that (12.8) (with $h \equiv c_0$, $c = |c_0|$) and (12.9) hold,

$$x_0 = \|\ell_0(1)\|_L, \qquad y_0 = \|\ell_1(1)\|_L,$$

and the problem (10.1), (10.2) with $h \equiv c_0$ has no solution. In particular, the strict inequality (14.3) cannot be replaced by the nonstrict one.

In Theorems 14.3 and 14.4, the conditions guaranteeing the unique solvability of the problem (10.1), (10.2) are established.

Theorem 14.3. *Let* $|\mu| \le |\lambda|$, *the inequality* (12.24) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}\big([a,b];R\big)$, *where* $c = |h(0)|$, *the inequality* (12.25) *holds. If, moreover,* (14.1) *is satisfied, where* α *is defined by* (14.2), *then the problem* (10.1), (10.2) *is uniquely solvable.*

Remark 14.3. The examples constructed in Subsection 14.3 (see On Remark 14.1, p. 280) also show that the strict inequality (14.1) cannot be replaced by the nonstrict one.

Theorem 14.4. *Let* $|\mu| \le |\lambda|$, *the condition* (12.26) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}\big([a,b];R\big)$, *where* $c = |h(0)|$, *the inequality* (12.27) *holds. If, moreover, the inequality* (14.3) *is satisfied, then the problem* (10.1), (10.2) *is uniquely solvable.*

Remark 14.4. The examples constructed in Subsection 14.3 (see On Remark 14.2, p. 284) also show that the strict inequality (14.3) cannot be replaced by the nonstrict one.

In the case, where $|\mu| \ge |\lambda|$, the following assertions hold.

Theorem 14.5. *Let* $|\mu| \ge |\lambda|$, $c \in R_+$, *the inequality* (12.8) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}\big([a,b];R\big)$ *the inequality* (12.9) *holds. If, moreover,*

$$\|\ell_0(1)\|_L < \beta(\lambda,\mu), \tag{14.4}$$

where

$$\beta(\lambda,\mu) = \begin{cases} -\frac{\lambda}{\mu} + 2\sqrt{1 - \|\ell_1(1)\|_L} & \text{if} \quad \|\ell_1(1)\|_L < 1 - \left(\frac{\lambda}{\mu}\right)^2 \\ \frac{\mu}{\lambda}\left(1 - \|\ell_1(1)\|_L\right) & \text{if} \quad \|\ell_1(1)\|_L \ge 1 - \left(\frac{\lambda}{\mu}\right)^2 \end{cases}, \tag{14.5}$$

then the problem (10.1), (10.2) *has at least one solution.*

Theorem 14.6. *Let* $|\mu| \ge |\lambda|$, $c \in R_+$, *the condition* (12.1) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}\big([a,b];R\big)$ *the inequality* (12.3) *holds. If, moreover,*

$$\|\ell_1(1)\|_L + \frac{\lambda}{\mu}\|\ell_0(1)\|_L < \frac{\lambda}{\mu}, \tag{14.6}$$

then the problem (10.1), (10.2) *has at least one solution.*

In Theorems 14.7 and 14.8, the conditions guaranteeing the unique solvability of the problem (10.1), (10.2) are established.

Theorem 14.7. *Let* $|\mu| \geq |\lambda|$, *the inequality* (12.26) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda\mu c}\big([a,b];R\big)$, *where* $c = |h(0)|$, *the inequality* (12.27) *holds. If, moreover,* (14.4) *is satisfied, where* β *is defined by* (14.5), *then the problem* (10.1), (10.2) *is uniquely solvable.*

Theorem 14.8. *Let* $|\mu| \geq |\lambda|$, *the condition* (12.24) *be fulfilled, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda\mu c}\big([a,b];R\big)$, *where* $c = |h(0)|$, *the inequality* (12.25) *holds. If, moreover, the inequality* (14.6) *is satisfied, then the problem* (10.1), (10.2) *is uniquely solvable.*

Remark 14.5. According to Remark 12.14 (see p. 211), Theorems 14.5–14.8 can be immediately derived from Theorems 14.1–14.4. Moreover, by virtue of Remarks 14.1–14.4, Theorems 14.5–14.8 are nonimprovable in an appropriate sense.

14.2. Proofs

First we will prove two lemmas.

Lemma 14.1. *Let* $|\mu| \leq |\lambda|$ *and the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. *If, moreover, the condition* (14.1) *holds, where* α *is defined by* (14.2), *then* $\ell \in \mathcal{A}^1\big(\lambda,\mu\big)$.

Proof. Let $q^* \in L\big([a,b];R_+\big)$, $c \in R_+$, and $u \in \widetilde{C}\big([a,b];R\big)$ satisfy (11.13) and (11.14) for $i = 1$. Define the number λ_0 by (12.45). We will show that (11.15) holds with

$$r = \begin{cases} \dfrac{\lambda_0\big(\|\ell_1(1)\|_L+1+\frac{\mu}{\lambda}\big)}{1-\|\ell_0(1)\|_L-\frac{1}{4}\big(\|\ell_1(1)\|_L+\frac{\mu}{\lambda}\big)^2} & \text{if } \ \|\ell_0(1)\|_L < 1-\big(\frac{\mu}{\lambda}\big)^2 \\ \dfrac{\lambda_0\big(\|\ell_1(1)\|_L+1+\frac{\mu}{\lambda}\big)}{1-\|\ell_0(1)\|_L-\frac{\mu}{\lambda}\|\ell_1(1)\|_L} & \text{if } \ \|\ell_0(1)\|_L \geq 1-\big(\frac{\mu}{\lambda}\big)^2 \end{cases}. \tag{14.7}$$

Obviously, u satisfies (12.47), where $\widetilde{q}$ is defined by (12.48). It is also evident that the inequalities (12.49) and (12.50) hold.

First suppose that u does not change its sign. According to (12.50), (7.1), and the assumption $\frac{\mu}{\lambda} \in \,]0,1]$, we obtain

$$|u(a)| \leq \frac{c}{|\lambda|}\,. \tag{14.8}$$

Choose $t_0 \in [a,b]$ such that

$$|u(t_0)| = \|u\|_C. \tag{14.9}$$

Due to (12.2) and (12.49), (12.47) implies

$$|u(t)|' \le \|u\|_C\, \ell_0(1)(t) + q^*(t) \quad \text{for} \quad t \in [a,b]. \tag{14.10}$$

The integration of (14.10) from a to t_0, on account of (12.2), (14.8), and (14.9), results in

$$\|u\|_C - \frac{c}{|\lambda|} \le \|u\|_C - |u(a)| \le \|u\|_C \int_a^{t_0} \ell_0(1)(s)ds + \int_a^{t_0} q^*(s)ds \le$$

$$\le \|u\|_C \|\ell_0(1)\|_L + \|q^*\|_L.$$

Thus, in view of (12.45), the inequality

$$\|u\|_C\,(1 - \|\ell_0(1)\|_L) \le \frac{c}{|\lambda|} + \|q^*\|_L \le \lambda_0(c + \|q^*\|_L)$$

holds and, consequently, on account of (14.1), (14.2), (14.7), and Remark 14.1, the estimate (11.15) holds.

Now suppose that u changes its sign. Define numbers M and m by (12.59) and choose $t_M, t_m \in [a,b]$ such that (12.60) is fulfilled. Obviously, $M > 0$, $m > 0$, and either (12.61) or (12.62) holds.

First suppose that (12.61) is satisfied. It is clear that there exists $\alpha_2 \in \,]t_m, t_M[$ such that

$$u(t) > 0 \quad \text{for } \alpha_2 < t \le t_M, \qquad u(\alpha_2) = 0. \tag{14.11}$$

Let

$$\alpha_1 = \inf\{t \in [a, t_m] : u(s) < 0 \text{ for } t \le s \le t_m\}.$$

Obviously,

$$u(t) < 0 \quad \text{for} \quad \alpha_1 < t \le t_m \tag{14.12}$$

and

$$\text{if} \quad \alpha_1 > a, \quad \text{then} \quad u(\alpha_1) = 0. \tag{14.13}$$

It follows from (7.1), (12.50), (14.12), and the assumption $\frac{\mu}{\lambda} \in \,]0,1]$ that

$$u(\alpha_1) \ge -\frac{\mu}{\lambda}[u(b)]_+ - \frac{c}{|\lambda|} \ge -\frac{\mu}{\lambda}M - \frac{c}{|\lambda|}\,. \tag{14.14}$$

The integration of (12.47) from α_1 to t_m and from α_2 to t_M, in view of (12.2), (12.49), (12.59), (12.60), (14.11)–(14.14), yields

$$m - \frac{\mu}{\lambda}M - \frac{c}{|\lambda|} \le m + u(\alpha_1) \le$$

$$\le M\int_{\alpha_1}^{t_m} \ell_1(1)(s)ds + m\int_{\alpha_1}^{t_m} \ell_0(1)(s)ds + \int_{\alpha_1}^{t_m} q^*(s)ds,$$

$$M \le M\int_{\alpha_2}^{t_M} \ell_0(1)(s)ds + m\int_{\alpha_2}^{t_M} \ell_1(1)(s)ds + \int_{\alpha_2}^{t_M} q^*(s)ds.$$

From the last two inequalities we obtain

$$\begin{gathered} m(1-C_1) \le M\left(A_1 + \frac{\mu}{\lambda}\right) + \|q^*\|_L + \frac{c}{|\lambda|}, \\ M(1-D_1) \le mB_1 + \|q^*\|_L, \end{gathered} \tag{14.15}$$

where

$$A_1 = \int_{\alpha_1}^{t_m} \ell_1(1)(s)ds, \qquad B_1 = \int_{\alpha_2}^{t_M} \ell_1(1)(s)ds,$$

$$C_1 = \int_{\alpha_1}^{t_m} \ell_0(1)(s)ds, \qquad D_1 = \int_{\alpha_2}^{t_M} \ell_0(1)(s)ds.$$

According to Remark 14.1, $\|\ell_0(1)\|_L < 1$, i.e., $C_1 < 1$ and $D_1 < 1$. By virtue of (12.45), the inequalities (14.15) imply

$$\begin{gathered} 0 < m(1-C_1)(1-D_1) \le \left(A_1 + \frac{\mu}{\lambda}\right)(mB_1 + \|q^*\|_L) + \|q^*\|_L + \\ + \frac{c}{|\lambda|} \le m\left(A_1 + \frac{\mu}{\lambda}\right)B_1 + \lambda_0(\|q^*\|_L + c)\left(\|\ell_1(1)\|_L + 1 + \frac{\mu}{\lambda}\right), \\ 0 < M(1-C_1)(1-D_1) \le B_1\left(M\left(A_1 + \frac{\mu}{\lambda}\right) + \|q^*\|_L + \frac{c}{|\lambda|}\right) + \\ + \|q^*\|_L \le M\left(A_1 + \frac{\mu}{\lambda}\right)B_1 + \lambda_0(\|q^*\|_L + c)\left(\|\ell_1(1)\|_L + 1 + \frac{\mu}{\lambda}\right). \end{gathered} \tag{14.16}$$

Obviously,

$$(1-C_1)(1-D_1) \ge 1 - (C_1 + D_1) \ge 1 - \|\ell_0(1)\|_L > 0. \tag{14.17}$$

If $\|\ell_0(1)\|_L \geq 1 - \left(\frac{\mu}{\lambda}\right)^2$, then, according to (14.1) and (14.2), we obtain $\|\ell_1(1)\|_L < \frac{\mu}{\lambda}$. Hence, $B_1 < \frac{\mu}{\lambda}$ and

$$\left(A_1 + \frac{\mu}{\lambda}\right) B_1 = A_1 B_1 + \frac{\mu}{\lambda} B_1 \leq \frac{\mu}{\lambda}(A_1 + B_1) \leq \frac{\mu}{\lambda}\|\ell_1(1)\|_L.$$

By the last inequality, (14.1), (14.2), and (14.17), from (14.16) we get

$$\begin{aligned} m &\leq r_0 \lambda_0 \left(\|\ell_1(1)\|_L + 1 + \frac{\mu}{\lambda}\right)(c + \|q^*\|_L), \\ M &\leq r_0 \lambda_0 \left(\|\ell_1(1)\|_L + 1 + \frac{\mu}{\lambda}\right)(c + \|q^*\|_L), \end{aligned} \tag{14.18}$$

where

$$r_0 = \left(1 - \|\ell_0(1)\|_L - \frac{\mu}{\lambda}\|\ell_1(1)\|_L\right)^{-1}. \tag{14.19}$$

Therefore, on account of (12.59), (14.7), (14.18), and (14.19), the estimate (11.15) holds.

If $\|\ell_0(1)\|_L < 1 - \left(\frac{\mu}{\lambda}\right)^2$, then by virtue of the inequality

$$\left(A_1 + \frac{\mu}{\lambda}\right) B_1 \leq \frac{1}{4}\left(A_1 + B_1 + \frac{\mu}{\lambda}\right)^2 \leq \frac{1}{4}\left(\|\ell_1(1)\|_L + \frac{\mu}{\lambda}\right)^2,$$

(14.1), (14.2), and (14.17), (14.16) implies

$$\begin{aligned} m &\leq r_1 \lambda_0 \left(\|\ell_1(1)\|_L + 1 + \frac{\mu}{\lambda}\right)(c + \|q^*\|_L), \\ M &\leq r_1 \lambda_0 \left(\|\ell_1(1)\|_L + 1 + \frac{\mu}{\lambda}\right)(c + \|q^*\|_L), \end{aligned} \tag{14.20}$$

where

$$r_1 = \left(1 - \|\ell_0(1)\|_L - \frac{1}{4}\left(\|\ell_1(1)\|_L + \frac{\mu}{\lambda}\right)^2\right)^{-1}. \tag{14.21}$$

Therefore, on account of (12.59), (14.7), (14.20), and (14.21), the estimate (11.15) is valid.

Now suppose that (12.62) is satisfied. It is clear that there exists $\alpha_4 \in]t_M, t_m[$ such that

$$u(t) < 0 \quad \text{for } \alpha_4 < t \leq t_m, \qquad u(\alpha_4) = 0. \tag{14.22}$$

Let

$$\alpha_3 = \inf\{t \in [a, t_M] : u(s) > 0 \text{ for } t \leq s \leq t_M\}.$$

Obviously,

$$u(t) > 0 \quad \text{for} \quad \alpha_3 < t \leq t_M \tag{14.23}$$

and

$$\text{if} \quad \alpha_3 > a, \quad \text{then} \quad u(\alpha_3) = 0. \tag{14.24}$$

From (7.1), (12.50), (14.23), and the assumption $\frac{\mu}{\lambda} \in]0,1]$ we get

$$u(\alpha_3) \leq \frac{\mu}{\lambda}[u(b)]_- + \frac{c}{|\lambda|} \leq \frac{\mu}{\lambda} m + \frac{c}{|\lambda|}\ . \tag{14.25}$$

The integration of (12.47) from α_3 to t_M and from α_4 to t_m, in view of (12.2), (12.49), (12.59), (12.60), (14.22)–(14.25), results in

$$M - \frac{\mu}{\lambda} m - \frac{c}{|\lambda|} \leq M - u(\alpha_3) \leq$$

$$\leq M \int_{\alpha_3}^{t_M} \ell_0(1)(s)ds + m \int_{\alpha_3}^{t_M} \ell_1(1)(s)ds + \int_{\alpha_3}^{t_M} q^*(s)ds,$$

$$m \leq M \int_{\alpha_4}^{t_m} \ell_1(1)(s)ds + m \int_{\alpha_4}^{t_m} \ell_0(1)(s)ds + \int_{\alpha_4}^{t_m} q^*(s)ds.$$

From the last two inequalities we obtain

$$\begin{gathered} M(1 - C_2) \leq m\left(A_2 + \frac{\mu}{\lambda}\right) + \|q^*\|_L + \frac{c}{|\lambda|}, \\ m(1 - D_2) \leq M B_2 + \|q^*\|_L, \end{gathered} \tag{14.26}$$

where

$$A_2 = \int_{\alpha_3}^{t_M} \ell_1(1)(s)ds, \qquad B_2 = \int_{\alpha_4}^{t_m} \ell_1(1)(s)ds,$$

$$C_2 = \int_{\alpha_3}^{t_M} \ell_0(1)(s)ds, \qquad D_2 = \int_{\alpha_4}^{t_m} \ell_0(1)(s)ds.$$

Due to Remark 14.1, $\|\ell_0(1)\|_L < 1$, i.e., $C_2 < 1$ and $D_2 < 1$. By virtue of (12.45), the inequalities (14.26) imply

$$\begin{gathered}0 < M(1-C_2)(1-D_2) \leq \left(A_2 + \frac{\mu}{\lambda}\right)(MB_2 + \|q^*\|_L) + \|q^*\|_L + \\ + \frac{c}{|\lambda|} \leq M\left(A_2 + \frac{\mu}{\lambda}\right)B_2 + \lambda_0(\|q^*\|_L + c)\left(\|\ell_1(1)\|_L + 1 + \frac{\mu}{\lambda}\right), \\ 0 < m(1-C_2)(1-D_2) \leq B_2\left(m\left(A_2 + \frac{\mu}{\lambda}\right) + \|q^*\|_L + \frac{c}{|\lambda|}\right) + \\ + \|q^*\|_L \leq m\left(A_2 + \frac{\mu}{\lambda}\right)B_2 + \lambda_0(\|q^*\|_L + c)\left(\|\ell_1(1)\|_L + 1 + \frac{\mu}{\lambda}\right).\end{gathered} \tag{14.27}$$

Obviously,

$$(1-C_2)(1-D_2) \geq 1 - (C_2 + D_2) \geq 1 - \|\ell_0(1)\|_L > 0. \tag{14.28}$$

If $\|\ell_0(1)\|_L \geq 1 - \left(\frac{\mu}{\lambda}\right)^2$, then, according to (14.1) and (14.2), we obtain $\|\ell_1(1)\|_L < \frac{\mu}{\lambda}$. Hence, $B_2 < \frac{\mu}{\lambda}$ and

$$\left(A_2 + \frac{\mu}{\lambda}\right)B_2 = A_2B_2 + \frac{\mu}{\lambda}B_2 \leq \frac{\mu}{\lambda}(A_2 + B_2) \leq \frac{\mu}{\lambda}\|\ell_1(1)\|_L.$$

By the last inequality, (14.1), (14.2), and (14.28), (14.27) implies (14.18), where r_0 is defined by (14.19). Therefore, on account of (12.59), (14.7), (14.18), and (14.19), the estimate (11.15) is valid.

If $\|\ell_0(1)\|_L < 1 - \left(\frac{\mu}{\lambda}\right)^2$, then by virtue of the inequality

$$\left(A_2 + \frac{\mu}{\lambda}\right)B_2 \leq \frac{1}{4}\left(A_2 + B_2 + \frac{\mu}{\lambda}\right)^2 \leq \frac{1}{4}\left(\|\ell_1(1)\|_L + \frac{\mu}{\lambda}\right)^2,$$

(14.1), (14.2), and (14.28), (14.27) implies (14.20), where r_1 is defined by (14.21). Therefore, on account of (12.59), (14.7), (14.20), and (14.21), the estimate (11.15) holds. □

Lemma 14.2. *Let* $|\mu| \leq |\lambda|$ *and the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. *If, moreover, the condition* (14.3) *holds, then* $\ell \in \mathcal{A}^2(\lambda, \mu)$.

Proof. Let $q^* \in L([a,b]; R_+)$, $c \in R_+$, and $u \in \widetilde{C}([a,b]; R)$ satisfy (11.13) and (11.14) for $i = 2$. Put

$$\mu_0 = \max\left\{1, \frac{1}{|\mu|}\right\}. \tag{14.29}$$

We will show that (11.15) holds, where

$$r = \frac{\mu_0(\mu\|\ell_0(1)\|_L + \lambda + \mu)}{\mu - \mu\|\ell_1(1)\|_L - \lambda\|\ell_0(1)\|_L}. \tag{14.30}$$

Obviously, u satisfies (12.47), where $\widetilde{q}$ is defined by (12.48). It is also evident that the inequalities (12.96) and (12.97) hold.

First suppose that u does not change its sign. According to (7.1), (12.97), and the assumption $\frac{\mu}{\lambda} \in]0,1]$, we obtain

$$|u(b)| \le \frac{c}{|\mu|}. \tag{14.31}$$

Choose $t_0 \in [a,b]$ such that (14.9) holds. Due to (12.2) and (12.96), (12.47) implies

$$-|u(t)|' \le \|u\|_C\, \ell_1(1)(t) + q^*(t) \quad \text{for} \quad t \in [a,b]. \tag{14.32}$$

The integration of (14.32) from t_0 to b, on account of (12.2), (14.9), and (14.31), results in

$$\|u\|_C - \frac{c}{|\mu|} \le \|u\|_C - |u(b)| \le \|u\|_C \int_{t_0}^{b} \ell_1(1)(s)ds + \int_{t_0}^{b} q^*(s)ds \le$$
$$\le \|u\|_C\|\ell_1(1)\|_L + \|q^*\|_L.$$

Thus, in view of (14.29), the inequality

$$\|u\|_C\,(1 - \|\ell_1(1)\|_L) \le \frac{c}{|\mu|} + \|q^*\|_L \le \mu_0(c + \|q^*\|_L)$$

holds and, consequently, on account of (7.1), (14.3), and (14.30), the estimate (11.15) holds.

Now suppose that u changes its sign. Define numbers M and m by (12.59) and choose $t_M, t_m \in [a,b]$ such that (12.60) is fulfilled. Obviously, $M > 0$, $m > 0$, and either (12.61) or (12.62) is valid.

First suppose that (12.62) holds. It is clear that there exists $\alpha_1 \in]t_M, t_m[$ such that

$$u(t) > 0 \quad \text{for } t_M \le t < \alpha_1, \qquad u(\alpha_1) = 0. \tag{14.33}$$

Let

$$\alpha_2 = \sup\{t \in [t_m, b] : u(s) < 0 \text{ for } t_m \le s \le t\}.$$

Obviously,

$$u(t) < 0 \quad \text{for} \quad t_m \le t < \alpha_2, \tag{14.34}$$

and

$$\text{if} \quad \alpha_2 < b, \quad \text{then} \quad u(\alpha_2) = 0. \tag{14.35}$$

From (7.1), (12.97), (14.34), and the assumption $\frac{\mu}{\lambda} \in]0,1]$ we obtain

$$u(\alpha_2) \ge -\frac{\lambda}{\mu}[u(a)]_+ - \frac{c}{|\mu|} \ge -\frac{\lambda}{\mu}M - \frac{c}{|\mu|}\,. \tag{14.36}$$

The integration of (12.47) from t_M to α_1 and from t_m to α_2, in view of (12.2), (12.59), (12.60), (12.96), (14.33)–(14.36), implies

$$M \le M\int_{t_M}^{\alpha_1} \ell_1(1)(s)ds + m\int_{t_M}^{\alpha_1} \ell_0(1)(s)ds + \int_{t_M}^{\alpha_1} q^*(s)ds,$$

$$m - \frac{\lambda}{\mu}M - \frac{c}{|\mu|} \le m + u(\alpha_2) \le$$

$$\le M\int_{t_m}^{\alpha_2} \ell_0(1)(s)ds + m\int_{t_m}^{\alpha_2} \ell_1(1)(s)ds + \int_{t_m}^{\alpha_2} q^*(s)ds.$$

From the last two inequalities we get

$$\begin{gathered} M(1-A_1) \le mC_1 + \|q^*\|_L, \\ m(1-B_1) \le M\left(D_1 + \frac{\lambda}{\mu}\right) + \|q^*\|_L + \frac{c}{|\mu|}, \end{gathered} \tag{14.37}$$

where

$$A_1 = \int_{t_M}^{\alpha_1} \ell_1(1)(s)ds, \qquad B_1 = \int_{t_m}^{\alpha_2} \ell_1(1)(s)ds,$$

$$C_1 = \int_{t_M}^{\alpha_1} \ell_0(1)(s)ds, \qquad D_1 = \int_{t_m}^{\alpha_2} \ell_0(1)(s)ds.$$

Due to (7.1) and (14.3), $A_1 < 1$, $B_1 < 1$. By virtue of (14.29), the inequalities (14.37) imply

$$\begin{aligned}
&0 < M(1-A_1)(1-B_1) \le C_1 \left(M\left(D_1 + \frac{\lambda}{\mu}\right) + \|q^*\|_L + \frac{c}{|\mu|}\right) + \\
&+\|q^*\|_L \le MC_1 \left(D_1 + \frac{\lambda}{\mu}\right) + \mu_0 \left(\|\ell_0(1)\|_L + 1 + \frac{\lambda}{\mu}\right)(\|q^*\|_L + c)\,, \\
&0 < m(1-A_1)(1-B_1) \le \left(D_1 + \frac{\lambda}{\mu}\right)(mC_1 + \|q^*\|_L) + \|q^*\|_L + \\
&+\frac{c}{|\mu|} \le mC_1 \left(D_1 + \frac{\lambda}{\mu}\right) + \mu_0 \left(\|\ell_0(1)\|_L + 1 + \frac{\lambda}{\mu}\right)(\|q^*\|_L + c)\,.
\end{aligned} \tag{14.38}$$

Obviously,

$$(1-A_1)(1-B_1) \ge 1 - (A_1 + B_1) \ge 1 - \|\ell_1(1)\|_L > 0. \tag{14.39}$$

According to (7.1), (14.3), and the assumption $\frac{\mu}{\lambda} \in \,]0,1]$, we get $\|\ell_0(1)\|_L < \frac{\lambda}{\mu}$. Hence, $C_1 < \frac{\lambda}{\mu}$ and

$$C_1 \left(D_1 + \frac{\lambda}{\mu}\right) = C_1 D_1 + \frac{\lambda}{\mu} C_1 \le \frac{\lambda}{\mu}(C_1 + D_1) \le \frac{\lambda}{\mu}\|\ell_0(1)\|_L.$$

By the last inequality, (14.3), and (14.39), from (14.38) we get

$$\begin{aligned}
M &\le r_0 \mu_0 \left(\mu \|\ell_0(1)\|_L + \lambda + \mu\right)(c + \|q^*\|_L), \\
m &\le r_0 \mu_0 \left(\mu \|\ell_0(1)\|_L + \lambda + \mu\right)(c + \|q^*\|_L),
\end{aligned} \tag{14.40}$$

where

$$r_0 = (\mu - \mu\|\ell_1(1)\|_L - \lambda\|\ell_0(1)\|_L)^{-1}. \tag{14.41}$$

Therefore, on account of (12.59), (14.30), (14.40), and (14.41), the estimate (11.15) holds.

Now suppose that (12.61) is valid. Obviously, there exists $\alpha_3 \in \,]t_m, t_M[$ such that

$$u(t) < 0 \quad \text{for} \quad t_m \le t < \alpha_3, \qquad u(\alpha_3) = 0. \tag{14.42}$$

Let

$$\alpha_4 = \sup\{t \in [t_M, b] : u(s) > 0 \text{ for } t_M \le s \le t\}.$$

It is clear that

$$u(t) > 0 \quad \text{for} \quad t_M \le t < \alpha_4, \tag{14.43}$$

and

$$\text{if} \quad \alpha_4 < b, \quad \text{then} \quad u(\alpha_4) = 0. \tag{14.44}$$

It follows from (7.1), (12.97), (14.43), and the assumption $\frac{\mu}{\lambda} \in]0,1]$ that

$$u(\alpha_4) \le \frac{\lambda}{\mu}[u(a)]_- + \frac{c}{|\mu|} \le \frac{\lambda}{\mu} m + \frac{c}{|\mu|}\,. \tag{14.45}$$

The integration of (12.47) from t_m to α_3 and from t_M to α_4, in view of (12.2), (12.59), (12.60), (12.96), (14.42)–(14.45), yields

$$m \le M \int_{t_m}^{\alpha_3} \ell_0(1)(s)ds + m \int_{t_m}^{\alpha_3} \ell_1(1)(s)ds + \int_{t_m}^{\alpha_3} q^*(s)ds,$$

$$M - \frac{\lambda}{\mu} m - \frac{c}{|\mu|} \le M - u(\alpha_4) \le$$

$$\le M \int_{t_M}^{\alpha_4} \ell_1(1)(s)ds + m \int_{t_M}^{\alpha_4} \ell_0(1)(s)ds + \int_{t_M}^{\alpha_4} q^*(s)ds.$$

From the last two inequalities we get

$$\begin{gathered} m(1 - A_2) \le M C_2 + \|q^*\|_L, \\ M(1 - B_2) \le m\left(D_2 + \frac{\lambda}{\mu}\right) + \|q^*\|_L + \frac{c}{|\mu|}, \end{gathered} \tag{14.46}$$

where

$$A_2 = \int_{t_m}^{\alpha_3} \ell_1(1)(s)ds, \qquad B_2 = \int_{t_M}^{\alpha_4} \ell_1(1)(s)ds,$$

$$C_2 = \int_{t_m}^{\alpha_3} \ell_0(1)(s)ds, \qquad D_2 = \int_{t_M}^{\alpha_4} \ell_0(1)(s)ds.$$

Due to (7.1) and (14.3), $A_2 < 1$ and $B_2 < 1$. By virtue of (14.29), the inequalities (14.46) imply

$$
\begin{gathered}
0 < m(1-A_2)(1-B_2) \le C_2\left(m\left(D_2+\frac{\lambda}{\mu}\right)+\|q^*\|_L+\frac{c}{|\mu|}\right)+ \\
+\|q^*\|_L \le mC_2\left(D_2+\frac{\lambda}{\mu}\right)+\mu_0\left(\|\ell_0(1)\|_L+1+\frac{\lambda}{\mu}\right)(\|q^*\|_L+c)\,, \\
0 < M(1-A_2)(1-B_2) \le \left(D_2+\frac{\lambda}{\mu}\right)(MC_2+\|q^*\|_L)+\|q^*\|_L+ \\
+\frac{c}{|\mu|} \le MC_2\left(D_2+\frac{\lambda}{\mu}\right)+\mu_0\left(\|\ell_0(1)\|_L+1+\frac{\lambda}{\mu}\right)(\|q^*\|_L+c)\,.
\end{gathered}
\tag{14.47}
$$

Obviously,

$$
(1-A_2)(1-B_2) \ge 1-(A_2+B_2) \ge 1-\|\ell_1(1)\|_L > 0. \tag{14.48}
$$

According to (7.1), (14.3), and the assumption $\frac{\mu}{\lambda} \in]0,1]$, we get $\|\ell_0(1)\|_L < \frac{\lambda}{\mu}$. Hence, $C_2 < \frac{\lambda}{\mu}$ and

$$
C_2\left(D_2+\frac{\lambda}{\mu}\right) = C_2D_2+\frac{\lambda}{\mu}C_2 \le \frac{\lambda}{\mu}(C_2+D_2) \le \frac{\lambda}{\mu}\|\ell_0(1)\|_L.
$$

By the last inequality, (14.3), and (14.48), (14.47) implies (14.40), where r_0 is defined by (14.41). Therefore, on account of (12.59), (14.30), (14.40), and (14.41), the estimate (11.15) is valid. □

Theorem 14.1 follows from Lemma 11.3 (see p. 195) and Lemma 14.1 (see p. 270). Theorem 14.2 follows from Lemma 11.3 (see p. 195) and Lemma 14.2 (see p. 275). Theorem 14.3 follows from Lemma 11.4 (see p. 196) and Lemma 14.1 (see p. 270). Theorem 14.4 follows from Lemma 11.4 (see p. 196) and Lemma 14.2 (see p. 275).

14.3. Comments and Examples

On Remark 14.1. Let $|\mu| \le |\lambda|$. It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin U$, then (x_0, y_0) belongs at least to one of the following sets:

$$
\begin{aligned}
U_1 &= \{(x,y) \in R_+ \times R_+ \ : \ 1 < x\}\,, \\
U_2 &= \left\{(x,y) \in R_+ \times R_+ \ : \ 1-\left(\frac{\mu}{\lambda}\right)^2 \le x \le 1,\ \frac{\lambda}{\mu}(1-x) \le y\right\}, \\
U_3 &= \left\{(x,y) \in R_+ \times R_+ \ : \ x < 1-\left(\frac{\mu}{\lambda}\right)^2,\ 2\sqrt{1-x}-\frac{\mu}{\lambda} \le y\right\}.
\end{aligned}
$$

Let $(x_0, y_0) \in U_1$ and let $\varepsilon \in \left]0, \frac{\mu}{\lambda}\right[$ be such that $x_0 - \varepsilon \geq 1$. Put $a = 0$, $b = 4$, $t_0 = 3 + \frac{\varepsilon}{1+\varepsilon}$,

$$p(t) = \begin{cases} 0 & \text{for } t \in [0,1[\\ -y_0 & \text{for } t \in [1,2[\\ x_0 - 1 - \varepsilon & \text{for } t \in [2,3[\\ 1 + \varepsilon & \text{for } t \in [3,4] \end{cases}, \quad \tau(t) = \begin{cases} t_0 & \text{for } t \in [0,3[\\ 4 & \text{for } t \in [3,4] \end{cases},$$

$$z(t) = \begin{cases} -\frac{\mu - \lambda\varepsilon}{\mu - (\mu - \lambda\varepsilon)t} & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,4] \end{cases}.$$

It is not difficult to verify that (12.204) holds and the problem (12.205) has the nontrivial solution

$$u(t) = \begin{cases} -(\mu - \lambda\varepsilon)t + \mu & \text{for } t \in [0,1[\\ \lambda\varepsilon & \text{for } t \in [1,3[\\ -\lambda(1+\varepsilon)(t-3) + \lambda\varepsilon & \text{for } t \in [3,4] \end{cases}.$$

Then, by Remark 1.1 (see p. 14), there exist $q_0 \in L\big([a,b]; R\big)$ and $c_0 \in R$ such that the problem (10.1), (10.2) with F and h given by (12.206) has no solution, while the conditions (12.1) and (12.3) are fulfilled, where ℓ_0, ℓ_1, q, and c are defined by (12.207).

Let $(x_0, y_0) \in U_2$. Put $a = 0$, $b = 4$,

$$p(t) = \begin{cases} 0 & \text{for } t \in [0,1[\\ -\frac{\lambda}{\mu}(1 - x_0) & \text{for } t \in [1,2[\\ x_0 & \text{for } t \in [2,3[\\ \frac{\lambda}{\mu}(1 - x_0) - y_0 & \text{for } t \in [3,4] \end{cases}, \quad \tau(t) = \begin{cases} 0 & \text{for } t \in [0,2[\\ 4 & \text{for } t \in [2,3[\\ 1 & \text{for } t \in [3,4] \end{cases}.$$

Obviously, (12.204) holds. Furthermore, define the operator $G \in K_{ab}$ by

$$G(v)(t) = \begin{cases} -v(t)|v(t)| & \text{for } t \in [0,1[\\ q_0(t) & \text{for } t \in [1,2[\\ 0 & \text{for } t \in [2,4] \end{cases},$$

where $q_0 \in L\big([a,b]; R\big)$ is such that

$$\int_1^2 q_0(s)ds \geq 1 + y_0 - \frac{\lambda}{\mu}(1 - x_0). \tag{14.49}$$

We will show that the problem (10.1), (10.2) with F and h given by (12.209) has no solution, while the conditions (12.1) and (12.3) are fulfilled, where ℓ_0, ℓ_1, q, and c are defined by (12.210).

Indeed, suppose on the contrary that u is a solution of the problem (10.1), (10.2) with F and h given by (12.209), i.e., the equalities (1.2_0) and (12.211) hold. From (12.211) we get

$$u(1) = \frac{u(0)}{1+|u(0)|}\,, \tag{14.50}$$

$$u(2) = u(1) - \frac{\lambda}{\mu}(1-x_0)u(0) + \int_1^2 q_0(s)ds\,, \tag{14.51}$$

$$u(3) = u(2) + u(4)x_0\,, \tag{14.52}$$

$$u(4) = u(3) - \left(y_0 - \frac{\lambda}{\mu}(1-x_0)\right)u(1)\,. \tag{14.53}$$

The equalities (14.51)–(14.53), in view of (1.2_0) and (7.1), result in

$$\int_1^2 q_0(s)ds = \left(y_0 - \frac{\lambda}{\mu}(1-x_0) - 1\right)u(1).$$

Hence, the last equality, together with (14.50), implies

$$\int_1^2 q_0(s)ds = \left(y_0 - \frac{\lambda}{\mu}(1-x_0) - 1\right)\frac{u(0)}{1+|u(0)|} \leq$$

$$\leq \left(y_0 - \frac{\lambda}{\mu}(1-x_0) + 1\right)\frac{|u(0)|}{1+|u(0)|} < 1 + y_0 - \frac{\lambda}{\mu}(1-x_0)\,,$$

which contradicts (14.49).

Let $(x_0, y_0) \in U_3$. Put $a = 0$, $b = 5$,

$$p(t) = \begin{cases} \frac{\mu}{\lambda} - \sqrt{1-x_0} & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,2[\\ -\sqrt{1-x_0} & \text{for } t \in [2,3[\\ x_0 & \text{for } t \in [3,4[\\ 2\sqrt{1-x_0} - \frac{\mu}{\lambda} - y_0 & \text{for } t \in [4,5] \end{cases}\,,$$

and

$$\tau(t) = \begin{cases} 5 & \text{for } t \in [0,2[\cup [3,4[\\ 1 & \text{for } t \in [2,3[\\ 2 & \text{for } t \in [4,5] \end{cases} .$$

Obviously, (12.204) holds. Furthermore, define the operator $G \in K_{ab}$ by

$$G(v)(t) = \begin{cases} 0 & \text{for } t \in [0,1[\cup [2,3[\cup [4,5] \\ -v(t)|v(t)| & \text{for } t \in [1,2[\\ q_0(t) & \text{for } t \in [3,4[\end{cases} ,$$

where $q_0 \in L\big([a,b];R\big)$ is such that

$$\int_1^2 q_0(s)ds \geq 1 + y_0 + \frac{\mu}{\lambda} - 2\sqrt{1-x_0}\,. \tag{14.54}$$

We will show that the problem (10.1), (10.2) with F and h given by (12.209) has no solution, while the conditions (12.1) and (12.3) are fulfilled, where ℓ_0, ℓ_1, q, and c are defined by (12.210).

Indeed, suppose on the contrary that u is a solution of the problem (10.1), (10.2) with F and h given by (12.209), i.e., the equalities (1.2_0) and (12.211) hold. From (12.211) we get

$$u(1) = u(0) + \left(\frac{\mu}{\lambda} - \sqrt{1-x_0}\right) u(5)\,, \tag{14.55}$$

$$u(2) = \frac{u(1)}{1+|u(1)|}\,, \tag{14.56}$$

$$u(3) = u(2) - u(1)\sqrt{1-x_0}\,, \tag{14.57}$$

$$u(4) = u(3) + u(5)x_0 + \int_3^4 q_0(s)ds\,, \tag{14.58}$$

$$u(5) = u(4) - \left(y_0 + \frac{\mu}{\lambda} - 2\sqrt{1-x_0}\right) u(2)\,. \tag{14.59}$$

The equalities (14.55) and (14.57)–(14.59), in view of (1.2_0) and (7.1), result in

$$\int_3^4 q_0(s)ds = \left(y_0 + \frac{\mu}{\lambda} - 2\sqrt{1-x_0} - 1\right) u(2).$$

Hence, the last equality, together with (14.56), implies

$$\int_3^4 q_0(s)ds = \Big(y_0 + \frac{\mu}{\lambda} - 2\sqrt{1-x_0} - 1\Big)\,\frac{u(1)}{1+|u(1)|} \le$$

$$\le \Big(y_0 + \frac{\mu}{\lambda} - 2\sqrt{1-x_0} + 1\Big)\,\frac{|u(1)|}{1+|u(1)|} < 1 + y_0 + \frac{\mu}{\lambda} - 2\sqrt{1-x_0}\,,$$

which contradicts (14.54).

On Remark 14.2. Let $|\mu| \le |\lambda|$. It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin S$, then (x_0, y_0) belongs at least to one of the following sets:

$$S_1 = \Big\{(x,y) \in R_+ \times R_+ \;:\; \frac{\mu}{\lambda} < x\Big\},$$
$$S_2 = \Big\{(x,y) \in R_+ \times R_+ \;:\; x \le \frac{\mu}{\lambda},\ -\frac{\lambda}{\mu}x + 1 \le y\Big\}.$$

Let $(x_0, y_0) \in S_1$ and $\varepsilon \in \,]0,1[$ be such that $x_0 - \frac{\mu}{\lambda} \ge \varepsilon$. Put $a = 0$, $b = 4$, $t_0 = \frac{\mu}{\mu+\lambda\varepsilon}$,

$$p(t) = \begin{cases} \frac{\mu}{\lambda} + \varepsilon & \text{for } t \in [0,1[\\ -y_0 & \text{for } t \in [1,2[\\ x_0 - \frac{\mu}{\lambda} - \varepsilon & \text{for } t \in [2,3[\\ 0 & \text{for } t \in [3,4] \end{cases},$$

$$z(t) = \begin{cases} 0 & \text{for } t \in [0,3[\\ \frac{1-\varepsilon}{(1-\varepsilon)(t-4)+1} & \text{for } t \in [3,4] \end{cases}, \quad \tau(t) = \begin{cases} 4 & \text{for } t \in [0,1[\\ t_0 & \text{for } t \in [1,4] \end{cases}.$$

It is not difficult to verify that (12.204) holds and the problem (12.205) has the nontrivial solution

$$u(t) = \begin{cases} (\mu + \lambda\varepsilon)t - \mu & \text{for } t \in [0,1[\\ \lambda\varepsilon & \text{for } t \in [1,3[\\ \lambda(1-\varepsilon)(t-4) + \lambda & \text{for } t \in [3,4] \end{cases}.$$

Then, by Remark 1.1 (see p. 14), there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (10.1), (10.2) with F and h given by (12.206) has no solution, while the conditions (12.8) and (12.9) are fulfilled, where ℓ_0, ℓ_1, q, and c are defined by (12.207).

Let $(x_0, y_0) \in S_2$. Put $a = 0$, $b = 4$,

$$p(t) = \begin{cases} x_0 & \text{for } t \in [0,1[\\ \frac{\lambda}{\mu}\, x_0 - 1 & \text{for } t \in [1,2[\\ 0 & \text{for } t \in [2,3[\\ 1 - y_0 - \frac{\lambda}{\mu}\, x_0 & \text{for } t \in [3,4] \end{cases}, \quad \tau(t) = \begin{cases} 4 & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,4] \end{cases}.$$

Obviously, (12.204) holds. Furthermore, define the operator $G \in K_{ab}$ by

$$G(v)(t) = \begin{cases} q_0(t) & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,2[\cup [3,4] \\ v(t)|v(t)| & \text{for } t \in [2,3[\end{cases},$$

where $q_0 \in L\big([a,b]; R\big)$ is such that

$$\int_0^1 q_0(s)ds \geq 1\,. \tag{14.60}$$

We will show that the problem (10.1), (10.2) with F and h given by (12.209) has no solution, while the conditions (12.8) and (12.9) are fulfilled, where ℓ_0, ℓ_1, q, and c are defined by (12.210).

Indeed, suppose on the contrary that u is a solution of the problem (10.1), (10.2) with F and h given by (12.209), i.e., the equalities (1.2_0) and (12.211) hold. From (12.211) we get

$$u(1) = u(0) + u(4)x_0 + \int_0^1 q_0(s)ds\,, \tag{14.61}$$

$$u(2) = u(1) - \left(1 - \frac{\lambda}{\mu}\, x_0\right) u(0)\,, \tag{14.62}$$

$$u(2) = \frac{u(3)}{1 + |u(3)|}\,. \tag{14.63}$$

The equalities (14.61) and (14.62), in view of (1.2_0) and (7.1), result in

$$\int_0^1 q_0(s)ds = u(2).$$

Hence, the last equality, together with (14.63), implies

$$\int_0^1 q_0(s)ds = \frac{u(3)}{1 + |u(3)|} \leq \frac{|u(3)|}{1 + |u(3)|} < 1\,,$$

which contradicts (14.60).

§15. Antiperiodic Type BVP for EDA

In this section, we will establish some consequences of the main results from §14 for the equation with deviating arguments (10.1′). Here we will also suppose that the inequality (7.1) is fulfilled.

In what follows we will use the notation

$$p_0(t) = \sum_{j=1}^{m} p_j(t), \qquad g_0(t) = \sum_{j=1}^{m} g_j(t) \quad \text{for} \quad t \in [a,b]$$

and we will suppose that the function $q \in K\big([a,b] \times R_+; R_+\big)$ is nondecreasing in the second argument and satisfies (10.5), i.e.,

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)ds = 0.$$

15.1. Existence and Uniqueness Theorems

In the case, where $|\mu| \leq |\lambda|$, the following statements hold.

Theorem 15.1. *Let* $|\mu| \leq |\lambda|$, $p_k, g_k \in L\big([a,b]; R_+\big)$ $(k = 1, \dots, m)$, $c \in R_+$, *the condition* (12.1) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.1) *hold. If, moreover,*

$$\|g_0\|_L < \gamma(\lambda, \mu), \tag{15.1}$$

where

$$\gamma(\lambda,\mu) = \begin{cases} -\frac{\mu}{\lambda} + 2\sqrt{1 - \|p_0\|_L} & \text{if} \quad \|p_0\|_L < 1 - \left(\frac{\mu}{\lambda}\right)^2 \\ \frac{\lambda}{\mu}\left(1 - \|p_0\|_L\right) & \text{if} \quad \|p_0\|_L \geq 1 - \left(\frac{\mu}{\lambda}\right)^2 \end{cases}, \tag{15.2}$$

then the problem (10.1′), (10.2) *has at least one solution.*

Remark 15.1. The examples constructed in Subsection 14.3 (see On Remark 14.1, p. 280) also show that the strict inequality (15.1) cannot be replaced by the nonstrict one.

Theorem 15.2. *Let* $|\mu| \le |\lambda|$, $p_k, g_k \in L([a,b];R_+)$ $(k = 1,\dots,m)$, $c \in R_+$, *the condition* (12.8) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.6) *hold. If, moreover,*

$$\int_a^b p_0(s)ds + \frac{\mu}{\lambda}\int_a^b g_0(s)ds < \frac{\mu}{\lambda}, \tag{15.3}$$

then the problem (10.1′), (10.2) *has at least one solution.*

Remark 15.2. The examples constructed in Subsection 14.3 (see On Remark 14.2, p. 284) also show that the strict inequality (15.3) cannot be replaced by the nonstrict one.

In Theorems 15.3 and 15.4, the conditions guaranteeing the unique solvability of the problem (10.1′), (10.2) are established.

Theorem 15.3. *Let* $|\mu| \le |\lambda|$, $p_k, g_k \in L([a,b];R_+)$ $(k = 1,\dots,m)$, *the condition* (12.24) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.18) *hold. If, moreover,* (15.1) *is fulfilled, where* γ *is defined by* (15.2), *then the problem* (10.1′), (10.2) *is uniquely solvable.*

Remark 15.3. The examples constructed in Subsection 14.3 (see On Remark 14.1, p. 280) also show that the strict inequality (15.1) cannot be replaced by the nonstrict one.

Theorem 15.4. *Let* $|\mu| \le |\lambda|$, $p_k, g_k \in L([a,b];R_+)$ $(k = 1,\dots,m)$, *the condition* (12.26) *be fulfilled, and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.19) *hold. If, moreover, the inequality* (15.3) *is fulfilled, then the problem* (10.1′), (10.2) *is uniquely solvable.*

Remark 15.4. The examples constructed in Subsection 14.3 (see On Remark 14.2, p. 284) also show that the strict inequality (15.3) cannot be replaced by the nonstrict one.

In the case, where $|\mu| \ge |\lambda|$, the following assertions hold.

Theorem 15.5. *Let* $|\mu| \ge |\lambda|$, $p_k, g_k \in L([a,b];R_+)$ $(k = 1,\dots,m)$, *the condition* (12.8) *be fulfilled and let on the set* $[a,b] \times R^{n+1}$ *the inequality* (13.6) *hold. If, moreover,*

$$\|p_0\|_L < \delta(\lambda,\mu), \tag{15.4}$$

where

$$\delta(\lambda,\mu)=\begin{cases}-\frac{\lambda}{\mu}+2\sqrt{1-\|g_0\|_L} & \text{if } \ \|g_0\|_L<1-\left(\frac{\lambda}{\mu}\right)^2\\ \frac{\mu}{\lambda}\left(1-\|g_0\|_L\right) & \text{if } \ \|g_0\|_L\geq 1-\left(\frac{\lambda}{\mu}\right)^2\end{cases}, \tag{15.5}$$

then the problem (10.1′), (10.2) *has at least one solution.*

Theorem 15.6. *Let* $|\mu|\geq|\lambda|$, $p_k,g_k\in L\big([a,b];R_+\big)$ $(k=1,\dots,m)$, *the condition* (12.1) *be fulfilled, and let on the set* $[a,b]\times R^{n+1}$ *the inequality* (13.1) *hold. If, moreover,*

$$\int\limits_a^b g_0(s)ds+\frac{\lambda}{\mu}\int\limits_a^b p_0(s)ds<\frac{\lambda}{\mu}, \tag{15.6}$$

then the problem (10.1′), (10.2) *has at least one solution.*

In Theorems 15.7 and 15.8, the conditions guaranteeing the unique solvability of the problem (10.1′), (10.2) are established.

Theorem 15.7. *Let* $|\mu|\geq|\lambda|$, $p_k,g_k\in L\big([a,b];R_+\big)$ $(k=1,\dots,m)$, *the condition* (12.26) *be fulfilled, and let on the set* $[a,b]\times R^{n+1}$ *the inequality* (13.19) *hold. If, moreover,* (15.4) *is fulfilled, where* δ *is defined by* (15.5), *then the problem* (10.1′), (10.2) *is uniquely solvable.*

Theorem 15.8. *Let* $|\mu|\geq|\lambda|$, $p_k,g_k\in L\big([a,b];R_+\big)$ $(k=1,\dots,m)$, *the condition* (12.24) *be fulfilled, and let on the set* $[a,b]\times R^{n+1}$ *the inequality* (13.18) *hold. If, moreover, the inequality* (15.6) *is fulfilled, then the problem* (10.1′), (10.2) *is uniquely solvable.*

Remark 15.5. According to Remark 12.14 (see p. 211), Theorems 15.5–15.8 can be derived from Theorems 15.1–15.4. Moreover, by virtue of Remarks 15.1–15.4, Theorems 15.5–15.8 are nonimprovable in an appropriate sense.

15.2. Proofs

Proof of Theorem 15.1. Obviously, the conditions (13.1), (15.1), and (15.2) yield the conditions (12.3), (14.1), and (14.2), where F, ℓ_0, and ℓ_1 are defined by (13.35). Consequently, the assumptions of Theorem 14.1 (see p. 266) are fulfilled. □

Proof of Theorem 15.2. Similarly to the proof of Theorem 15.1 one can show that the assumptions of Theorem 14.2 (see p. 267) are satisfied. □

Proof of Theorem 15.3. Obviously, the conditions (13.18), (15.1), and (15.2) yield the conditions (12.25), (14.1), and (14.2), where F, ℓ_0, and ℓ_1 are defined by (13.35). Consequently, the assumptions of Theorem 14.3 (see p. 269) are fulfilled. □

Proof of Theorem 15.4. Similarly to the proof of Theorem 15.3 one can show that the assumptions of Theorem 14.4 (see p. 269) are satisfied. □

Suplementary Remarks

The main ideas of the results presented in Chapter II can be found in [23, 25, 28, 29], where the special case of the boundary condition (10.2) with $\lambda = 1$ is considered.

Theorems 12.1, 12.3, 12.7, and 12.9 are proved in [25], Theorems 12.4 and 12.10 are proved in [29], Theorems 12.5 and 12.11 are proved in [28], and Theorems 14.1–14.4 one can find in [23].

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