Q_p SPACES FOR WEIGHTED MÖBIUS ACTIONS

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Abstract. We extend the classical theory of Q_p-spaces of Aulaskari, Xiao and Zhao in two directions: first, by considering more general construction of such spaces; and second, by considering invariance with respect to weighted Möbius actions. In addition, related results for the associated Bloch-type spaces are also established.

1. Introduction

Let $\mathbf{D}$ be the unit disc in the complex plane $\mathbf{C}$. For $-\infty < p < \infty$, a holomorphic function f is said to belong to the space Q_p if

$$\sup_{x\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 \Big(1 - \Big|\frac{x-z}{1-\overline{x}z}\Big|^2\Big)^p \, dz < \infty, \tag{1}$$

the square root of the last quantity being, by definition, the (semi)norm in Q_p. Here dz denotes the Lebesgue area measure. Since any Möbius map ϕ (i.e. biholomorphic self-map of $\mathbf{D}$) is of the form $\phi(z) = \epsilon\dfrac{x-z}{1-\overline{x}z}$, with $|\epsilon| = 1$ and $x \in \mathbf{D}$, the quantity (1) can be rewritten as

$$\begin{aligned}
&\sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} |f'(z)|^2 \, (1-|\phi(z)|^2)^p \, dz \\
&\qquad = \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \Delta|f|^2(z) \, (1-|\phi(z)|^2)^p \, dz \\
&\qquad = \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} (\widetilde{\Delta}|f|^2)(z) \, (1-|\phi(z)|^2)^p \, d\mu(z) \\
&\qquad = \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \widetilde{\Delta}|f\circ\phi(z)|^2 \, (1-|z|^2)^p \, d\mu(z), \\
&\qquad = \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \Delta|f\circ\phi(z)|^2 \, (1-|z|^2)^p \, dz, \\
&\qquad = \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} |(f\circ\phi)'(z)|^2 \, (1-|z|^2)^p \, dz,
\end{aligned} \tag{2}$$

where $\widetilde{\Delta} = (1-|z|^2)^2 \dfrac{\partial^2}{\partial z \partial\overline{z}}$ and $d\mu(z) = \dfrac{dz}{(1-|z|^2)^2}$ are the $\mathrm{Aut}(\mathbf{D})$-invariant Laplacian and the $\mathrm{Aut}(\mathbf{D})$-invariant measure on $\mathbf{D}$, respectively, and $\mathrm{Aut}(\mathbf{D})$ stands for the group of all Möbius maps. (Note that we are using the normalization $\Delta = \partial\overline{\partial}$ for the Euclidean Laplacian, which differs from the usual one by a factor of 4.) From

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the last formula it is apparent that $f \in Q_p$ implies $f \circ \phi \in Q_p$ and f and $f \circ \phi$ have the same norm in Q_p, for all $\phi \in \mathrm{Aut}(\mathbf{D})$. That is, the space Q_p is *Möbius invariant* under the unweighted composition

$$f \mapsto f \circ \phi, \qquad \phi \in \mathrm{Aut}(\mathbf{D}). \tag{3}$$

The spaces Q_p were introduced in 1995 by Aulaskari, Xiao and Zhao [AXZ], who showed that

$$\begin{array}{rcl} p > 1 & \Longrightarrow & Q_p = \mathcal{B}, \quad \text{the Bloch space}, \\ p = 1 & \Longrightarrow & Q_p = BMOA, \\ 0 \le p_1 < p_2 \le 1 & \Longrightarrow & Q_{p_1} \subsetneq Q_{p_2}, \\ p = 0 & \Longrightarrow & Q_p = \mathcal{D}, \quad \text{the Dirichlet space}, \\ p < 0 & \Longrightarrow & Q_p = \{\text{constants}\}. \end{array} \tag{4}$$

Thus the Q_p spaces provide a whole range of Möbius-invariant function spaces on $\mathbf{D}$ lying strictly between the Dirichlet space on the one hand, and $BMOA$ and the Bloch space

$$\mathcal{B} = \{f \text{ holomorphic on } \mathbf{D} : \sup_{z \in \mathbf{D}} (1 - |z|^2)|f'(z)| < \infty\}$$

on the other hand.

The Q_p spaces subsequently attracted a lot of attention; see e.g. the book by Xiao [Xiao] and the references therein. They were generalized to the unit ball $\mathbf{B}^d \subset \mathbf{C}^d$ in 1998 by Ouyang, Yang and Zhao [OYZ], and, more recently, to bounded symmetric domains in $\mathbf{C}^n$ by Arazy and the author [AE]. Other generalizations include the spaces Q_K of Wulan, Essen and Zhu [EW] [WuZh], where $(1 - |\frac{x-z}{1-\bar{x}z}|^2)^p$ in (1) is replaced by $K(-\log|\frac{x-z}{1-\bar{x}z}|)$ with a more general function K; the spaces $F(p, q, s)$, $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$ of Zhao [Zhao] and Rättyä [Ratt], consisting of holomorphic functions f on $\mathbf{D}$ satisfying

$$\sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\phi(z)|^2)^s \, dz < \infty; \tag{5}$$

and the spaces $Q(n, p, \alpha)$, $0 < p < \infty$, $-1 < \alpha < \infty$, of Zhu [Zhu1], consisting of holomorphic functions f on $\mathbf{D}$ satisfying

$$\sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} |(f \circ \phi)^{(n)}(z)|^p (1 - |z|^2)^\alpha \, dz < \infty. \tag{6}$$

The spaces $Q(n, p, \alpha)$ are again, by their very definition, Möbius invariant under the unweighted composition (3), and in fact $Q(1, 2, p) = Q_p$ by the last line of (2). The spaces $F(p, q, s)$ are invariant if $p = q + 2$, but not in general (however, see further below); and similarly, $F(2, 0, p) = Q_p$.

In this paper, we extend all the above in two directions: first, we consider Möbius invariance under the weighted action

$$U_\phi^{(\lambda)} : f \mapsto f \circ \phi \cdot (\phi')^{\lambda/2}, \qquad \lambda \in \mathbf{R}, \tag{7}$$

which reduces to (3) for $\lambda = 0$; and second, we consider a more general construction of the corresponding Q_p-type spaces.

The weighted actions (7) have been around and of interest from the point of view of automorphic forms (see e.g. the book by Kra [Kra]) and also represent the simplest example of holomorphic discrete series representations of semisimple

Lie groups (for $\lambda > 1$) and their analytic continuation (for $\lambda > 0$); see Rossi and Vergne [RoVe] (cf. also Section 5 below). It turns out that the case when λ is a nonpositive integer are somewhat special, in that the space $\mathcal{P}_{\le -\lambda}$ of polynomials of degree not exceeding $-\lambda$ is invariant under the action (7), and is thus most of the time trivially contained in the spaces that we consider.

We pause to note that, although the spaces $F(p,q,s)$ — as mentioned above — are not in general Möbius invariant under the unweighted action (3), it turns out that the space of first derivatives of $F(p,q,s)$ functions,

$$\begin{aligned} E(p,q,s) &:= \{f' : f \in F(p,q,s)\} \\ &= \{f \in \mathcal{H}(\mathbf{D}) : \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} |f(z)|^p (1-|z|^2)^q (1-|\phi(z)|^2)^s \, dz < \infty\} \end{aligned} \tag{8}$$

is invariant under the weighted action (7) where $\lambda = 2\frac{q+2}{p} > 0$; see (4.1) and Proposition 4.3 in [Zhao]. Cf. also the comment after Theorem 8 below.

As for the construction of our Q-type spaces, consider, quite generally, a map q from the Frechet space $\mathcal{H}(\mathbf{D})$ of holomorphic functions on $\mathbf{D}$ into the interval $[0,+\infty]$ with the following properties:

(i) there exists $C > 0$ such that for all $f, g \in \mathcal{H}(\mathbf{D})$, $Cq[f+g] \le q[f] + q[g]$;
(ii) there exists $p > 0$ such that for all $f \in \mathcal{H}(\mathbf{D})$ and $c \in \mathbf{C}$, $q[cf] = |c|^p q[f]$ (with $0.(+\infty) = 0$).

Let $Q^{(\lambda)}_{[q]}$ be the space of all $f \in \mathcal{H}(\mathbf{D})$ such that

$$\|f\|^{(\lambda)}_{[q]} := \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} q[U^{(\lambda)}_\phi f]^{1/p} < \infty. \tag{9}$$

Then $Q^{(\lambda)}_{[q]}$ is a vector space and $\|\cdot\|^{(\lambda)}_{[q]}$ is a (quasi-)seminorm on $Q^{(\lambda)}_{[q]}$.

It should be noted that there is some ambiguity in the definition (7), connected with the choice of the branch of the power of ϕ'; strictly speaking, when $\lambda/2$ is not an integer, one should define $U^{(\lambda)}_\phi$ for ϕ not in $\mathrm{Aut}(\mathbf{D})$ but in its universal cover $\widetilde{\mathrm{Aut}(\mathbf{D})}$. However, since (ii) implies

$$q[\epsilon f] = q[f] \qquad \text{whenever} \epsilon \in \mathbf{C}, |\epsilon| = 1, \tag{10}$$

the choice of power plays no role and can thus be ignored.

As examples of the quantities q in (9), we have the following.

Example 1. $q[f] = |f^{(k)}(0)|$, where k is a nonnegative integer. The corresponding space $Q^{(\lambda)}_{[q]}$ will be denoted by $\mathcal{E}^{(\lambda)}_k$:

$$\mathcal{E}^{(\lambda)}_k := \{f \in \mathcal{H}(\mathbf{D}) : \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} |(U^{(\lambda)}_\phi f)^{(k)}(0)| < \infty\},$$

and called the (k,λ)-Bloch space. For $\lambda = 0$, $\mathcal{E}^{(0)}_1$ is the ordinary Bloch space, and one can show that $\mathcal{E}^{(0)}_k = \mathcal{E}^{(0)}_1$ for all $k \ge 1$. For $k = 0$, $\mathcal{E}^{(0)}_0 = H^\infty$, the space of bounded analytic functions.

Example 2. $q[f] = \int_{\mathbf{D}} |f^{(k)}|^p \, d\mu_\rho$, where $p > 0$, k is a nonnegative integer, and $d\mu_\rho$, $\rho \in \mathbf{R}$, is the measure

$$d\mu_\rho(z) = (1-|z|^2)^\rho \, dz,$$

with dz the Lebesgue area measure normalized so that $\mathbf{D}$ has total mass 1. For $\lambda = 0$, the corresponding spaces $Q^{(0)}_{[q]}$ coincide with Zhu's spaces $Q(k,p,\rho)$.

Note that for $p=2$, one has $|f^{(k)}|^2 = \Delta^k |f|^2$, so the corresponding spaces are

$$Q^{(\lambda)}_{k,\rho} := \{f \in \mathcal{H}(\mathbf{D}) : \|f\|^2_{k,\rho} := \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \Delta^k |U^{(\lambda)}_\phi f|^2 \, d\mu_\rho < \infty\}, \tag{11}$$

which reduce to the Q_ρ from (1) for $\lambda = 0$, $k = 1$.

The last example, as well as (6), are generalizations of the last two lines in (2); the next example is about a generalization of the third line from the bottom in (2). Consider, quite generally, a constant coefficient linear differential operator L on $\mathbf{C}$ which is invariant under rotations, i.e.

$$L(f_\epsilon) = (Lf)_\epsilon, \qquad \forall \epsilon \in \mathbf{C}, |\epsilon| = 1, \tag{12}$$

where

$$f_\epsilon(z) := f(\epsilon z).$$

Then the (linear) differential operator $\widetilde{L}$ on $\mathbf{D}$ defined by

$$(\widetilde{L} f)(\phi(0)) := (1 - |\phi(0)|^2)^{-\lambda} (L(V^{(\lambda)}_\phi f))(0) \tag{13}$$

is invariant under the action

$$V^{(\lambda)}_\phi : f \mapsto f \circ \phi \cdot |\phi'|^\lambda, \qquad \phi \in \mathrm{Aut}(\mathbf{D}). \tag{14}$$

(Thus $\widetilde{L}$ depends also on λ, though this is not reflected by the notation.) Indeed, (13) precisely says that $(V^{(\lambda)}_\phi \widetilde{L} f)(0) = (L V^{(\lambda)}_\phi f)(0)$, and it follows from the composition law

$$V^{(\lambda)}_\phi V^{(\lambda)}_\psi = V^{(\lambda)}_{\psi\circ\phi}$$

that

$$V^{(\lambda)}_\phi \widetilde{L} = \widetilde{L} V^{(\lambda)}_\phi, \qquad \forall \phi \in \mathrm{Aut}(\mathbf{D}). \tag{15}$$

Note also that, thanks to (12), the right-hand side of (13) remains unchanged when ϕ is replaced by ϕ_ϵ, $|\epsilon| = 1$, so that it indeed does not depend on ϕ but only on $\phi(0)$.

Taking in particular for L the power Δ^k of the Laplacian, we get the invariant differential operators $\widetilde{\Delta^k}$, $k = 0, 1, 2, \dots$, on $\mathbf{D}$. Note that for f holomorphic,

$$V^{(\lambda)}_\phi |f|^p = |U^{(2\lambda/p)}_\phi f|^p, \tag{16}$$

thus, in particular, for any $a \in \mathbf{D}$ and $\phi \in \mathrm{Aut}(\mathbf{D})$ with $\phi(0) = a$,

$$(\widetilde{\Delta^k} |f|^p)(a) = (1 - |a|^2)^{-\lambda} |(U^{(2\lambda/p)}_\phi f)^{(k)}(0)|^p \ge 0 \tag{17}$$

for any f holomorphic, and the following definition of q therefore makes sense.

Example 3. $q[f] = \int_{\mathbf{D}} \widetilde{\Delta^k} |f|^p \, d\mu_{\rho-2}$, where $p > 0$, k is a nonnegative integer, and $\rho \in \mathbf{R}$. In particular, for $p = 2$, the corresponding Q spaces are

$$\widetilde{Q}^{(\lambda)}_{k,\rho} := \{f \in \mathcal{H}(\mathbf{D}) : \|f\|^2_{\widetilde{k,\rho}} := \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \widetilde{\Delta^k} |U^{(\lambda)}_\phi f|^2 \, d\mu_{\rho-2} < \infty\}. \tag{18}$$

For $\lambda = 0$ and $k = 1$, $\widetilde{\Delta}$ is just the invariant Laplacian $(1-|z|^2)^2\Delta$, so $\widetilde{Q}^{(0)}_{\widetilde{1,\rho}} = Q^{(0)}_{1,\rho}$ $(= Q_\rho)$. (This equality, of course, was the reason for the change from $d\mu_\rho$ in (11) to $d\mu_{\rho-2}$ in (18).)

It is immediate from the definitions that

$$\rho_1 \le \rho_2 \implies Q^{(\lambda)}_{k,\rho_1} \subset Q^{(\lambda)}_{k,\rho_2},\ \widetilde{Q}^{(\lambda)}_{k,\rho_1} \subset \widetilde{Q}^{(\lambda)}_{k,\rho_2}.$$

Example 4. $q[f] = \int_{\mathbf{D}} \Delta|f|^2(z)\, K(-\log|z|)\, dz$, where $K : [0,\infty) \to [0,\infty)$ is a nondecreasing function, not identically zero. For $\lambda = 0$, the corresponding Q-space coincides with the Q_K from [WuZh].

More generally we can take

$$q[f] = \int_{\mathbf{D}} \Delta^k |f|^2(z)\, K(-\log|z|)\, dz,$$

or

$$q[f] = \int_{\mathbf{D}} \widetilde{\Delta^k} |f|^2(z)\, K(-\log|z|)\, dz,$$

with k a nonnegative integer, and get the corresponding Q-spaces $Q^{(\lambda)}_{k,K}$ and $\widetilde{Q}^{(\lambda)}_{k,K}$, respectively.

For $p = 2$, the last four examples turn out to be just special cases of the following one. Assume that in addition to (i) and (ii) above, q in addition satisfies

(iii) $C = 1$, $p = 2$ and $q[f]$ satisfies the parallelogram law

$$q[f+g] + q[f-g] = 2(q[f] + q[g]);$$

(iv) $q[f_\epsilon] = q[f]$, for all $\epsilon \in \mathbf{C}$, $|\epsilon| = 1$.

Then, by (iii), the set $\{f \in \mathcal{H}(\mathbf{D}) : q[f] < \infty\}$ becomes a (semi-definite) inner product space under the inner product obtained by polarizing $q[f]$; in an appropriate sense, it is therefore given in terms of the Taylor coefficients f_j of f by

$$q[f] = \sum_{j,k=0}^{\infty} q_{jk} f_j \overline{f}_k, \qquad f(z) = \sum_{j=0}^{\infty} f_j z^j,$$

with some positive-semidefinite matrix $\{q_{jk}\}$. The condition (iv) then implies that $q_{jk} = 0$ for $j \ne k$. Thus $q[f] = \sum_j s_j |f_j|^2$, with $s_j = q_{jj}$, and $s_j = +\infty$ interpreted to mean that $q[f] < \infty$ only if $f_j = 0$. Heuristically, this is our motivation for the next example.

Example 5. Let $\mathbf{s} = (s_0, s_1, s_2, \dots)$ be a sequence of numbers $s_j \in [0, +\infty]$, and set $q[f] = \sum_j s_j |f_j|^2$ (with $s_j|0|^2$ interpreted as 0 if $s_j = +\infty$), where f_j are the Taylor coefficients of f. Thus the corresponding Q-space is

$$Q^{(\lambda)}_{\mathbf{s}} := \{f \in \mathcal{H}(\mathbf{D}) : \|f\|^2_{\mathbf{s}} := \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \sum_j s_j |(U^{(\lambda)}_\phi f)_j|^2 < \infty\}.$$

Plainly the spaces $\mathcal{E}^{(\lambda)}_k$ correspond to $\mathbf{s} = \mathbf{e}_k$, the sequence with $s_k = 1$ and $s_j = 0\ \forall j \ne k$. We will identify the sequences $\mathbf{s}$ corresponding to the spaces $\widetilde{Q}^{(\lambda)}_{k,\rho}$ and $Q^{(\lambda)}_{k,\rho}$ in Sections 3 and 4 below, respectively.

We will write $\mathbf{s} \lesssim \mathbf{s}'$ if there exists a finite constant $c > 0$ such that $s_j \le c s'_j$ for all j, and $\mathbf{s} \asymp \mathbf{s}'$ if $\mathbf{s} \lesssim \mathbf{s}'$ and $\mathbf{s}' \lesssim \mathbf{s}$. Again, clearly

$$\mathbf{s} \lesssim \mathbf{s}' \implies Q^{(\lambda)}_{\mathbf{s}'} \hookrightarrow Q^{(\lambda)}_{\mathbf{s}}, \tag{19}$$

and $\mathbf{s} \asymp \mathbf{s}'$ implies that $Q^{(\lambda)}_{\mathbf{s}} = Q^{(\lambda)}_{\mathbf{s}'}$, with equivalent norms.

Our main results are the following. For k a nonnegative integer and $\nu \in \mathbf{R}$, we introduce the sequence

$$\mathbf{s}[k,\nu] = \{s_j\} \text{ where } s_j = \begin{cases} 0 & \text{if } j < k, \\ (j+1)^\nu & \text{if } j \geq k, \end{cases} \tag{20}$$

and we denote by $Q^{(\lambda)}_{\mathbf{s}[k,\nu]}$ the corresponding space $Q^{(\lambda)}_{\mathbf{s}}$. Finally, denote

$$L^{\infty,\nu}_{\text{hol}}(\mathbf{D}) = \{f \in \mathcal{H}(\mathbf{D}) : \sup_{z\in\mathbf{D}} (1-|z|^2)^\nu |f(z)| < \infty\}.$$

Note that $L^{\infty,\nu}_{\text{hol}}(\mathbf{D}) = \{0\}$ for $\nu < 0$, by the maximum principle.

Theorem 6. (i) *If $-\lambda$ is not a nonnegative integer, then $\mathcal{E}^{(\lambda)}_k = L^{\infty,\lambda/2}_{\text{hol}}(\mathbf{D})$ for all $k = 0, 1, 2, \dots$.*

(ii) *If $-\lambda = m$ is a nonnegative integer, then*

$$\mathcal{E}^{(\lambda)}_k = L^{\infty,\lambda/2}_{\text{hol}}(\mathbf{D}) \qquad \text{for } k = 0, 1, \dots, m,$$
$$\mathcal{E}^{(\lambda)}_k = \mathcal{E}^{(\lambda)}_{m+1} \qquad \text{for } k > m,$$

and

$$\mathcal{E}^{(\lambda)}_{m+1} = \{f \in \mathcal{H}(\mathbf{D}) : f^{(m/2)} \in \mathcal{B}\} \qquad \text{for } m \text{ even}, \tag{21}$$

$$\mathcal{E}^{(\lambda)}_{m+1} = \{f \in \mathcal{H}(\mathbf{D}) : f^{(\frac{m-1}{2})} \in \mathcal{B}_{1/2}\} \qquad \text{for } m \text{ odd}, \tag{22}$$

with equivalent norms.

Here $\mathcal{B}_\alpha$, $0 < \alpha \leq 1$, denotes the weighted Bloch space

$$\mathcal{B}_\alpha = \{f \in \mathcal{H}(\mathbf{D}) : \sup_{z\in\mathbf{D}} (1-|z|^2)^\alpha |f'(z)| < \infty\},$$

and $\mathcal{B} = \mathcal{B}_1$ is the ordinary Bloch space.

Theorem 7. *For $-\lambda$ not a nonnegative integer,*

(i) *$\widetilde{Q}^{(\lambda)}_{k,\rho} = \{0\}$ if $\rho \leq 1$ or $\rho < \lambda$,*

(ii) *$\widetilde{Q}^{(\lambda)}_{k,\rho} = L^{\infty,\lambda/2}_{\text{hol}}(\mathbf{D})$ of $\rho > \max(1, 1+\lambda)$,*

(iii) *for $\rho > 1$ and $\lambda \leq \rho \leq \lambda + 1$, the spaces $\widetilde{Q}^{(\lambda)}_{k,\rho}$ are strictly increasing with ρ; in fact (cf. (8)), $\widetilde{Q}^{(\lambda)}_{k,\rho} = E(2, \lambda - 2, \rho - \lambda)$,*

(iv) *$\widetilde{Q}^{(\lambda)}_{k,\rho} = Q^{(\lambda)}_{\mathbf{s}[0,1-\rho]}$ for $\rho > 1$.*

Thus (ii)-(iii) also give a description of $Q^{(\lambda)}_{\mathbf{s}[0,\nu]}$ for $\nu < 0$.

Theorem 8. *For $-\lambda = m$ a nonnegative integer,*

(i) *$\widetilde{Q}^{(\lambda)}_{k,\rho} = \{0\}$ if $k \leq m$ and $\rho \leq 1$,*

(ii) *$\widetilde{Q}^{(\lambda)}_{k,\rho} = L^{\infty,\lambda/2}_{\text{hol}}(\mathbf{D})$ if $k \leq m$ and $\rho > 1$,*

(iii) *$\widetilde{Q}^{(\lambda)}_{k,\rho} = \mathcal{P}_{\leq m}$ trivially (i.e. with seminorm identically zero) if $k > m$ and $\rho < -m$,*

(iv) *$\widetilde{Q}^{(\lambda)}_{k,\rho} = \mathcal{E}^{(\lambda)}_{m+1}$ if $k > m$ and $\rho > -m + 1$,*

(v) *for $k > m$ and $-m \leq \rho \leq -m + 1$, the spaces $\widetilde{Q}^{(\lambda)}_{k,\rho}$ are strictly increasing with ρ; in fact,*

$$f \in \widetilde{Q}^{(\lambda)}_{k,\rho} \iff f^{(m)} \in F(2, m, \rho + m),$$

(vi) $\widetilde{Q}^{(\lambda)}_{k,\rho} = Q^{(\lambda)}_{\mathbf{s}[m+1,1-\rho]}$ *for* $\rho > -2m-1$.

Note that (iv) implies that the space of m-th primitives of $F(2,m,s)$, $0 \le s \le 1$, is Möbius invariant under (7) with $\lambda = -m$; this seems not to have been noticed in the literature.

For $\lambda \in \mathbf{C}$ and k a nonnegative integer, denote by

$$(\lambda)_k := \lambda(\lambda+1)\dots(\lambda+k-1), \qquad (\lambda)_0 := 1,$$

the Pochhammer symbol (raising factorial). Thus $-\lambda \in \{0,1,\dots,k-1\}$ is precisely equivalent to $(\lambda)_k = 0$.

Theorem 9. (i) *If* $(\lambda)_k \neq 0$ *and* $\lambda < 0$, *then* $Q^{(\lambda)}_{k,\rho} = \{0\}$ *for all* ρ.

(ii) $Q^{(\lambda)}_{k,\rho} = \mathcal{E}^{(\lambda)}_k$ *for* $\rho > 2k-1+\lambda$.

(iii) *If* $(\lambda)_k = 0$, *then* $Q^{(\lambda)}_{k,\rho} = \mathcal{P}_{\le -\lambda}$ *trivially for* $\rho < 2k+\lambda-2$.

(iv) *If* $(\lambda)_k \neq 0$, *then* $Q^{(\lambda)}_{k,\rho} = \{0\}$ *for* $\rho < 2k+\lambda-2$.

(v) *If* $(\lambda)_k = 0$ *and* $2k+\lambda-2 \le \rho \le 2k+\lambda-1$, *then* $Q^{(\lambda)}_{k,\rho}$ *are strictly increasing with* ρ*; in fact,* $Q^{(\lambda)}_{k,\rho} = \widetilde{Q}^{(\lambda)}_{k,\rho-2k+2}$ *are described by the preceding theorem.*

(vi) *If* $(\lambda)_k \neq 0$, $\lambda \ge 0$ *and* $\rho \in (2k-1,\infty) \cap [2k+\lambda-2, 2k+\lambda-1]$, *then* $Q^{(\lambda)}_{k,\rho}$ *are strictly increasing with* ρ*; in fact,* $Q^{(\lambda)}_{k,\rho} = \widetilde{Q}^{(\lambda)}_{k,\rho-2k+2}$ *are described by the two preceding theorems.*

(vii) $Q^{(\lambda)}_{k,\rho} = Q^{(\lambda)}_{\mathbf{s}[k,2k-\rho-1]}$ *for* $\rho > -1$. *Also,*

$$Q^{(\lambda)}_{\mathbf{s}[k,2k-\rho-1]} = \begin{cases} Q^{(\lambda)}_{\mathbf{s}[0,2k-\rho-1]} & \text{if} (\lambda)_k \neq 0, \\ Q^{(\lambda)}_{\mathbf{s}[1-\lambda,2k-\rho-1]} & \text{if } (\lambda)_k = 0. \end{cases} \tag{23}$$

The only case left out in the last theorem thus is $(\lambda)_k \neq 0$, $\lambda \ge 0$ and $2k+\lambda-2 \le \rho \le 2k-1$ where $k \ge 1$. (For $k=0$, we have straight from their definition $Q^{(\lambda)}_{k,\rho} = \widetilde{Q}^{(\lambda)}_{k,\rho+2}$, which are described by the previous two theorems.)

The proofs of Theorems 6–8 are given in Section 3, after fixing some preliminaries in Section 2. The proof of Theorem 9 is given in Section 4. Some final remarks and comments are collected in the last section, Section 5.

Notation. Recall that any $\phi \in \mathrm{Aut}(\mathbf{D})$ can be written as $\phi(z) = \epsilon \frac{a-z}{1-\overline{a}z}$ with $|\epsilon| = 1$ and $a \in \mathbf{D}$. We use the shorthand

$$\phi_a(z) := \frac{a-z}{1-\overline{a}z}$$

for the Möbius map interchanging 0 and a, and abbreviate $U^{(\lambda)}_{\phi_a}$ just to $U^{(\lambda)}_a$. Similarly, the rotation $z \mapsto \epsilon z$ will be denoted by $U^{(\lambda)}_\epsilon$. For convenience, we also fix the choice of the branches of the powers of ϕ' in (7) as follows:

$$U^{(\lambda)}_\epsilon : f \mapsto f_\epsilon, \qquad U^{(\lambda)}_a : f \mapsto f \circ \phi_a \cdot \frac{(1-|a|^2)^{\lambda/2}}{(1-\overline{a}z)^\lambda},$$

with the principal branch of $(1-z)^\lambda$, $z \in \mathbf{D}$. The symbol $\mathbf{N} = \{0,1,2,\dots\}$ denotes the set of all nonnegative integers, and $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ the unit circle. For $f \in \mathcal{H}(\mathbf{D})$, $(f)_j$ or just f_j denotes the j-th Taylor coefficient of f (so that $f(z) = \sum_{j=0}^\infty f_j z^j$). The symbols $\mathcal{P}_m$ and $\mathcal{P}_{\le m}$ denote the space of polynomials on $\mathbf{D}$ of degree equal to m and not exceeding m, respectively. Finally, we fix the weight parameter λ from now on, and write only $U_\phi, V_\phi, \mathcal{E}_k, Q_{k,\rho}, \widetilde{Q}_{k,\rho}$ etc. for $U_\phi^{(\lambda)}, V_\phi^{(\lambda)}, \mathcal{E}_k^{(\lambda)}, Q_{k,\rho}^{(\lambda)}, \widetilde{Q}_{k,\rho}^{(\lambda)}$, etc., respectively.

2. Preliminaries

The following proposition is common knowledge, but we include a proof for completeness.

Proposition 10. (i) *For $\nu > -1$, there exists a finite constant c_ν such that*

$$\sup_{z\in\mathbf{D}}(1-|z|^2)^{\nu+1}|f'(z)| \le c_\nu \sup_{z\in\mathbf{D}}(1-|z|^2)^\nu|f(z)| \tag{24}$$

for all $f\in\mathcal{H}(\mathbf{D})$.

(ii) *For $\nu>0$,*

$$\sup_{z\in\mathbf{D}}(1-|z|^2)^\nu|f(z)-f(0)| \le \frac{2^\nu}{\nu}\sup_{z\in\mathbf{D}}(1-|z|^2)^{\nu+1}|f'(z)| \tag{25}$$

for all $f\in\mathcal{H}(\mathbf{D})$.

Proof. (i) If the supremum on the right-hand side of (24) is finite, then $f\in L^1_{\mathrm{hol}}(\mathbf{D}, d\mu_\nu)$, so by the reproducing property of the weighted Bergman kernel [Zhu2, Proposition 4.23],

$$f(z) = (\nu+1)\int_{\mathbf{D}} \frac{f(w)}{(1-z\overline{w})^{\nu+2}}\,d\mu_\nu(w).$$

Differentiating under the integral sign gives

$$f'(z) = (\nu+1)(\nu+2)\int_{\mathbf{D}} \frac{\overline{w}f(w)(1-|w|^2)^\nu}{(1-z\overline{w})^{\nu+3}}\,dw$$

so

$$|f'(z)| \le (\nu+1)(\nu+2)\Big(\sup_{w\in\mathbf{D}}(1-|w|^2)^\nu|f(w)|\Big)\int_{\mathbf{D}}\frac{dw}{|1-z\overline{w}|^{\nu+3}}.$$

By the classical Forelli-Rudin estimates [Zhu2, Lemma 3.10], the last integral is majorized by $(1-|z|^2)^{-\nu-1}$, and the claim follows.

(ii) Denoting the supremum on the right-hand side by c, we have

$$\begin{aligned}|f(z)-f(0)| = \Big|\oint_{[0,z]} f'\Big| &\le c\int_0^{|z|}\frac{dt}{(1-t^2)^{\nu+1}} \\ &\le c\int_0^{|z|}\frac{dt}{(1-t)^{\nu+1}} = \frac{c}{\nu}\Big[\frac{1}{(1-|z|)^\nu}-1\Big] \le \frac{c2^\nu/\nu}{(1-|z|^2)^\nu}.\end{aligned}$$

□

For $\lambda=0$, the next proposition yields a more precise version of Lemma 1 from [Zhu1] (with the constants explicitly evaluated).

Proposition 11. *For $f\in\mathcal{H}(\mathbf{D})$,*

$$(U_a f)^{(k)}(z) = (-1)^k\sum_{j=0}^k (1-|a|^2)^{\frac{\lambda}{2}+j} f^{(j)}(\phi_a(z))\frac{(-\overline{a})^{k-j}(\lambda+j)_{k-j}}{(1-\overline{a}z)^{k+j+\lambda}}\binom{k}{j}. \tag{26}$$

Proof. Pulling out the factor $(1-|a|^2)^{\lambda/2}$, we will prove a more general formula for

$$I_{k,\lambda}[f] := \Big(f\Big(\frac{a-z}{1-bz}\Big)(1-bz)^{-\lambda}\Big)^{(k)};$$

the claim will then follow just by taking $b=\overline{a}$. By the Leibniz rule,

$$I_{1,\lambda}[f] = -f'\Big(\frac{a-z}{1-bz}\Big)\frac{1-ba}{(1-bz)^{\lambda+2}} + f\Big(\frac{a-z}{1-bz}\Big)\frac{\lambda b}{(1-bz)^{\lambda+1}}.$$

Taking k-th derivative on both sides yields

$$I_{k+1,\lambda}[f] = -(1-ba)I_{k,\lambda+2}[f'] + \lambda b I_{k,\lambda+1}[f].$$

From this recurrence formula, together with $I_{0,\lambda}[f] = f(\frac{a-z}{1-bz})(1-bz)^{-\lambda}$, we get by induction

$$I_{k,\lambda}[f] = (-1)^k \sum_{j=0}^{k} (1-ba)^j f^{(j)}\Big(\frac{a-z}{1-bz}\Big)\frac{(-b)^{k-j}}{(1-bz)^{k+j+\lambda}} p_{jk}(\lambda),$$

with some polynomials $p_{jk}(\lambda)$ in λ. To evaluate the latter, we take $f(z)=z^m$ (with $0\le m\le k$), $a=0$ and evaluate at $z=0$. The right-hand side of the last formula then reduces to $(-1)^k m!(-b)^{k-m}p_{mk}(\lambda)$, while the left-hand side is

$$\begin{aligned}\Big(\Big(\frac{-z}{1-bz}\Big)^m(1-bz)^{-\lambda}\Big)^{(k)}\Big|_{z=0} &= \Big(\frac{(-z)^m}{(1-bz)^{m+\lambda}}\Big)^{(k)}\Big|_{z=0} \\ &= \Big(\sum_{j=0}^{\infty}\frac{(-1)^m(m+\lambda)_j}{j!} b^j z^{m+j}\Big)^{(k)}\Big|_{z=0} \\ &= (-1)^m \frac{(m+\lambda)_{k-m}}{(k-m)!} k! b^{k-m}.\end{aligned}$$

It follows that $p_{km}(\lambda) = (m+\lambda)_{k-m}\binom{k}{m}$, proving the assertion. □

Setting $z=0$ in the last proposition, we get in particular

$$(U_a f)^{(k)}(0) = (-1)^k \sum_{j=0}^{k}(1-|a|^2)^{\frac{\lambda}{2}+j} f^{(j)}(a)(-\overline{a})^{k-j}(\lambda+j)_{k-j}\binom{k}{j} \tag{27}$$

for $f\in\mathcal{H}(\mathbf{D})$. Together with (16) (with $p=2$) and (13), this yields also a formula for the weighted invariant k-Laplacians $\widetilde{\Delta^k}$.

Corollary 12. *We have*

$$\widetilde{\Delta^k} = \sum_{j,l=0}^{k}(1-|z|^2)^{j+l}(-\overline{z})^{k-j}(-z)^{k-l}(\lambda+j)_{k-j}(\lambda+l)_{k-l}\binom{k}{j}\binom{k}{l}\partial^j\overline{\partial}^l.$$

For $\lambda=0$ and $k=1$, this of course recovers the formula $\widetilde{\Delta} = (1-|z|^2)^2\Delta$ for the unweighted invariant Laplacian on $\mathbf{D}$.

Note that if $-\lambda = m\in\mathbf{N}$ and $k>m$, then $(\lambda+j)_{k-j}=0$ for all $j\le m$; we thus get another consequence of the last proposition.

Corollary 13. *If $-\lambda = m\in\mathbf{N}$, then the space $\mathcal{P}_{\le m}$ of polynomials of degree not exceeding m is invariant under U_ϕ, $\phi\in\mathrm{Aut}(\mathbf{D})$. Also, $\widetilde{\Delta^k}|f|^2\equiv 0$ for any $f\in\mathcal{P}_{\le m}$ and $k>m$.*

Proof. By (26), $(U_\phi f)^{(k)}\equiv 0$ for $f\in\mathcal{P}_{\le m}$ and $k>m$. □

Next we turn to the behavior of the measures $d\mu_\rho$ under U_ϕ.

Proposition 14. *For $\rho \in \mathbf{R}$ and F nonnegative measurable,*

$$\int_{\mathbf{D}} V_\phi F \, d\mu_\rho = \int F \, |(\phi^{-1})'|^{2-\lambda+\rho} \, d\mu_\rho. \tag{28}$$

In particular, the measure $d\mu_{\lambda-2}$ is invariant under V_ϕ.

Proof. Denoting temporarily $\phi^{-1} =: \psi$, and noting (by the chain rule) that $\phi' \circ \psi \cdot \psi' \equiv 1$, we have by the change of variable $z \mapsto \psi(z)$

$$\begin{aligned}
\int_{\mathbf{D}} V_\phi F \, d\mu_\rho &= \int_{\mathbf{D}} F \circ \phi \cdot |\phi'|^\lambda \, (1-|z|^2)^\rho \, dz \\
&= \int_{\mathbf{D}} F \, |\phi' \circ \psi|^\lambda (1-|\psi|^2)^\rho |\psi'|^2 \, dz \\
&= \int_{\mathbf{D}} F \, |\psi'|^{2-\lambda} (1-|\psi|^2)^\rho \, dz = \int_{\mathbf{D}} F \, |\psi'|^{2-\lambda+\rho} \, d\mu_\rho,
\end{aligned}$$

since $1-|\psi|^2 = (1-|z|^2)|\psi'|$ for any $\psi \in \mathrm{Aut}(\mathbf{D})$. $\square$

As a corollary, we have for any $f \in \mathcal{H}(\mathbf{D})$, using (15),

$$\begin{aligned}
\int_{\mathbf{D}} \widetilde{\Delta}^k |U_\phi f|^2 \, d\mu_{\lambda-2} &= \int_{\mathbf{D}} \widetilde{\Delta}^k V_\phi |f|^2 \, d\mu_{\lambda-2} = \int_{\mathbf{D}} V_\phi \widetilde{\Delta}^k |f|^2 \, d\mu_{\lambda-2} \\
&= \int_{\mathbf{D}} \widetilde{\Delta}^k |f|^2 \, d\mu_{\lambda-2}.
\end{aligned} \tag{29}$$

The space of all $f \in \mathcal{H}(\mathbf{D})$ for which the last integral is finite — that is, $\widetilde{Q}_{k,\lambda}$ — is therefore a (higher-order) Möbius invariant Dirichlet space.

The next proposition gives a rigorous version of the heuristic argument in the paragraph before Example 5 in the Introduction, which applies, in particular, to all our spaces $Q_{k,\rho}$, $\widetilde{Q}_{k,\rho}$, $Q_{k,K}$ and $\widetilde{Q}_{k,K}$.

Proposition 15. *Let L be a rotation-invariant linear differential operator on $\mathbf{D}$ such that $L|f|^2 \geq 0 \ \forall f \in \mathcal{H}(\mathbf{D})$, $d\mu$ a rotation invariant measure on $\mathbf{D}$, and set*

$$q[f] = \int_{\mathbf{D}} L|f|^2 \, d\mu.$$

Then the corresponding Q-space equals $Q_{\mathbf{s}}$ for the sequence given by

$$s_j = \int_{\mathbf{D}} L|z^j|^2 \, d\mu. \tag{30}$$

Proof. Expanding f into the Taylor series, we get by uniform convergence, for any $0 < r < 1$,

$$\int_{r\mathbf{D}} L|f|^2 \, d\mu = \sum_{j,k} f_j \overline{f}_k \int_{r\mathbf{D}} L(z^j \overline{z}^k) \, d\mu.$$

Since both L and $d\mu$ are rotation invariant, the last integral vanishes for $j \neq k$. Letting $r \nearrow 1$, we thus get

$$q[f] = \sum_j |f_j|^2 \int_{\mathbf{D}} L|z^j|^2 \, d\mu,$$

which coincides with $q_{\mathbf{s}}[f]$ for $\mathbf{s}$ as indicated. $\square$

Note that the hypothesis on L in the last proposition is plainly satisfied for $L = \Delta^k$, since

$$\Delta^k |f|^2 = |f^{(k)}|^2 \geq 0 \qquad \forall f \in \mathcal{H}(\mathbf{D});$$

as well as for $L = \widetilde{\Delta^k}$, in view of (17).

Proposition 16. (i) *If $s_n = +\infty$ for some n and $f \in Q_{\mathbf{s}}$, then $(U_\phi f)_n = 0$ for all $\phi \in \mathrm{Aut}(\mathbf{D})$. In particular, if $s_n = +\infty$ $\forall n > m$, then $Q_{\mathbf{s}} \subset \mathcal{P}_{\leq m}$.*
(ii) *If $s_n > 0$ for some n, then $Q_{\mathbf{s}} \hookrightarrow \mathcal{E}_n$.*

Proof. Part (i) is immediate from the definitions. For (ii), we have for any $f \in Q_{\mathbf{s}}$ and $\phi \in \mathrm{Aut}(\mathbf{D})$,

$$\|f\|_{\mathbf{s}}^2 = \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} q_{\mathbf{s}}[U_\phi f] = \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \sum_j s_j |(U_\phi f)_j|^2 \geq \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} s_n |(U_\phi f)_n|^2.$$

Since $(U_\phi f)_n = \frac{1}{n!}(U_\phi f)^{(n)}(0)$, this gives

$$\sup_{\phi \in \mathrm{Aut}(\mathbf{D})} |(U_\phi f)^{(n)}(0)|^2 \leq \frac{n!^2}{s_n} \|f\|_{\mathbf{s}}^2,$$

so $\|f\|_{\mathcal{E}_n} \leq \frac{n!}{\sqrt{s_n}} \|f\|_{\mathbf{s}}$, proving the claim. ☐

Recall that for $m \in \mathbf{N}$, we denoted by $\mathbf{e}_m$ the sequence given by $s_j = 0$ for $j \neq m$, $s_m = 1$.

Proposition 17. *Assume that for some $m, n \in \mathbf{N}$, $s_n > 0$ and $\mathcal{E}_n = \mathcal{E}_m$. Then $Q_{\mathbf{s}} = Q_{\mathbf{s}+\mathbf{e}_m}$.*

Proof. Since $\mathbf{s} + \mathbf{e}_m \geq \mathbf{s}$, obviously $Q_{\mathbf{s}+\mathbf{e}_m} \hookrightarrow Q_{\mathbf{s}}$, by (19). On the other hand, from $s_n > 0$ we have, by part (ii) of the last proposition, $Q_{\mathbf{s}} \hookrightarrow \mathcal{E}_n$, so if $\mathcal{E}_n = \mathcal{E}_m$, then $Q_{\mathbf{s}} \hookrightarrow \mathcal{E}_m$, i.e.

$$\|f\|_{\mathcal{E}_m} \leq c\|f\|_{\mathbf{s}} \qquad \forall f \in \mathcal{H}(\mathbf{D}). \tag{31}$$

Hence

$$\begin{aligned}
\|f\|_{\mathbf{s}+\mathbf{e}_m}^2 &= \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} (q_{\mathbf{s}}[U_\phi f] + q_{\mathbf{e}_m}[U_\phi f]) \\
&\leq \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} q_{\mathbf{s}}[U_\phi f] + \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} q_{\mathbf{e}_m}[U_\phi f] \\
&= \|f\|_{\mathbf{s}}^2 + \|f\|_{\mathbf{e}_m}^2 \leq (1 + c^2)\|f\|_{\mathbf{s}}^2,
\end{aligned}$$

implying that $Q_{\mathbf{s}} \hookrightarrow Q_{\mathbf{s}+\mathbf{e}_m}$. ☐

Corollary 18. *If $\mathcal{E}_m = \mathcal{E}_{m+1}$ then $Q_{\mathbf{s}[m+1,\nu]} = Q_{\mathbf{s}[m,\nu]}$ $\forall \nu \in \mathbf{R}$.*
Consequently, if $\mathcal{E}_m = \mathcal{E}_j$ $\forall j \geq m$, then $Q_{\mathbf{s}[j,\nu]} = Q_{\mathbf{s}[m,\nu]}$ $\forall j \geq m$ $\forall \nu \in \mathbf{R}$.

Proof. Applying the last proposition to $\mathbf{s} = \mathbf{s}[m+1, \nu]$, $n = m+1$ gives the first assertion. The second assertion follows from the first. ☐

3. The spaces $\mathcal{E}_k$ and $\widetilde{Q}_{k,\rho}$

In this section we treat the (k,λ)-Bloch spaces $\mathcal{E}_k$ and the Q-spaces $\widetilde{Q}_{k,\rho}$ defined using the invariant Laplacians; recall that these consist of all $f \in \mathcal{H}(\mathbf{D})$ for which the quantities

$$\|f\|_{\mathcal{E}_k} := \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} |(U_\phi f)^{(k)}(0)|$$

and

$$\|f\|_{\widetilde{k,\rho}} := \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \Big(\int_{\mathbf{D}} \widetilde{\Delta^k} |U_\phi f|^2 \, d\mu_{\rho-2} \Big)^{1/2} \tag{32}$$

are finite, respectively. Here is an alternative characterization of the spaces $\mathcal{E}_k$.

Proposition 19. *$\mathcal{E}_k = \{f \in \mathcal{H}(\mathbf{D}) : \sup_{z\in\mathbf{D}} (1-|z|^2)^\lambda \widetilde{\Delta^k} |f(z)|^2 < \infty\}$, the last supremum being equal to $\|f\|^2_{\mathcal{E}_k}$.*

Proof. By (13), (16) and the definition of $\widetilde{\Delta^k}$,

$$\begin{aligned}|(U_\phi f)^{(k)}(0)|^2 &= (\Delta^k V_\phi |f|^2)(0) = (\widetilde{\Delta^k} V_\phi |f|^2)(0) \\ &= V_\phi(\widetilde{\Delta^k}|f|^2)(0) = (1-|\phi(0)|^2)^\lambda (\widetilde{\Delta^k}|f|^2)(\phi(0)),\end{aligned}$$

and the assertion follows. □

In particular, for $k=0$ we get $\|f\|_{\mathcal{E}_0} = \sup_{z\in\mathbf{D}} (1-|z|^2)^{\lambda/2} |f(z)|$, so

$$\mathcal{E}_0 = L^{\infty,\lambda/2}_{\mathrm{hol}}(\mathbf{D}) \tag{33}$$

with equality of norms.

Proposition 20. *For $\rho > 1+\lambda$, we have $\mathcal{E}_k \hookrightarrow \widetilde{Q}_{k,\rho}$.*

Proof. Since $d\mu_\nu$ is a finite measure for $\nu > -1$, we get for $\rho - 2 - \lambda > -1$

$$\begin{aligned}\int_{\mathbf{D}} \widetilde{\Delta^k} |f|^2 \, d\mu_{\rho-2} &= \int_{\mathbf{D}} (1-|z|^2)^\lambda \widetilde{\Delta^k} |f|^2 \, d\mu_{\rho-2-\lambda} \\ &\le \mu_{\rho-2-\lambda}(\mathbf{D}) \sup_{z\in\mathbf{D}} (1-|z|^2)^\lambda \widetilde{\Delta^k}|f|^2 \\ &= \mu_{\rho-2-\lambda}(\mathbf{D}) \|f\|^2_{\mathcal{E}_k}\end{aligned}$$

by the preceding proposition. Replacing f by $U_\phi f$, noting that $\|U_\phi f\|_{\mathcal{E}_k} = \|f\|_{\mathcal{E}_k}$, and taking supremum over all $\phi \in \mathrm{Aut}(\mathbf{D})$ yields $\|f\|_{\widetilde{k,\rho}} \le \mu_{\rho-2-\lambda}(\mathbf{D})^{1/2} \|f\|_{\mathcal{E}_k}$, as desired. □

Our next task will be the examination of the coefficients s_j in (30), where — in view of (32) — we need to take $d\mu = d\mu_{\rho-2}$ and $L = \widetilde{\Delta^k}$. Recall that the hypergeometric function ${}_2F_1$ with parameters $a, b, c, -c \notin \mathbf{N}$, is defined for $z \in \mathbf{D}$ by

$${}_2F_1\Big(\begin{matrix} a, b \\ c \end{matrix}\Big| z\Big) = \sum_{j=0}^\infty \frac{(a)_j (b)_j}{(c)_j j!} z^j.$$

If a (or b) is a nonpositive integer, then the series terminates; in that case one can allow even parameters c with $-c \in \mathbf{N}$, as long as $c \le a$ (or $c \le b$).

Lemma 21. *For $k, m \in \mathbf{N}$, one has*

$$\widetilde{\Delta}^k |z|^{2m} = \Big(\sum_{j=0}^{\min(m,k)} (1-t)^j t^{\frac{m+k}{2}-j} (-m)_j (\lambda+j)_{k-j} \binom{k}{j} \Big)^2 \tag{34}$$

$$= t^{m-k} \Big(\sum_{j=0}^{k} \frac{(-k)_j}{j!} (m+\lambda)_j (\lambda+j)_{k-j} (1-t)^j \Big)^2, \tag{35}$$

where, for the sake of brevity, we have set $t := |z|^2$.

If $(\lambda)_k \neq 0$, this can be written more neatly as

$$\widetilde{\Delta}^k |z|^{2m} = (\lambda)_k^2 t^{m+k} {}_2F_1 \Big(\begin{matrix} -m, -k \\ \lambda \end{matrix} \Big| 1 - \frac{1}{t} \Big)^2 \tag{36}$$

$$= (\lambda)_k^2 t^{m-k} {}_2F_1 \Big(\begin{matrix} -k, m+\lambda \\ \lambda \end{matrix} \Big| 1 - t \Big)^2. \tag{37}$$

Proof. Applying Corollary 12 to $|z|^{2m}$ yields

$$\begin{aligned} \widetilde{\Delta}^k |z|^{2m} &= \Big| \sum_{j=0}^{\min(m,k)} (1-|z|^2)^j \frac{m!}{(m-j)!} z^{m-j} (-\overline{z})^{k-j} (\lambda+j)_{k-j} \binom{k}{j} \Big|^2 \\ &= \Big| z^{m+k} \sum_{j=0}^{\min(m,k)} (1-|z|^2)^j (-m)_j |z|^{-2j} (\lambda+j)_{k-j} \binom{k}{j} \Big|^2, \end{aligned}$$

which is (34). For $(\lambda)_k \neq 0$, one can write

$$(\lambda+j)_{k-j} = \frac{(\lambda)_k}{(\lambda)_j}, \tag{38}$$

while $\binom{k}{j} = (-1)^j \frac{(-k)_j}{j!}$; thus we can continue with

$$\widetilde{\Delta}^k |z|^{2m} = (\lambda)_k^2 t^{m+k} \Big(\sum_{j=0}^{\min(m,k)} (1-t)^j \frac{(-m)_j (-k)_j}{j! (\lambda)_j} (-t)^j \Big)^2,$$

which is (36). Using the transformation formula for hypergeometric functions [BE, 2.1.4(22)]

$${}_2F_1 \Big(\begin{matrix} a, b \\ c \end{matrix} \Big| z \Big) = (1-z)^{-b} {}_2F_1 \Big(\begin{matrix} c-a, b \\ c \end{matrix} \Big| \frac{z}{z-1} \Big),$$

the last expression becomes

$$\widetilde{\Delta}^k |z|^{2m} = (\lambda)_k^2 t^{m-k} {}_2F_1 \Big(\begin{matrix} -k, m+\lambda \\ \lambda \end{matrix} \Big| 1-t \Big)^2,$$

which is (37). Expanding the hypergeometric function back into series and using again (38) yields (35). Finally, since (34) and (35) both are polynomials in λ, their equality for $(\lambda)_k \neq 0$ implies that they actually coincide for all λ. $\square$

Denote

$$\widetilde{c}_{jk\rho} := \int_{\mathbf{D}} \widetilde{\Delta}^k |z|^{2j} \, d\mu_{\rho-2}(z). \tag{39}$$

Thus, by Proposition 15, $\widetilde{Q}_{k,\rho} = Q_{\mathbf{s}}$ for the sequence $\mathbf{s} = \{\widetilde{c}_{jk\rho}\}_{j=0}^{\infty}$.

Lemma 22. *Let $k, j \in \mathbf{N}$.*

(i) *If* $-\lambda \in \{0,1,\dots,k-1\}$ *and* $j \le -\lambda$, *then* $\widetilde{c}_{jk\rho} = 0$ *for all* $\rho \in \mathbf{R}$.
(ii) *If* $-\lambda \in \{0,1,\dots,k-1\}$ *and* $j > -\lambda$, *then*

$$\widetilde{c}_{jk\rho} = \begin{cases} \text{finite positive} & \rho > 2\lambda - 1, \\ +\infty & \rho \le 2\lambda - 1. \end{cases}$$

(iii) *If* $-\lambda \notin \{0,1,\dots,k-1\}$, *then, for all* j,

$$\widetilde{c}_{jk\rho} = \begin{cases} \text{finite positive} & \rho > 1, \\ +\infty & \rho \le 1. \end{cases}$$

Note that the condition $-\lambda \in \{0,1,\dots,k-1\}$ is equivalent just to $(\lambda)_k = 0$.

Proof. (i) In this case, as already noted in Corollary 13, $\widetilde{\Delta^k}|z|^{2j} \equiv 0$, and the conclusion follows trivially.

(ii) In this case $(\lambda + j)_{k-j} = 0$ for $0 \le j \le -\lambda$, while $(\lambda + j)_{k-j} > 0$ for $1 - \lambda \le j \le k$. Thus by (34), $\widetilde{\Delta^k}|z|^{2j}$ is a nonnegative smooth radial function on $\mathbf{D}$ which behaves as $(1-|z|^2)^{2-2\lambda}$ as $|z| \nearrow 1$. The integral (39) is therefore finite positive for $\rho - 2 + (2-2\lambda) > -1$, i.e. $\rho > 2\lambda - 1$, and equals $+\infty$ otherwise.

(iii) In this case $(\lambda+j)_{k-j} \ne 0$ for all j; so by (34) again, $\widetilde{\Delta^k}|z|^{2j}$ is a nonnegative smooth radial function on $\mathbf{D}$ which tends to the a nonzero boundary value $(\lambda)_k^2$ at $|z| = 1$. It follows that the integral (39) is finite positive for $\rho - 2 > -1$, i.e. $\rho > 1$, and equals $+\infty$ otherwise. □

We will say that a set $\mathcal{P}$ is contained in $\widetilde{Q}_{k,\rho}$ trivially if $\|f\|_{\widetilde{k,\rho}} = 0$ for all $f \in \mathcal{P}$.

Corollary 23. *Let* $k \in \mathbf{N}$.

(i) *If* $(\lambda)_k = 0$, *then*

$$\mathcal{P}_{\le -\lambda} \hookrightarrow \widetilde{Q}_{k,\rho} \text{ trivially}, \ \forall \lambda \in \mathbf{R}, \tag{40}$$

$$\widetilde{Q}_{k,\rho} = \mathcal{P}_{\le -\lambda} \text{ for } \rho \le 2\lambda - 1, \tag{41}$$

$$\mathcal{E}_k \hookrightarrow \widetilde{Q}_{k,\rho} \hookrightarrow \mathcal{E}_j \text{ for } j > -\lambda \text{ and } \rho > \max(1+\lambda, 2\lambda - 1). \tag{42}$$

(ii) *If* $(\lambda)_k \ne 0$, *then*

$$\widetilde{Q}_{k,\rho} = \{0\} \text{ for } \rho \le 1, \tag{43}$$

$$\mathcal{E}_k \hookrightarrow \widetilde{Q}_{k,\rho} \hookrightarrow \mathcal{E}_j \quad \forall j \in \mathbf{N}, \text{ for } \rho > \max(1, 1+\lambda). \tag{44}$$

Proof. (i) The first claim (40) is immediate from part (i) of the last lemma, together with the fact that $\mathcal{P}_{\le -\lambda}$ is Möbius invariant by Corollary 13. Using part (i) of Proposition 16, (41) follows. For (42), we have $\widetilde{Q}_{k,\rho} \hookrightarrow \mathcal{E}_j \ \forall \rho > 2\lambda - 1$ by part (ii) of the last lemma and part (ii) of Proposition 16; while $\mathcal{E}_k \hookrightarrow \widetilde{Q}_{k,\rho}$ for $\rho > 1 + \lambda$ by Proposition 20.

(ii) Using part (iii) of the last lemma, part (i) of Proposition 16 immediately gives (43), while part (ii) of the same gives $\widetilde{Q}_{k,\rho} \hookrightarrow \mathcal{E}_j$ for all j $\forall \rho > 1$. Combining the latter with Proposition 20 yields (44). □

Theorem 24. (i) *For* $-\lambda \notin \mathbf{N}$, *we have* $\mathcal{E}_k = L^{\infty,\lambda/2}_{\text{hol}}(\mathbf{D})$ *for all* k.

(ii) *For* $-\lambda \in \mathbf{N}$,

$$L^{\infty,\lambda/2}_{\text{hol}}(\mathbf{D}) = \mathcal{E}_0 = \mathcal{E}_1 = \dots = \mathcal{E}_{-\lambda} \hookrightarrow \mathcal{E}_{1-\lambda} = \mathcal{E}_j \quad \forall j > -\lambda.$$

Note that $L^{\infty,\lambda/2}_{\rm hol}(\mathbf{D})=\{0\}$ for $\lambda<0$, by the maximum principle.

Proof. (i) If $-\lambda\notin\mathbf{N}$, then $(\lambda)_k\neq 0\ \forall k$, so choosing $\rho>\max(1,1+\lambda)$ we get from (44)

$$\mathcal{E}_k\hookrightarrow\mathcal{E}_j \qquad \forall j,k\in\mathbf{N}.$$

By (33), we are done.

(ii) If $-\lambda\in\mathbf{N}$, we have $(\lambda)_k=0\ \forall k>-\lambda$, and $(\lambda)_k\neq 0$ for $k\le-\lambda$. Choosing $\rho>\max(1,1+\lambda,2\lambda-1)$, we thus have

$$\begin{aligned}\mathcal{E}_k\hookrightarrow\mathcal{E}_j &\quad \text{if } j,k>-\lambda \text{ by (42), and}\\ \mathcal{E}_k\hookrightarrow\mathcal{E}_j &\quad \text{if } k\le-\lambda,\ j\in\mathbf{N} \text{ by (44).}\end{aligned}$$

Recalling (33) again, the conclusion follows. $\square$

For $\lambda>0$, part (i) of the last theorem was proved as Lemma 4.6 in [Zhao] (by different method).

Note that the last theorem says that if the sum (27) is bounded as $a\in\mathbf{D}$ — which is precisely the condition for f to belong to $\mathcal{E}_k$ — then in fact each term of this sum separately is bounded. (Note that for $(\lambda)_k=0$, only the terms with $1-\lambda\le j\le k$ are nonzero.) (Cf. also the proof of Proposition 41 below.)

We complete our characterization of (k,λ)-Bloch spaces by giving a description of the remaining unknown space, namely, $\mathcal{E}_{1-\lambda}$.

Proposition 25. *For $-\lambda=m\in\mathbf{N}$, the norm in $\mathcal{E}_{m+1}$ is equivalent to*

$$\begin{aligned}&\sup_{z\in\mathbf{D}}(1-|z|^2)|f^{(\frac{m}{2}+1)}(z)| \quad \textit{if } m \textit{ is even,}\\ &\sup_{z\in\mathbf{D}}(1-|z|^2)^{1/2}|f^{(\frac{m+1}{2})}(z)| \quad \textit{if } m \textit{ is odd.}\end{aligned} \tag{45}$$

Proof. For $\lambda=-m$, $k=m+1$ we get from (27)

$$(U_af)^{(k)}(0)=(-1)^k(1-|a|^2)^{\frac{\lambda}{2}+k}f^{(k)}(a)=(-1)^k(1-|a|^2)^{\frac{m}{2}+1}f^{(m+1)}(a).$$

Thus

$$\|f\|_{\mathcal{E}_{m+1}}=\sup_{a\in\mathbf{D}}(1-|a|^2)^{\frac{m}{2}+1}|f^{(m+1)}(a)|.$$

The equivalence of this seminorm to (45) now follows by repeated application of Proposition 10. $\square$

Proposition 26. (i) *For $-\lambda\notin\mathbf{N}$,*

$$\begin{aligned}\widetilde{Q}_{k,\rho}&=\{0\} \quad \textit{for } \rho\le 1,\\ \widetilde{Q}_{k,\rho}&=L^{\infty,\lambda/2}_{\rm hol}(\mathbf{D}) \quad \textit{for } \rho>\max(1,1+\lambda).\end{aligned}$$

(ii) *For $-\lambda\in\mathbf{N}$,*

$$\begin{aligned}&\widetilde{Q}_{k,\rho}=\{0\} \quad \textit{if } k\le-\lambda,\ \rho\le 1,\\ &\widetilde{Q}_{k,\rho}=L^{\infty,\lambda/2}_{\rm hol}(\mathbf{D}) \quad \textit{if } k\le-\lambda,\ \rho>1,\\ &\widetilde{Q}_{k,\rho}=\mathcal{P}_{\le-\lambda} \textit{ trivially if } k>-\lambda,\ \rho\le 2\lambda-1,\\ &\mathcal{P}_{\le-\lambda}\overset{\text{trivially}}{\hookrightarrow}\widetilde{Q}_{k,\rho}\hookrightarrow\mathcal{E}_{1-\lambda} \quad \textit{if } k>-\lambda,\ 2\lambda-1<\rho\le\lambda+1,\\ &\mathcal{P}_{\le-\lambda}\overset{\textit{trivially}}{\hookrightarrow}\widetilde{Q}_{k,\rho}=\mathcal{E}_{1-\lambda} \quad \textit{if } k>-\lambda,\ \rho>\lambda+1.\end{aligned}$$

Proof. (i) This is immediate from (43) and (44), together with our description of the spaces $\mathcal{E}_k$ in part (i) of the last theorem.

(ii) For $k \le -\lambda$, we still have $(\lambda)_k \ne 0$, so the assertion again follows in the same way as for (i), noting only that $\max(1, 1+\lambda) = 1$ since $\lambda \le 0$. For $k > -\lambda$, we have $(\lambda)_k = 0$, so the first assertion is just (41), the second follows from (40) and (42) (with $j = 1 - \lambda$, noting that $2\lambda - 1 < \lambda + 1$ since $\lambda \le 0$), and the third from the same together with Proposition 20. □

By the last proposition, the remaining interesting intervals for ρ are thus $1 < \rho \le 1 + \lambda$ for $\lambda > 0$, and $2\lambda - 1 < \rho \le \lambda + 1$ for $-\lambda \in \mathbf{N}$ and $k > -\lambda$. We proceed to resolve these cases.

Recall that $\widetilde{Q}_{k,\rho} = Q_{\mathbf{s}}$ for the sequence $s_j = \widetilde{c}_{jk\rho}$ given by (39).

Proposition 27. *If* $(\lambda)_k \ne 0$*, then*

$$\widetilde{Q}_{k,\rho} = \begin{cases} Q_{\mathbf{s}[0,1-\rho]} & \text{for } \rho > 1, \\ \{0\} & \text{for } \rho \le 1. \end{cases}$$

If $(\lambda)_k = 0$*, then*

$$\widetilde{Q}_{k,\rho} = \begin{cases} \mathcal{P}_{\le -\lambda} & \text{for } \rho \le 2\lambda - 1, \\ Q_{\mathbf{s}[1-\lambda,1-\rho]} & \text{for } \rho > 2\lambda - 1. \end{cases}$$

(Here we are using the notation (20).)

Note that by Corollary 18 and Theorem 24,

$$Q_{\mathbf{s}[1-\lambda,1-\rho]} = Q_{\mathbf{s}[k,1-\rho]} \tag{46}$$

if $(\lambda)_k = 0$.

Proof. The cases of $\rho \le 1$, $(\lambda)_k \ne 0$ and $\rho \le 2\lambda - 1$, $(\lambda)_k = 0$ are already covered by Proposition 26; we thus only need to show that when $(\lambda)_k \ne 0$, $\rho > 1$ or $(\lambda)_k = 0$, $\rho > 2\lambda - 1$, $j > -\lambda$ — which, by Lemma 22, are precisely the cases when $\widetilde{c}_{jk\rho}$ is finite and positive — the constants $\widetilde{c}_{jk\rho}$ satisfy

$$\widetilde{c}_{mk\rho} \asymp m^{1-\rho} \qquad \text{as } m \to +\infty.$$

By (35), we have

$$\begin{aligned} \widetilde{c}_{mk\rho} &= \int_{\mathbf{D}} \widetilde{\Delta^k} |z|^{2m} \, d\mu_{\rho-2} \\ &= \int_0^1 t^{m-k} \Big(\sum_{j=0}^k \frac{(-k)_j}{j!} (m+\lambda)_j (\lambda+j)_{k-j} (1-t)^j \Big)^2 (1-t)^{\rho-2} \, dt \\ &= \sum_{j,l=0}^k \frac{(-k)_j}{j!} \frac{(-k)_l}{l!} (m+\lambda)_j (m+\lambda)_l (\lambda+j)_{k-j} (\lambda+l)_{k-l} \\ &\qquad \times \int_0^1 t^{m-k} (1-t)^{j+l+\rho-2} \, dt. \end{aligned} \tag{47}$$

If $(\lambda)_k = 0$, the sum involves only $j, l \ge 1 - \lambda$ (the other terms are zero). Thus in both cases above $j + l + \rho - 2 > -1$, so the last integral exists and equals

$$\int_0^1 t^{m-k} (1-t)^{j+l+\rho-2} \, dt = \frac{(m-k)! \Gamma(j+l+\rho-1)}{\Gamma(j+l+\rho+m-k)} \sim \Gamma(j+l+\rho-1) m^{1-\rho-j-l}$$

as $m \to \infty$, by Stirling's formula. Since $(m+\lambda)_j \sim m^j$ as $m \to \infty$, we thus get $\widetilde{c}_{mk\rho} \sim cm^{1-\rho}$ as $m \to \infty$, with

$$c = \sum_{j,l=0}^{k} \frac{(-k)_j}{j!} \frac{(-k)_l}{l!} (\lambda+j)_{k-j} (\lambda+l)_{k-l} \Gamma(j+l+\rho-1).$$

Now

$$\begin{aligned} c &= \sum_{j,l=0}^{k} \frac{(-k)_j}{j!} \frac{(-k)_l}{l!} (\lambda+j)_{k-j} (\lambda+l)_{k-l} \int_0^\infty e^{-t} t^{j+l+\rho-2}\, dt \\ &= \int_0^\infty \Big| \sum_{j=0}^{k} \frac{(-k)_j}{j!} (\lambda+j)_{k-j} t^j \Big|^2 e^{-t} t^{\rho-2}\, dt > 0 \end{aligned}$$

since the polynomial given by the last sum does not vanish identically. Thus indeed $\widetilde{c}_{mk\rho} \asymp m^{1-\rho}$ as $m \to \infty$, completing the proof. □

The following proposition gives a characterization of $\widetilde{Q}_{k,\rho}$ analogous to (1).

Proposition 28. *For any $k \in \mathbf{N}$ and $\rho \in \mathbf{R}$,*

$$\|f\|^2_{\widetilde{k,\rho}} = \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \widetilde{\Delta^k} |f|^2 \, (1-|\phi|^2)^{\rho-\lambda} \, d\mu_{\lambda-2}. \tag{48}$$

Thus $f \in \widetilde{Q}_{k,\rho}$ if and only if the last supremum is finite.

Proof. Using the invariance of $\widetilde{\Delta^k}$ and $d\mu_{\lambda-2}$ (Proposition 14),

$$\begin{aligned} \|f\|^2_{\widetilde{k,\rho}} &= \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \widetilde{\Delta^k}(V_\phi |f|^2) \, d\mu_{\rho-2} \\ &= \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} (1-|z|^2)^{\rho-\lambda} \widetilde{\Delta^k}(V_\phi |f|^2) \, d\mu_{\lambda-2} \\ &= \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} (1-|z|^2)^{\rho-\lambda} V_\phi(\widetilde{\Delta^k} |f|^2) \, d\mu_{\lambda-2} \\ &= \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} V_\phi[(1-|\phi^{-1}|^2)^{\rho-\lambda} \widetilde{\Delta^k} |f|^2] \, d\mu_{\lambda-2} \\ &= \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} (1-|\phi^{-1}|^2)^{\rho-\lambda} \widetilde{\Delta^k} |f|^2 \, d\mu_{\lambda-2}, \end{aligned}$$

where we noted that $V_\phi(uv) = (u \circ \phi)(V_\phi v)$ for any u, v. □

Proposition 29. *If $\rho < \lambda$, then $\widetilde{Q}_{k,\rho} = \{f \in \mathcal{H}(\mathbf{D}) : \widetilde{\Delta^k}|f|^2 \equiv 0\}$. That is, for $\rho < \lambda$, $\widetilde{Q}_{k,\rho} = \{0\}$ if $(\lambda)_k \neq 0$, and $\widetilde{Q}_{k,\rho} = \mathcal{P}_{\leq -\lambda}$ trivially if $(\lambda)_k = 0$.*

Proof. Assume there is $f \in \widetilde{Q}_{k,\rho}$ with $\widetilde{\Delta^k}|f|^2(y) > 0$ for some $y \in \mathbf{D}$; replacing f by $U_y f$, we may assume without loss of generality that $y = 0$. By continuity, there exist $0 < r < 1$ and $\delta > 0$ such that $\widetilde{\Delta^k}|f|^2 \geq \delta > 0$ for $|z| \leq r$. Thus by (48),

$$\begin{aligned} \|f\|^2_{\widetilde{k,\rho}} &= \sup_{a \in \mathbf{D}} \int_{\mathbf{D}} (1-|\phi_a|^2)^{\rho-\lambda} \widetilde{\Delta^k} |f|^2 \, d\mu_{\lambda-2} \\ &\geq \delta \sup_{a \in \mathbf{D}} \int_{r\mathbf{D}} (1-|\phi_a|^2)^{\rho-\lambda} \, d\mu_{\lambda-2} \end{aligned}$$

$$= \delta \sup_{a\in\mathbf{D}} (1-|a|^2)^{\rho-\lambda} \int_{r\mathbf{D}} \frac{(1-|z|^2)^{\rho-2}}{|1-\overline{a}z|^{2(\rho-\lambda)}}\,dz$$
$$\geq (1-r^2)^{\max(0,\rho-2)} \delta \sup_{a\in\mathbf{D}} (1-|a|^2)^{\rho-\lambda} \int_{r\mathbf{D}} |1-\overline{a}z|^{2(\lambda-\rho)}\,dz$$
$$\geq (1-r^2)^{\max(0,\rho-2)} \min((1-r)^{2(\lambda-\rho)},(1+r)^{2(\lambda-\rho)}) \delta \sup_{a\in\mathbf{D}} (1-|a|^2)^{\rho-\lambda}.$$

It therefore follows that $(1-|a|^2)^{\rho-\lambda}$ is bounded as $a \in \mathbf{D}$, which means that $\rho \geq \lambda$. This proves the first assertion.

If $\widetilde{\Delta^k}|f|^2 \equiv 0$, then by (39) and the proof of Proposition 15,

$$0 = \int_{\mathbf{D}} \widetilde{\Delta^k}|f|^2\,d\mu_{\rho-2} = \sum_j |f_j|^2 \widetilde{c}_{jk\rho},$$

implying that $f_j = 0$ unless $\widetilde{c}_{jk\rho} = 0$. By Lemma 22, this means that $f_j = 0$ $\forall j$ if $(\lambda)_k \neq 0$, and $f_j = 0$ $\forall j > -\lambda$ if $(\lambda)_k = 0$, which proves the second assertion. □

Since $2\lambda - 1 < \lambda$ for $\lambda \leq 0$, the last proposition restricts the unresolved cases to $\rho \in (1,\infty) \cap [\lambda, \lambda+1]$ for $\lambda > 0$, and $\lambda \leq \rho \leq \lambda + 1$ for $-\lambda \in \mathbf{N}$, $k > -\lambda$.

Proposition 30. *For $(\lambda)_k \neq 0$ and $\rho > 1$,*

$$\widetilde{Q}_{k,\rho} = E(2, \lambda-2, \rho-\lambda)$$

(cf. (8)*).*

Proof. By Proposition 27, $\widetilde{Q}_{k,\rho} = Q_{\mathbf{s}[0,1-\rho]} = \widetilde{Q}_{0,\rho}$; and by (48),

$$\|f\|^2_{\widetilde{0,\rho}} = \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} |f(z)|^2 (1-|z|^2)^{\lambda-2} (1-|\phi(z)|^2)^{\rho-\lambda}\,dz = \|f\|^2_{E(2,\lambda-2,\rho-\lambda)}.$$

□

Here we remark that the spaces $F(p,q,s)$ — and, hence, also our space $E(p,q,s)$ of primitives of functions in $F(p,q,s)$ — are usually defined for $q > -2$ and $s \geq 0$; however the definitions (5) and (8), of course, make sense for any $q, s \in \mathbf{R}$. For any fixed $\phi \in \mathrm{Aut}(\mathbf{D})$, the product $(1-|z|^2)^q (1-|\phi(z)|^2)^s$ is $\asymp (1-|z|^2)^{q+s}$ as $|z| \nearrow 1$; from the subharmonicity of $|f|^p$ it therefore follows that the integral in (8) can exist only if $q + s > -1$ (unless $f \equiv 0$). From the last corollary and Proposition 29, we infer that also $E(2,q,s) = \{0\}$ if $s < 0$; for the record, we give a direct (and different) proof of this fact.

Proposition 31. *$E(p,q,s) = \{0\}$ if $s < 0$, for any $p > 0$ and $q \in \mathbf{R}$.*

Proof. Assume that $\|f\|_{E(p,q,s)} =: c < \infty$. From

$$c \geq \int_{\mathbf{D}} |f|^p (1-|\phi_a|^2)^s\,d\mu_q = \int_{\mathbf{D}} |f(z)|^p \frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}\,d\mu_{q+s}(z)$$

we get

$$\int_{\mathbf{D}} \Big| \frac{f(z)}{(1-\overline{a}z)^{2s/p}} \Big|^p\,d\mu_{q+s}(z) \leq c(1-|a|^2)^{-s}. \tag{49}$$

If $s < 0$, letting $|a| \nearrow 1$ shows, by Fatou's lemma, that

$$(1-\overline{a}z)^{-2s/p} f(z) = 0, \qquad \forall z \in \mathbf{D},\ a \in \mathbf{T}.$$

Since, for each fixed $z \in \mathbf{D}$, the left-hand side is a holomorphic function of $\overline{a} \in \mathbf{D}$, it follows by the maximum principle that the left-hand side vanishes in fact for all $a, z \in \mathbf{D}$. Taking $a = 0$ gives $f \equiv 0$. $\square$

It turns out that at least in our case $p = 2$, the space $E(p, q, s)$ is trivial also for $q < -2$. For this, we need the following proposition, which is of interest in its own right.

Proposition 32. *Let $f(z) = \sum_{j=0}^{\infty} f_j z^j$ belong to $\widetilde{Q}_{k,\rho}$. Then $f_j z^j \in \widetilde{Q}_{k,\rho}$ $\forall j$.*

Proof. By rotation invariance, we have for each $\epsilon \in \mathbf{T}$ and $\phi \in \mathrm{Aut}(\mathbf{D})$

$$\|f\|^2_{\widetilde{k,\rho}} = \|f_\epsilon\|^2_{\widetilde{k,\rho}} \geq \int_{\mathbf{D}} \widetilde{\Delta^k} |f_\epsilon|^2 \, (1 - |\phi|^2)^{\rho - \lambda} \, d\mu_{\lambda-2}$$

by (48). Integrating over ϵ gives, by uniform convergence,

$$\begin{aligned} \|f\|^2_{\widetilde{k,\rho}} &\geq \int_{\mathbf{D}} \widetilde{\Delta^k} \Big(\int_{\mathbf{T}} |f_\epsilon|^2 \, d\epsilon \Big) \, (1 - |\phi|^2)^{\rho - \lambda} \, d\mu_{\lambda-2} \\ &= \int_{\mathbf{D}} \sum_{j=0}^{\infty} \widetilde{\Delta^k} |f_j z^j|^2 \, (1 - |\phi|^2)^{\rho - \lambda} \, d\mu_{\lambda-2}. \end{aligned}$$

Since $\widetilde{\Delta^k} |g|^2 \geq 0$ for any $g \in \mathcal{H}(\mathbf{D})$, it follows that

$$\|f\|^2_{\widetilde{k,\rho}} \geq \int_{\mathbf{D}} \widetilde{\Delta^k} |f_j z^j|^2 \, (1 - |\phi|^2)^{\rho - \lambda} \, d\mu_{\lambda-2}$$

for any fixed j. Taking supremum over all $\phi \in \mathrm{Aut}(\mathbf{D})$ yields $\|f\|^2_{\widetilde{k,\rho}} \geq \|f_j z^j\|^2_{\widetilde{k,\rho}}$, proving the claim. $\square$

Corollary 33. *$E(2, q, s) = \{0\}$ if $q < -2$, for any $s \in \mathbf{R}$.*

Proof. By the preceding proposition, if there is a nonzero $f \in E(2, q, s) = \widetilde{Q}^{(q+2)}_{0,s+q+2}$, then $z^m \in E(2, q, s)$ for some $m \in \mathbf{N}$. Thus by (49)

$$\int_{\mathbf{D}} \Big| \frac{z^m}{(1 - \overline{a} z)^s} \Big|^2 \, d\mu_{q+s}(z) \leq \frac{c}{(1 - |a|^2)^s} \qquad \forall a \in \mathbf{D}$$

with $c := \|z^m\|^2_{E(2,q,s)} < \infty$. Thus $q + s > -1$ and, expanding $(1 - \overline{a} z)^{-s}$ by the binomial theorem and integrating term by term,

$$\sum_{j=0}^{\infty} \frac{(s)_j^2}{j!^2} |a|^{2j} \int_{\mathbf{D}} |z^{m+j}|^2 \, d\mu_{q+s}(z) \leq \frac{c}{(1 - |a|^2)^s},$$

that is,

$$\sum_{j=0}^{\infty} \frac{(s)_j^2}{j!^2} \frac{(m+j)! \Gamma(q+s+1)}{\Gamma(q+s+m+j+2)} |a|^{2j} \leq \frac{c}{(1 - |a|^2)^s}.$$

By Stirling's formula, the coefficients at $|a|^{2j}$ is $\asymp (j+1)^{s-q-3}$ as $j \to \infty$, hence the last sum behaves as $(1-|a|^2)^{q+2-s}$ if $q+2-s < 0$, as $-\log(1-|a|^2)$ if $q+2-s = 0$, and is bounded if $q+2-s > 0$. Since in our case $q+2-s = 2q+2-(q+s) < 2q+3 < -1$, we conclude that

$$(1 - |a|^2)^{q+2-s} \lesssim (1 - |a|^2)^{-s}.$$

However, this means that $q + 2 \geq 0$, contradicting the hypothesis. $\square$

Note that for $q=-2$, we have by Proposition 26(ii) $E(2,q,s)=\widetilde{Q}^{(0)}_{0,s}=\{0\}$ for $s\le 1$ and $E(2,q,s)=L^{\infty,0}_{\text{hol}}(\mathbf{D})=H^\infty$ for $s>1$.

We are now ready to complete our discussion of the first of the remaining cases, namely, $\widetilde{Q}_{k,\rho}$ with $\lambda>0$ and $\rho\in(1,\infty)\cap[\lambda,\lambda+1]$.

Theorem 34. *For $-\lambda\notin\mathbf{N}$ and $\rho\in(1,\infty)\cap[\lambda,\lambda+1]$, the spaces $\widetilde{Q}_{k,\rho}$ are nontrivial and strictly increasing with ρ, i.e. $\widetilde{Q}_{k,\rho}\subsetneq\widetilde{Q}_{k,\rho'}$ if $\rho<\rho'$.*

Proof. This follows from Proposition 30 and the corresponding property of $F(p,q,s)$ spaces, cf. Theorem 5.5 in [Zhao]; however, since the proof there concerns $F(p,q,s)$ rather than $E(p,q,s)$, we briefly repeat the details here. Consider a lacunary series $f(z)=\sum_k a_k z^{2^k}$. On the one hand, we have

$$\begin{aligned}\|f\|^2_{\widetilde{0,\rho}} \ge \int_{\mathbf{D}}|f|^2\,d\mu_{\rho-2} &\asymp \sum_j |f_j|^2(j+1)^{1-\rho} \qquad \text{by Proposition 27}\\ &\asymp \sum_k |a_k|^2 2^{(1-\rho)k}.\end{aligned}$$

On the other hand, we have for $\phi=\epsilon\phi_a\in\operatorname{Aut}(\mathbf{D})$

$$\begin{aligned}&\int_{\mathbf{D}}|f|^2(1-|\phi|^2)^{\rho-\lambda}\,d\mu_{\lambda-2}\\ &\quad\le \int_{\mathbf{D}}\Big(\sum_j |f_j||z|^j\Big)^2\frac{(1-|z|^2)^{\rho-2}(1-|a|^2)^{\rho-\lambda}}{|1-\bar az|^{2(\rho-\lambda)}}\,dz\\ &\quad=\int_0^1\Big(\sum_j|f_j|r^j\Big)^2(1-r^2)^{\rho-2}(1-|a|^2)^{\rho-\lambda}\Big(\sum_j\frac{(\rho-\lambda)_j^2}{j!^2}|a|^{2j}r^{2j}\Big)\,r\,dr\end{aligned}$$

by integrating in polar coordinates. As $\frac{(\rho-\lambda)_j^2}{j!^2}\asymp(j+1)^{2(\rho-\lambda-1)}$ by Stirling's formula, the last sum is majorized by

$$\begin{cases}(1-r|a|)^{2\lambda+1-2\rho} & \text{if } 2\lambda+1-2\rho<0,\\ -\log(1-r|a|) & \text{if } 2\lambda+1-2\rho=0,\\ 1 & \text{if } 2\lambda+1-2\rho>0;\end{cases}$$

in our case of $0\le\rho-\lambda\le1$, this is in turn majorized by $(1-|a|r)^{\lambda-\rho}$. Since $\frac{1-|a|^2}{1-|a|r}\le 2$, we get

$$\int_{\mathbf{D}}|f|^2(1-|\phi|^2)^{\rho-\lambda}\,d\mu_{\lambda-2}\lesssim\int_0^1\Big(\sum_j|f_j|r^j\Big)^2(1-r^2)^{\rho-2}\,dr.$$

By [AXZ, Lemma 3], for our lacunary f and $\rho>1$ the last integral is majorized by $\sum_k 2^{-(\rho-1)k}|a_k|^2$. Altogether, we thus see that

$$f\in\widetilde{Q}_{0,\rho}\iff\sum_k|a_k|^2 2^{(1-\rho)k}<\infty.$$

Taking $a_k=2^{(\rho_1-1)k/2}$ therefore gives an f which belongs to $\widetilde{Q}_{0,\rho}$ for $\rho>\rho_1$, but not to $\widetilde{Q}_{0,\rho_1}$. □

It remains to tackle the case $\lambda\le\rho\le\lambda+1$ and $-\lambda\in\mathbf{N}$, $k>-\lambda$ (i.e. $(\lambda)_k=0$).

Theorem 35. *If $(\lambda)_k = 0$ and $\lambda \le \rho \le \lambda + 1$, then*

$$\widetilde{Q}_{k,\rho} = \{f \in \mathcal{H}(\mathbf{D}) : f^{(-\lambda)} \in F(2, -\lambda, \rho - \lambda)\} \tag{50}$$

with equivalent seminorms. Consequently, the spaces $\widetilde{Q}_{k,\rho}$ are nontrivial, do not depend on k (as long as $k > -\lambda$) and are strictly increasing with ρ, i.e. $\widetilde{Q}_{k,\rho} \subsetneq \widetilde{Q}_{k,\rho'}$ if $\rho < \rho'$.

Proof. By Proposition 27, we have $\widetilde{Q}_{k,\rho} = Q_{\mathbf{s}[1-\lambda,1-\rho]}$; since the latter does not depend on k, we may assume that $k = 1 - \lambda$. Then by Corollary 12

$$\widetilde{\Delta^k} = (1 - |z|^2)^{2k} \Delta^k,$$

so by (48)

$$\|f\|^2_{\widetilde{k,\rho}} = \sup_{\phi \in \mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} |f^{(k)}|^2 \, (1 - |\phi|^2)^{\rho - \lambda} \, d\mu_{2k+\lambda-2}.$$

Since $2k + \lambda - 2 = -\lambda$, we thus get

$$f \in \widetilde{Q}_{k,\rho} \iff f^{(k-1)} \in F(2, -\lambda, \rho - \lambda)$$

with equivalent seminorms, proving the first claim.

The second claim again follows from the corresponding property of the $F(p, q, s)$ spaces, cf. [Zhao, Theorem 5.5]: namely, for a lacunary series $f(z) = \sum_j a_j z^{n_j}$ with $\inf \frac{n_{j+1}}{n_j} > 1$, one has $f \in F(2, -\lambda, \rho - \lambda)$ if and only if $\sum_j |a_j|^2 n_j^{2\lambda+1-\rho} < \infty$; hence, $f \in \widetilde{Q}_{k,\rho}$ if and only if $\sum_j |a_j|^2 n_j^{1-\rho} < \infty$, and one constructs the required examples as in the preceding proof. □

Remark 36. For $\lambda = -1$, one can use Theorem 3.2 in [Ratt] to conclude that (50) is equivalent to

$$\widetilde{Q}^{(-1)}_{k,\rho} = F(2, -1, \rho + 1), \qquad \rho \ge -1, k \ge 2. \quad \square$$

The material amassed above establishes also our theorems on the spaces $\mathcal{E}_k$ and $\widetilde{Q}_{k,\rho}$ from the Introduction.

Proof of Theorem 6. This is just Theorem 24, together with the description of $\mathcal{E}_{1-\lambda}$, $-\lambda \in \mathbf{N}$, from Proposition 25. □

Proof of Theorem 7. (i) This follows from Proposition 26(i) and Proposition 29.
(ii) This is contained in Proposition 26(i).
(iii) This follows from Proposition 30 and Theorem 34.
(iv) This is contained in the first part of Proposition 27. □

Proof of Theorem 8. (i) and (ii) are the first two assertions of Proposition 26(ii).
(iii) This follows from the second part of Proposition 29.
(iv) This is the fifth assertion in Proposition 26(ii).
(v) This is Theorem 35.
(vi) This is contained in the second part of Proposition 27. □

We conclude by remarking that Theorem 34 (together with Proposition 30) and Theorem 35 also give a description of the "invariant Dirichlet space" $\widetilde{Q}_{k,\lambda}$ from (29): namely, for $\lambda > 1$ this is just the weighted Bergman space $E(2, \lambda-2, 0) = L^2_{\mathrm{hol}}(\mathbf{D}, d\mu_{\lambda-2})$; for $-\lambda = m \in \mathbf{N}$, it is the holomorphic Sobolev space $W^{\frac{m}{2}+1}_{\mathrm{hol}}(\mathbf{D})$ of order $\frac{m}{2} + 1$; for other values of λ, it reduces just to the constant zero.

4. The spaces $Q_{k,\rho}$

In this section we treat the spaces $Q_{k,\rho}$; recall that these were defined using ordinary Laplacians, namely they consists of all $f \in \mathcal{H}(\mathbf{D})$ for which the quantity

$$\|f\|^2_{k,\rho} = \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \Delta^k |U_\phi f|^2 \, d\mu_\rho = \sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} |(U_\phi f)^{(k)}|^2 \, d\mu_\rho$$

is finite. For $k=0$, clearly

$$Q_{0\rho} = \widetilde{Q}_{0,\rho+2}. \tag{51}$$

By (26), we also have

$$\|f\|^2_{k,\rho} = \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \Big| \sum_{j=0}^k (1-|a|^2)^{\frac{\lambda}{2}+j} \frac{f^{(j)}(\phi_a(z))(-\overline{a})^{k-j}}{(1-\overline{a}z)^{k+j+\lambda}} (\lambda+j)_{k-j} \binom{k}{j} \Big|^2 \, d\mu_\rho(z). \tag{52}$$

Proposition 37. *For $f \in \mathcal{H}(\mathbf{D})$,*

$$\|f\|^2_{k,\rho} = \sup_{a\in\mathbf{D}} (1-|a|^2)^{\rho+2-2k-\lambda} \int_{\mathbf{D}} \Big| \sum_{j=0}^k \frac{f^{(j)}(y)(-\overline{a})^{k-j}}{(1-\overline{a}y)^{\rho+2-k-j-\lambda}} (\lambda+j)_{k-j} \binom{k}{j} \Big|^2 \, d\mu_\rho(y). \tag{53}$$

Proof. Make the change of variable $z = \phi_a(y)$ in (52), and use the facts that $1-\overline{a}\phi_a(y) = \frac{1-|a|^2}{1-\overline{a}y}$ and

$$d\mu_\rho(\phi_a(y)) = \frac{(1-|a|^2)^{\rho+2}}{|1-\overline{a}y|^{2\rho+4}} \, d\mu_\rho(y).$$

□

The next proof borrows from the proof of Theorem 5 in [Zhu1].

Proposition 38. *For $\rho < 2k+\lambda-2$,*

$$Q_{k,\rho} = \begin{cases} \{0\} & \text{if } (\lambda)_k \neq 0, \\ \mathcal{P}_{\le -\lambda} \text{ trivially} & \text{if } (\lambda)_k = 0. \end{cases}$$

Proof. For $f \in Q_{k,\rho}$, (53) implies

$$\int_{\mathbf{D}} \Big| \sum_{j=0}^k \frac{f^{(j)}(y)(-\overline{a})^{k-j}}{(1-\overline{a}y)^{\rho+2-k-j-\lambda}} (\lambda+j)_{k-j} \binom{k}{j} \Big|^2 \, d\mu_\rho(y) \le \frac{\|f\|^2_{k,\rho}}{(1-|a|^2)^{\rho+2-2k-\lambda}}.$$

If $\rho+2-2k-\lambda < 0$, then letting $|a| \nearrow 1$ we obtain, by Fatou's lemma,

$$\int_{\mathbf{D}} \Big| \sum_{j=0}^k \frac{f^{(j)}(y)(-\overline{a})^{k-j}}{(1-\overline{a}y)^{\rho+2-k-j-\lambda}} (\lambda+j)_{k-j} \binom{k}{j} \Big|^2 \, d\mu_\rho(y) = 0 \qquad \forall a \in \mathbf{T}.$$

Consequently, the sum vanishes identically for $a \in \mathbf{T}$ and $z \in \mathbf{D}$. Since for each $y \in \mathbf{D}$, the sum is a holomorphic function of $\overline{a}$ in some neighborhood of $\overline{\mathbf{D}}$, it follows by the maximum principle that it vanishes for all $a, y \in \mathbf{D}$. Writing this as

$$0 \equiv (-\overline{a})^k (1-\overline{a}y)^{k+\lambda-\rho-2} \sum_{j=0}^k \Big(\frac{1}{\overline{a}} - y\Big)^j f^{(j)}(y)(\lambda+j)_{k-j} \binom{k}{j}$$

and noting that the last sum is a polynomial in $\frac{1}{a} - y$, it follows that

$$(\lambda + j)_{k-j} f^{(j)} \equiv 0 \qquad \forall j = 0, \dots, k. \tag{54}$$

If $(\lambda)_k \neq 0$, taking $j = 0$ shows that $f \equiv 0$, so $Q_{k,\rho} = \{0\}$. If $(\lambda)_k = 0$, (54) precisely means that $f^{(1-\lambda)} \equiv 0$, so $Q_{k,\rho} \subset \mathcal{P}_{\leq -\lambda}$. Conversely, if $(\lambda)_k = 0$, then by Corollary 13 U_ϕ maps $\mathcal{P}_{\leq -\lambda}$ into itself, hence $(U_\phi f)^{(k)} \equiv 0$ for all $f \in \mathcal{P}_{\leq -\lambda}$, so $\mathcal{P}_{\leq -\lambda} \subset Q_{k,\rho}$ trivially. □

Recall from Proposition 15 that $Q_{k,\rho} = Q_{\mathbf{s}}$ for $\mathbf{s} = \{c_{jk\rho}\}_{j=0}^\infty$, where

$$c_{jk\rho} := \int_{\mathbf{D}} \Delta^k |z|^{2j} \, d\mu_\rho(z). \tag{55}$$

Lemma 39. *For $j, k \in \mathbf{N}$ and $\rho \in \mathbf{R}$,*

$$c_{jk\rho} = \begin{cases} 0 & \text{if } k > j, \ \forall \rho \in \mathbf{R}, \\ +\infty & \text{if } k \leq j, \ \rho \leq -1, \\ \frac{j!^2}{(j-k)!(\rho+1)_{j-k+1}} & \text{if } k \leq j, \ \rho > -1. \end{cases}$$

Proof. For $k > j$, the integrand in (55) vanishes identically. For $k \leq j$, we have $\Delta^k |z|^{2j} = \frac{j!^2}{(j-k)!^2} |z|^{2j-2k}$, and

$$\int_{\mathbf{D}} |z|^{2j-2k} \, d\mu_\rho(z) = \int_0^1 t^{j-k} (1-t)^\rho \, dt = \frac{(j-k)! \Gamma(\rho+1)}{\Gamma(\rho+j-k+2)}$$

for $\rho > -1$, while for $\rho \leq -1$ the integral is infinite. The assertion follows. □

Corollary 40. *We have $Q_{k,\rho} \subset \mathcal{P}_{\leq k}$ trivially if $\rho \leq -1$, while*

$$Q_{k,\rho} = Q_{\mathbf{s}[k, 2k-\rho-1]} \qquad \text{for } \rho > -1. \tag{56}$$

Proof. Immediate from the last lemma, since for $j \geq k$

$$\frac{j!^2}{(j-k)!(\rho+1)_{j-k+1}} \asymp (j+1)^{2k-\rho-1}$$

by Stirling's formula. □

By the last corollary and Proposition 16(ii), we have

$$Q_{k,\rho} \hookrightarrow \mathcal{E}_k \qquad \forall \rho \in \mathbf{R} \tag{57}$$

(and trivially if $\rho \leq -1$).

Proposition 41. *For $\rho > 2k - 1 + \lambda$, $\mathcal{E}_k \hookrightarrow Q_{k,\rho}$.*

Proof. Let $f \in \mathcal{E}_k$. By Theorem 24, for $(\lambda)_k \neq 0$ we have $\mathcal{E}_j = L^{\infty, \lambda/2}_{\mathrm{hol}}(\mathbf{D})$ for $j = 0, 1, \dots, k$. Since, by Proposition 19 and (27),

$$f \in \mathcal{E}_m \iff \sum_{j=0}^m (1-|z|^2)^{\frac{\lambda}{2}+j} f^{(j)}(z) (-\overline{z})^{k-j} (\lambda+j)_{k-j} \binom{k}{j} \in L^\infty(\mathbf{D}), \tag{58}$$

it follows by an induction argument (starting from $j = 0$) that in fact

$$(1-|z|^2)^{\frac{\lambda}{2}+j} f^{(j)}(z) \in L^\infty(\mathbf{D}) \qquad \text{for all } 0 \leq j \leq k. \tag{59}$$

Now for $a \in \mathbf{D}$, by the Minkowski inequality,

$$\Big(\int_{\mathbf{D}} |(U_a f)^{(k)}|^2 \, d\mu_\rho \Big)^{1/2}$$

$$= \Big(\int_{\mathbf D}\Big|\sum_{j=0}^k (1-|a|^2)^{\frac\lambda2+j}\frac{f^{(j)}(\phi_a(z))(-\overline a)^{k-j}}{(1-\overline az)^{j+k+\lambda}}(\lambda+j)_{k-j}\binom kj\Big|^2\,d\mu_\rho\Big)^{1/2}$$

$$\le \sum_{j=0}^k (1-|a|^2)^{\frac\lambda2+j}|\overline a^{k-j}(\lambda+j)_{k-j}|\binom kj\Big(\int_{\mathbf D}\Big|\frac{f^{(j)}(\phi_a(z))}{(1-\overline az)^{j+k+\lambda}}\Big|^2\,d\mu_\rho\Big)^{1/2}.$$

By (59),

$$\Big|\frac{f^{(j)}(\phi_a(z))}{(1-\overline az)^{j+k+\lambda}}\Big| \le c\frac{(1-|\phi_a(z)|^2)^{-\frac\lambda2-j}}{|1-\overline az|^{j+k+\lambda}} \le c\frac{(1-|z|^2)^{-\frac\lambda2-j}(1-|a|^2)^{-\frac\lambda2-j}}{|1-\overline az|^{k-j}},$$

so the last integral is majorized by

$$(1-|a|^2)^{-\lambda-2j}\int_{\mathbf D}\frac{(1-|z|^2)^{\rho-\lambda-2j}}{|1-\overline az|^{2k-2j}}\,dz. \tag{60}$$

For $\rho > 2k-1+\lambda$, one has $\rho-\lambda-2j \ge \rho-\lambda-2k > -1$, so the integral in (60) exists and, by the classical Forelli-Rudin estimates [Zhu2, Lemma 3.10], is bounded as $a\in\mathbf D$. Hence (60) is majorized by $(1-|a|^2)^{-\lambda-2j}$, and

$$\Big(\int_{\mathbf D}|(U_af)^{(k)}|^2\,d\mu_\rho\Big)^{1/2} \lesssim 1,$$

i.e. the left-hand side is bounded on $\mathbf D$. Thus $f\in Q_{k,\rho}$ and $\mathcal E_k \hookrightarrow Q_{k,\rho}$.

For $(\lambda)_k = 0$, the sum in (58) involves only $1-\lambda\le j\le k$ (the other terms vanish since $(\lambda_j)_{k-j}=0$), while, by Theorem 24 again, $\mathcal E_j = \mathcal E_{1-\lambda}$ for $1-\lambda\le j\le k$. Since (by (58) one more time) $f\in\mathcal E_{1-\lambda} \iff (1-|z|^2)^{1-\lambda/2}f^{(1-\lambda)}(z)\in L^\infty(\mathbf D)$, it again follows by a simple induction argument (starting from $j=1-\lambda$) that

$$(1-|z|^2)^{\frac\lambda2+j}f^{(j)}(z)\in L^\infty(\mathbf D) \qquad \text{for all } 1-\lambda\le j\le k.$$

Using this in the place of (59) the argument above (with the sums now extending only over $1-\lambda\le j\le k$) works without change, with the same conclusion. □

Corollary 42. *For $\rho > 2k-1+\lambda$, $Q_{k,\rho}=\mathcal E_k$, with equivalent seminorms.*

Proof. Immediate from (57) and the last proposition. □

In combination with Proposition 38, we thus see that the interesting range for $Q_{k,\rho}$ spaces is

$$2k+\lambda-2\le\rho\le 2k+\lambda-1. \tag{61}$$

Note that if $(\lambda)_k\ne0$ and $\lambda<0$, then $\mathcal E_k=\{0\}$ by Theorem 24, hence by (57) $Q_{k,\rho}=\{0\}$ too in that case, for all ρ. Below, we are able to handle the range (61) when either $(\lambda)_k=0$, or $(\lambda)_k\ne0$ and $\rho>2k-1$; the case of (61) for $(\lambda)_k\ne0$, $\lambda\ge0$ and $2k+\lambda-2\le\rho\le2k-1$ (this can only happen for $0\le\lambda\le1$) thus, unfortunately, remains open.

Proposition 43. *For $(\lambda)_k=0$ and $\rho-2k-\lambda\in[-2,-1]$, the spaces $Q_{k,\rho}$ are nontrivial and strictly increasing with ρ, i.e. $Q_{k,\rho}\subsetneq Q_{k,\rho'}$ if $\rho<\rho'$.*

Proof. Set $\rho=2k-2+\beta$, with $\lambda\le\beta\le\lambda+1$. Note that $2k+\lambda-2\ge2(1-\lambda)+\lambda-2=-\lambda\ge0$, since $-\lambda\in\mathbf N$ and $k\ge1-\lambda$ by hypothesis. In particular, $\rho>-1$, so by (56)

$$Q_{k,\rho}=Q_{\mathbf s[k,1-\beta]}.$$

On the other hand, by Proposition 27 and (46), we see that

$$\widetilde{Q}_{k,\beta} = Q_{\mathbf{s}[1-\lambda,1-\beta]} = Q_{\mathbf{s}[k,1-\beta]}.$$

(Note that $\rho \geq -\lambda \geq \lambda > 2\lambda - 1$ since $\lambda \leq 0$.) Thus $Q_{k,\rho} = \widetilde{Q}_{k,\beta}$. The desired conclusion therefore follows by Theorem 35. ☐

Proposition 44. *For* $(\lambda)_k \neq 0$ *and* $\rho \in (2k-1,\infty) \cap [2k+\lambda-2, 2k+\lambda-1]$, *the spaces* $Q_{k,\rho}$ *are nontrivial and strictly increasing with* ρ, *i.e.* $Q_{k,\rho} \subsetneq Q_{k,\rho'}$ *if* $\rho < \rho'$.

Proof. By Theorem 24, for $(\lambda)_k \neq 0$ we have $\mathcal{E}_0 = \mathcal{E}_1 = \cdots = \mathcal{E}_k$. Applying Corollary 18 repeatedly, we thus see that $Q_{\mathbf{s}[k,\nu]} = Q_{\mathbf{s}[0,\nu]}$ $\forall \nu \in \mathbf{R}$. Hence by (56), as $\rho > -1$ by hypothesis,

$$Q_{k,\rho} = Q_{\mathbf{s}[0,2k-\rho-1]}.$$

On the other hand, setting again $\rho = 2k - 2 + \beta$ with $\lambda \leq \beta \leq \lambda + 1$, from Proposition 27 we have

$$\widetilde{Q}_{k,\beta} = Q_{\mathbf{s}[0,1-\beta]}$$

provided that $\beta > 1$, i.e. $\rho > 2k-1$. Thus under the hypothesis of our proposition, we have $Q_{k,\rho} = \widetilde{Q}_{k,\beta}$, with $\beta = \rho + 2 - 2k$ satisfying $\beta \in [\lambda, \lambda+1]$. The conclusion then follows by Theorem 34. ☐

Again, after all the preparations above, we are ready to prove Theorem 9 from the Introduction.

Proof of Theorem 9. (i) As already noted, this is immediate from Theorem 24 and (57).

(ii) This is Proposition 41.

(iii) and (iv) are just Proposition 38.

(v) This is Proposition 43.

(vi) This is Proposition 44.

(vii) This is Corollary 40, while (23) was established in course of the proof of Propositions 43 and 44. ☐

5. Concluding remarks

5.1. **L^p-variants.** As in [Zhu1], one could consider also the "L^p-variants" of our "L^2" objects investigated so far; that is, we might define the spaces $\widetilde{Q}_{pk\rho}$, $Q_{pk\rho}$, etc., as consisting of all functions $f \in \mathcal{H}(\mathbf{D})$ for which the quantities

$$\sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} (\widetilde{\Delta^k}|U_\phi f|^2)^{p/2}\, d\mu_{\rho-2},$$

$$\sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} (\Delta^k|U_\phi f|^2)^{p/2}\, d\mu_{\rho},$$

or perhaps even

$$\sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \widetilde{\Delta^k}|U_\phi f|^p\, d\mu_{\rho-2},$$

$$\sup_{\phi\in\mathrm{Aut}(\mathbf{D})} \int_{\mathbf{D}} \Delta^k|U_\phi f|^p\, d\mu_{\rho}$$

are finite. We have not tried to see whether our methods could be extended to any of these situations.

5.2. **Heuristics.** The construction of $\widetilde{Q}_{k,\rho}$, $Q_{k,\rho}$ with $k \geq 2$ of course draws upon the established principle of function theory on the disc, which says that "a holomorphic function $g(z)$ can in many respects be replaced by $(1-|z|^2)g'(z)$, only the latter is better"; an excellent place to see this approach in action for Bergman spaces is the book by Zhao and Zhu [ZhaZh]. From this perspective, our spaces $Q_{k,\rho}$ should "essentially be equal to $Q_{k+1,\rho+2}$, only the latter is better". In particular, $Q_{k,\rho}$ should depend only on the difference $\rho - 2k$, as soon as k is sufficiently large. Our results above demonstrate that this is indeed the case. Similarly for $\widetilde{Q}_{k,\rho}$, and for the fact that $\widetilde{Q}_{k,\rho}$ coincide with $Q_{k,\rho+2k-2}$ in the cases which are of greatest interest.

5.3. **Inclusions.** We have not examined possible inclusions among the various spaces $Q_{k,\rho}$ and $\widetilde{Q}_{k,\rho}$ with different values of the weight parameter λ. It was shown in Proposition 4.8 in [Zhao] that if X is a functional Banach space on $\mathbf{D}$ which is invariant under $U^{(\lambda)}$ with $\lambda > \alpha > 0$, then X cannot be invariant under $U^{(\alpha)}$. We expect similar result could be obtained (in the same way) also for our Q-spaces here, and without the restriction of positivity of λ and α.

5.4. **Bounded symmetric domains.** Most of the results in Section 3 likely extend also to the case when the disc $\mathbf{D}$ is replaced by any irreducible bounded symmetric domain $\Omega \subset \mathbf{C}^n$, $n \geq 1$. Referring the reader to [AE] for the various prerequisites on bounded symmetric domains and for the notation, let, for a signature $\mathbf{m}$, $K_{\mathbf{m}}$ denote the reproducing kernel of the space $\mathcal{P}_{\mathbf{m}}$ of polynomials in the Peter-Weyl decomposition, and let $\mathbf{\Delta}^{(\lambda)}_{\mathbf{m}} = \mathbf{\Delta}_{\mathbf{m}}$ be the differential operator which coincides with $K_{\mathbf{m}}(\partial, \partial)$ at the origin and is invariant under the weighted action

$$V^{(\lambda)}_\phi : f \mapsto (f \circ \phi) \cdot |\operatorname{Jac}_\phi|^{2\lambda/p}, \qquad \phi \in \operatorname{Aut}(\Omega),$$

where $\lambda \in \mathbf{R}$ and p is the genus of Ω. With

$$U^{(\lambda)}_\phi : f \mapsto (f \circ \phi) \cdot \operatorname{Jac}_\phi^{\lambda/p}$$

the associated action on holomorphic functions (so that $V^{(\lambda)}_\phi |f|^2 = |U^{(\lambda)}_\phi f|^2$ for $f \in \mathcal{H}(\Omega)$), we define

$$\mathcal{E}^{(\lambda)}_{\mathbf{m}} = \{f \in \mathcal{H}(\Omega) : \sup_\phi K_{\mathbf{m}}(\partial, \partial) |U^{(\lambda)}_\phi f|^2(0) < \infty\},$$

$$\widetilde{Q}_{\mathbf{m},\rho} = \{f \in \mathcal{H}(\Omega) : \sup_\phi \int_\Omega \mathbf{\Delta}_{\mathbf{m}} |U_\phi f|^2 \, d\mu_{\rho-p} < \infty\},$$

$$Q_{\mathbf{m},\rho} = \{f \in \mathcal{H}(\Omega) : \sup_\phi \int_\Omega K_{\mathbf{m}}(\partial, \partial) |U_\phi f|^2 \, d\mu_\rho < \infty\},$$

where $\mu_\rho(z) = h(z, z)^\rho \, dz$ where h is the Jordan triple determinant. Also, for a sequence $\mathbf{s} = \{s_{\mathbf{m}}\}$ of numbers $s_{\mathbf{m}} \in [0, +\infty]$ indexed by the signatures $\mathbf{m}$, let

$$Q_{\mathbf{s}} = \{f \in \mathcal{H}(\Omega) : \sup_\phi \sum_{\mathbf{m}} s_{\mathbf{m}} \|(U_\phi f)_{\mathbf{m}}\|^2_F < \infty\}$$

where $g_{\mathbf{m}}$ denotes the component in the Peter-Weyl decomposition $g = \sum_{\mathbf{m}} g_{\mathbf{m}}$ of a holomorphic function g, and $\| \cdot \|_F$ stands for the Fischer-Fock norm. Using the methods of [AE], it should be possible to extend our results in Sections 3–4 also to the spaces above. The author hopes to possibly return to this topis in a future work.

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