CONNECTION AND CURVATURE ON BUNDLES OF BERGMAN AND HARDY SPACES

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Abstract. We consider a complex domain $D \times V$ in the space $\mathbb{C}^m \times \mathbb{C}^n$ and a family of weighted Bergman spaces on V defined by a weight $e^{-k\phi(z,w)}$ for a pluri-subharmonic function $\phi(z, w)$ with a quantization parameter k. The weighted Bergman spaces define an infinite dimensional Hermitian vector bundle over the domain D. We consider the natural covariant differentiation ∇_Z on the sections, namely the unitary Chern connections preserving the Bergman norm. We prove a Dixmier trace formula for the curvature of the unitary connection and we find the asymptotic expansion for the curvatures $R^{(k)}(Z, Z)$ for large k and for the induced connection $[\nabla_Z^{(k)}, T_f^{(k)}]$ on Toeplitz operators T_f. In the special case when the domain D is the Siegel domain and the weighted Bergman spaces are the Fock spaces we find the exact formula for $[\nabla_Z^{(k)}, T_f^{(k)}]$ as Toeplitz operators. This generalizes earlier work of J.E. Andersen in Comm. Math. Phys. 255 (2005), 727–745. Finally, we also determine the formulas for the curvature and for the induced connection in the general case of $D \times V$ replaced by a general strictly pseudoconvex domain $\mathcal{V} \subset \mathbb{C}^m \times \mathbb{C}^n$ fibered over a domain $D \subset \mathbb{C}^m$.

1. Introduction

There has been much interest in the study of bundles of infinite-dimensional Hilbert spaces in complex analysis and in quantization. As in the case of finite-dimensional Hermitian holomorphic vector bundles we would like to study the Chern connection and the curvature tensor, if they exist. Another relevant question of special interests in quantization is whether it is possible to introduce an integer parameter m with $\frac{1}{m}$ interpreted as the Planck constant and to study then the induced connection of sections of endomorphisms of the bundles. The most studied case might the bundle of Fock spaces on $\mathbb{C}^n$ over the Siegel space D; see e.g. [10] and Section 5 below. The periodic version of the Fock space are the flat bundles over abelian variaties covered by $\mathbb{C}^n$, which is an important topic in quantization [1]. We may also replace the Siegel-Jacobi space $D \times \mathbb{C}^n$ by a domain $\mathcal{V} \subset \mathbb{C}^{m+n}$ over a domain D and consider the corresponding Bergman and Hardy spaces over the fibers $\mathcal{V}_z$ of $z \in D$. In the present paper we shall study systematically connections and curvatures of these bundles by using Toeplitz operators.

Consider first a product domain $D \times V \subset \mathbb{C}^m \times \mathbb{C}^n$ and let ϕ be a plurisubharmonic function on $D \times V$. The Bergman spaces $L^2_h(V, e^{-\phi})$ on V with respect to the weight $e^{-\phi(z,\cdot)}$ form a holomorphic Hermitian vector bundle over D in an appropriate sense. In [2] the curvature is computed and it is proved that for a strictly pseudo-convex bounded domain V the curvature operator satisfies the Nagano positivity using the Hörmander estimate for solutions of the $\bar\partial$-equation. In the present paper we shall compute the Dixmier trace of the curvature operator, more precisely we prove that the curvature operator in this case is a Fredholm operator of the form $T_0 + T_1$ where T_0 is an invertible Toeplitz operator and T_1 is a compact operator, and we compute the Dixmier trace of T_1. We consider also the

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covariant differentiation $[\nabla_Z, T_f]$ on the Toeplitz operators T_f and prove likewise a Dixmier trace formula. Indeed the covariant differentiation $[\nabla_Z, T_f]$ is a natural generalization of the curvature operator $R(Z, \bar{Z}) = [\nabla_Z, \bar{\partial}_Z]$. We treat then the general case of fibrations over D by strictly pseudo-convex bounded domains. We find the curvature operator as Toeplitz operators and find its principal symbol.

We note that yet another important case is when $\mathcal{T}$ is the infinite dimensional Teichmüller space with the fibers being the unit disc whose complex structure changes by the quasi-conformal mappings parameterizing $\mathcal{T}$; see [11]. There are many important and difficult analytical problems in this case. We hope our study here will also shed light on this subject of infinite dimensional Teichmüller space. For a fiberation $\mathcal{X} \to \mathcal{T}$ of Kähler manifold $\mathcal{X}$ with compact fibers and a line bundle over $\mathcal{X}$ a general formulation for the study of variations of Bergman kernels is through the relative canonical bundle, namely the variation of the cohomology space $H^0(\mathcal{K} \otimes L|_{\mathcal{X}_t})$ on the fiber space $\mathcal{X}_t$. This has been studied extensively; see [2, 12] and references therein.

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2. Preliminaries

We formulate a convenient setup for vector bundles of Bergman spaces on complex domains and fix notation. There has been a lot of study of the general case of bundles of infinite dimensional Hilbert spaces and their connections; see e.g. [9] and references therein.

2.1. **Toeplitz operators on Bergman spaces.** Let $V \subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$. Let $L^2(V, e^{-\phi})$ be the L^2-space on V with respect to the measure $e^{-\phi(w)}dm(w)$, where $dm(w)$ is the Euclidean measure on $\mathbb{C}^n$, and $L^2_h(V, e^{-\phi})$ the Bergman space of holomorphic functions. Let $P : L^2(V, e^{-\phi}) \to L^2_h(V, e^{-\phi})$ be the Bergman projection. The Toeplitz operator $T_f : L^2_h(V, e^{-\phi}) \to L^2_h(V, e^{-\phi})$ and the Hankel operator $H_f : L^2_h(V, e^{-\phi}) \to L^2_h(V, e^{-\phi})^\perp$ with symbol $f \in L^\infty(V) = L^\infty(V, dm)$ are defined by $T_f = PM_f$, $H_f = (I - P)M_f$, where M_f is the operator of multiplication by f.

2.2. **Differentiation formulas for fiber integrations.** Let $\mathcal{V} \subset \mathbb{C}^{m+n}$ be a bounded domain with smooth boundary fibred over a domain $D \subset \mathbb{C}^m$. Let ρ be a defining function for $\mathcal{V}$, $\mathcal{V} = \{(z, w) \in \mathbb{C}^{m+n}; \rho(z, w) < 0\}$. The coordinates will be written as $z = (z^\alpha), w = (w^j)$, the $(1, 0)$-tangent vectors as Z, W and the $(0, 1)$-tangent vectors as $\overline{Z}, \overline{W}$. We assume that the projection $\pi : \mathbb{C}^{m+n} \to \mathbb{C}^m$ on $\mathcal{V}$ is of rank m and each $\mathcal{V}_z = \pi^{-1}(z)$ is a connected domain. In particular each $\mathcal{V}_z$ is bounded with smooth boundary. Let ν denote the surface measure on $\partial\mathcal{V}$. We fix $z_0 \in D$ and consider the differentiation by ∂_α of integrations $\int_{\mathcal{V}_z} f(z, w)dm(w)$ and $\int_{\partial\mathcal{V}_z} f(z, w)d\nu(w)$ of a function f along the fiber $\mathcal{V}_z$ and its boundary $\partial\mathcal{V}_z$. For simplicity we assume $z_0 = 0 \in D$.

Lemma 2.1. Suppose there is a $(1, 0)$-vector field $L_\alpha = \partial_\alpha + U_\alpha = \partial_\alpha + \sum_j U^j_\alpha \frac{\partial}{\partial w^j}$ such that $L_\alpha(\rho) = 0$ on $\partial\mathcal{V}$, i.e. L_α is tangential to the boundary $\partial\mathcal{V}$ and $\pi_*(L_\alpha + U_\alpha) = \partial_\alpha$. Then we have

$$\partial_\alpha \int_{\mathcal{V}_z} f(z, w)dm(w) = \int_{\mathcal{V}_z} (L_\alpha + \operatorname{div}(U_\alpha)) f dm(w)$$

and

$$\partial_\alpha \int_{\partial\mathcal{V}_z} f(z, w)\nu = \int_{\partial\mathcal{V}_z} (L_\alpha + \operatorname{div}_\nu(L_\alpha)) f\nu$$

where $\operatorname{div}(U_\alpha)$ is the divergence of U_α with respect to the Euclidean measure $dm(w)$ on $\mathcal{V}_z$ and $\operatorname{div}_\nu(L_\alpha)$ is the divergence of the tangential vector field L_α with respect to the area form ν on $\partial\mathcal{V}_z$, i.e., defined by the Lie derivative of ν, $\operatorname{Lie}(L_\alpha)\nu = \operatorname{div}_\nu(L_\alpha)\nu$.

Proof. This is presumably well-known, see e.g. [12]; we sketch a proof here for completeness. It is sufficient to consider the case when D is the unit disk in $\mathbb{C}$, so $z = x + iy$. Consider the real vector field ∂_x and a tangential lift $H = \partial_x + U$ of ∂, $d\rho(H) = 0$ when $\rho = 0$. The local diffeomorphism, say $\exp(xH)$, generated by the non-vanishing vector field H preserves the level set $\rho = 0$ and thus maps $\mathcal{V}_0$ to $\mathcal{V}_x$, since $\pi_*(H) = \partial_x$, namely $\exp(xH) : \{0\} \times \mathcal{V}_0 \to \{x\} \times \mathcal{V}_x$. We have then,

$$\int_{\mathcal{V}_x} f(x, w)dm(w) = \int_{\mathcal{V}_0} f(\exp(xH)(0, w))J(\exp(xH))(w)dm(w)$$

where $J(\exp(xH))(w)$ is the Jacobian of $\exp(xH)$ above in the vertical direction. Performing differentiation $\frac{d}{dx}$ and evaluating at $x = 0$ we find

$$\frac{d}{dx}\int_{\mathcal{V}_x} f(x, w)dm(w) = \int_{\mathcal{V}_0} (Hf(w) + \operatorname{div}(U)f)\, dm(w)$$

since $\frac{d}{dx}J(\exp(xH)) = \operatorname{div}(U)$ is the divergence of $H = \partial_x + U$ with respect to the volume dm at $x = 0$. The same argument works for any vector field $a\partial_x + b\partial_y$. The first claim follows by taking complexification. The second claim is almost the same. $\square$

Denote $L_{\bar\alpha} = \overline{L_\alpha} = \bar\partial_\alpha + \sum_j \overline{U_\alpha^j}\frac{\partial}{\partial \bar w^j}$. Taking the complex conjugate of the above formula we obtain the similar differentiation formula for $L_{\bar\alpha}$.

2.3. **Bundles of Hilbert spaces of holomorphic functions, Hermitian connection and Curvature.** Let $\mathcal{V} \subset \mathbb{C}^{n+m}$ be a bounded domain fibered over $D \subset \mathbb{C}^n$ as above. Suppose for each $z \in D$ there is a Hilbert space E_z of holomorphic functions on $\mathcal{V}_z$. For such families we shall define a notion of Hermitian bundles and curvature operator. A general treatment is to view E_z as Bergman space of $(n, 0)$-forms but we shall adapt a rather elementary and ad-hoc approach. On the other hand our definition includes families of Hilbert spaces whose norms are not defined in terms of measures; see Remark 2.4 below.

Definition 2.2. Let $E = \{E_z, z \in D\}$ be a family of Hilbert spaces E_z of holomorphic functions on $\mathcal{V}_z = \pi^{-1}(z)$.

(1) The family $E = \{E_z\}$ is called a holomorphic bundle of Hilbert spaces over D if for each $z_0 \in D$ there is a neighborhood $U_0 \subset D$ of z_0, a linear space $F(U_0)$ of holomorphic functions on $\pi^{-1}(U_0) \subset \mathcal{V}$ such that the subspace $F_z := F(U_0)|_z = \{u(z, \cdot); u \in F(U_0)\}$ is dense in E_z, and the coefficients $u_k(z, w)$ in the Taylor expansion of $u(z, w)$ in z near z_0,

$$u(z, w) = u(z_0, w) + u_1(z, w) \cdot (z - z_0) + \cdots + u_k(z, w) \cdot \odot^k(z - z_0) + \dots$$

are all in E_z, $z \in U_0$. Here $u_k(z, w)$ takes value in the symmetric tensors $\odot^k\mathbb{C}^n$ and $u \cdot v$ is the standard pairing in $\odot^k\mathbb{C}^n$. The union of all spaces $F(U_0)$ will be denoted by $\mathcal{O}(E)$ and will be called the space of locally holomorphic sections of E.

(2) A smooth section u of the bundle E near z_0 is defined as a function $u = u(z, w) \in C^\infty(\pi^{-1}(U_0))$ for a neighborhood U_0 of z_0, such that $u(z, \cdot)$ is in the dense subspace F_z and all derivatives in z are in F_z, $\partial_1^{k_1} \cdots \partial_n^{k_n}\bar\partial_1^{l_1} \cdots \bar\partial_n^{l_1}u(z, \cdot) \in F_z$, $z \in U_0$. The space of smooth sections will be denoted by $\Gamma(E)$. Note that $\mathcal{O}(E) \subset \Gamma(E)$.

(3) A connection on $\Gamma(E)$ is defined as a linear operator $X \to \nabla_X : \Gamma(E) \to \Gamma(E)$, for vector fields X on D such that

$$\nabla_X(fg) = f\nabla_X g + (Xf)g$$

and

$$\nabla_{fX} g = f \nabla_X g,$$

$f \in C^\infty(D)$ and $g \in \Gamma(E)$ whenever all the quantities are in $\Gamma(E)$. ∇ is called a *Chern connection* if

$$\nabla_{\bar{Z}} u = 0, \quad \partial_Z \langle u, v\rangle = \langle \nabla_Z u, v\rangle$$

for all $u, v \in \mathcal{O}(E)$, and $(1,0)$-vector fields Z on D.

(4) The curvature operator $R(Z, \bar{Z})$ at z, $Z \in T_z^{(1,0}(D)$, is defined as a linear operator on the dense subspace $F_z \subset E_z$ by

$$\bar{\partial}_Z \partial_Z \langle u, v\rangle = -\langle R(Z, \bar{Z}) u(z_0) v(z_0)\rangle + \langle \nabla_Z u, \nabla_Z v\rangle$$

for all $u, v \in \mathcal{O}(E)$.

Remark 2.3. When E is a finite-dimensional holomorphic Hermitian vector bundle over D it is an elementary fact that the curvature is determined by the formula above. We claim that in our case the curvature operator $(R(Z, \bar{Z})u)(z_0)$ is well-defined and depends only on $u(z_0)$, as in the finite-dimensional case. To see this it is enough to take D to be the unit disk and $z_0 = 0$. Let $u(z)$ be a holomorphic section near 0, such that $u(0) = 0$, and write u as

$$u(z) = u(0) + z u_1(z) = z u_1(z),$$

for some holomorphic function $u_1(z) = u_1(z, w)$. We prove that $(R(\partial, \bar{\partial})u)(0) = 0$. Let $v(z)$ be any holomorphic section near 0, and perform the differentiation

$$\partial\langle u, v\rangle = \partial\langle z u_1, v\rangle = \langle z \nabla u_1 + u_1(z), v\rangle = z\langle \nabla u_1, v\rangle + \langle u_1, v\rangle,$$

and

$$\bar{\partial}\partial\langle u, v\rangle = z\bar{\partial}\langle \nabla u_1, v\rangle + \bar{\partial}\langle u_1, v\rangle = z\bar{\partial}\langle \nabla u_1, v\rangle + \langle u_1, \nabla v\rangle. \tag{2.1}$$

Here we have used the fact that u_1 is holomorphic and $\bar{\partial}\langle u_1, v\rangle = \overline{\partial\langle v, u_1\rangle}$. Evaluating at 0 gives $\bar{\partial}\partial\langle u, v\rangle(0) = \langle u_1(0), \nabla v(0)\rangle$. On the other hand $\nabla u = \nabla(z u_1) = u_1 + z\nabla u_1$ so that

$$\langle \nabla u(0), \nabla v(0)\rangle = \langle u_1(0), \nabla v(0)\rangle.$$

It follows from the definition of the curvature that $\langle (R(\partial, \bar{\partial})u)(0), v(0)\rangle = 0$, for all v. By the density assumption we have $(R(\partial, \bar{\partial})u)(0) = 0$, proving our claim.

There are many examples where the above assumption is satisfied, for example when each $\mathcal{V}_z$ is a Reinhardt domain and E is the bundle of Bergman or Hardy spaces where the dense subspace F can be taken to be the space of polynomials in $\mathbb{C}^n$. For general domains $\mathcal{V}$ and the family of Bergman spaces on $\mathcal{V}_z$ some related questions have been studied; see e.g. [3, Lemma 3.4].

Example 2.4. A natural family of Bergman space on the unit disc D is the following: Consider the unit ball $B \in \mathbb{C}^{1+n}$ as fibered over the unit disk D, the fiber being the ball $B_z = \{w \in \mathbb{C}^n; |w|^2 \le 1 - |z|^2\}$. Let E_z be the Hilbert space of holomorphic functions on B_z with the reproducing kernel

$$\frac{(1 - |z|^2)^{1+\alpha}}{(1 - |z|^2 - w w')^{1+\alpha+n}}.$$

For $\alpha > -1$ this corresponds to the Bergman space $L^2_a(B_z, e^{-\phi} dm(w))$, $e^{-\phi} = (1 - |z|^2 - |w|^2)^\alpha$.

3. Trace formula for curvature of bundles of Bergman spaces. The product case

3.1. **Connection and curvature on bundles of Bergman spaces.** We consider first the case when $\mathcal{V} = D \times V$ is a product domain where D and V are open domains in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. This case is somewhat easier than the general case of fiberations in the next section, and we give a separate treatment here. In particular we obtain a Dixmier trace formula. Some other results can be obtained as corollaries of the general case.

We assume further that the domain $V = \{z \in \mathbb{C}^n, \rho(z) < 0\}$ is a strongly pseudo-convex bounded domain in $\mathbb{C}^n$ with smooth boundary, with ρ being a strictly plurisubharmonic function on a neighborhood of the closure of V.

Let $\phi(z, w)$ be a smooth plurisubharmonic function on $D \times V$. The Hessian of $\phi(z, w)$ will be written as $\phi_{\alpha\bar{\beta}}$, $\phi_{\alpha\bar{j}}$, $\phi_{j\bar{k}}$, etc., and the inverse of $(\phi_{j\bar{k}})$ as $(\phi^{k\bar{j}})$. We write the function $\phi(z, w)$ of w as $\phi(z)$, $\phi(z)(w) = \phi(z, w)$.

We consider the families $L^2(V, e^{-\phi(z)})$, $L^2_h(V, e^{-\phi(z)})$ of L^2-spaces and Bergman spaces and view $L^2_z = L^2(V, e^{-\phi(z)}) \to z \in D$, $E_z = L^2_h(V, e^{-\phi(z)}) \to z \in D$ as bundles of L^2 and Bergman spaces on D; we denote them by L^2 and E respectively. Note that E is a holomorphic bundle of Hilbert spaces in the sense of Definition 2.2 (though L^2 is not, although still being a bundle of Hilbert spaces in an obvious sense with a connection and curvature obtained by differentiation of L^2-integration). In the rest of this paper we shall consider only the connections ∇ such that $\nabla_{\overline{Z}} u(z, w) = \partial_{\overline{Z}} u(z, w)$.

The Chern connections on L^2 and E are

$$\begin{aligned} \nabla^{L^2}_Z &= \partial_Z - \partial_Z \phi, \\ \nabla^E_Z &= P \nabla^{L^2}_Z . \end{aligned} \tag{3.1}$$

It is a straightforward computation that the curvature $R^{L^2}(Z, \overline{Z})$ of ∇^{L^2} is then

$$R^{L^2}(Z, \overline{Z}) = \partial_Z \overline{\partial}_Z \phi,$$

namely the multiplication operator by $\partial_Z \overline{\partial}_Z \phi$. We shall be only interested in the curvature R^E.

The following proposition is essentially proved in [2], we provide a detailed proof here.

Proposition 3.1. The curvature of the vector bundle E is given by

$$R(Z, \overline{Z})u = T_{\partial_Z \partial_{\overline{Z}} \phi} - H^*_{\partial_Z \phi} H_{\partial_Z \phi}.$$

Proof. We perform differentiations according to the definition, for $u \in \mathcal{O}(E)$:

$$\partial_{\overline{Z}} \partial_Z \|u\|^2 = -\langle R^{L^2}(Z, \overline{Z})u, u\rangle + \|\nabla^{L^2}_Z u\|^2.$$

On the other hand the same computation with u being viewed as section of E gives

$$\partial_{\overline{Z}} \partial_Z \|u\|^2 = -\langle R^E(Z, \overline{Z})u, u\rangle + \|\nabla^E_Z u\|^2.$$

Thus

$$\begin{aligned} \langle R^E(Z, \overline{Z})u, u\rangle &= \langle R^{L^2}(Z, \overline{Z})u, u\rangle - \left(\|\nabla^{L^2}_Z u\|^2 - \|\nabla^E_Z u\|^2\right) \\ &= \langle T_{\partial_Z \partial_{\overline{Z}} \phi} u, u\rangle - \|(I - P)\nabla^{L^2}_Z u\|^2. \end{aligned}$$

The curvature tensor $R^E(Z, Z)u$ at $z = z_0$ depends only on $u(z_0)$ so we can choose $u = u(z_0)$ a constant section. Consequently $\nabla^{L^2}_Z = -\partial_Z \phi$ and we find

$$\langle R^E(Z, \overline{Z})u, u\rangle = \langle T_{\partial_Z \partial_{\overline{Z}} \phi} u, u\rangle - \langle H^*_{\partial_Z \phi} H_{\partial_Z \phi} u, u\rangle.$$

□

It seems interesting to note that the curvature operator $R_{\alpha\bar\alpha}$ is the sum of a positive invertible operator $T_{\partial_Z\bar\partial_Z\phi}$ and a compact operator, namely a Fredholm operator. Moreover the compact operator is in the Dixmier class and we can compute its Dixmier trace; see e.g [5] for the general theory of Dixmier trace. Recall that the boundary of a smoothly-bounded strictly pseudoconvex domain Ω comes equipped with the dual Levi form $\mathcal{L}_{\partial\Omega}$ on $T^{(0,1)}\partial\Omega$, so that, in particular, for any $f, g \in C^\infty(\partial\Omega)$ one has the function $\mathcal{L}(\bar\partial_b f, \bar\partial_b g)$, where $\bar\partial_b$ is the boundary Cauchy-Riemann operator on $\partial\Omega$ (see [7]).

Theorem 3.2. Let $n \ge 2$. Suppose V is a strongly pseudo-convex domain with smooth boundary ∂V. Suppose further that ϕ is smooth on $D \times \overline{V}$, where $\overline{V}$ is the closure of V. Then we have

$$\mathrm{Tr}_\omega(T_{\partial_Z\bar\partial_Z\phi} - R(Z,\overline{Z}))^n = \frac{1}{n!(2\pi)^n}\int_{\partial V} \mathcal{L}_{\partial V}(\bar\partial_b\partial_Z\phi, \bar\partial_b\partial_Z\phi)^n\ \eta\wedge(d\eta)^{n-1}$$

where $\eta = \mathrm{Im}(\partial\rho) = \frac{\partial\rho - \bar\partial\rho}{2i}$.

Proof. By the last proposition, $R(Z,\overline{Z}) = T_{\partial_Z\bar\partial_Z\phi} - H^*_{\partial_Z\phi}H_{\partial_Z\phi}$. The Dixmier trace formula then follows from [7, Theorem 11]. □

The curvature tensor R can be viewed as an operator on the space $L^2_h(V, e^{-\phi(z,\cdot)})\otimes T^{(1,0)}D$, via the Hermitian form $R(u\otimes\partial_j, v\otimes\partial_j) = \langle R(\partial_j,\bar\partial_j)u, v\rangle$. Similarly the dual Levi form $\mathcal{L}(\bar\partial_b\partial_j\phi, \bar\partial_b\partial_k\phi)$ can be viewed as a bivector and we let $\mathcal{L}^n$ be its power. Thus $\mathrm{Tr}_\omega(T_{\partial_Z\bar\partial_Z\phi} - R(Z,\overline{Z}))^n$ is an element of the space of symmetric tensors $\odot^n(T^{(1,0)}(D)\otimes\overline{T^{(1,0)}(D)})$ and the above result can also be written as

$$\mathrm{Tr}_\omega(T_{\partial\bar\partial\phi} - R)^n = \frac{1}{n!(2\pi)^n}\int_{\partial V} \mathrm{tr}\,\mathcal{L}^n(\bar\partial_b\partial\phi, \bar\partial_b\partial\phi)\ \eta\wedge(d\eta)^{n-1}.$$

We finally remark that from Proposition 3.1 one gets very simple "semiclassical" asymptotics of the curvature when the potential ϕ is rescaled to $m\phi$ with $m\to+\infty$ interpreted as the reciprocal of the Planck constant: namely,

$$R^{(m)}(Z,\overline{Z}) = mT^{(m)}_{\partial_Z\partial_{\overline{Z}}\phi} - m^2(H^{(m)}_{\partial_Z\phi})^*H^{(m)}_{\partial_Z\phi}.$$

The product $(H^{(m)}_{\partial_Z\phi})^*H^{(m)}_{\partial_Z\phi} = T^{(m)}_{|\partial_Z\phi|^2} - T^{(m)}_{\bar\partial_Z\phi}T^{(m)}_{\partial_Z\phi}$ involves the product of two Toeplitz operators, and can further be expanded as a sum of Toeplitz operators using the standard techniques [6]:

$$(H^{(m)}_g)^*H^{(m)}_f = \sum_{j=1}^\infty m^{-j}T_{\mathcal{C}_j(f,g)} \tag{3.2}$$

in operator norm, with some bidifferential operators $\mathcal{C}_j$, where $\mathcal{C}_1(f,g) = \mathcal{L}_{\partial\mathcal{V}}(\bar\partial_b f, \bar\partial_b g)$. Consequently,

$$R^{(m)}(Z,\overline{Z}) = mT^{(m)}_{\partial_Z\partial_{\overline{Z}}\phi - \mathcal{L}(\bar\partial_b\partial_Z\phi, \bar\partial_b\partial_Z\phi)} - \sum_{j=2}^\infty m^{2-j}T^{(m)}_{\mathcal{C}_j(\partial_Z\phi,\partial_Z\phi)},$$

with the cochains $\mathcal{C}_j$ above.

3.2. **Induced connection on Toeplitz operators.** We recall [1] that the induced connection on sections T of $\mathrm{End}(\mathcal{W})$ of a connection ∇ on a complex bundle $\mathcal{W}$ is

$$\nabla^{\mathrm{ind}}_Z T = [\nabla_Z, T].$$

For any $f = f(z,w)$ in $C^\infty(D\times V)$ we denote, with some abuse of notation, $M_f = M_{f(z,\cdot)}$ and $T_f = T_{f(z,\cdot)}$ the multiplication operator by $f(z,\cdot)$ on $L^2(V, e^{-\phi(z,\cdot)})$ and respectively the corresponding Toeplitz operator on the Bergman space $L^2_h(V, e^{-\phi(z,\cdot)})$. We define the connection ∇^{ind}_Z on the sections $z \mapsto T_{f(z,\cdot)}$ formally as above with ∇ being the $(1,0)$-part in (3.1).

The next lemma justifies the definition. Let $K(z; w, w')$ be the Bergman kernel for $L^2_h(V, e^{-\phi(z,\cdot)})$ for each fixed $z \in D$.

Lemma 3.3. The induced connection $\nabla^{\mathrm{ind}}_Z T_f$ on sections $T_f = T_{f(z,\cdot)}$ of Toeplitz operators is a bounded operator for each z and is given by

$$\nabla^{\mathrm{ind}}_Z T_f = T_{\partial_Z f} - H^*_{\overline{f}} H_{\partial_Z \phi}, \qquad \partial^{Ind}_{\overline{Z}} T_f = T_{\partial_{\overline{Z}} f} - H^*_{\overline{\partial_{\overline{Z}}\phi}} H_f.$$

Proof. We prove the first formula for $\nabla^{\mathrm{ind}}_Z T_f$, the second for $\partial^{\mathrm{ind}}_{\overline{Z}}$ is done similarly. We have

$$\nabla^{\mathrm{ind}}_Z T_f = [P\partial_Z P - T_{\partial_Z \phi}, T_f] = [P\partial_Z P, T_f] + [T_f, T_{\partial_Z \phi}],$$

and

$$[P\partial_Z P, T_f] = P\partial_Z T_f - T_f P\partial_Z P.$$

The operator $g \mapsto \nabla^{\mathrm{ind}}_Z T_f g$ depends only on g at z and we can take a section $g = g(w)$ independent of z, so that $\partial_Z g = 0$. Thus

$$[P\partial_Z P, T_f]g = P\partial_Z T_f g - T_f P\partial_Z P g = P\partial_Z T_f g.$$

We perform the differentiation ∂_Z on the Toeplitz operator

$$T_f g(w) = \int_V f(z,u) K(z; w, u) e^{-\phi(z;u)} g(u) dm(u)$$

and get

$$\begin{aligned}
&\langle \partial_Z T_f g, h\rangle \\
&= \int_V \int_V \partial_Z \Big[f(z,u)) K(z; w, u) e^{-\phi(z;u)} g(u) \Big]\, dm(u) \overline{h(w)} e^{-\phi(z;w)}\, dm(w) \\
&= \int_V \int_V \partial_Z \Big[f(z,u)) K(z; w, u) e^{-\phi(z;u)} g(u) \overline{h(w)} e^{-\phi(z;w)} \Big]\, dm(u)\, dm(w) \\
&\qquad - \int_V \int_V f(z,u)) K(z; w, u) e^{-\phi(z;u)} g(u) \partial_Z \Big[\overline{h(w)} e^{-\phi(z;w)} \Big]\, dm(u)\, dm(w) \\
&= \partial_Z \langle T_f g, h\rangle \\
&\qquad + \int_V \int_V f(z,u)) K(z; w, u) e^{-\phi(z;u)} g(u) \overline{h(w)} \partial_Z \phi(z; w) e^{-\phi(z;w)}\, dm(u)\, dm(w) \\
&= \int_V \partial_Z (f e^{-\phi}) g \overline{h}\, dm + \int_V (T_f g)(\partial_Z \phi) \overline{h} e^{-\phi}\, dm \\
&= \langle (T_{e^\phi \partial_Z (f e^{-\phi})} + T_{\partial_Z \phi} T_f) g, h\rangle.
\end{aligned}$$

Thus $\partial_Z T_f = T_{\partial_Z f - f\partial_Z \phi} + T_{\partial_Z \phi} T_f$ (hence, in particular, $P\partial_Z T_f = \partial_Z T_f$) and

$$\nabla^{\mathrm{ind}}_Z T_f = [T_f, T_{\partial_Z \phi}] + T_{\partial_Z f - f\partial_Z \phi} + T_{\partial_Z \phi} T_f = T_{\partial_Z f - f\partial_Z \phi} + T_f T_{\partial_Z \phi} = T_{\partial_Z f} - H^*_{\overline{f}} H_{\partial_Z \phi}.$$

This completes the proof. ∎

Note that, by the main result of [7], the operators $H^*_{\overline{f}} H_{\partial_Z \phi}$ and $H^*_{\overline{\partial_{\overline{Z}}\phi}} H_f$ above again belong to the Lorentz class $\mathcal{S}^{n,\infty}$ (more precisely: upon identifying holomorphic functions on V with their distributional boundary values on ∂V, these operators become generalized Toeplitz operators of order -1 on the Hardy space) and we have formulas for the Dixmier trace of their n-th power in terms of the dual Levi form.

Recall that the boundary Poisson bracket (cf. [7]) on ∂V is given by

$$\{f,g\}_b := \mathcal{L}_{\partial V}(\bar\partial_b g, \bar\partial_b \bar f) - \mathcal{L}_{\partial V}(\bar\partial_b f, \bar\partial_b \bar g).$$

Theorem 3.4. Suppose V is a strongly pseudo-convex domain with smooth boundary ∂V, $V=\{w\in\mathbb{C}^n, r(z,w)<0\}$, and $r(z,w)=0$, $\partial_w r(z,w)\neq 0$ on ∂V. Then we have

$$\operatorname{Tr}_\omega \nabla_Z^{\rm ind}[T_f,T_g]^n = \frac{1}{(n-1)!(2\pi)^n}$$
$$\int_{\partial V}(\{\partial_Z f, g\}_b + \{f,\partial_Z g\}_b)\{f,g\}_b^{n-1}\,\eta\wedge(d\eta)^{n-1}.$$

Proof. We compute the commutator

$$\nabla_Z^{\rm ind}[T_f,T_g]^n = [\nabla_Z,[T_f,T_g]^n] = \sum_{j=1}^n [T_f,T_g]^{j-1}[\nabla_Z,[T_f,T_g]][T_f,T_g]^{n-j},$$

and

$$[\nabla_Z,[T_f,T_g]] = [[\nabla_Z,T_f],T_g] + [T_f,[\nabla_Z,T_g]].$$

The commutator $[\nabla_Z,T_f]$ is computed in Lemma 3.3 and equals $T_{\partial_Z f} - H^*_{\bar f} H_{\partial_Z \phi}$ with $H^*_{\bar f} H_{\partial_Z\phi}$ being (after passing again to the Hardy space via boundary values) a Toeplitz operator of degree -1; similarly for the second term $[T_f,[\nabla_Z,T_g]]$. Thus

$$\nabla_Z^{\rm ind}[T_f,T_g]^n = \sum_{j=1}^n [T_f,T_g]^{j-1}([T_{\partial_Z f},T_g] + [T_f,T_{\partial_Z g}])[T_f,T_g]^{n-j} + U$$

with the rest term U being a Toeplitz operator of degree $-n-1$. The rest follows then from Theorem 11 in [7]. □

We again observe that Lemma 3.3 gives very simple "semiclassical" asymptotics for the induced connection when the potential ϕ is rescaled to $m\phi$ and $m\to+\infty$: namely,

$$\nabla_Z^{\rm ind}T_f = T_{\partial_Z f} - mH^*_{\bar f}H_{\partial_Z\phi}, \qquad \partial_{\overline{Z}}^{Ind}T_f = T_{\partial_{\overline{Z}} f} - mH^*_{\overline{\partial_{\overline{Z}}\phi}}H_f.$$

Applying (3.2) to the products of Hankel operators, as before, yields

$$\nabla_Z^{\rm ind}T_f = T_{\partial_Z f - \mathcal{L}(\bar\partial_b\partial_Z\phi, \bar\partial_b \bar f)} - \sum_{j=2}^\infty m^{1-j} T_{\mathcal{C}_j(\partial_Z\phi,\bar f)},$$

and similarly for $\partial_{\overline{Z}}^{Ind}T_f$.

We remark that the curvature of the induced connection is identically zero, by the Jacobi identity.

4. Trace formula for curvature of bundles of Hardy and Bergman spaces. The general case

We consider now the case of a holomorphic fibration $\pi:\mathcal{V}\to D$ with strongly pseudoconvex bounded domains $\mathcal{V}_z=\pi^{-1}(z)$, $z\in D$, generalizing the previous case with $\mathcal{V}=D\times V$. In [12] a curvature formula is found by following the earlier approach in [2] using $(n,0)$-forms. We shall use our elementary definition above and derive a curvature formula using Toeplitz operators. Our formula is different from that in [12] and in particular we prove that the curvature operator is a Toeplitz operator of order 1 for general fiberations whose principal symbol is positive.

We assume now that $\mathcal{V}=\{(z,w)\in\mathbb{C}^{n+m};\rho(z,w)<0\}$ is a bounded strongly pseudo-convex domain in $\mathbb{C}^{n+m}$ fibered over a domain $D\subset\mathbb{C}^n$, with ρ a strongly pluri-subharmonic function defined in a neighborhood of the closure of $\mathcal{V}$. We write $\phi=-\log(-\rho)$. It might possible to choose a different ϕ independent of the defining function ρ but we shall fix this choice in this section. We recall the following lemma from [4].

Lemma 4.1. The vector fields

$$L_\alpha = \partial_\alpha + U^j_\alpha \partial_j = \partial_\alpha - \phi_{\alpha\bar{k}} \phi^{j\bar{k}} \partial_j$$

are smooth on the closure of $\mathcal{V}$ and are tangential on $\partial\mathcal{V}$.

This is easily verified using the explicit formulas

$$\phi^{\bar{k}j} = (-\rho)\Big(\rho^{\bar{k}j} + \frac{\rho^j \rho^{\bar{k}}}{\rho - |\partial\rho|^2}\Big)$$

and

$$U^j_\alpha = \frac{\rho^j \rho_\alpha}{\rho - |\partial\rho|^2} - \rho_{\alpha\bar{k}} \rho^{\bar{k}j} - \frac{\rho^j \rho^{\bar{k}} \rho_{\alpha\bar{k}}}{\rho - |\partial\rho|^2}. \tag{4.1}$$

Here $\rho^{\bar{k}j}$ is the inverse matrix to $\rho_{j\bar{k}}$, $\rho^j := \rho^{\bar{k}j}\rho_{\bar{k}}$, $\rho^{\bar{k}} := \rho^{\bar{k}j}\rho_j$, $|\partial\rho|^2 := \rho^j \rho_j$ is the norm of $\partial\rho$ with respect to $(\partial\bar{\partial}\rho)^{-1} = (\rho^{\bar{k}j})$, and we are (as always) using the usual summation convention of automatically summing over any index that appears twice.

4.1. **Bundle of Hardy spaces.** Throughout this subsection, we will denote by the same letter a holomorphic function in $\mathcal{V}_z$ and its boundary value on $\partial\mathcal{V}_z$ (for ease of notation).

We keep our previous notation for the L^2-spaces $L^2(\partial\mathcal{V}_z)$ on the boundary $\partial\mathcal{V}_z$ with respect to the surface measure, and for the Hardy spaces $L^2_h(\partial\mathcal{V}_z)$ of boundary values of holomorphic functions in $\mathcal{V}_z$, with $\Pi = \Pi_z$ the Szegö projection of L^2 onto L^2_h. For $f \in C^\infty(\partial\mathcal{V}_z)$, the Toeplitz operator T_f on the Hardy space is defined by $T_f u = \Pi(fu)$, and the Hankel operator H_f from the Hardy space into its orthogonal complement is defined by $H_f u = (I - \Pi)(fu)$. (We will not consider the Toeplitz and Hankel operators on Bergman spaces in this subsection, so the use of the same notation T_f, H_f for them should cause no confusion.)

Furthermore, for a pseudo-differential operator A on $L^2(\partial\mathcal{V}_z)$, $T_A = \Pi A \Pi$ is the (generalized) Toeplitz operator on $L^2_h(\partial\mathcal{V})$, sometimes written as $T(A)$ for typographical reasons. The order of T_A is defined as the order of A, and the symbol $\sigma(T_A)$ of T_A is defined as the restriction of the symbol of A to the subset

$$\Sigma_z := \{(x, t\eta_x) : \ x \in \partial\mathcal{V}_z, \ t > 0\}$$

of the cotangent bundle $T^*\partial\mathcal{V}_z$; here η is the one-form $\eta = \operatorname{Im} \rho(z) = \frac{1}{2i}(\partial\rho(z) - \bar{\partial}\rho(z))$ (which thus depends on z, although this is not reflected by the notation). The reader is referred i.e. to Section 2.1 of [7] for more details on generalized Toeplitz operators.

We also define generalized Hankel operators with pseudo-differential symbols by $H_A = (I - \Pi)A\Pi$. Note that $T^*_A = T_{A^*}$ and $T_{AB} - T_A T_B = H^*_{A^*} H_B$. Note that the operators T_f are still recovered as T_A for A the operator of multiplication by f; in particular, T_f is of order zero and $\sigma(T_f) = f$. It also follows from the proof of [7, Theorem 9] that for $f, g \in C^\infty(\partial\mathcal{V}_z)$, the product $H^*_g H_f = T_{fg} - T_g T_f$ is of order -1 with principal symbol

$$\sigma(H^*_g H_f)(x, t\eta_x) = \tfrac{1}{t} \mathcal{L}_{\partial\mathcal{V}_z}(\bar{\partial}_b f, \bar{\partial}_b g)(x).$$

A special case of the above operators are, in fact, the differentiations ∂_j in the fiber variables. Namely, with the notation $\mathbf{K}$ for the Poisson extension operator (assigning to a function u on $\partial\mathcal{V}_z$ the harmonic function $\mathbf{K}u$ on $\mathcal{V}_z$ with boundary values u) and r for its inverse (i.e. the operator of taking the boundary values of a harmonic function), it is known that $r\partial_j\mathbf{K}$ (which is well-defined since the derivative $\partial_j\mathbf{K}u$ of a harmonic function $\mathbf{K}u$ is again harmonic) is a pseudo-differential operator Z_j of order 1 on the boundary. On the cone Σ, the symbol of Z_j satisfies

$$\sigma(Z_j)(x,t\eta_x) \equiv \sigma(T_{Z_j})(x,t\eta_x) = t\rho_j. \tag{4.2}$$

By the very definition of Z_j,

$$T_{Z_j} = \partial_j.$$

Since the derivative of a holomorphic function is again holomorphic, we have $Z_j\Pi = \Pi Z_j \Pi$, i.e. T_{Z_j} is just the restriction of Z_j to the Hardy space; in particular, $H_{Z_j} = 0$, and

$$T_{AZ_j} = T_A T_{Z_j} = T_A \partial_j, \qquad H_A T_{Z_j} = H_{AZ_j}$$

for any generalized Teoplitz operator T_A and generalized Hankel operator H_A.

In this subsection we consider the bundle H of Hardy spaces $H_z = L^2_h(\partial\mathcal{V}_z)$. Setting as before $U_\alpha = U^j_\alpha \partial_j$, so that $L_\alpha = \partial_\alpha + U_\alpha$, we also denote by

$$d_\alpha := \operatorname{div}_\nu L_\alpha$$

the divergence of the tangential operator L_α with respect to the surface measure on $\partial\mathcal{V}_z$.

By Lemma 2.1, we then get for $u, v \in \mathcal{O}(E)$,

$$\partial_\alpha \int_{\partial\mathcal{V}_z} u\overline{v}\, d\nu = \int_{\partial\mathcal{V}_z} (L_\alpha + d_\alpha)(u\overline{v})\, d\nu = \int_{\partial\mathcal{V}_z} (L_\alpha + d_\alpha)u\,\overline{v}\, d\nu$$

since $L_\alpha \overline{v} = \overline{L_{\overline{\alpha}} v} = 0$ by the holomorphy of v. As in Section 3.1, it transpires that the Chern connection on H is given by

$$\nabla^H_\alpha u = \Pi(L_\alpha + d_\alpha)u. \tag{4.3}$$

Our main result is that the associated curvature is a generalized Toeplitz operator of order 1.

Theorem 4.2. The curvature of ∇^H is given by

$$R_{\alpha\bar\beta} = -T_{(L_{\bar\beta} U^j_\alpha) Z_j} - T_{L_{\bar\beta} d_\alpha} - H^*_{U^k_\beta Z_k + d_\beta} H_{U^j_\alpha Z_j + d_\alpha}.$$

Proof. By the definition of curvature and (4.3), for $u, v \in \mathcal{O}(H)$,

$$\langle R_{\alpha\overline{\beta}} u, v\rangle_z = \langle \Pi(L_\alpha + d_\alpha)u, \Pi(L_\beta + d_\beta)v\rangle_z - \partial_{\overline{\beta}}\partial_\alpha \langle u, v\rangle_z.$$

Using Lemma 2.1, we have

$$\begin{aligned}
\partial_{\overline{\beta}}\partial_\alpha \langle u, v\rangle_z &= \partial_{\overline{\beta}}\partial_\alpha \int_{\partial\mathcal{V}_z} u\overline{v}\, d\nu \\
&= \int_{\partial\mathcal{V}_z} (L_{\overline{\beta}} + d_{\overline{\beta}})(L_\alpha + d_\alpha)(u\overline{v})\, d\nu \\
&= \int_{\partial\mathcal{V}_z} (L_{\overline{\beta}} + d_{\overline{\beta}})[(L_\alpha + d_\alpha)u \,\cdot\, \overline{v}]\, d\nu \\
&= \int_{\partial\mathcal{V}_z} [L_{\overline{\beta}}(L_\alpha + d_\alpha)u \,\cdot\, \overline{v}] \\
&\qquad\qquad + [(L_\alpha + d_\alpha)u \,\cdot\, \overline{(L_\beta + d_\beta)v}]\, d\nu \\
&= \langle \Pi(L_{\overline{\beta}}(L_\alpha + d_\alpha)u), v\rangle_z + \langle (L_\alpha + d_\alpha)u, (L_\beta + d_\beta)v\rangle_z.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle R_{\alpha\overline{\beta}} u, v\rangle_z &= -\langle (I - \Pi)(L_\alpha + d_\alpha)u, (I - \Pi)(L_\beta + d_\beta\rangle_z - \langle \Pi(L_{\overline{\beta}}(L_\alpha + d_\alpha)u), v\rangle_z \\
&= -\langle \Pi(L_{\overline{\beta}}(L_\alpha + d_\alpha)u), v\rangle_z - \langle H_{L_\alpha + d_\alpha} u, H_{L_\beta + d_\beta} v\rangle_z.
\end{aligned}$$

Now

$$\begin{aligned} H_{L_\alpha} u &= (I-\Pi)(\partial_\alpha u + U^j_\alpha \partial_j u) \\ &= (I-P)(U^j_\alpha \partial_j u) = H_{U^j_\alpha} \partial_j u \\ &= H_{U^j_\alpha} T_{Z_j} u \equiv H_{U^j_\alpha Z_j} u, \end{aligned}$$

since $\partial_\alpha u$ is holomorphic. Similarly, $\Pi(L_{\bar\beta} d_\alpha u) = \Pi(L_{\bar\beta} d_\alpha) u = T_{L_{\bar\beta} d_\alpha} u$ and

$$\begin{aligned} \Pi(L_{\bar\beta} L_\alpha u) &= \Pi L_{\bar\beta}(\partial_\alpha u + U^j_\alpha \partial_j u) \\ &= \Pi L_{\bar\beta}(U^j_\alpha \partial_j u) = \Pi(L_{\bar\beta} U^j_\alpha)\partial_j u \\ &= T_{L_{\bar\beta} U^j_\alpha} T_{Z_j} u \equiv T_{(L_{\bar\beta} U^j_\alpha) Z_j} u, \end{aligned}$$

and the assertion follows. □

We proceed to examine the principal symbol of the curvature operator.

Theorem 4.3. (i) $H^*_{U^k_\beta Z_k + d_\beta} H_{U^j_\alpha Z_j + d_\alpha}$ is a generalized Toeplitz operator of order 1, whose principal symbol, however, vanishes; consequently, it is in fact a generalized Toeplitz operator of order zero (hence, in particular, bounded).

(ii) $T_{(L_{\bar\beta} U^j_\alpha) Z_j}$ is a generalized Toeplitz operator of order 1, and the matrix of principal symbols $\{\sigma(-T_{(L_{\bar\beta} U^j_\alpha) Z_j})\}^m_{\alpha,\beta=1}$ is positive definite.

Altogether, we thus see that $(R_{\alpha\bar\beta})$ is a matrix of generalized Toeplitz operators of order 1 with positive-definite principal symbol; the positivity of the matrix $(R_{\alpha\bar\beta})$ is generally defined as Nagano positivity [2].

Proof. (i) The claim concerning the order is immediate from the formula $H^*_{A^*} H_B = T_{AB} - T_A T_B$ and the properties of generalized Toeplitz operators (noting that both $U^j_\alpha Z_j + d_\alpha$ and $U^k_\beta Z_k + d_\beta$ are pseudo-differential operators of order 1 on the boundary).

To compute the principal symbol, note that H_{d_β} and H_{d_α} are of order zero, hence it is enough to compute the principal symbol of

$$H^*_{U^k_\beta Z_k} H_{U^j_\alpha Z_j} = T^*_{Z_k} H^*_{U^k_\beta} H_{U^j_\alpha} T_{Z_j},$$

which equals, by (4.2),

$$t\rho_{\bar k} \mathcal{L}_{\partial\mathcal{V}_z}(\bar\partial_b U^j_\alpha, \bar\partial_b U^k_\beta)\rho_j = t\mathcal{L}_{\partial\mathcal{V}_z}(\rho_j \bar\partial_b U^j_\alpha, \rho_k \bar\partial_b U^k_\beta). \tag{4.4}$$

Now on the boundary, we have $0 = L_\alpha \rho = \rho_\alpha + U^j_\alpha \rho_j$, and since $\bar\partial_b$ depends only on boundary values, this gives, by the Leibniz rule,

$$\rho_j \bar\partial_b U^j_\alpha = -\bar\partial_b \rho_\alpha - U^j_\alpha \bar\partial_b \rho_j .$$

Hence the principal symbol equals (for brevity we write just $\mathcal{L}$ for $\mathcal{L}_{\partial\mathcal{V}_z}$ throughout the rest of this proof)

$$\begin{aligned} &t\mathcal{L}(\bar\partial_b\rho_\alpha + U^j_\alpha \bar\partial_b \rho_j, \bar\partial_b \rho_\beta + U^k_\beta \bar\partial_b \rho_k) \\ &= t\Big[\mathcal{L}(\bar\partial_b\rho_\alpha, \bar\partial_b\rho_\beta) + \mathcal{L}(\bar\partial_b\rho_\alpha, \bar\partial_b\rho_\ell)\overline{U^\ell_\beta} + U^m_\alpha \mathcal{L}(\bar\partial_b\rho_m, \bar\partial_b\rho_\beta) + U^m_\alpha \mathcal{L}(\bar\partial_b\rho_m, \bar\partial_b\rho_\ell)\overline{U^\ell_\beta}\Big] \end{aligned} \tag{4.5}$$

Recall now from [7, page 618] that, quite generally,

$$\mathcal{L}(\bar\partial_b f, \bar\partial_b g) = \begin{bmatrix} \bar\partial g \\ 0 \end{bmatrix}^* \begin{bmatrix} \partial\bar\partial\rho & \bar\partial\rho \\ \partial\rho & 0 \end{bmatrix}^{-1} \begin{bmatrix} \bar\partial f \\ 0 \end{bmatrix}.$$

Computing the inverse of the middle matrix, this gives explicitly, in our notation,

$$\mathcal{L}(\bar\partial_b f, \bar\partial_b g) = \partial_{\bar k} f\Big(\rho^{\bar k j} - \frac{\rho^{\bar k}\rho^j}{|\partial\rho|^2}\Big)\partial_j \bar g. \tag{4.6}$$

Thus, in particular,

$$\begin{aligned}
\mathcal{L}(\bar\partial_b\rho_\alpha, \bar\partial_b\rho_\beta) &= \rho_{\alpha\bar k}\Big(\rho^{\bar k j} - \frac{\rho^{\bar k}\rho^j}{|\partial\rho|^2}\Big)\rho_{j\bar\beta}, \\
\mathcal{L}(\bar\partial_b\rho_\alpha, \bar\partial_b\rho_\ell) &= \rho_{\alpha\bar\ell} - \frac{\rho_{\alpha\bar k}\rho^{\bar k}\rho_{\bar\ell}}{|\partial\rho|^2}, \\
\mathcal{L}(\bar\partial_b\rho_m, \bar\partial_b\rho_\beta) &= \rho_{m\bar\beta} - \frac{\rho_m\rho^j\rho_{j\bar\beta}}{|\partial\rho|^2}, \\
\mathcal{L}(\bar\partial_b\rho_m, \bar\partial_b\rho_\ell) &= \rho_{m\bar\ell} - \frac{\rho_m\rho_{\bar\ell}}{|\partial\rho|^2}.
\end{aligned}$$

Next, from (4.1) we have

$$U_\alpha^m = -\frac{\rho^m\rho_\alpha}{|\partial\rho|^2} - \rho_{\alpha\bar s}\rho^{\bar s m} + \frac{\rho^m\rho^{\bar s}\rho_{\alpha\bar s}}{|\partial\rho|^2} \qquad \text{on } \partial\mathcal{V}. \tag{4.7}$$

Hence

$$\begin{aligned}
U_\alpha^m\mathcal{L}(\bar\partial_b\rho_m, \bar\partial_b\rho_\beta) &= \Big(\rho_{m\bar\beta} - \frac{\rho_m\rho^j\rho_{j\bar\beta}}{|\partial\rho|^2}\Big)\Big(\frac{\rho^m\rho^{\bar s}\rho_{\alpha\bar s}}{|\partial\rho|^2} - \rho_{\alpha\bar s}\rho^{\bar s m} - \frac{\rho^m\rho_\alpha}{|\partial\rho|^2}\Big) \\
&= \frac{\rho_{m\bar\beta}\rho^m\rho^{\bar s}\rho_{\alpha\bar s}}{|\partial\rho|^2} - \rho_{m\bar\beta}\rho_{\alpha\bar s}\rho^{\bar s m} - \frac{\rho_{m\bar\beta}\rho^m\rho_\alpha}{|\partial\rho|^2} - \frac{\rho^j\rho_{j\bar\beta}\rho^{\bar s}\rho_{\alpha\bar s}}{|\partial\rho|^2} + \frac{\rho^j\rho_{j\bar\beta}\rho_{\alpha\bar s}\rho^{\bar s}}{|\partial\rho|^2} + \frac{\rho_\alpha\rho^j\rho_{j\bar\beta}}{|\partial\rho|^2} \\
&= \rho_{\alpha\bar s}\Big(-\rho^{\bar s m} + \frac{\rho^m\rho^{\bar s}}{|\partial\rho|^2}\Big)\rho_{m\bar\beta} = -\mathcal{L}(\bar\partial_b\rho_\alpha, \bar\partial_b\rho_\beta),
\end{aligned}$$

since on the middle line the first term cancels the fourth and the third term cancels the sixth. Similarly,

$$\overline{U_\beta^\ell}\mathcal{L}(\bar\partial_b\rho_\alpha, \bar\partial_b\rho_\ell) = -\mathcal{L}(\bar\partial_b\rho_\alpha, \bar\partial_b\rho_\beta)$$

and

$$U_\alpha^m\mathcal{L}(\bar\partial_b\rho_m, \bar\partial_b\rho_\ell)\overline{U_\beta^\ell} = \mathcal{L}(\bar\partial_b\rho_\alpha, \bar\partial_b\rho_\beta).$$

Consequently, in (4.5) the first and fourth terms cancel the second and third, and the symbol vanishes, as claimed.

(ii) Using again the formula (4.2) for the symbol of T_{Z_j}, we have

$$\sigma(T_{(L_{\bar\beta}U_\alpha^j)Z_j})(x, t\eta_x) = t\rho_j(x)L_{\bar\beta}U_\alpha^j(x).$$

By the Leibniz rule,

$$\rho_j L_{\bar\beta}U_\alpha^j = L_{\bar\beta}(\rho_j U_\alpha^j) - U_\alpha^j L_{\bar\beta}(\rho_j).$$

Now $L_{\bar\beta}$ is a tangential operator, so the boundary values of $L_{\bar\beta}(\rho_j U_\alpha^j)$ depend only on the boundary values of $\rho_j U_\alpha^j$; but since $L_\alpha\rho = 0$ there, we have

$$0 = L_\alpha\rho = \rho_\alpha + U_\alpha^j\rho_j,$$

or $\rho_j U^j_\alpha = -\rho_\alpha$ on the boundary. Thus on the boundary,

$$\begin{aligned}
-\rho_j L_{\bar\beta} U^j_\alpha &= L_{\bar\beta}(\rho_\alpha) + U^j_\alpha L_{\bar\beta}(\rho_j) \\
&= \rho_{\alpha\bar\beta} + \overline{U^k_\beta}\rho_{\alpha\bar k} + U^j_\alpha \rho_{j\bar\beta} + U^j_\alpha \overline{U^k_\beta}\rho_{j\bar k} \\
&= \left[I, U^j_\alpha\right] \begin{bmatrix} \rho_{\alpha\bar\beta} & \rho_{\alpha\bar k} \\ \rho_{j\bar\beta} & \rho_{j\bar k}\end{bmatrix} \begin{bmatrix} I \\ \overline{U^k_\beta}\end{bmatrix}.
\end{aligned}$$

By the hypothesis of strict plurisubharmonicity of ρ on the closure of $\mathcal{V}$, the last 2×2 block matrix is positive definite; hence also the whole product of the three block matrices is positive definite, proving the claim. □

Remark 4.4. By an analogous computation as in the proof of part (i) above, one can derive the following explicit formula for the last symbol:

$$\sigma(T_{(L_{\bar\beta}U^j_\alpha)Z_j}) = \rho_{\alpha\bar\beta} + \rho_{\alpha\bar\ell}\Big(\frac{\rho^m\rho^{\bar\ell}}{|\partial\rho|^2} - \rho^{\bar\ell m}\Big)\rho_{m\bar\beta} + \frac{\rho_\alpha\rho_{\bar\beta} - \rho_{m\bar\beta}\rho^m\rho_\alpha - \rho_{\alpha\bar\ell}\rho^{\bar\ell}\rho_{\bar\beta}}{|\partial\rho|^2},$$

which however is not very enlightening, nor is it immediate that it is positive definite.

Example 4.5. We work out the various quantities in this subsection for the situation of the ball $\mathcal{V} = \mathbf{B}^{n+m}$ from Example 2.4 with $\alpha = 1$. By (4.1), upon a small computation,

$$U^j_\alpha = -\frac{\bar z_\alpha}{1-|z|^2} w_j.$$

Hence $H_{U^j_\alpha} = -\frac{\bar z_\alpha}{1-|z|^2} H_{w_j} = 0$ since w_j is holomorphic.

To compute d_α, observe that, by rotational symmetry in the fibers, it must be constant on each $\partial\mathcal{V}_z$. Applying Lemma 2.1 to the function constant one therefore gives

$$\partial_\alpha \nu(\partial\mathcal{V}_z) = d_\alpha \nu(\partial\mathcal{V}_z),$$

or $d_\alpha = \partial_\alpha \log\nu(\partial\mathcal{V}_z)$. As in our case $\nu(\partial\mathcal{V}_z) = \text{const.}(1-|z|^2)^{(2m-1)/2}$, we get

$$d_\alpha = -(m-\tfrac12)\frac{\bar z_\alpha}{1-|z|^2}.$$

Further computations give

$$\begin{aligned}
L_{\bar\beta} d_\alpha = \partial_{\bar\beta} d_\alpha &= -(m-\tfrac12)\partial_{\bar\beta}\partial_\alpha \log\frac{1}{1-|z|^2}, \\
L_{\bar\beta} U^j_\alpha \partial_j u &= -\Big(\partial_{\bar\beta}\partial_\alpha \log\frac{1}{1-|z|^2}\Big) R_w u,
\end{aligned}$$

where R_w denotes the radial derivative in the w-variables. Hence $H_{d_\alpha} = 0$ and thus both Hankel operators in Theorem 4.2 vanish, while the Toeplitz operator there become

$$R_{\alpha\bar\beta} = \Big(\partial_{\bar\beta}\partial_\alpha \log\frac{1}{1-|z|^2}\Big)(R_w + m - \tfrac12).$$

Note that the matrix $\{\partial_{\bar\beta}\partial_\alpha \log\frac{1}{1-|z|^2}\}^n_{\alpha,\beta=1}$ is nothing else but the standard invariant metric on $\mathbf{B}^n$ (hence, in particular, positive definite, in full accordance with part (ii) of the last theorem).

We conclude by computing the induced connection $\nabla^{\text{ind}} = [\nabla, \cdot]$ on Toeplitz operators. We deal only with the holomorphic part, leaving $\bar\partial^{\text{ind}}$ to the reader.

Theorem 4.6. We have
$$\nabla^{\rm ind}_\alpha T_f = T_{L_\alpha f} - H^*_{\bar f} H_{U_\alpha+d_\alpha}.$$
Furthermore, the second term is actually an operator of order -1 (hence, belonging to the Lorentz ideal $\mathcal{S}^{n,\infty}$).

Proof. By the definition of the induced connection, we have for $g\in\mathcal{O}(H)$
$$\begin{aligned}\nabla^{\rm ind}_\alpha T_f g &= [\Pi(L_\alpha+d_\alpha)\Pi, T_f]g = [\Pi\partial_\alpha\Pi, T_f]g + [T_{U_\alpha+d_\alpha}, T_f]g \\ &= \partial_\alpha T_f g - T_f\partial_\alpha g + [T_{U_\alpha+d_\alpha}, T_f]g,\end{aligned}$$
where we dropped some Π's on the last line since ∂_α preserves $\mathcal{O}(H)$. Now, recalling the definition of the Chern connection and using Lemma 2.1, for $g,h\in\mathcal{O}(H)$,
$$\begin{aligned}\langle\partial_\alpha T_f g, h\rangle &= \langle(\nabla^H_\alpha - U_\alpha - d_\alpha)T_f g, h\rangle \\ &= \partial_\alpha\langle T_f g, h\rangle - \langle T_{U_\alpha+d_\alpha}T_f g, h\rangle \\ &= \int_{\partial\mathcal{V}_z}(L_\alpha+d_\alpha)(fg\bar h)\,d\nu - \langle T_{U_\alpha+d_\alpha}T_f g, h\rangle \\ &= \int_{\partial\mathcal{V}_z}((L_\alpha+d_\alpha)(fg))\bar h\,d\nu - \langle T_{U_\alpha+d_\alpha}T_f g, h\rangle \\ &= \int_{\partial\mathcal{V}_z}(((L_\alpha f) + fL_\alpha + fd_\alpha)g)\bar h\,d\nu - \langle T_{U_\alpha+d_\alpha}T_f g, h\rangle \\ &= \langle T_{(L_\alpha f)+f(L_\alpha+d_\alpha)}g, h\rangle - \langle T_{U_\alpha+d_\alpha}T_f g, h\rangle,\end{aligned}$$
implying that $\partial_\alpha T_f = T_{(L_\alpha f)+f(L_\alpha+d_\alpha)} - T_{U_\alpha+d_\alpha}T_f$. Consequently,
$$\begin{aligned}\nabla^{\rm ind}_\alpha T_f &= T_{(L_\alpha f)+f(L_\alpha+d_\alpha)} - T_{U_\alpha+d_\alpha}T_f - T_f\partial_\alpha + [T_{U_\alpha+d_\alpha}, T_f] \\ &= T_{(L_\alpha f)+f(U_\alpha+d_\alpha)} - T_{U_\alpha+d_\alpha}T_f + [T_{U_\alpha+d_\alpha}, T_f] \\ &= T_{(L_\alpha f)+f(U_\alpha+d_\alpha)} - T_f T_{U_\alpha+d_\alpha} \\ &= T_{L_\alpha f} + H^*_{\bar f}H_{U_\alpha+d_\alpha},\end{aligned}$$
proving the first claim. The second term in the last expression is a generalized Toeplitz operator of order $0+1-1=0$, with principal symbol
$$\sigma(H^*_{\bar f}H_{U_\alpha}) = \sigma(H^*_{\bar f}H_{U^m_\alpha}T_{\partial_m}) = \rho_m\mathcal{L}(\bar\partial_b U^m_\alpha, \bar\partial_b f),$$
which equals, by the same argument as in the proof of part (i) of the last theorem,
$$-\mathcal{L}(\bar\partial_b\rho_\alpha + U^m_\alpha\bar\partial_b\rho_m, \bar\partial_b f). \tag{4.8}$$
Now, using again (4.6),
$$\mathcal{L}(\bar\partial_b\rho_\alpha, \bar\partial_b f) = \partial_j\bar f\Big(\rho^{\bar k j} - \frac{\rho^{\bar k}\rho^j}{|\partial\rho|^2}\Big)\rho_{\alpha\bar k},$$
while
$$\begin{aligned}U^m_\alpha\mathcal{L}(\bar\partial_b\rho_m, \bar\partial_b f) &= \partial_j\bar f\Big(\rho^{\bar k j} - \frac{\rho^{\bar k}\rho^j}{|\partial\rho|^2}\Big)\rho_{m\bar k}U^m_\alpha \\ &= \partial_j\bar f\Big(U^j_\alpha - \frac{\rho_m\rho^j}{|\partial\rho|^2}U^m_\alpha\Big) \\ &= \partial_j\bar f\Big(U^j_\alpha + \frac{\rho_\alpha\rho^j}{|\partial\rho|^2}\Big) \qquad \text{as } \rho_m U^m_\alpha + \rho_\alpha = L_\alpha\rho = 0 \\ &= -\rho_{\alpha\bar s}\Big(\rho^{\bar s j} - \frac{\rho^{\bar s}\rho^j}{|\partial\rho|^2}\Big)\partial_j\bar f,\end{aligned}$$

where the last equality follows from (4.7). Thus the two summands in (4.8) cancel each other, proving the second claim. ☐

From the second part of the last theorem, it follows in particular that $(H^*_{\bar f} H_{U_\alpha + d_\alpha})^n$ is Dixmier traceable. We have not tried to compute its Dixmier trace.

4.2. **Curvature formula for bundle of Bergman spaces.** We now consider the bundle E of Bergman spaces $E_z = L^2_h(\mathcal{V}_z, e^{-\phi(z)} dm)$. Keeping our previous notation

$$L_\alpha = \partial_\alpha + U_\alpha, \qquad U_\alpha := U^j_\alpha \partial_j, \qquad U^j_\alpha = -\phi_{\alpha\bar k}\phi^{j\bar k},$$

we again denote for brevity

$$d_\alpha := \operatorname{div} U_\alpha = \partial_k(U^k_\alpha).$$

(Note: this is a different quantity than the function d_α in the preceding subsection; however there should be no danger of confusion.) By Lemma 2.1, we then have, for $u, v \in \mathcal{O}(E)$,

$$\partial_\alpha \int_{\mathcal{V}_z} u\bar v e^{-\phi}\, dm = \int_{\mathcal{V}_z} (L_\alpha + d_\alpha)(u\bar v e^{-\phi})\, dm = \int_{\mathcal{V}_z} \bar v e^{-\phi}(L_\alpha + d_\alpha - L_\alpha \phi) u \, dm.$$

It follows, as in Section 3.1, that the Chern connection on E is given by

$$\nabla^E_\alpha u = P(L_\alpha + d_\alpha - L_\alpha\phi)u. \tag{4.9}$$

Similarly as for the Hardy space, we will need a generalization of Toeplitz operators on the Bergman space whose symbols are allowed to be not only functions but differential operators: namely, quite generally, for a differential operator L on a complex domain Ω with coefficients smooth on the closure of Ω we define the Toeplitz operator T_L on a weighted Bergman space $L^2_h(\Omega, e^{-\phi})$ by $T_L u := P(Lu)$. Of course, T_L is in general only a densely defined, closed unbounded operator. The simplest examples of such Toeplitz operators are the coordinate differentiations $T_{\partial_k} = \partial_k$.

Generalized Hankel operators with $H_L : L^2_h \to L^2 \ominus L^2_h$ are defined analogously as $H_L u := (I - P)(Lu)$.

Note that it is not in general true that $T^*_L = T_{L^*}$, as also witnessed by the following lemma.

Lemma 4.7. Assume that the weight $-e^{-\phi} = \rho$ is a defining function for Ω. Then $T^*_{\partial_k} = -T_{2\rho_{\bar k}} T^{-1}_\rho$.

Proof. This is well-known, but we include a proof for completeness. Assume that $u, v \in L^2_h(\Omega, e^{-\phi})$ are smooth up to the boundary $\partial\Omega$. By the Stokes theorem (ν_k stands for the appropriate component of the outward unit normal on the boundary),

$$\int_\Omega \partial_k(u\bar v \rho^2)\, dm = \int_{\partial\Omega} u\bar v\rho^2 \nu_k = 0,$$

since by hypothesis ρ vanishes on the boundary. By the Leibniz rule, it follows that

$$0 = \int_\Omega (-\rho\partial_k u - 2u\rho_k)\bar v e^{-\phi}\, dm.$$

Since this holds for all u, v in a dense subset, we conclude that

$$T_\rho T_{\partial_k} + 2T_{\rho_k} = 0,$$

or $T_{\partial_k} = -T^{-1}_\rho T_{2\rho_k}$. Taking adjoints completes the proof. ☐

Identifying holomorphic functions with their boundary values — or, somewhat more precisely, using the Poisson extension operator $\mathbf{K}$ and the operator r of restriction to the boundary — one can transfer operators T on the Bergman space $L^2_h(\Omega, e^{-\phi})$ to operators $rT\mathbf{K}$ on the Hardy space $L^2_h(\partial\Omega)$ from the preceding section. Abusing language, we will say that T is a (generalized) Toeplitz operator (of order k and with leading symbol σ) on the Bergman space if $rT\mathbf{K}$ is a generalized Toeplitz operator (of order k and with symbol σ) on the Hardy space. It is then a famous result due to Boutet de Monvel that ordinary Toeplitz operators T_f, with f smooth on the closure of Ω, are Toeplitz operators of order zero in the above sense, with symbol $\sigma(x,\xi) = f(x)$, $x \in \partial\Omega$; and if f vanishes on $\partial\Omega$ to order k, that T_f is actually of order $-k$. In particular, if $-e^{-\phi} = \rho$ is a defining function, T_ρ is a Toeplitz operator of order -1 with principal symbol

$$\sigma(T_\rho)(x, t\eta_x) = -2/t.$$

The product $H_g^* H_f$ of two Hankel operators, with f, g smooth on the closure of Ω, being equal to $T_{\overline{f}g} - T_{\overline{g}} T_f$, is a generalized Toeplitz operator of order -1 (more generally, of order $-k-q-1$ if f, g vanish on the boundary to orders k, q, respectively), with principal symbol again given by (cf. [7, Theorem 9])

$$\sigma(H_g^* H_f)(x, t\eta_x) = t\mathcal{L}_{\partial\Omega}(\bar\partial_b f, \bar\partial_b g)(x), \tag{4.10}$$

for $t > 0$ and $x \in \partial\Omega$. Furthermore, T_{∂_k} is a generalized Toeplitz operator of order 1, with symbol

$$\sigma(T_{\partial_k})(x, t\eta_x) = t\rho_k|_{\partial\Omega}. \tag{4.11}$$

The reader is again referred e.g. to Section 2 in [7] and the references therein for more details about all the facts just mentioned.

All the above notions apply, in particular, to our domains $\mathcal{V}_z$. We then have the following theorem. Note that the Toeplitz operator in the formula below is of order 1, as is the product of the two Hankel operators.

Theorem 4.8. The curvature of ∇^E is given by

$$R_{\alpha\overline{\beta}} = T_{L_{\bar\beta} L_\alpha \phi - L_{\bar\beta} d_\alpha - (L_{\bar\beta} U_\alpha^j)\partial_j} - H^*_{U_\beta + d_\beta - L_\beta\phi} H_{U_\alpha + d_\alpha - L_\alpha \phi}.$$

Proof. By the definition and (4.9), for $u, v \in \mathcal{O}(E)$,

$$\langle R_{\alpha\overline{\beta}} u, v\rangle_z = \langle P(L_\alpha + d_\alpha - L_\alpha\phi)u, P(L_\beta + d_\beta - L_\beta\phi)v\rangle_z - \partial_{\bar\beta}\partial_\alpha \langle u, v\rangle_z.$$

Using again Lemma 2.1, we have

$$\begin{aligned}
\partial_{\bar\beta}\partial_\alpha\langle u, v\rangle_z &= \partial_{\bar\beta}\partial_\alpha \int_{\mathcal{V}_z} u\overline{v} e^{-\phi}\, dm \\
&= \int_{\mathcal{V}_z} (L_{\bar\beta} + d_{\bar\beta})(L_\alpha + d_\alpha)(u\overline{v}e^{-\phi})\, dm \\
&= \int_{\mathcal{V}_z} (L_{\bar\beta} + d_{\bar\beta})[(L_\alpha + d_\alpha - L_\alpha\phi)u \cdot \overline{v}e^{-\phi}]\, dm \\
&= \int_{\mathcal{V}_z} L_{\bar\beta}(L_\alpha + d_\alpha - L_\alpha\phi)u \cdot \overline{v}e^{-\phi} \\
&\qquad\qquad + (L_\alpha + d_\alpha - L_\alpha\phi)u \cdot \overline{(L_\beta + d_\beta - L_\beta\phi)v} \cdot e^{-\phi}]\, dm \\
&= \langle T_{L_{\bar\beta}(L_\alpha + d_\alpha - L_\alpha\phi)} u, v\rangle_z + \langle (L_\alpha + d_\alpha - L_\alpha\phi)u, (L_\beta + d_\beta - L_\beta\phi)v\rangle_z.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle R_{\alpha\overline{\beta}} u, v\rangle_z &= -\langle (I-P)(L_\alpha + d_\alpha - L_\alpha\phi)u, (I-P)(L_\beta + d_\beta - L_\beta\phi)v\rangle_z - \langle T_{L_{\bar\beta}(L_\alpha + d_\alpha - L_\alpha\phi)} u, v\rangle_z \\
&= \langle T_{L_{\bar\beta}(L_\alpha\phi - d_\alpha - L_\alpha)} u, v\rangle_z - \langle H_{L_\alpha + d_\alpha - L_\alpha\phi} u, H_{L_\beta + d_\beta - L_\beta\phi} v\rangle_z.
\end{aligned}$$

Note that since $L_{\bar\beta}$ involves only anti-holomorphic differentiations, we have $L_{\bar\beta} L_\alpha u = L_{\bar\beta}(\partial_\alpha u + U_\alpha^j \partial_j u) = (L_{\bar\beta} U_\alpha^j)\partial_j u$, and similarly $H_{L_\alpha} = H_{U_\alpha}$, $H_{L_\beta} = H_{U_\beta}$. The assertion follows. ∎

From the formula $H_L^* H_K = T_{L^*K} - T_{L^*}T_K$, valid for any pair of differential operators L, K, it follows that $H^*_{U_\beta} H_{U_\alpha}$ differs from $H^*_{U_\beta+d_\beta-L_\beta\phi} H_{U_\alpha+d_\alpha-L_\alpha\phi}$ by an operator of order zero (hence bounded); similarly $T_{L_{\bar\beta}L_\alpha\phi-L_{\bar\beta}d_\alpha}$ is a bounded operator (recall that $L_\alpha\phi$ and U^j_α are smooth on the closure of $\mathcal{V}_z$, see the preceding section). The next theorem therefore describes the leading order terms of $R_{\alpha\bar\beta}$.

Theorem 4.9. (i) $H^*_{U_\beta} H_{U_\alpha}$ is a generalized Toeplitz operator of order 1, whose principal symbol however vanishes, so that it is in fact of order zero (hence, in particular, bounded).

(ii) $T_{(L_{\bar\beta}U^j_\alpha)\partial_j}$ is a generalized Toeplitz operator of order 1, and the matrix of principal symbols $\{\sigma(-T_{(L_{\bar\beta}U^j_\alpha)\partial_j})\}^n_{\alpha,\beta=1}$ is positive definite.

Proof. (i) By (4.11) and (4.10),

$$\sigma(H^*_{U_\beta} H_{U_\alpha}) = \sigma(T^*_{\partial_k} H^*_{U^k_\beta} H_{U^j_\alpha} T_{\partial_j}) = t\rho_{\bar k}\mathcal{L}(\bar\partial_b U^j_\alpha, \bar\partial_b U^k_\beta)\rho_j = t\mathcal{L}(\rho_j\bar\partial_b U^j_\alpha, \rho_k\bar\partial_b U^k_\beta).$$

However, this is the same expression as (4.4) we had for the Hardy bundle in the preceding subsection, and we showed in the proof of Theorem 4.3 that it vanishes identically.

(ii) From (4.11),

$$\sigma(T_{(L_{\bar\beta}U^j_\alpha)\partial_j})(x, t\eta_x) = t\rho_j(x) L_{\bar\beta} U^j_\alpha(x)$$

for x on the boundary. However, this is exactly the same expression as in the proof of part (ii) of Theorem 4.3, and thus we immediately get the conclusion by the argument used there. □

Altogether, we thus see that the curvature $R_{\alpha\bar\beta}$ is again a matrix of generalized Toeplitz operators of order 1, with principal symbol which is positive definite — and is, remarkably, the same as for the Hardy bundle in the preceding subsection.

Example 4.10. We again work out the various quantities in this subsection for the situation of the ball $\mathcal{V} = \mathbf{B}^{n+m}$ from Example 2.4. As we already saw in Example 4.5,

$$U^j_\alpha = -\frac{\bar z_\alpha}{1-|z|^2} w_j.$$

Hence

$$U_\alpha = -\frac{\bar z_\alpha}{1-|z|^2} R_w,$$

where as before R_w denotes the radial derivative in the w-variables. The latter preserves holomorphic functions, so in particular $H_{U_\alpha} = 0$. Further computations give

$$\begin{aligned} d_\alpha &= -m\frac{\bar z_\alpha}{1-|z|^2}, \\ L_\alpha\phi &= \frac{\bar z_\alpha}{1-|z|^2}, \\ L_{\bar\beta} d_\alpha &= \partial_{\bar\beta} d_\alpha = -m\partial_{\bar\beta}\partial_\alpha \log\frac{1}{1-|z|^2}, \end{aligned}$$

while we already saw in Example 4.5 that

$$L_{\bar\beta} U^j_\alpha \partial_j u = -\Big(\partial_{\bar\beta}\partial_\alpha \log\frac{1}{1-|z|^2}\Big) R_w u.$$

Hence $H_{d_\alpha} = H_{L_\alpha\phi} = 0$ and both Hankel operators in Theorem 4.8 vanish, while the Toeplitz operator there gives

$$R_{\alpha\bar\beta} = \Big(\partial_{\bar\beta}\partial_\alpha \log\frac{1}{1-|z|^2}\Big)(R_w + m + 1).$$

We again witness the appearance of the matrix $\{\partial_{\bar\beta}\partial_\alpha \log \frac{1}{1-|z|^2}\}_{\alpha,\beta=1}^n$ representing the standard invariant metric on $\mathbf{B}^n$ (hence, in particular, positive definite, in full accordance with part (ii) of the last theorem), except the term $R_w + m - \frac12$ for the Hardy space now becomes $R_w + m + 1$.

As in the previous subsection, we conclude by giving a formula for the induced connection on Toeplitz operators. We again deal only with the holomorphic part.

Theorem 4.11. For the induced connection $\nabla^{\text{ind}} T = [\nabla^E, T]$, we have

$$\nabla^{\text{ind}} T_f = T_{L_\alpha f} + H^*_{\bar f} H_{U_\alpha + d_\alpha - L_\alpha\phi}.$$

The second term is a generalized Toeplitz operator of order -1, in particular, it is a compact operator in the Lorentz ideal $\mathcal{S}^{n,\infty}$.

Proof. The proof is exactly the same as for the Hardy case, with trivial modifications. Namely, by the definition of the induced connection, we have for $g \in \mathcal{O}(E)$

$$\begin{aligned}\nabla^{\text{ind}}_\alpha T_f g &= [P(L_\alpha + d_\alpha - L_\alpha\phi)P, T_f]g = [P\partial_\alpha P, T_f]g + [T_{U_\alpha+d_\alpha-L_\alpha\phi}, T_f]g \\ &= \partial_\alpha T_f g - T_f\partial_\alpha g + [T_{U_\alpha+d_\alpha-L_\alpha\phi}, T_f]g,\end{aligned}$$

where we dropped some P's on the last line since ∂_α preserves $\mathcal{O}(E)$. Now, recalling the definition of the Chern connection and using Lemma 2.1, for $g, h \in \mathcal{O}(E)$,

$$\begin{aligned}\langle \partial_\alpha T_f g, h\rangle &= \langle (\nabla^E_\alpha - U_\alpha - d_\alpha + L_\alpha\phi)T_f g, h\rangle \\ &= \partial_\alpha \langle T_f g, h\rangle - \langle T_{U_\alpha+d_\alpha-L_\alpha\phi} T_f g, h\rangle \\ &= \int_{\mathcal{V}_z} (L_\alpha + d_\alpha)(fg\bar h e^{-\phi})\, dm - \langle T_{U_\alpha+d_\alpha-L_\alpha\phi} T_f g, h\rangle \\ &= \int_{\mathcal{V}_z} (e^\phi (L_\alpha + d_\alpha)(fge^{-\phi}))\bar h e^{-\phi}\, dm - \langle T_{U_\alpha+d_\alpha-L_\alpha\phi} T_f g, h\rangle \\ &= \int_{\mathcal{V}_z} (((L_\alpha f) + fL_\alpha + fd_\alpha - fL_\alpha\phi)g)\bar h\, dm - \langle T_{U_\alpha+d_\alpha-L_\alpha\phi} T_f g, h\rangle \\ &= \langle T_{(L_\alpha f)+f(L_\alpha+d_\alpha-L_\alpha\phi)} g, h\rangle - \langle T_{U_\alpha+d_\alpha-L_\alpha\phi} T_f g, h\rangle,\end{aligned}$$

implying that $\partial_\alpha T_f = T_{(L_\alpha f)+f(L_\alpha+d_\alpha-L_\alpha\phi)} - T_{U_\alpha+d_\alpha-L_\alpha\phi}T_f$. Consequently,

$$\begin{aligned}\nabla^{\text{ind}}_\alpha T_f &= T_{(L_\alpha f)+f(L_\alpha+d_\alpha-L_\alpha\phi)} - T_{U_\alpha+d_\alpha-L_\alpha\phi}T_f - T_f\partial_\alpha + [T_{U_\alpha+d_\alpha-L_\alpha\phi}, T_f] \\ &= T_{(L_\alpha f)+f(U_\alpha+d_\alpha-L_\alpha\phi)} - T_{U_\alpha+d_\alpha-L_\alpha\phi}T_f + [T_{U_\alpha+d_\alpha-L_\alpha\phi}, T_f] \\ &= T_{(L_\alpha f)+f(U_\alpha+d_\alpha-L_\alpha\phi)} - T_f T_{U_\alpha+d_\alpha-L_\alpha\phi} \\ &= T_{L_\alpha f} + H^*_{\bar f} H_{U_\alpha+d_\alpha-L_\alpha\phi},\end{aligned}$$

proving the first claim. The second term in the last expression is a generalized Toeplitz operator of order $0+1-1=0$, whose principal symbol coincides with the one from Theorem 4.6, which we have shown to vanish. This settles also the second claim. ∎

Again, we have not tried to compute the Dixmier trace of $(H^*_{\bar f} H_{U_\alpha+d_\alpha-L_\alpha\phi})^n$.

Note that replacing ϕ by $m\phi$ (which leaves U_α, d_α and L_α unchanged) and letting $m \to +\infty$, we also get simple "semiclassical" expansions for the curvature and for the induced connection.

5. Fock bundle

We study now another important fibration and family of Bergman spaces, namely the domain V being the space $\mathbb{C}^n$ with a family of Fock spaces parameterized by the Siegel domain D. One may consider further the quotients of $\mathbb{C}^n$ by lattices $\mathbb{Z}_z^{2n}, z \in D$, and the corresponding spaces of holomorphic sections of flat bundles, the so-called theta bundle; see [1]. However we will restrict ourself to the Fock bundle, much computations can be extended to that case.

5.1. **Siegel-Jacobi domain.** We let $V = \mathbb{C}^n$ and D be the Siegel domain

$$D := \{z = z^t \in M_{n,n}(\mathbb{C}); y = \operatorname{Im} z > 0\}. \tag{5.1}$$

The product $D \times V$ is also called the Siegel-Jacobi space. Let

$$\boxed{\phi(z, w) = -\log\det(\operatorname{Im} z) + 2\pi\langle(\operatorname{Im} z)^{-1} \operatorname{Im} w, \operatorname{Im} w\rangle = -\log\det y + 2\pi \left(y^{-1}v, v\right)}. \tag{5.2}$$

Here $(w, t) = \sum_j w^j t^j$ is the bilinear form on $\mathbb{C}^n$. The Hessian $\partial\bar\partial\phi$ is given by

$$\partial_{Z_1}\partial_{\overline{Z}_2}\phi = \frac{1}{4}\left(y^{-\frac{1}{2}}Z_1y^{-\frac{1}{2}}, y^{-\frac{1}{2}}\overline{Z_2}y^{-\frac{1}{2}}\right) + \pi\left(Q_{y^{-1}}(Z_1, \overline{Z_2})y^{-1}v, y^{-1}v\right) \tag{5.3}$$

$$\partial_Z\partial_{\overline{W}}\phi = -\pi\left(y^{-1}Zy^{-1}v, \overline{W}\right), \ \partial_{W_1}\partial_{\overline{W}_2}\phi = \pi\left(y^{-1}W_1, \overline{W}_2\right), \tag{5.4}$$

where $Q_{y^{-1}}$ denotes the (Jordan theoretic) sesqui-linear operator

$$Q_{y^{-1}}(Z_1, \overline{Z}_2) = \frac{1}{2}(Z_1y^{-1}\overline{Z}_2 + \overline{Z}_2y^{-1}Z_1).$$

The positivity of the Hessian follows simply by Cauchy-Schwarz inequality.

5.2. **Fock bundle over the Siegel domain.** For each $z \in D$ we let $\mathcal{F}_z(\mathbb{C}^n)$ be the Fock space of holomorphic functions h on $\mathbb{C}^n$ equipped with the inner product

$$\|h\|^2 = \|h\|_z^2 = \int_{\mathbb{C}^n} |h(w)|^2 d\mu_z(w) < \infty,$$

where

$$d\mu_z(w) = e^{-\phi(z,w)}dm(w) = (\det\operatorname{Im} z)e^{-2\pi\left((\operatorname{Im} z)^{-1}v,v\right)}dm(w), \ w = u + iv.$$

The reproducing kernel is well-known and is given by

$$K(z; w, t) = \det(y)^{-2}\exp\left(-\frac{\pi}{2}\left(y^{-1}(w - \bar t), w - \bar t\right)\right), \ w, t \in \mathbb{C}^n.$$

It can be easily checked that the family $\mathcal{F} = \{\mathcal{F}_z(\mathbb{C}^n)\}_{z\in D}$ forms then a Hermitian bundle over D according to Definition 2.2.

Remark 5.1. In [10] a different normalization for the measure $d\mu_z(w) = e^{-\phi}dm(w)$ is chosen, with $\det\operatorname{Im} z$ above replaced by $(\det\operatorname{Im} z)^{\frac{1}{2}}$, our choice of ϕ is made so that it is plurisubharmonic on the total space in consistence with Section 3 above. In particular the formula for the covariant differentiation ∇_Z below is different from that in [10]. We remark also that we may introduce the weights $\det(\operatorname{Im} z)e^{-\left(y^{-1}v,v\right)}$ on the total space $D \times V$ and consider the Bergman-Fock space of holomorphic functions $h(z, w)$ on $D \times V$ with respect to the weight; the corresponding reproducing kernel is

$$\det\left(\frac{z - \bar{z}'}{2i}\right)^{-\frac{1}{2}}\exp\left(-\frac{\pi}{2}\left(\left(\frac{z - \bar{z}'}{2i}\right)^{-1}(w - \bar t), w - \bar t\right)\right).$$

In particular each such $h(z, w)$ for fixed z is in the Fock space $\mathcal{F}_z(\mathbb{C}^n)$ and can be viewed as global holomorphic section of the Fock bundle.

5.3. **Connection and Curvature on the Fock bundle.** We introduce the holomorphic wave operator

$$\Box_Z h = \partial_Z h - \frac{1}{4\pi i} \sum_{j,k=1}^{n} Z_{jk} \partial_j \partial_k h,$$

and its constant shift

$$\nabla_Z h = \frac{i}{4} \left(y^{-1}, Z \right) h + \Box_Z h.$$

This can be written also as

$$\nabla = \frac{1}{4i} \left(y^{-1}, dz \right) \mathrm{I} + \partial_{jk} \otimes dz_{jk} - \frac{1}{4\pi i} \sum_{j,k}^{n} \partial_j \partial_k \otimes dz_{jk},$$

taking values on differential forms on D.

Lemma 5.2. The Chern connection on the Fock bundle is given by ∇_Z.

Proof. The connection is $\nabla_Z = \partial_Z - T_{\partial_Z \phi}$ with the symbol $\partial_Z \phi$ of the Toeplitz operator $T_{\partial_Z \phi}$ being

$$\partial_Z \phi(z,t) = -\frac{1}{2i} \left(Z, y^{-1} \right) + \pi i \left(y^{-1} Z y^{-1} s, s \right), \quad t = r + is.$$

Thus

$$T_{\partial_Z \phi} h(w) = -\frac{1}{2i} \left(Z, y^{-1} \right) h + \pi \int_{\mathbb{C}^n} \left(y^{-1} Z y^{-1} s, s \right) K(z; w, t) h(t) d\mu_z(t). \tag{5.5}$$

We claim that the Toeplitz operator $T_{\partial_Z \phi}$ is

$$T_{\partial_Z \phi} h(w) = \frac{1}{4\pi i} \left(Z \partial, \partial \right) h - \frac{1}{4i} \left(Z, y^{-1} \right) h. \tag{5.6}$$

Indeed differentiating the reproducing formula $h(w) = \langle (h(\cdot), K(z; \cdot, w) \rangle$ we find

$$\begin{aligned} &\partial_j \partial_k h(w) \\ &= \int_{\mathbb{C}^n} \left(\pi^2 \left(y^{-1} e_j, w - \bar{t} \right) \left(y^{-1} e_k, w - \bar{t} \right) - \pi \left(y^{-1} e_k, e_j \right) \right) K(z; w, t) h(t) d\mu_z(t) \\ &= \pi^2 \left(y^{-1} e_j \otimes y^{-1} e_k, \int_{\mathbb{C}^n} (w - \bar{t})^2 K(z; w, t) h(t) d\mu_z(t) \right) \\ &\quad - \pi \int_{\mathbb{C}^n} \left(y^{-1} e_k, e_j \right) K(z; w, t) h(t) d\mu_z(t), \end{aligned} \tag{5.7}$$

where we have written $u^2 = u \otimes u$ and extended the bilinear product on $\mathbb{C}^n$ to the tensor product $\mathbb{C}^n \otimes \mathbb{C}^n$. Write $w - \bar{t} = (w - t) + (t - \bar{t})$ and $(w - \bar{t})^2 = (w - t)^2 + (t - \bar{t}) \otimes (w - t) + (w - t) \otimes (t - \bar{t}) + (t - \bar{t})^2$. The corresponding integrations in (5.7) involving the first three terms are vanishing due to the reproducing kernel property that

$$\int_{\mathbb{C}^n} (w_j - t_j) h(t) K(z; w, t) d\mu_z(t) = 0, \quad \int_{\mathbb{C}^n} (w_j - t_j)(w_k - t_k) h(t) K(z; w, t) d\mu_z(t) = 0.$$

Thus

$$
\begin{aligned}
&\sum_{jk} Z_{jk}\partial_j\partial_k h(w) \\
&= \sum_{jk} Z_{jk}\int_{\mathbb{C}^n}\left(\pi^2\left(y^{-1}e_j\otimes y^{-1}e_k,(t-\bar{t})^2\right)-\pi\left(y^{-1}e_k,e_j\right)\right)K(z;w,t)h(t)\,d\mu_z(t) \\
&= \int_{\mathbb{C}^n}\left(-4\pi^2\left(y^{-1}Zy^{-1}s,s\right)-\pi\left(y^{-1},Z\right)\right)K(z;w,t)h(t)\,d\mu_z(t) \\
&= -4\pi^2\int_{\mathbb{C}^n}\left(y^{-1}Zy^{-1}s,s\right)K(z;w,t)h(t)\,d\mu_z(t)-\pi\left(y^{-1},Z\right)h(w).
\end{aligned}
$$

Here we have used the simple fact that

$$\sum Z_{jk}\left(y^{-1}e_j\otimes y^{-1}e_k,s\otimes s\right)=\left(y^{-1}Zy^{-1},s\otimes s\right)=\left(y^{-1}Zy^{-1}s,s\right).$$

Hence

$$\frac{1}{4\pi i}\sum_{jk}Z_{jk}\partial_j\partial_k h(w)=i\pi\int_{\mathbb{C}^n}\left(y^{-1}Zy^{-1}s,s\right)K(z;w,t)h(t)\,d\mu_z(t)-\frac{1}{4i}\left(y^{-1},Z\right)h(w).$$

Comparing this with (5.5) we obtain (5.6). □

Proposition 5.3. The curvature of Fock bundle is given by

$$R(Z,\overline{Z})=\frac{1}{8}\left(y^{-1}Zy^{-1},\overline{Z}\right)\mathrm{Id}.$$

i.e., it is proportional to the Siegel metric $\left(y^{-1}Zy^{-1},\overline{Z}\right)$ on D.

Proof. This follows by a direct computation. We take a section $h(z)=h(z,w)$ holomorphic on the total space with $h(z)$ being in the Fock space $\mathcal{F}_z$, and compute

$$R(Z,\overline{Z})h=[\nabla_Z,\partial_{\overline{Z}}]h=-\partial_{\overline{Z}}\nabla_Z h=-\frac{1}{4i}\partial_{\overline{Z}}\left(y^{-1},Z\right)h=\frac{1}{8}\left(y^{-1}Zy^{-1},\overline{Z}\right)h.$$

□

Next we find the induced connection on sections of Toeplitz operators. For that purpose we introduce the Jordan product

$$T\circ S=\frac{1}{2}(TS+ST)$$

for any two operators.

Proposition 5.4. The induced connection of ∇D on T_f is

$$
\begin{aligned}
\nabla_Z^{ind}T_f=[\nabla_Z,T_f]&=T_{\partial_Z f}+\frac{i}{2}\operatorname{tr}(y^{-1}Zy^{-1})T_f+[T_f,T_{\partial_Z\phi}]+\pi i T_{(y^{-1}Zy^{-1}v,v)f)} \\
&\quad-\frac{\pi}{4i}\left(T_f\circ T_{(y^{-1}Zy^{-1}w,w)}-\sum_{j=1}^n T_{(\bar{w},e_j)f}\circ T_{(y^{-1}Zy^{-1}w,e_j)}\right)
\end{aligned}
\tag{5.8}
$$

Proof. The commutator is $[\nabla_Z,T_f]=[\partial_Z-T_{\partial_Z\phi},T_f]=[\partial_Z,T_f]+[T_f,T_{\partial_Z\phi}]$. Writing the Toeplitz operator T_f as an integral operator

$$T_f g(z,t)=\int f(z,w)g(z,w)K(z;t,w)d\mu_z(w)$$

we find

$$[\partial_Z, T_f]g = T_{\partial_Z f}g + \int f(z,w)g(z,w)K(z;t,w)$$
$$\left(-\operatorname{tr}(y^{-1}Z) - \pi i(y^{-1}Zy^{-1}v, v) - \frac{\pi i}{4}(y^{-1}Zy^{-1}(t-\bar{w}), t-\bar{w})\right) d\mu_z(w).$$

The integrals above can all be written in terms of Toeplitz operators and we omit the computations here. □

Finally if we replace ϕ by $m\phi$ then $\nabla_Z^{ind} T_f^{(m)}$ has a formal expansion $\sum_j (\frac{1}{m})^j T_{f_j}^{(m)}$ in m in terms of Toeplitz operators T_{f_j} with symbols f_j being differential operators acting on f; the expansion of product of Toeplitz operators has been very well studied [6, 8].

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