THE M-HARMONIC DIRICHLET SPACE ON THE BALL

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Abstract. We describe the Dirichlet space of M-harmonic functions, i.e. functions annihilated by the invariant Laplacian on the unit ball of the complex n-space, as the limit of the analytic continuation (in the spirit of Rossi and Vergne) of the corresponding weighted Bergman spaces. Characterizations in terms of tangential derivatives are given, and the associated inner product is shown to be Moebius invariant. The pluriharmonic and harmonic cases are also briefly treated.

1. Introduction

Let $\mathbf{B}^n$ be the unit ball in the complex n-space $\mathbf{C}^n$, $n \geq 1$, and consider the standard weighted Bergman spaces

$$\mathcal{A}_s(\mathbf{B}^n) := \{f \in L^2(\mathbf{B}^n, d\mu_s) : \ f \text{ is holomorphic on } \mathbf{B}^n\}$$

of holomorphic functions on $\mathbf{B}^n$ square-integrable with respect to the measure

$$d\mu_s(z) := \frac{\Gamma(s+n+1)}{\pi^n \Gamma(s+1)} (1-|z|^2)^s \, dz, \qquad s > -1, \tag{1}$$

where dz denotes the Lebesgue volume on $\mathbf{C}^n$. The restriction on s ensures that these spaces are nontrivial, and the factor $\frac{\Gamma(s+n+1)}{\pi^n\Gamma(s+1)}$ makes $d\mu_s$ a probability measure, so that the function $\mathbf{1}$ (constant one) has unit norm. It is well known that $\mathcal{A}_s$ is a reproducing kernel Hilbert space, with reproducing kernel

$$K_s^{\mathrm{hol}}(x,y) = (1-\langle x,y\rangle)^{-s-n-1}. \tag{2}$$

In terms of the Taylor coefficients, a holomorphic function $f(z) = \sum_\nu f_\nu z^\nu$ on $\mathbf{B}^n$ belongs to $\mathcal{A}_s$ if and only if

$$\|f\|_s^2 := \sum_\nu |f_\nu|^2 \frac{\nu!\Gamma(n+s+1)}{\Gamma(n+s+|\nu|+1)} < +\infty \tag{3}$$

and the sum then coincides with the squared norm in $\mathcal{A}_s$. (Here the sum runs over all multi-indices ν, and we are employing the usual multi-index notations.) Alternatively, in terms of the homogeneous components $f_m(z) := \sum_{|\nu|=m} f_\nu z^\nu$,

$$\|f\|_s^2 = \sum_{m=0}^\infty \frac{\Gamma(n+s+1)\Gamma(n+m)}{\Gamma(n)\Gamma(n+s+m+1)} \|f_m\|^2_{\partial\mathbf{B}^n} = \sum_{m=0}^\infty \frac{(n)_m}{(n+s+1)_m} \|f_m\|^2_{\partial\mathbf{B}^n}, \tag{4}$$

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where $\|f\|_{\partial\mathbf{B}^n}$ denotes the norm in the space $L^2(\partial\mathbf{B}^n, d\sigma)$ with respect to the normalized surface measure $d\sigma$ on $\partial\mathbf{B}^n$. Here $(x)_m := x(x+1)\dots(x+m-1)$ denotes the usual Pochhammer symbol (rising factorial).

It is a remarkable fact — which prevails in the much more general context of bounded symmetric domains, constituting the "analytic continuation" of the principal series representations of certain semisimple Lie groups, cf. Rossi and Vergne [VR] — that the weighted Bergman kernels $K_s^{\rm hol}(x,y)$, $s > -1$, continue to be positive definite kernels in the sense of Aronszajn [Ar] for all $s \ge -n-1$, yielding thus an "analytic continuation" of the spaces $\mathcal{A}_s$. (One calls the interval $[-n-1, +\infty)$ the *Wallach set* of $\mathbf{B}^n$.) For $s > -n-1$, the norm in $\mathcal{A}_s$ is still given by (3) and (4). For $s = -n-1$, the kernel (2) becomes constant one, and the corresponding reproducing kernel Hilbert space thus reduces just to the constants. However, a much more interesting space arises as the "residue" of $\mathcal{A}_s$ at $s = -n-1$: namely, the limit

$$\lim_{s\searrow -n-1} \frac{K_s^{\rm hol}(x,y)-1}{s+n+1} = \log\frac{1}{1-\langle x,y\rangle} =: K_\circ^{\rm hol}(x,y) \tag{5}$$

is a positive definite kernel on $\mathbf{B}^n \times \mathbf{B}^n$, and the associated reproducing kernel Hilbert space — denoted $\mathcal{A}_\circ$ — consists of all f holomorphic on $\mathbf{B}^n$ for which

$$\begin{aligned} \|f\|_\circ^2 &:= \sum_m \lim_{s\searrow -n-1} \frac{(n)_m}{(n+s+1)_m} \|f_m\|^2_{\partial\mathbf{B}^n} \\ &= \sum_m m\frac{(n)_m}{m!}\|f_m\|^2_{\partial\mathbf{B}^n} = \sum_\nu |\nu|\frac{\nu!}{|\nu|!}|f_\nu|^2 < +\infty, \end{aligned} \tag{6}$$

and $\|f\|_\circ$ gives the semi-norm on $\mathcal{A}_\circ$. This space is nothing else but the familiar *Dirichlet space* on $\mathbf{B}^n$, see e.g. Chapter 6.4 in Zhu [Zh], where it is furthermore shown that the space $\mathcal{A}_\circ$ and the above semi-inner product are *Moebius invariant*, in the sense that $f \in \mathcal{A}_\circ \implies f\circ\phi \in \mathcal{A}_\circ$ and

$$\langle f, g\rangle_\circ = \langle f\circ\phi, g\circ\phi\rangle_\circ$$

for any biholomorphic self-map ϕ of $\mathbf{B}^n$.

The goal of the present paper is to exhibit the M-harmonic analogue of the construction above.

Recall that a function on $\mathbf{B}^n$ is called *Moebius-harmonic* (or *invariantly harmonic*), or *M-harmonic* for short, if it is annihilated by the invariant Laplacian

$$\widetilde{\Delta} = 4(1-|z|^2)\sum_{j,k=1}^n (\delta_{jk} - z_j\overline{z}_k)\frac{\partial^2}{\partial z_j \partial\overline{z}_k}. \tag{7}$$

It is standard (see e.g. Rudin [Ru], Stoll [St], or Chapter 6 in Krantz [Kr]) that $\widetilde{\Delta}$ commutes with biholomorphic self-maps (Moebius maps) of the ball:

$$\widetilde{\Delta}(f\circ\phi) = (\widetilde{\Delta}f)\circ\phi, \qquad \forall f \in C^2(\mathbf{B}^n), \phi \in \operatorname{Aut}(\mathbf{B}^n);$$

and, accordingly, f is M-harmonic if and only if $f\circ\phi$ is, for any $\phi \in \operatorname{Aut}(\mathbf{B}^n)$. The class of M-harmonic functions lies in a way on the crossroads between the holomorphic and the harmonic functions: it resembles the latter in the sense that it is preserved by complex conjugation, while resembling the former by reflecting the complex structure inherent in the invariance of the Laplacian $\widetilde{\Delta}$. One again has the *M-harmonic weighted Bergman spaces*

$$\mathcal{M}_s := \{f \in L^2(\mathbf{B}^n, d\mu_s) : \ f \text{ is } M\text{-harmonic on } \mathbf{B}^n\}$$

which are nontrivial if and only if $s > -1$. The role of the Taylor coefficients or, rather, homogeneous components from the holomorphic case is now played by the decomposition into "bi-graded spherical harmonics". Namely, under the action of the group $U(n)$ of unitary linear maps of $\mathbf{C}^n$, the space $L^2(\partial\mathbf{B}^n, d\sigma)$ decomposes into irreducible components

$$L^2(\partial\mathbf{B}^n, d\sigma) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}^{pq}, \tag{8}$$

where $\mathcal{H}^{pq}$ is the space of restrictions to the sphere of harmonic polynomials on $\mathbf{C}^n$ homogeneous of degree p in z and of degree q in $\overline{z}$. Performing such a decompisition on each sphere $|z| \equiv$ const. leads to the analogous Peter-Weyl decomposition

$$\mathcal{M}_s = \bigoplus_{p,q} \mathbf{H}^{pq},$$

where the space $\mathbf{H}^{pq}$ of "solid harmonics"

$$\mathbf{H}^{pq} = \{f \in C(\overline{\mathbf{B}^n}) : \ f \text{ is } M\text{-harmonic on } \mathbf{B}^n \text{ and } f|_{\partial\mathbf{B}^n} \in \mathcal{H}^{pq}\},$$

and the norm of $f = \sum_{p,q} f_{pq}$, $f_{pq} \in \mathbf{H}^{pq}$, is given by

$$\|f\|_s^2 = \sum_{p,q=0}^{\infty} C_{pq}(s) \|f_{pq}\|^2_{\partial\mathbf{B}^n}, \tag{9}$$

with the coefficients $C_{pq}(s)$ — the counterparts of the $\frac{(n)_m}{(n+s+1)_m}$ from the holomorphic case — given by an explicit formula involving hypergeometric functions; see Section 2 below for the details. Finally, the reproducing kernel of the space $\mathcal{M}_s$, $s > -1$, is given by

$$K_s(z, w) = \sum_{p,q} \frac{K_{pq}(z, w)}{C_{pq}(s)}, \tag{10}$$

where $K_{pq}(zw)$ is the reproducing kernel of $\mathbf{H}^{pq}$ (with the inner product inherited from $L^2(\partial\mathbf{B}^n, d\sigma)$), for which there is again an explicit formula. Now it has been shown in Section 6.3 of [EY] that, exactly as in the holomorphic case, $K_s(z, w)$ extends to a holomorphic function of s on $\operatorname{Re} s > -n-1$, continues to be a positive definite kernel on $\mathbf{B}^n \times \mathbf{B}^n$ for all $s \geq -n-1$ (and only for these s), and the norm in the corresponding reproducing kernel Hilbert space — the "analytic continuation" of $\mathcal{M}_s$ — for $s > -n-1$ is still given by (9). (Thus the "M-harmonic Wallach set' of $\mathbf{B}^n$ is again the interval $[-n-1, +\infty)$.)

Motivated by the considerations for the holomorphic case, we are now interested in the limit as $s \searrow -n-1$. In contrast to the holomorphic case, this can now be done in *three* ways.

(a) We simply take $s = -n-1$ in (10). The kernel $K_s(z, w)$ reduces to constant one, and the corresponding reproducing kernel Hilbert space thus again reduces just to the constants, with $\|\mathbf{1}\| = 1$. This is the trivial case.

(b) As in the holomorphic case, next we take

$$\lim_{s\searrow -n-1} \frac{K_s(z,w)-1}{n+s+1} = \log\frac{1}{|1-\langle x,y\rangle|^2} =: K_\circ(x,y). \tag{11}$$

This is a positive definite kernel, and the corresponding reproducing kernel Hilbert space — denoted $\mathcal{M}_\circ$ — consists precisely of the orthogonal sum of the holomorphic Dirichlet space $\mathcal{A}_\circ$ above and its complex conjugate $\overline{\mathcal{A}_\circ}$ (the constants being counted, of course, only once), with the (semi-)norm given by

$$\|f+\overline{g}\|_\circ^2 := \|f\|_\circ^2 + \|g\|_\circ^2. \tag{12}$$

In some sense, one can perhaps view $\mathcal{M}_\circ$ as the *pluriharmonic Dirichlet space.*

(c) Finally, we can take

$$\lim_{s\searrow -n-1} \frac{K_s(x,y)-1-(n+s+1)K_\circ(x,y)}{(n+s+1)^2} =: K_{\circ\circ}(x,y), \tag{13}$$

which is a positive-definite kernel on $\mathbf{B}^n\times\mathbf{B}^n$, with the (semi-)norm in the corresponding reproducing kernel Hilbert space — denoted $\mathcal{M}_{\circ\circ}$ — given by

$$\|f\|_{\circ\circ}^2 := \sum_{p,q} \lim_{s\searrow -n-1} (n+s+1)^2 C_{pq}(s)\|f_{pq}\|^2_{\partial\mathbf{B}^n} \tag{14}$$

(hence, this time, all pluriharmonic functions get zero norm). This is, by definition, the *M-harmonic Dirichlet space.*

The occurrence of case (c) arises from the fact that the coefficient functions $C_{pq}(s)$ now turn out to have a double pole at $s=-n-1$ (in contrast to the single pole of $\frac{(n)_m}{(n+s+1)_m}$ in the holomorphic situation). Again, this phenomenon is clearly reminiscent of the "composition series" arising in the theory of analytic continuation of holomorphic discrete series representations mentioned above, cf. Faraut and Koranyi [FK]. Note also that the case (c) disappears completely when $n=1$; thus $\mathcal{M}_{\circ\circ}$ is relevant only for $\mathbf{B}^n$ with $n\geq 2$.

Our first result is a direct formula for the semi-norm in $\mathcal{M}_{\circ\circ}$.

Corollary. (Corollary 5) *In terms of the Peter-Weyl decomposition $f=\sum_{p,q} f_{pq}$, $f_{pq}\in\mathbf{H}^{pq}$,*

$$\|f\|_{\circ\circ}^2 = \sum_{p,q} \frac{(p)_n(q)_n}{\Gamma(n)^2}\|f_{pq}\|^2_{\partial\mathbf{B}^n}.$$

By polarization, this of course implies also the corresponding formula

$$\langle f,g\rangle_{\circ\circ} = \sum_{p,q} \frac{(p)_n(q)_n}{\Gamma(n)^2}\langle f_{pq},g_{pq}\rangle_{\partial\mathbf{B}^n} \tag{15}$$

for the semi-inner product in $\mathcal{M}_{\circ\circ}$.

Next we give a description of $\mathcal{M}_{\circ\circ}$ that does not involve the Peter-Weyl components. It is easy to see that the averaging operator

$$\Pi_0 f(r\zeta) := \int_{\partial\mathbf{B}^n} f(r\eta)\,d\sigma(\eta), \qquad 0\leq r<1, \zeta\in\partial\mathbf{B}^n,$$

(this can also be written as $\Pi_0 f(z) = \int_{U(n)} f(kz)\,dk$, where dk stands for the normalized Haar measure on the compact group $U(n)$) is just the projection

$$f \longmapsto f(0)\mathbf{1}$$

of M-harmonic functions onto the subspace $\mathbf{H}^{00}$ of constants. Similarly, one can give projections Π and $\overline{\Pi}$ onto the subspaces $\bigoplus_p \mathbf{H}^{p0}$ and $\bigoplus_q \mathbf{H}^{0q}$ of the holomorphic and anti-holomorphic functions, respectively (explicit formulas for Π and $\overline{\Pi}$ will be given in Section 2 below). Hence, we also have

$$P := \Pi + \overline{\Pi} - \Pi_0,$$

the projection onto the subspace of pluriharmonic functions, and

$$Q := I - P,$$

the projection onto their orthogonal complement $\bigoplus_{p,q\geq 1} \mathbf{H}^{pq}$.

(All these projections are automatically orthogonal with respect to any $U(n)$-invariant inner product, but make sense also in complete generality on the vector space $\mathcal{M}$ of all M-harmonic functions on $\mathbf{B}^n$.)

Consider now the tangential vector fields

$$L_{jk} := \overline{z}_j \partial_k - \overline{z}_k \partial_j, \qquad \overline{L}_{jk} := z_j \overline{\partial}_k - z_k \overline{\partial}_j, \qquad 1 \leq j,k \leq n, \quad j \neq k,$$

and denote by $\mathcal{L}_m$, $1 \leq m \leq 2n(n-1)$, the collection of all these operators (in some fixed order). Finally, for any function f on $\mathbf{B}^n$ and $0 < r < 1$, let f_r be the function on $\partial\mathbf{B}^n$ defined by

$$f_r(\zeta) := f(r\zeta)$$

and denote

$$\|f\|^2_{\mathrm{Hardy}} := \sup_{0<r<1} \|f_r\|^2_{\partial\mathbf{B}^n}.$$

Theorem. (Theorem 9) *If f is M-harmonic on $\mathbf{B}^n$, $n \geq 2$, then $f \in \mathcal{M}_{\circ\circ}$ if and only if*

$$\sum_{j_1,j_2,\dots,j_n=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_n}(Qf)\|^2_{\mathrm{Hardy}} < +\infty,$$

and the square root of the left-hand side is a seminorm equivalent to $\|f\|_{\circ\circ}$.

As a consequence, we obtain the following complete analogues of the holomorphic case.

Corollary. (Corollary 10) *The space $\mathcal{M}_{\circ\circ}$ is Moebius invariant: $f \in \mathcal{M}_{\circ\circ}$ implies $f \circ \phi \in \mathcal{M}_{\circ\circ}$ for any $\phi \in \mathrm{Aut}(\mathbf{B}^n)$.*

Theorem. (Theorem 15) *The semi-inner product* (15) *is Moebius invariant:*

$$\langle f, g\rangle_{\circ\circ} = \langle f\circ\phi, g\circ\phi\rangle_{\circ\circ} \qquad \forall f,g \in \mathcal{M}_{\circ\circ}, \forall\phi \in \mathrm{Aut}(\mathbf{B}^n).$$

The usual proof of the analogue of the last corollary for the holomorphic case (cf. Zhu [Zh], Theorem 6.13) relies on the use of radial derivatives and their generalizations; this approach unfortunately breaks down in the M-harmonic situation. Similarly, the usual proof of the last theorem in the holomorphic case relies on explicit computations involving Taylor coefficients (cf. Zhu [Zh], Theorem 6.15), which becomes hopeless for $\mathcal{M}_{\circ\circ}$; we instead use an argument employing analytic continuation. In both cases, our approach here can be used to give a new description of the classical Dirichlet space on the ball and a new proof of the invariance of the Dirichlet inner product in the holomorphic case.

Additionally, the methods just mentioned apply also to the pluriharmonic Dirichlet space $\mathcal{M}_\circ$, which seems to have received basically no attention at all in the literature. The second result below apparently has no counterpart in the M-harmonic case.

Theorem. (Theorem 16) *If f is pluriharmonic on $\mathbf{B}^n$, $n \ge 2$, then $f \in \mathcal{M}_\circ$ if and only if*

$$\sum_{j_1,j_2,\dots,j_n=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_n} f\|^2_{\text{Hardy}} < +\infty$$

if and only if

$$\sum_{j_1,j_2,\dots,j_{n+k+1}=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_{n+k+1}} f\|^2_k < +\infty,$$

for some (equivalently, any) nonnegative integer k.

Furthermore, the square root of the above quantities is a seminorm equivalent to $\|f\|_\circ$.

Here $\|\cdot\|_s$ denotes, more generally, the norm in $L^2(\mathbf{B}^n, d\mu_s)$, $s > -1$ (i.e. not only on its holomorphic or M-harmonic subspaces).

Let

$$\mathcal{N} := \sum_{j=1}^n z_j\partial_j + \overline{z}_j\overline{\partial}_j$$

denote the radial derivative operator on $\mathbf{B}^n$; note than $\mathcal{N}f$ is pluriharmonic whenever f is.

Theorem. (Theorem 17) *If f is pluriharmonic on $\mathbf{B}^n$, $n \ge 1$, then $f \in \mathcal{M}_\circ$ if and only if*

$$\|\mathcal{N}^m f\|^2_{2m-n-1} < +\infty$$

for some (equivalently, any) integer $m > \frac{n}{2}$. Furthermore, the square root of the left-hand side is a seminorm equivalent to $\|f\|_\circ$.

For f holomorphic the last two theorems, of course, give criteria for f to belong to the ordinary Dirichlet space $\mathcal{A}_\circ$ on $\mathbf{B}^n$; the second one is then common knowledge, and though the first of them must surely also be known to experts in the field, the authors were unable to pinpoint a specific reference to the literature.

Finally, for the sake of completeness, we give details also for the context of *harmonic* functions, where the corresponding *harmonic Dirichlet space* appears in the literature under various names. Namely, consider this time the unit ball B^n of $\mathbf{R}^n$, $n \ge 2$, and for any $s > -1$ let

$$\mathcal{H}_s(B^n) := \{f \in L^2(B^n, d\rho_s) : \ f \text{ is harmonic on } B^n\}$$

be the *weighted harmonic Bergman space* of all harmonic functions on B^n square-integrable with respect to the measure

$$d\rho_s(x) := \frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)}(1-|x|^2)^s\,dx,$$

where dx denotes the Lebesgue volume on $\mathbf{R}^n$. The restriction on s ensures that these spaces are nontrivial, and the factor $\frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)}$ makes $d\rho_s$ a probability measure, so that $\|\mathbf{1}\| = 1$. For each $p \geq 0$, denote by $\mathbf{H}^p$ the space of harmonic polynomials on $\mathbf{R}^n$ homogeneous of degree p. Any harmonic function f on B^n then admits a (unique) decomposition

$$f = \sum_{p=0}^{\infty} f_p, \qquad f_p \in \mathbf{H}^p, \tag{16}$$

and the space $\mathcal{H}_s$ decomposes as

$$\mathcal{H}_s = \bigoplus_{p=0}^{\infty} \mathbf{H}^p, \tag{17}$$

with the norm given by

$$\|f\|_s^2 = \sum_{p=0}^{\infty} \frac{(\frac{n}{2})_p}{(\frac{n}{2}+s+1)_p} \|f_p\|_{\partial B^n}, \tag{18}$$

where $\|\cdot\|_{\partial B^n}$ denotes the norm in $L^2(\partial B^n, d\sigma)$ with respect to the normalized surface measure $d\sigma$ on the unit sphere ∂B^n. The reproducing kernel of $\mathcal{H}_s$ is given by

$$K_s^{\text{harm}}(x,y) = \sum_p \frac{(\frac{n}{2}+s+1)_p}{(\frac{n}{2})_p} Z_p(x,y), \tag{19}$$

where the *zonal harmonic* $Z_p(x,y)$ is the reproducing kernel of $\mathbf{H}^p$ (with respect to the norm $\|\cdot\|_{\partial B^n}$). It follows that $K_s^{\text{harm}}(x,y)$ extends as a holomorphic function of s to the entire complex plane, and continues to be a positive definite kernel on $B^n \times B^n$ for any $s \geq -\frac{n}{2}-1$ (and only for these s). For $s > -\frac{n}{2}-1$, the norm in the corresponding reproducing kernel Hilbert spaces (still denoted by $\mathcal{H}_s$) is still given by the formula (18). For $s = -\frac{n}{2}-1$, (19) again reduces just to constant one, and the corresponding reproducing kernel Hilbert space thus reduces to the constants; while the limit

$$\lim_{s\searrow -\frac{n}{2}-1} \frac{K_s^{\text{harm}}(x,y)-1}{s+\frac{n}{2}+1} =: K_\square^{\text{harm}}(x,y)$$

is a positive definite kernel on $B^n \times B^n$, corresponding to the reproducing kernel Hilbert space with the (semi)norm given by

$$\|f\|_\square^2 := \sum_p \lim_{s\searrow -\frac{n}{2}-1} (s+\tfrac{n}{2}+1) \frac{(\frac{n}{2})_p}{(\frac{n}{2}+s+1)_p} \|f_p\|_{\partial B^n} = \sum_p p \frac{(\frac{n}{2})_p}{p!} \|f_p\|_{\partial B^n}. \tag{20}$$

This space — denoted $\mathcal{H}_\square$ — is the *harmonic Dirichlet space* on B^n.

Let X_{jk}, $j,k = 1,\dots,n$, $j \neq k$, denote the tangential vector fields

$$X_{jk} = x_j\partial_k - x_k\partial_j$$

on $\mathbf{R}^n$, and denote by $\mathcal{X}_m$, $1 \leq m \leq n(n-1)$, the collection of all these operators (in some fixed order).

Theorem. (Theorem 19) *If f is harmonic on B^n, $n \geq 2$, then $f \in \mathcal{H}_\square$ if and only if*

$$\sum_{j_1,\dots,j_m=1}^{n(n-1)} \|\mathcal{X}_{j_1} \dots \mathcal{X}_{j_m} f\|^2_{2m-\frac{n}{2}-1} < +\infty$$

for some (equivalently, any) integer $m > \frac{n}{4}$. Furthermore, the square root of the left-hand side is a seminorm equivalent to $\|f\|_\square$.

This time there is no Moebius invariance, since the Moebius self-maps of B^n do not preserve harmonicity for $n > 2$.

We review the necessary background material in Section 2, then present the basic properties of the M-harmonic Dirichlet space $\mathcal{M}_{\circ\circ}$ in Section 3. Moebius invariance is discussed in Section 4. The pluriharmonic Dirichlet space is treated in Section 5, and the harmonic case in Section 6.

Throughout the paper, the notation

$$A \asymp B$$

means that

$$cA \leq B \leq \frac{1}{c} A$$

for some $0 < c \leq 1$ independent of the variables in question. The symbols $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z}_j}$, commonly abbreviated just to ∂_j and $\overline{\partial}_j$, respectively, stand for the usual Wirtinger operators on $\mathbf{C}^n$; similarly on $\mathbf{R}^n$, ∂_k stands for $\frac{\partial}{\partial x_k}$. For typesetting reasons, the inner product $\langle x, y \rangle$ in $\mathbf{C}^n$ is sometimes also denoted by $x \cdot \overline{y}$. Finally, $\mathbf{Z}, \mathbf{N}, \mathbf{R}$ and $\mathbf{C}$ denote the sets of all integers, all nonnegative integers, all real and all complex numbers, respectively.

2. Notation and preliminaries

The stabilizer of the origin $0 \in \mathbf{B}^n$ in $\mathrm{Aut}(\mathbf{B}^n)$ is the group $U(n)$ of all unitary transformations of $\mathbf{C}^n$; that is, of all linear operators U that preserve inner products:

$$\langle Uz, Uw \rangle = \langle z, w \rangle \qquad \forall z, w \in \mathbf{C}^n.$$

Each $U \in U(n)$ maps the unit sphere $\partial \mathbf{B}^n$ onto itself, and the surface measure $d\sigma$ on $\partial \mathbf{B}^n$ is invariant under U. It follows that the composition with elements of $U(n)$,

$$T_U : f \mapsto f \circ U^{-1}, \tag{21}$$

is a unitary representation of $U(n)$ on $L^2(\partial \mathbf{B}^n, d\sigma)$. We will need the decomposition of this representation into irreducible subspaces. These turn out to be given by *bigraded spherical harmonics* $\mathcal{H}^{pq}$; the standard sources for this are Rudin [Ru, Sections 12.1–12.2], or Krantz [Kr, Sections 6.6–6.8], with basic ingredients going back to Folland [Fo].

Namely, for integers $p, q \geq 0$, let $\mathcal{H}^{pq}$ be vector space of restrictions to $\partial \mathbf{B}^n$ of harmonic polynomials $f(z, \overline{z})$ on $\mathbf{C}^n$ which are homogeneous of degree p in z and homogeneous of degree q in $\overline{z}$. Then $\mathcal{H}^{pq}$ is invariant under the action (21) of $U(n)$, is $U(n)$-irreducible (i.e. has no proper $U(n)$-invariant subspace) and

$$L^2(\partial \mathbf{B}^n, d\sigma) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}^{pq}. \tag{22}$$

Furthermore, if T is a linear operator on $L^2(\partial\mathbf{B}^n, d\sigma)$ commuting with the action (21) i.e. $T(f\circ U) = (Tf)\circ U$ for all $U \in U(n)$ and $f \in L^2(\partial\mathbf{B}^n, d\sigma)$), then T is diagonalized by the decomposition (21), i.e. T maps each $\mathcal{H}^{pq}$ into itself and $T|\mathcal{H}^{pq} = c_{pq} I|\mathcal{H}^{pq}$ for some complex constants c_{pq}, where I denotes the identity operator.

Since each space $\mathcal{H}^{pq}$ is finite-dimensional, the evaluation functional $f \mapsto f(\zeta)$ at each $\zeta \in \partial\mathbf{B}^n$ is automatically continuous on it; it follows that $\mathcal{H}^{pq}$ — with the inner product inherited from $L^2(\partial\mathbf{B}^n, d\sigma)$ — has a reproducing kernel. This reproducing kernel turns out to be given by $H^{pq}(\zeta\cdot\overline{\eta})$, where for $n \geq 2$

$$
\begin{aligned}
H^{pq}(z) = {} & \frac{(-1)^q(n+p+q-1)(n+p-2)!}{(n-1)!q!(p-q)!} \\
& \times z^{p-q}{}_2F_1\left(\begin{matrix}-q, n+p-1\\ p-q+1\end{matrix}\middle| |z|^2\right) \qquad \text{for } p \geq q,
\end{aligned} \tag{23}
$$

while $H^{pq}(z) = H^{qp}(\overline{z})$ for $p < q$. For $n = 1$, the spaces $\mathcal{H}^{pq}$ reduce just to $\{0\}$ if $pq \neq 0$, while $\mathcal{H}^{p0} = \mathbf{C}z^p$, $\mathcal{H}^{0q} = \mathbf{C}\overline{z}^q$ and $H^{p0}(z) = z^p$, $H^{0q}(z) = \overline{z}^q$; note that the formula (23) still works for $n = 1$ and $pq = 0$.

Denote

$$
\begin{aligned}
S^{pq}(r) &:= r^{p+q}{}_2F_1\left(\begin{matrix}p, q\\ p+q+n\end{matrix}\middle| r^2\right) \Big/ {}_2F_1\left(\begin{matrix}p, q\\ p+q+n\end{matrix}\middle| 1\right) \\
&= \frac{\Gamma(p+n)\Gamma(q+n)}{\Gamma(n)\Gamma(p+q+n)} r^{p+q}{}_2F_1\left(\begin{matrix}p, q\\ p+q+n\end{matrix}\middle| r^2\right).
\end{aligned} \tag{24}
$$

Then for each $f \in \mathcal{H}^{pq}$, the (unique) solution to the Dirichlet problem $\widetilde{\Delta}u = 0$ on $\mathbf{B}^n$, $u|_{\partial\mathbf{B}^n} = f$ is given by

$$
u(r\zeta) = S^{pq}(r) f(\zeta), \qquad 0 \leq r \leq 1,\ \zeta \in \partial\mathbf{B}^n. \tag{25}
$$

Many of the formulas above originate in [Fo].

For each $p, q \geq 0$, let $\mathbf{H}^{pq}$ be the space of all functions on $\mathbf{B}^n$ of the form (25) with $f \in \mathcal{H}^{pq}$. In other words, while $\mathcal{H}^{pq}$ is the space of spherical harmonics on the sphere $\partial\mathbf{B}^n$, $\mathbf{H}^{pq}$ is the associated space of "solid" M-harmonic functions on $\mathbf{B}^n$. With the inner product inherited from $L^2(\partial\mathbf{B}^n, d\sigma)$, each $\mathbf{H}^{pq}$ is thus a finite-dimensional Hilbert space of M-harmonic functions on $\mathbf{B}^n$, unitarily isomorphic to the space $\mathcal{H}^{pq}$ via the isomorphism (25), and with reproducing kernel

$$
K^{pq}(r\zeta, R\xi) := S^{pq}(r) S^{pq}(R) H^{pq}(\zeta\cdot\overline{\xi}). \tag{26}
$$

It was shown in Proposition 1 in [EY] that if H is any Hilbert space of M-harmonic functions on $\mathbf{B}^n$ whose inner product is invariant under rotations (i.e. $f \in H$, $U \in U(n)$ imply $f\circ U \in H$ and $\|f\circ U\|_H = \|f\|_H$), then the following is true:

- for each $p, q \geq 0$, $H \cap \mathbf{H}^{pq}$ is either $\{0\}$ or all of $\mathbf{H}^{pq}$;
- if $\mathbf{H}^{pq} \subset H$, then

$$
\langle f, g\rangle_H = c_{pq}\langle f, g\rangle_{\partial\mathbf{B}^n} \qquad \forall f, g \in \mathbf{H}^{pq}
$$

  with some constant c_{pq}, $0 < c_{pq} < +\infty$ (independent of f, g);
- the reproducing kernel of H is given by

$$
K_H(x, y) = \sum_{p,q:\mathbf{H}^{pq}\subset H} \frac{K_{pq}(x, y)}{c_{pq}}, \tag{27}
$$

with the sum converging locally uniformly on $\mathbf{B}^n \times \mathbf{B}^n$, as well as in H as a function of x for each fixed y, or vice versa.

One can also formally define $c_{pq} := +\infty$ if $\mathbf{H}^{pq} \not\subset H$; then H contains precisely those $\mathbf{H}^{pq}$ for which c_{pq} is finite, and in (27) the summation can be extended over all $p, q \geq 0$, with the usual convention that $1/\infty := 0$.

One can also allow semi-Hilbert spaces, i.e. with semi-definite (semi-)norm instead of norm; then the above still holds, except that c_{pq} can be zero for some p, q and (27) is the reproducing kernel not for H but only for the subspace $H_0 := \bigoplus\{\mathbf{H}^{pq} : c_{pq} > 0\}$.

For the weighted M-harmonic Bergman spaces of all M-harmonic functions in $L^2(\mathbf{B}^n, (1-|z|^2)^s\,dz)$, $s > -1$, an explicit formula for the $c_{pq} =: c_{pq}(s)$ was given in (69) in [EY]. Renormalizing so as to pass to our normalized measures from (1), and our spaces $\mathcal{M}_s$ from the Introduction, it was shown in Section 6.3 of [EY] that the corresponding constants $C_{pq}(s) = c_{pq}(s)/c_{00}(s)$ are given by

$$C_{pq}(s) = \frac{1}{\Gamma(n)} \int_0^1 G_{pq}^{(n)}(t)\,(1-t)^{n+s}\,dt, \tag{28}$$

where G_{pq} is the function

$$G_{pq}(t) := \frac{\Gamma(n+p)^2\Gamma(n+q)^2}{\Gamma(n)^2\Gamma(n+p+q)^2} t^{p+q+n-1} {}_2F_1\Big(\begin{matrix} p, q \\ n+p+q \end{matrix}\Big| t\Big)^2$$

(see formula (98) there). Furthermore, it was shown there that $G_{pq}^{(n)}$ is positive and continuous on $(0,1)$, with a finite value at the origin and $G_{pq}^{(n)}(t) = O(\log\frac{1}{1-t})$ as $t \nearrow 1$, from which it follows that (28) furnishes an analytic continuation of $C_{pq}(s)$ to $\operatorname{Re} s > -n-1$ and $0 < C_{pq}(s) < +\infty$ for $s \in (-n-1, +\infty)$. The functions

$$K_s(z, w) = \sum_{p,q} \frac{K_{pq}(z, w)}{C_{pq}(s)}$$

thus continue to be positive definite kernels on $\mathbf{B}^n \times \mathbf{B}^n$ in the sense of Aronszajn [Ar], and the norm in the associated reproducing kernel Hilbert spaces — still denoted $\mathcal{M}_s$ — is given by

$$\|f\|_s^2 = \sum_{p,q} C_{pq}(s) \|f_{pq}\|^2_{\partial B^n}, \qquad s > -n-1,$$

for an M-harmonic function f with Peter-Weyl decomposition $f = \sum_{p,q} f_{pq}$, $f_{pq} \in \mathbf{H}^{pq}$.

We conclude this section by describing the projections onto the Peter-Weyl components $\mathbf{H}^{pq}$. From the reproducing property of $H^{qp}(\zeta \cdot \overline{\eta})$ and (25), we have, for any M-harmonic function f on $\mathbf{B}^n$,

$$f_{pq}(r\zeta) = \int_{\partial B^n} f(r\eta) H^{pq}(\zeta \cdot \overline{\eta})\,d\sigma(\eta) \qquad \forall \zeta \in \partial B^n, \forall r \in (0,1). \tag{29}$$

In particular, for $p = q = 0$,

$$\Pi_0 f(r\zeta) = \int_{\partial B^n} f(r\eta)\,d\sigma(\eta), \qquad 0 \leq r < 1,\ \zeta \in \partial\mathbf{B}^n, \tag{30}$$

is the projection onto the constants $\mathbf{H}^{00}$.

Proposition 1. *For any f M-harmonic on $\mathbf{B}^n$, the limit*

$$\Pi f(r\zeta) := \lim_{R\nearrow 1} \int_{\partial B^n} \frac{f(r\eta)}{(1-R\langle\zeta,\eta\rangle)^n}\, d\sigma(\eta) \tag{31}$$

exists, and Πf equals the projection of f onto the subspace $\mathcal{A} := \bigoplus_{P=0}^{\infty} \mathbf{H}^{p0}$ of holomorphic functions on $\mathbf{B}^n$.

Proof. Expanding $(1-R\langle\zeta,\eta\rangle)^{-n}$ by the binomial formula shows that the integral equals

$$\begin{aligned}
\int_{\partial B^n} f(r\eta) \sum_{j=0}^{\infty} R^j \frac{(n)_j}{j!} \langle\zeta,\eta\rangle^j \, d\sigma(\eta)& \\
&= \int_{\partial B^n} f(r\eta) \sum_{j=0}^{\infty} R^j H^{j0}(\zeta\cdot\overline{\eta})\, d\sigma(\eta) \qquad \text{by (23)} \\
&= \sum_j R^j f_{j0}(r\zeta) \qquad \text{by (29)} \\
&= \sum_j f_{j0}(Rr\zeta) = \Pi f(Rr\zeta),
\end{aligned}$$

the interchange of the summation and integration signs being justified by the locally uniform convergence. Letting $R \nearrow 1$, the claim follows. □

We remark that it is, of course, not possible to interchange the limit and the integral in (31), since $(1-\langle\zeta,\cdot\rangle)^{-n}$ is not integrable over $\partial\mathbf{B}^n$, for any $\zeta \in \partial\mathbf{B}^n$.

Taking complex conjugates in (31), one gets also the projection $\overline{\Pi}$ onto the subspace of anti-holomorphic functions, and the projection

$$P := \Pi + \overline{\Pi} - \Pi_0$$

onto the subspace $\mathcal{P} := \bigoplus_{pq=0} \mathbf{H}^{pq}$ of pluriharmonic functions on $\mathbf{B}^n$.

For later use, we denote by

$$Q := I - P$$

the projection onto the orthogonal complement $\bigoplus_{pq>0} \mathbf{H}^{pq}$ ("the M-harmonic functions with no pluriharmonic component").

3. M-harmonic Dirichlet space

For an M-harmonic function $f = \sum_{p,q} f_{pq}$, $f_{pq} \in \mathbf{H}^{pq}$, on $\mathbf{B}^n$, we have by (21) and (25)

$$\|f\|_s^2 = \sum_{p,q} C_{pq}(s) \|f_{pq}\|^2_{\partial\mathbf{B}^n} \tag{32}$$

for any $s > -1$, with

$$\begin{aligned}
C_{pq}(s) &:= \frac{\Gamma(n+s+1)}{\pi^n\Gamma(s+1)} \int_0^1 S^{pq}(r)^2 \frac{2\pi^n}{\Gamma(n)} (1-r^2)^s r^{2n-1}\, dr \\
&= \frac{(s+1)_n}{\Gamma(n)} \int_0^1 S^{pq}(\sqrt{t})^2 t^{n-1}\, (1-t)^s\, dt \\
&= \frac{(s+1)_n}{\Gamma(n)} \int_0^1 G_{pq}(t)\, (1-t)^s\, dt, \qquad (33)
\end{aligned}$$

where G_{pq} is the function

$$G_{pq}(t) := t^{p+q+n-1} \frac{\Gamma(n+p)^2\Gamma(n+q)^2}{\Gamma(n)^2\Gamma(n+p+q)^2} {}_2F_1\Big(\begin{matrix} p, q \\ n+p+q \end{matrix}\Big| t\Big)^2. \tag{34}$$

(See [EY] for the details.)

The content of the following lemma is standard; we include the short proof for the sake of completeness.

Lemma 2. *Let $F(z) = \sum_{k=0}^\infty F_k z^k$ be a holomorphic function on the disc $|z| < R$, $R > 0$. Then for any $\delta \in (0, R)$,*

(a) *the integral*

$$\mathcal{I}(s) := \int_{1-\delta}^1 F(1-t)(1-t)^s \, dt, \qquad s > -1,$$

extends to a holomorphic function of s on the entire complex plane $\mathbf{C}$, except for possible simple poles at $s = -j-1$, $j = 0, 1, 2, \dots$, with residues F_j;

(b) *for $m = 1, 2, \dots$, the integral*

$$\mathcal{I}_m(s) := \int_{1-\delta}^1 F(1-t)\Big(\log \frac{1}{1-t}\Big)^m (1-t)^s \, dt, \qquad s > -1,$$

extends to a holomorphic function of s on the entire complex plane $\mathbf{C}$, except for possible poles of multiplicity $m+1$ at $s = -j-1$, $j = 0, 1, 2, \dots$, of strength $m! F_j$.

Proof. (a) From the Taylor expansion, we have for any $N = 0, 1, 2, \dots$,

$$F(z) = F_0 + F_1 z + \dots + F_{N-1} z^{N-1} + z^N G_N(z),$$

with G_N holomorphic on $|z| < R$. By uniform convergence,

$$\mathcal{I}(s) = \sum_{j=0}^{N-1} F_j \frac{\delta^{s+j+1}}{s+j+1} + \int_{1-\delta}^1 G_N(1-t)(1-t)^{s+N} \, dt. \tag{35}$$

The j-th summand in the sum is holomorphic on $\mathbf{C}$ except for a simple pole at $s = -j-1$ with residue F_j, while the integral is a holomorphic function on $\operatorname{Re} s > -N-1$. As N was arbitrary, the claim follows.

(b) Differentiating (35) m times with respect to s yields

$$\mathcal{I}_m(s) = \sum_{j=0}^{N-1} F_j \frac{m!\delta^{s+j+1}}{(s+j+1)^{m+1}} + \int_{1-\delta}^1 G_N(1-t)\Big(\log \frac{1}{1-t}\Big)^m (1-t)^{s+N} \, dt, \tag{36}$$

and the claim again follows. □

Note from (33) that $C_{00}(s) \equiv 1$ has no poles, while

$$C_{p0}(s) = C_{0p}(s) = \frac{(n)_p}{(n+s+1)_p} \tag{37}$$

has simple poles at $s = -n-1, \dots, -n-p$.

Proposition 3. *Let $pq > 0$. Then $C_{pq}(s)$ extends to a holomorphic function of s on the entire $\mathbf{C}$, except for double poles at $s = -n-1, -n-2, \dots, -2n$ and triple poles at $s = -2n-1-j$, $j = 0, 1, 2, \dots$. The double pole at $s = -n-1$ has strength $(p)_n(q)_n/\Gamma(n)^2$.*

Proof. Since G_{pq} is continuous on the unit disc, the integral

$$\int_0^{1-\delta} G_{pq}(t)(1-t)^s\,dt$$

is a holomorphic function of s on the entire complex plane, for any $0<\delta<1$. For the integral from $1-\delta$ to 1, formula (12) in [BE, §2.10] tells us that

$$\frac{\Gamma(n+p)\Gamma(n+q)}{\Gamma(n)\Gamma(n+p+q)}\,{}_2F_1\left(\begin{matrix}p,q\\n+p+q\end{matrix}\middle| t\right)=A_0(1-t)+A_1(1-t)(1-t)^n\log\frac{1}{1-t}$$

for $|\arg(1-t)|<\pi$, with

$$\begin{aligned}A_0(z)&:=\sum_{j=0}^{n-1}\frac{(p)_j(q)_j}{(1-n)_j j!}z^j+\sum_{j=0}^{\infty}\frac{(-1)^n\Gamma(n+p+j)\Gamma(n+q+j)}{j!(j+n)!\Gamma(n)\Gamma(p)\Gamma(q)}z^{n+j}h''_j,\\A_1(z)&:=\sum_{j=0}^{\infty}\frac{(-1)^n\Gamma(n+p+j)\Gamma(n+q+j)}{j!(j+n)!\Gamma(n)\Gamma(p)\Gamma(q)}z^j\end{aligned}$$

holomorphic on $|z|<1$; here

$$h''_j:=\psi(j+1)+\psi(j+n+1)-\psi(j+n+p)-\psi(j+n+q)$$

where $\psi:=\Gamma'/\Gamma$ is the digamma function. It follows that

$$G_{pq}(t)=B_0(1-t)+B_1(1-t)(1-t)^n\log\frac{1}{1-t}+B_2(t)(1-t)^{2n}\Big(\log\frac{1}{1-t}\Big)^2,$$

with

$$\begin{aligned}B_0(z)&=(1-z)^{p+q+n-1}A_0(z)^2,\\B_1(z)&=2(1-z)^{p+q+n-1}A_0(z)A_1(z),\qquad\text{and}\\B_2(z)&=(1-z)^{p+q+n-1}A_1(z)^2\end{aligned}$$

holomorphic on $|z|<1$. Applying the lemma, we see that

$$\int_{1-\delta}^1 G_{pq}(t)(1-t)^s\,dt$$

extends to an entire function of s, except for possible simple poles at $s=-1,-2,\dots,-n$, possible double poles at $s=-n-1,-n-2,\dots,-2n$ and possible triple poles at $s=-2n-1,-2n-2,\dots$. The strength of the double pole at $s=-n-1$ is

$$1!A_1(0)=\frac{(-1)^n\Gamma(n+p)\Gamma(n+q)}{n!\Gamma(n)\Gamma(p)\Gamma(q)}=\frac{(-1)^n}{n!\Gamma(n)}(p)_n(q)_n. \tag{38}$$

Finally, passing from $\int_0^1 G_{pq}(t)(1-t)^s\,dt$ to $C_{pq}(s)$, the factor $\frac{(s+1)_n}{\Gamma(n)}$ cancels the simple poles at $s=-1,-2,\dots,-n$, while (38) gets multiplied by $\frac{(s+1)_n}{\Gamma(n)}\Big|_{s=-n-1}=\frac{(-1)^n n!}{\Gamma(n)}$. □

Remark 4. A variant of the last proof can be given by first integrating by parts to get

$$C_{pq}(s)=\frac{1}{\Gamma(n)}\int_0^1 G_{pq}^{(n)}(t)(1-t)^{n+s}\,dt \tag{39}$$

for any $s>-n-1$ and $p+q>0$; see Section 6.3 in [EY]. This has the advantage of showing that $C_{pq}(s)$ is positive for any $s>-n-1$, and also is a decreasing function of s on this interval. Since we are not interested in positivity or monotonicity at the moment, it was simpler to apply Lemma 2 directly. $\square$

Denoting the analytic continuation still by $C_{pq}(s)$, we thus see that

$$C^{\circ\circ}_{pq} := \lim_{s\searrow -n-1} (s+n+1)^2 C_{pq}(s)$$

exists for all $p, q \geq 0$, and equals

$$C^{\circ\circ}_{pq} = \begin{cases} 0 & \text{if } pq = 0 \\ \dfrac{(p)_n (q)_n}{\Gamma(n)^2} & \text{if } pq > 0. \end{cases} \tag{40}$$

We thus arrive at the following corollary.

Corollary 5. *In terms of the Peter-Weyl decomposition $f = \sum_{p,q} f_{pq}$, $f_{pq} \in \mathbf{H}^{pq}$, of an M-harmonic function f on $\mathbf{B}^n$,*

$$\|f\|^2_{\circ\circ} = \sum_{p,q} \frac{(p)_n (q)_n}{\Gamma(n)^2} \|f_{pq}\|^2_{\partial \mathbf{B}^n}.$$

Definition 6. We call the space of M-harmonic functions $f = \sum_{p,q} f_{pq}$, $f_{pq} \in \mathbf{H}^{pq}$, on $\mathbf{B}^n$ for which $\|f\|^2_{\circ\circ} < +\infty$ the *M-harmonic Dirichlet space*, denoted $\mathcal{M}_{\circ\circ}$. $\square$

We also denote by

$$\mathcal{M}_{\circ\circ,0} := \{f \in \mathcal{M}_{\circ\circ} : f_{pq} = 0 \text{ if } pq = 0\}$$

the subspace of all functions in $\mathcal{M}_{\circ\circ}$ which have no pluriharmonic component (i.e. $Pf = 0$). The quantity $\|f\|_{\circ\circ}$ is a seminorm on $\mathcal{M}_{\circ\circ}$ and a norm on $\mathcal{M}_{\circ\circ,0}$; abusing the language, we will often speak just of a "norm". Obviously, $\mathcal{M}_{\circ\circ}$ contains all the spaces $\mathbf{H}^{pq}$, $p, q \geq 0$, and their span is dense in it.

Remark 7. The space $\mathcal{M}_{\circ\circ,0}$ has reproducing kernel

$$K^{\circ\circ}(z,w) = \Gamma(n)^2 \sum_{p,q \geq 1} \frac{K_{pq}(z,w)}{(p)_n (q)_n}.$$

The authors do not know if this sum can be evaluated explicitly. $\square$

4. Moebius invariance

The following facts likely are again quite standard, but we include the short proof for completeness. Recall that for a function f on $\mathbf{B}^n$ and $0 < r < 1$, f_r denotes the function on $\partial \mathbf{B}^n$ defined by $f_r(\zeta) := f(r\zeta)$.

Lemma 8. *Let $f = \sum_{p,q} f_{pq}$, $f_{pq} \in \mathbf{H}^{pq}$, be an M-harmonic function on $\mathbf{B}^n$. Then the following three conditions are equivalent:*

$$\sup_{0<r<1} \|f_r\|^2_{\partial \mathbf{B}^n} < +\infty; \tag{41}$$

$$\textit{a finite } \lim_{r \nearrow 1} \|f_r\|^2_{\partial \mathbf{B}^n} \textit{ exists}; \tag{42}$$

$$\sum_{p,q} \|f_{pq}\|^2_{\partial \mathbf{B}^n} < +\infty. \tag{43}$$

The three quantities above are then equal, and the sum and the sequence

$$\sum_{p,q} f_{pq} \tag{44}$$

$$\lim_{r\nearrow 1} f_r \tag{45}$$

then converge to the same function — denoted f^* — in $L^2(\partial\mathbf{B}^n, d\sigma)$; f^* can thus be interpreted as the "boundary value" of f.

We denote the quantity (41)–(43) by $\|f\|^2_{\text{Hardy}}$, and call the space of all M-harmonic f for which it is finite the *M-harmonic Hardy space* $\mathcal{M}_{\text{Hardy}}$. We remark that $\|f\|_{\text{Hardy}}$ actually coincides with $\|f\|_{-1}$, and the reproducing kernel of $\mathcal{M}_{\text{Hardy}}$ was computed explicitly in [EY].

Abusing the notation slightly, we will sometimes write just $f|_{\partial\mathbf{B}^n}$, or even f, instead of f^*, and just $\|f\|_{\partial\mathbf{B}^n}$ instead of $\|f^*\|_{\partial\mathbf{B}^n} = \|f\|_{\text{Hardy}}$.

Proof. From (21) and (25), we have

$$\|f_r\|^2_{\partial\mathbf{B}^n} = \sum_{pq} |S^{pq}(r)|^2 \|f_{pq}\|^2_{\partial\mathbf{B}^n}.$$

Since $S^{pq}(r)$ are nondecreasing (strictly increasing for $pq > 0$) functions of $r \in (0,1)$, with $S^{pq}(1) = 1$, it follows that the limit (42) coincides with the supremum (41); and by the Lebesgue Monotone Convergence Theorem, they are both equal to (43). This settles the first part. For the second, note that (43) means, by (21), that the partial sums of the series (44) form a Cauchy sequence; since L^2 is complete, they must have a limit f^*, and $\||f^*\|^2_{\partial\mathbf{B}^n} = \sum_{p,q} \|f_{pq}\|^2_{\partial\mathbf{B}^n}$. By (21) and (25) once again,

$$\|f_r - f^*\|^2_{\partial\mathbf{B}^n} = \sum_{p,q} (1 - S^{pq}(r)^2) \|f_{pq}\|^2_{\partial\mathbf{B}^n}.$$

However, the right-hand side tends to zero again by the Lebesgue Monotone Convergence Theorem, proving that $f_r \to f^*$. □

Recall that in the polar coordinates $z = r\zeta$ on $\mathbf{C}^n$ ($r > 0$, $\zeta \in \partial\mathbf{B}^n$), the Euclidean Laplacian Δ is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\text{sph}},$$

where Δ_{sph} is the *spherical Laplacian*, which involves only differentiations with respect to the ζ variables. In particular, the value of $\Delta_{\text{sph}}\phi$ on a sphere $|z|$ =const. depends only on the values of the function ϕ on that sphere. Another operator with this property is the *complex normal derivative* (or *Reeb vector field*)

$$\mathcal{R} := \sum_{j=1}^n \Big(z_j \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \Big).$$

The operator Δ_{sph} can be expressed explicitly as

$$\Delta_{\text{sph}} = -\mathcal{R}^2 + \sum_{j,k=1}^n (L_{jk}\overline{L}_{jk} + \overline{L}_{jk}L_{jk})$$

where $L_{jk}, \overline{L}_{jk}$ are the tangential vector fields

$$L_{jk} := \overline{z}_j \frac{\partial}{\partial z_k} - \overline{z}_k \frac{\partial}{\partial z_j}, \qquad \overline{L}_{jk} := z_j \frac{\partial}{\partial \overline{z}_k} - z_k \frac{\partial}{\partial \overline{z}_j}.$$

Both Δ_{sph} and $\mathcal{R}$ commute with the action of $U(n)$, i.e. $\Delta_{\text{sph}}(\phi \circ U) = (\Delta_{\text{sph}}\phi) \circ U$ for any $U \in U(n)$, and similarly for $\mathcal{R}$. (In fact, the algebra of all $U(n)$-invariant linear differential operators on $\partial\mathbf{B}^n$ is generated by Δ_{sph} and $\mathcal{R}$, but we will not need this fact.) From the irreducibility of the multiplicity-free decomposition (22), it follows by abstract theory that $\Delta_{\rm sph}$ and $\mathcal{R}$ map each $\mathcal{H}^{pq}$ (and $\mathbf{H}^{pq}$) into itself and actually reduce on it to a multiple of the identity. Evaluation on e.g. the element $\zeta_1^p\overline{\zeta}_2^q \in \mathcal{H}^{pq}$ (for $n \geq 2$) shows that, explicitly,

$$\begin{aligned} \Delta_{\rm sph}|\mathcal{H}^{pq} &= -(p+q)(p+q+2n-2)I|\mathcal{H}^{pq}, \\ \mathcal{R}|\mathcal{H}^{pq} &= (p-q)I|\mathcal{H}^{pq} \end{aligned} \tag{46}$$

(which prevail also for $n=1$ by the remarks on $\mathcal{H}^{pq}$ when $n=1$ after (25); in that case $\Delta_{\rm sph} = -\mathcal{R}^2$). Let $\mathcal{L}_m$, $1 \leq m \leq 2n(n-1)$, denote the collection of all the operators $L_{jk}, \overline{L}_{jk}$, $j,k=1,\dots,n$, $j \neq k$, in some (fixed) order.

Theorem 9. *If f is M-harmonic on $\mathbf{B}^n$, $n \geq 2$, then $f \in \mathcal{M}_{\circ\circ}$ if and only if*

$$\sum_{j_1,j_2,\dots,j_n=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_n}(Qf)\|^2_{\partial\mathbf{B}^n} < +\infty, \tag{47}$$

and the square root of the left-hand side is a seminorm equivalent to $\|f\|_{\circ\circ}$.

Proof. Since $-\overline{L_{jk}}$ is the adjoint of L_{jk} in $L^2(\partial\mathbf{B}^n, d\sigma)$, we have for any $g \in C^2(\partial\mathbf{B}^n)$

$$\sum_{j=1}^{2n(n-1)} \|\mathcal{L}_j g\|^2_{\partial\mathbf{B}^n} = -\sum_{j,k=1}^{n} \langle (L_{jk}\overline{L}_{jk} + \overline{L}_{jk}L_{jk})g, g\rangle_{\partial\mathbf{B}^n} = \langle (-\Delta_{\rm sph} - \mathcal{R}^2)g, g\rangle_{\partial\mathbf{B}^n}.$$

If $g = \sum_{p,q} g_{pq}$, $g_{pq} \in \mathcal{H}^{pq}$, is the Peter-Weyl decomposition (21) of g, we thus have by (46)

$$\sum_{j=1}^{2n(n-1)} \|\mathcal{L}_j g\|^2_{\partial\mathbf{B}^n} = \sum_{p,q} [4pq + (2n-2)(p+q)] \|g_{pq}\|^2_{\partial\mathbf{B}^n}.$$

Iterating this formula, we obtain

$$\sum_{j_1,j_2,\dots,j_m=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_m} g\|^2_{\partial\mathbf{B}^n} = \sum_{p,q} [4pq + (2n-2)(p+q)]^m \|g_{pq}\|^2_{\partial\mathbf{B}^n}, \tag{48}$$

for any $m = 0,1,2,\dots$. Now for $p,q \geq 1$ and $n \geq 2$ obviously

$$\begin{aligned} [4pq + (2n-2)(p+q)]^n &\asymp [4pq + (2n-2)(p+q) + 1]^n \\ &\asymp (p+1)^n (q+1)^n \\ &\asymp (p)_n (q)_n. \end{aligned}$$

Abusing notation by denoting by the same letter Q also the orthogonal projection in $L^2(\partial\mathbf{B}^n, d\sigma)$ onto $\bigoplus_{p,q\geq 1} \mathcal{H}^{pq}$, we thus have

$$\sum_{j_1,j_2,\dots,j_n=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_n}(Qg)\|^2_{\partial\mathbf{B}^n} \asymp \sum_{p,q=1}^{\infty} \frac{(p)_n (q)_n}{\Gamma(n)^2} \|g_{pq}\|^2_{\partial\mathbf{B}^n}.$$

Applying this to $g = f_r$ and using the last lemma, the assertion follows. $\square$

We remark that, strictly speaking, in (47) we should write more correctly

$$\sup_{0<r<1} \sum_{j_1,j_2,\dots,j_n=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_n}(Qf_r)\|^2_{\partial\mathbf{B}^n} < +\infty, \tag{49}$$

as $\mathcal{L}_j f$ need not be harmonic in general when f is; however, as $(\mathcal{L}_j f)_r = \mathcal{L}_j(f_r)$ (since $\mathcal{L}_j$ are tangential operators) and $(Qf)_r = Q(f_r)$ (from (31)), the proof shows that the last supremum actually coincides with $\|(-\Delta_{\rm sph} - \mathcal{R}^2)^{n/2} f\|^2_{\rm Hardy}$, and the function $(-\Delta_{\rm sph} - \mathcal{R}^2)^{n/2} f$ (defined in the sense of functional calculus for self-adjoint operators) is M-harmonic by (46). We thus take the liberty to use the shorthand indicated in the sentence before the proof of Lemma 8.

Yet another reformulation of the last theorem is as follows. Consider the weak-maximal operator X acting from $L^2(\partial\mathbf{B}^n)$ into the Cartesian product of $2n^2(n-1)$ copies of $L^2(\partial\mathbf{B}^n)$ by

$$g \longmapsto \{\mathcal{L}_{j_1} \dots \mathcal{L}_{j_n} g\}_{j_1,\dots,j_n=1}^{2n(n-1)};$$

that is, the domain of X consists of all $g \in L^2(\partial\mathbf{B}^n)$ for which all the $\mathcal{L}_{j_1} \dots \mathcal{L}_{j_n} g$ exist in the sense of distributions and belong to $L^2(\partial\mathbf{B}^n)$. (In other words, $X = Y^*$ where Y is the restriction of the formal adjoint $X^\dagger$ of X to $\bigoplus^{2n^2(n-1)} C^\infty(\partial\mathbf{B}^n)$.) Then $f \in \mathcal{M}$ belongs to $\mathcal{M}_{\circ\circ}$ if and only if $(Qf)^* \in \operatorname{Dom}(X)$, and $\|X(Qf)^*\|$ is a seminorm equivalent to $\|f\|_{\circ\circ}$.

Corollary 10. *The space $\mathcal{M}_{\circ\circ}$ is Moebius invariant: $f \in \mathcal{M}_{\circ\circ}$ implies $f\circ\phi \in \mathcal{M}_{\circ\circ}$ for any $\phi \in \operatorname{Aut}(\mathbf{B}^n)$.*

Proof. Let $\phi \in \operatorname{Aut}(\mathbf{B}^n)$ and $f \in \mathcal{M}_{\circ\circ}$. We need to show that the sum in (47) with $f \circ \phi$ in the place of f is finite. Note that

$$Q(f\circ\phi) = Q((Qf)\circ\phi) + Q((Pf)\circ\phi) = Q((Qf)\circ\phi),$$

since composition with ϕ preserves holomorphy and, hence, plurisubharmonicity. Note further that, by (48), for any $g \in L^2(\partial\mathbf{B}^n, d\sigma)$,

$$\sum_{j_1,j_2,\dots,j_m=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_m} Qg\|^2_{\partial\mathbf{B}^n} = \sum_{p,q\ge 1} [4pq + (2n-2)(p+q)]^m \|g_{pq}\|^2_{\partial\mathbf{B}^n}$$
$$\le \sum_{p,q\ge 0} [4pq + (2n-2)(p+q)]^m \|g_{pq}\|^2_{\partial\mathbf{B}^n} = \sum_{j_1,j_2,\dots,j_m=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_m} g\|^2_{\partial\mathbf{B}^n};$$

hence it is enough to show that, in fact, even the sum in (47) with $(Qf)\circ\phi$ in the place of Qf is finite.

Observe that the tangential vector-fields $\mathcal{L}_m$, $m = 1,\dots,2n(n-1)$, span (very redundantly) the entire complex tangent space to ∂B^n. Thus for any differentiable function g on $\partial\mathbf{B}^n$, $\sum_m \|\mathcal{L}_m g\|^2_{\partial\mathbf{B}^n} \asymp \|\nabla_{\rm ct} g\|^2$, the norm-square of the restriction $\nabla_{\rm ct} g$ of the tensor ∇g to the complex tangent space of $\partial\mathbf{B}^n$ in the sense of complex geometry. Now for any vector field X on $\partial\mathbf{B}^n$, one has $X(g\circ\phi) = d\phi(X)g$. Since ϕ maps the sphere $\partial\mathbf{B}^n$ onto itself, the derived map $d\phi$ maps the real tangent space of $\partial\mathbf{B}^n$ into itself. As ϕ is holomorphic, $d\phi$ is complex linear, hence $d\phi$ maps also the complex tangent space (consisting of all vectors Y such that both Y and iY are real tangent) of $\partial\mathbf{B}^n$ into itself. Finally, $d\phi|\partial\mathbf{B}^n$ is a smoothly varying map on the compact manifold $\partial\mathbf{B}^n$ (hence, in particular, so is its Jacobian). Consequently,

$$\|\nabla_{\rm ct}(g\circ\phi)\|^2_{\partial\mathbf{B}^n} = \|d\phi(\nabla_{\rm ct})g\|^2_{\partial\mathbf{B}^n} \le C_\phi \|\nabla_{\rm ct} g\|^2_{\partial\mathbf{B}^n}$$

with some finite C (independent of g). Iterating this argument, it transpires that

$$\|\nabla^n_{\rm ct}(g\circ\phi)\|^2_{\partial\mathbf{B}^n} \le C^n_\phi \|\nabla_{\rm ct} g\|^2_{\partial\mathbf{B}^n}.$$

Passing from $\nabla^n_{\rm ct}$ back to the $\mathcal{L}_m$, the last inequality reads

$$\sum_{j_1,j_2,\dots,j_n=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_n}(g\circ\phi)\|^2_{\partial\mathbf{B}^n} \le C^n_\phi \sum_{j_1,j_2,\dots,j_n=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_n}g\|^2_{\partial\mathbf{B}^n}.$$

Taking $g=Qf$, the proof is complete. □

Remark 11. The authors suspect that, for any fixed $s>-n-1$,

$$C_{pq}(s) \asymp [(p+1)(q+1)]^{-s-1} \tag{50}$$

uniformly for all $p,q\in\mathbf{N}$. If this is rue, then the proof of Theorem 9 shows that the condition (47) is further equivalent to

$$\sum_{j_1,j_2,\dots,j_{n+k+1}=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_{n+k+1}}(Qf)\|^2_k < +\infty$$

for some (equivalently, any) nonnegative integer k. See the proof of Theorem 16 below for the details. The authors showed in Theorem 11 in [EY] that (50) holds when p,q tend to infinity with he ratio p/q fixed, but were unable to get a uniform estimate. □

Remark 12. Another consequence of (50) would be an extension of the definition of $\mathcal{M}_s$ from the original range $s>-1$, and our “analytic continuation” to $s>-n-1$, to all real s. Namely, denoting

$$\mathcal{M}_{\#s} := \{f=\sum_{p,q} f_{pq},\ f_{pq}\in\mathbf{H}^{pq}:\ \sum_{p,q}\frac{\|f_{pq}\|^2_{\partial\mathbf{B}^n}}{(p+1)^{s+1}(q+1)^{s+1}} =: \|f\|^2_{\#s} < +\infty\}$$

we would then have

$$\mathcal{M}_{\#s} = \mathcal{M}_s \quad \text{for } s>-n-1, \text{ with equivalent norms,}$$

by (50), and

$$Q\mathcal{M}_{\#s} = Q\mathcal{M}_{\circ\circ} \quad \text{for } s=-n-1,$$

with equivalent norms on $M_{\circ\circ,0}$, by (40). Since evidently $\mathcal{M}_{\#s}\subset\mathcal{M}_{\#s'}$ continuously for $s<s'$, it would also follow that

$$\mathcal{M}_{\circ\circ,0}\subset\mathcal{M}_s \qquad \forall s>-n-1,$$

which inclusion the current authors are unable to verify.

(We believe the lower bound in (50) can be obtained from the inequality

$$\frac{\Gamma(n+p)\Gamma(n+q)}{\Gamma(n)\Gamma(n+p+q)}\,{}_2F_1\Big(\begin{matrix}p,q\\ p+q+n\end{matrix}\Big|t\Big) \ge \frac{t^{pq}}{3^{1/4}},$$

which seems to be true but we have not been able to prove it. We also have no clue how to get the upper bound in (50).) □

A priori, it is not evident that

$$\|f\|^2_{\circ\circ} = \sum_{P,q}\|f_{pq}\|^2_{\partial\mathbf{B}^n}\lim_{s\searrow-n-1}(n+s+1)^2C_{pq}(s)$$

coincides with

$$\lim_{s\searrow-n-1}(n+s+1)^2\sum_{P,q}C_{pq}(s)\|f_{pq}\|^2_{\partial\mathbf{B}^n} = \lim_{s\searrow-n-1}(n+s+1)^2\|f\|^2_s$$

— none of the standard conditions for interchanging the limit and the summation seems to apply. However, by Fatou's lemma, we at least always have

$$\|f\|^2_{\circ\circ} \le \liminf_{s\searrow -n-1} (n+s+1)^2 \|f\|^2_s, \tag{51}$$

with equality, of course, when the sums above are finite (i.e. for f with only finitely many nonzero $\mathbf{H}^{pq}$-components). In other words, if we introduce the space

$$\mathcal{M}' := \{f \in \mathcal{M} : f \in \mathcal{M}_s \ \forall s > -n-1 \text{ and a finite } \lim_{s\searrow -n-1} (n+s+1)^2 \|f\|^2_s \text{ exists}\}, \tag{52}$$

then plainly the last limit is a (semi-)norm on $\mathcal{M}'$, the limit

$$\langle f, g\rangle' := \lim_{s\searrow -n-1} (n+s+1)^2 \langle f, g\rangle_s$$

exists for any $f, g \in \mathcal{M}'$ and makes $\mathcal{M}'$ into a (semi-)inner product space, and by the remarks above,

$$\mathcal{M}' \subset \mathcal{M}_{\circ\circ} \text{ continuously} \tag{53}$$

while the algebraic span of $\mathbf{H}^{pq}$, $p, q \ge 0$, is contained in $\mathcal{M}'$ and the norms $\|\cdot\|'$ and $\|\cdot\|_{\circ\circ}$ coincide on it. It follows that $\mathcal{M}_{\circ\circ}$ is just the completion of $\mathcal{M}'$ with respect to the above norm.

Remark 13. It follows from (39) in Remark 4 above that $\|f\|^2_s$ is actually a non-increasing function of $s > -n-1$, for any $f \in \mathcal{M}$. In particular, one has

$$\mathcal{M}_s \subset \mathcal{M}_{s'} \quad \text{if } s' > s > -n-1. \quad \square$$

Remark 14. Up to the authors' knowledge, it seems to be an interesting open problem whether $\lim_{s\searrow -n-1}(n+s+1)^2\|f\|^2_s$ actually always exists for $f \in \mathcal{M}_{\circ\circ}$ and coincides with $\|f\|^2_{\circ\circ}$ (that is, whether $\mathcal{M}' = \mathcal{M}_{\circ\circ}$). The analogous assertion in the context of the ordinary holomorphic Dirichlet space holds: namely, by (4),

$$(n+s+1)\|f\|^2_s = (n+s+1)\|f_0\|^2_{\partial\mathbf{B}^n} + \sum_{j\ge 1} \frac{(n)_j}{(n+s+2)_{j-1}} \|f_j\|^2_{\partial\mathbf{B}^n},$$

and as $s \searrow -n-1$, the last sum tends to $\sum_{j\ge1} \frac{(n)_j}{(j-1)!}\|f_j\|^2_{\partial\mathbf{B}^n} = \|f\|^2_\circ$ by the Lebesgue Monotone Convergence Theorem. $\square$

Theorem 15. *The inner product in $\mathcal{M}_{\circ\circ}$ is Moebius invariant:*

$$\langle f, g\rangle_{\circ\circ} = \langle f\circ\phi, g\circ\phi\rangle_{\circ\circ} \tag{54}$$

for any $f, g \in \mathcal{M}_{\circ\circ}$ and $\phi \in \mathrm{Aut}(\mathbf{B}^n)$.

Proof. Any $\phi \in \mathrm{Aut}(\mathbf{B}^n)$ can be written in the form $\phi = U \circ \phi_a \circ V$, where $U, V \in U(n)$ while ϕ_a denotes the geodesic symmetry (i.e. $\phi_a \circ \phi_a = \mathrm{id}$ and ϕ_a has only an isolated fixed-point) interchanging the origin $0 \in \mathbf{B}^n$ with the point $(a, 0, 0, \dots, 0) \in \mathbf{B}^n$, $0 \le a < 1$. Since both $\mathcal{M}_{\circ\circ}$ and its inner product $\langle\cdot,\cdot\rangle_{\circ\circ}$ are $U(n)$-invariant (by their very construction), it is enough to prove the assertion for $\phi = \phi_a$. Furthermore, since we know from Corollary 10 (or, rather, from its proof) that the composition operator $f \mapsto f\circ\phi$ is continuous on $\mathcal{M}_{\circ\circ}$, it is further enough to prove the assertion for f, g in a dense subset of $\mathcal{M}_{\circ\circ}$. In particular, by linearity, we may assume that $f \in \mathbf{H}^{pq}$ and $g \in \mathbf{H}^{p'q'}$ for some $p, q, p', q' \in \mathbf{N}$.

We will show that under all these hypotheses, $\langle f\circ\phi_a, g\circ\phi_a\rangle'$ exists for all $0\le a<1$ and does not depend on a. By the observations in the paragraph before the theorem, this will complete the proof.

Fix $0<\rho<1$. Recall that the measure

$$d\tau(z) := \frac{dz}{(1-|z|^2)^{n+1}}$$

on $\mathbf{B}^n$ is invariant under ϕ, and also

$$1-|\phi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-az_1|^2}.$$

By the change of variable $z\mapsto\phi_a(z)$, we thus have, for any $s>-1$,

$$\begin{aligned}\langle f\circ\phi_a, g\circ\phi_a\rangle_s &= \frac{(s+1)_n}{\pi^n}\int_{\mathbf{B}^n}(f\overline{g})(\phi_a(z))(1-|z|^2)^{s+n+1}\,d\tau(z)\\ &= \frac{(s+1)_n}{\pi^n}\int_{\mathbf{B}^n}(f\overline{g})(z)(1-|\phi_a(z)|^2)^{s+n+1}\,d\tau(z)\\ &= \int_{\mathbf{B}^n}(f\overline{g})(z)\Big(\frac{1-a^2}{|1-az_1|^2}\Big)^{n+s+1}\,d\mu_s(z).\end{aligned}$$

Passing to the polar coordinate $z=r\zeta$, with $0\le r<1$ and $\zeta\in\partial\mathbf{B}^n$, we can continue with

$$= \frac{(s+1)_n}{\pi^n}\int_0^1\frac{2\pi^n}{\Gamma(n)}\int_{\partial\mathbf{B}^n}(f\overline{g})(r\zeta)\Big(\frac{1-a^2}{|1-ar\zeta_1|^2}\Big)^{n+s+1}(1-r^2)^s r^{2n-1}\,d\zeta\,dr, \tag{55}$$

that is, using (25).

$$= \frac{(s+1)_n}{\pi^n}\int_0^1 G(a,r)(1-r^2)^s r^{2n-1}\,dr,$$

$$\text{where } G(a,r) := \frac{2\pi^n}{\Gamma(n)}S^{pq}(r)S^{p'q'}(r)\int_{\partial\mathbf{B}^n}(f\overline{g})(\zeta)\Big(\frac{1-a^2}{|1-ar\zeta_1|^2}\Big)^{n+s+1}\,d\zeta.$$

Carrying out the ζ integration shows that $G(a,r)$ is a holomorphic function of $|a|<\rho$ and $|r|<1/\rho$. Invoking Lemma 2, it thus follows in the same way as in the proof of Proposition 3 that $\langle f\circ\phi_a, g\circ\phi_a\rangle_s$ extends to a holomorphic function of $|a|<\rho$ and $s\in\mathbf{C}$, except for at most double poles at $s=-n-1,-n-2,\dots,-2n$ and at most triple poles at $s=-2n-j-1$, $j\in\mathbf{N}$. Consequently, the function $(s+n+1)^2\langle f\circ\phi_a, g\circ\phi_a\rangle_s$ extends to a holomorphic function of $|a|<\rho$ and $s\in\mathbf{C}$ except for poles as above, excluding $s=-n-1$ where it assumes a finite value. In particular (taking $f=g$), this means that $f\circ\phi_a, g\circ\phi_a\in\mathcal{M}'$ for all $0\le a<\rho$, and the inner product $\langle f\circ\phi_a, g\circ\phi_a\rangle'$ is a smooth function of these a.

Finally, it is legitimate to differentiate under the integral sign in (55), yielding, for $s>-1$,

$$\begin{aligned}\frac{\partial}{\partial a}\langle f\circ\phi_a, g\circ\phi_a\rangle_s &= \frac{(s+1)_n}{\pi^n}\int_0^1\frac{2\pi^n}{\Gamma(n)}\int_{\partial\mathbf{B}^n}(f\overline{g})(\zeta)S^{pq}(r)S^{p'q'}(r)\times\\ &\quad (n+s+1)\Big(\frac{1-a^2}{|1-ar\zeta_1|^2}\Big)^{n+s}\Big[\frac{\partial}{\partial a}\frac{1-a^2}{|1-ar\zeta_1|^2}\Big](1-r^2)^s r^{2n-1}\,d\zeta\,dr.\end{aligned} \tag{56}$$

Repeating the argument above, it transpires that for all $0\le a<\rho$,

$$\frac{\partial}{\partial a}(n+s+1)^2\langle f\circ\phi_a, g\circ\phi_a\rangle_s = (n+s+1)F_a(s), \tag{57}$$

where $F_a(s)$ is a holomorphic function of s except for at most triple poles at $s = -2n-1-j$, $j \in \mathbf{N}$, and at most double poles at $s = -n-2, \dots, -2n$; in particular, $F_a(s)$ is holomorphic near $s = -n-1$ and assumes a finite value there. Hence, thanks to the factor $n+s+1$ in (57),

$$\frac{\partial}{\partial a} \langle f \circ \phi_a, g \circ \phi_a \rangle' = 0 \quad \text{for } 0 \le a < \rho.$$

Since ρ was arbitrary, it follows that $\langle f \circ \phi_a, g \circ \phi_a \rangle' = \langle f \circ \phi_0, g \circ \phi_0 \rangle' = \langle f, g \rangle'$ for all $0 \le a < 1$, completing the proof. $\square$

5. The pluriharmonic Dirichlet space

For $pq = 0$, the coefficients $C_{p0}(s) = C_{0p}(s)$ have only a single pole at $s = -n-1$ (cf. (37)) for $p \neq 0$, with residue

$$C^\circ_{p0} := \lim_{s \searrow -n-1} (n+s+1) C_{p0}(s) = \frac{(n)_p}{\Gamma(p)},$$

while $C^\circ_{00} := \lim_{s \searrow -n-1} (n+s+1) C_{00}(s) = 0$. Accordingly, $\mathcal{M}_\circ$ consists only of pluriharmonic functions, with (semi-)norm

$$\|f\|^2_\circ := \sum_{p=1}^{\infty} p \frac{(n)_p}{p!} (\|f_{p0}\|^2_{\partial \mathbf{B}^n} + \|f_{0p}\|^2_{\partial \mathbf{B}^n}).$$

In other words,

$$\mathcal{M}_\circ = \mathcal{A}_\circ \oplus \overline{\mathcal{A}_\circ}$$

is just the orthogonal sum of the usual holomorphic Dirichlet space $\mathcal{A}_\circ$ and its complex conjugate.

The result below, parallel to Theorem 9 for the M-harmonic case, is likely folk lore, but the authors are unaware of a specific reference.

Theorem 16. *If f is pluriharmonic on $\mathbf{B}^n$, $n \ge 2$, then $f \in \mathcal{M}_\circ$ if and only if*

$$\sum_{j_1, j_2, \dots, j_n = 1}^{2n(n-1)} \|\mathcal{L}_{j_1} \mathcal{L}_{j_2} \dots \mathcal{L}_{j_n} f\|^2_{\text{Hardy}} < +\infty \tag{58}$$

if and only if

$$\sum_{j_1, j_2, \dots, j_{n+k+1} = 1}^{2n(n-1)} \|\mathcal{L}_{j_1} \mathcal{L}_{j_2} \dots \mathcal{L}_{j_{n+k+1}} f\|^2_k < +\infty, \tag{59}$$

for some (equivalently, any) nonnegative integer k.

Proof. As we have seen in (48) in the proof of Theorem 9, (58) equals, for $f = \sum_{p,q} f_{pq}$ with $f_{pq} \in \mathbf{H}^{pq}$,

$$\sum_{j_1, j_2, \dots, j_n = 1}^{2n(n-1)} \|\mathcal{L}_{j_1} \mathcal{L}_{j_2} \dots \mathcal{L}_{j_n} f\|^2_{\partial \mathbf{B}^n} = \sum_{p,q} [4pq + (2n-2)(p+q)]^n \|f_{pq}\|^2_{\partial \mathbf{B}^n}.$$

Since f is now pluriharmonic, the right-hand side reduces just to

$$\sum_{p=1}^{\infty} [(2n-2)p]^n (\|f_{p0}\|^2_{\partial \mathbf{B}^n} + \|f_{0p}\|^2_{\partial \mathbf{B}^n}).$$

As $n \geq 2$ by hypothesis, we have

$$[(2n-2)p]^n \asymp p^n \asymp \frac{\Gamma(n+p)}{\Gamma(n)\Gamma(p)} = C^\circ_{p0}$$

for all p, and the first claim follows.

For the second claim, denote again, for any function f on $\mathbf{B}^n$, $f_r(\zeta) := f(r\zeta)$ for $0 < r < 1$ and $\zeta \in \partial\mathbf{B}^n$. Using (48) for $g = f_r$ yields

$$\sum_{j_1,j_2,\dots,j_m=1}^{2n(n-1)} \|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_m} f_r\|^2_{\partial\mathbf{B}^n} = \sum_{p,q} [4pq + (2n-2)(p+q)]^m S^{pq}(r)^2 \|f_{pq}\|^2_{\partial\mathbf{B}^n},$$

since $(f_r)_{pq}(\zeta) = S^{pq}(r) f_{pq}(\zeta)$ by (25). As $\mathcal{L}_j$, being tangential, do not act on the r variable, we also have

$$\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_m} f_r = (\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_m} f)_r.$$

Hence for any $s > -1$,

$$\begin{aligned}
\|\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_m} f\|^2_s &= \frac{(s+1)_n}{\pi^n} \int_0^1 \frac{2\pi^n}{\Gamma(n)} \|(\mathcal{L}_{j_1}\mathcal{L}_{j_2}\dots\mathcal{L}_{j_m} f)_r\|^2_{\partial\mathbf{B}^n} (1-r^2)^s r^{2n-1}\,dr \\
&= \frac{(s+1)_n}{\Gamma(n)} \sum_{p,q} [4pq + (2n-2)(p+q)]^m \|f_{pq}\|^2_{\partial\mathbf{B}^n} \int_0^1 S^{pq}(\sqrt{t})^2 t^{n-1}(1-t)^s\,dt \\
&= \sum_{p,q} [4pq + (2n-2)(p+q)]^m \|f_{pq}\|^2_{\partial\mathbf{B}^n} C_{pq}(s).
\end{aligned}$$

Specializing now to the current pluriharmonic case $pq = 0$, we again have for all $p \geq 1$

$$[(2n-2)p]^m C_{p0}(s) \asymp p^m C_{p0}(s) \asymp p^{m-s-1} \asymp p^{m-n-s-1} C^\circ_{p0}.$$

Hence for $s = k$ and $m = n + k + 1$, with any $k = 0, 1, 2, \dots$,

$$[(2n-2)p]^{n+k+1} C_{p0}(k) \asymp C^\circ_{p0} \qquad \forall p \in \mathbf{N}$$

(for $p = 0$, both sides vanish), and the second claim follows. □

The following simple result seems to have no counterpart in the M-harmonic case.

Theorem 17. *If f is pluriharmonic on $\mathbf{B}^n$, $n \geq 1$, then $f \in \mathcal{M}_\circ$ if and only if*

$$\|\mathcal{N}^m f\|^2_{2m-n-1} < +\infty$$

for some (equivalently, any) integer $m > \frac{n}{2}$. Furthermore, the square root of the left-hand side is a seminorm equivalent to $\|f\|_\circ$.

Proof. By straightforward inspection,

$$\mathcal{N} f_{p0} = p f_{p0}, \quad \mathcal{N} f_{0p} = p f_{0p} \qquad \forall p \geq 0.$$

Consequently,

$$\|\mathcal{N}^m f_{pq}\|^2_{\partial\mathbf{B}^n} = (p+q)^{2m} \|f_{pq}\|^2_{\partial\mathbf{B}^n} \qquad \text{for } pq = 0,$$

and, as in the preceding proof, for any pluriharmonic f,

$$\begin{aligned}
\|\mathcal{N}^m f\|^2_s &= \sum_p p^{2m} C_{p0}(s)(\|f_{p0}\|^2_{\partial\mathbf{B}^n} + \|f_{0p}\|^2_{\partial\mathbf{B}^n}) \\
&\asymp p^{2m-s-n-1} C^\circ_{p0}(\|f_{p0}\|^2_{\partial\mathbf{B}^n} + \|f_{0p}\|^2_{\partial\mathbf{B}^n})
\end{aligned}$$

$$\asymp \|f\|_\circ^2 \quad \text{if } 2m = n+s+1,$$

completing the proof. □

6. The harmonic Dirichlet space

The situation in the harmonic case is pretty similar as for the pluriharmonic functions in the preceding section, so we will be brief. For all $p \geq 0$, let $\mathbf{H}^p$ be the space of harmonic polynomials on $\mathbf{R}^n$, $n \geq 2$, homogeneous of degree p, and let $\mathcal{H}^p$ be the space of restrictions of elements of $\mathbf{H}^p$ to the unit sphere ∂B^n. We refer to [ABR], especially Chapter 5, for the Peter-Weyl decomposition

$$L^2(\partial B^n, d\sigma) = \bigoplus_{p=0}^{\infty} \mathcal{H}^p \tag{60}$$

under the action of the orthogonal group $O(n)$ of rotations of $\mathbf{R}^n$, and the associated decomposition

$$\mathcal{H} = \bigoplus_p \mathbf{H}^p \tag{61}$$

of the space of all harmonic functions on the unit ball B^n of $\mathbf{R}^n$ into the direct sum of the $\mathbf{H}^p$: namely, any harmonic function f on B^n can be uniquely written as

$$f = \sum_{p=0}^{\infty} f_p, \qquad f_p \in \mathbf{H}^p, \tag{62}$$

with the sum converging uniformly on compact subsets. Here $d\sigma$ now stands for the normalized surface measure on ∂B^n. The *weighted harmonic Bergman space*

$$\mathcal{H}_s(B^n) := \{f \in L^2(B^n, d\rho_s) : \ f \text{ is harmonic on } B^n\}$$

consists of all harmonic functions on B^n square-integrable with respect to the measure

$$d\rho_s(x) := \frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)}(1-|x|^2)^s\,dx, \qquad s > -1, \tag{63}$$

where dx denotes the Lebesgue volume on $\mathbf{R}^n$. The restriction on s ensures that these spaces are nontrivial, and the factor $\frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)}$ makes $d\rho_s$ a probability measure, so that $\|\mathbf{1}\| = 1$. For f as in (62), we have by the orthogonality in (60)

$$\begin{aligned}
\|f\|_s^2 &= \frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)} \int_0^1 \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_{\partial B^n} |f(r\zeta)|^2\,d\sigma(\zeta)\,(1-r^2)^s r^{n-1}\,dr \\
&= \frac{\Gamma(\frac{n}{2}+s+1)}{\Gamma(\frac{n}{2})\Gamma(s+1)} \int_0^1 \sum_{p=0}^{\infty} r^{2p}\|f_p\|_{\partial B^n}^2 (1-r^2)^s\,2r^{n-1}\,dr \\
&= \frac{\Gamma(\frac{n}{2}+s+1)}{\Gamma(\frac{n}{2})\Gamma(s+1)} \sum_p \|f_p\|_{\partial B^n}^2 \int_0^1 t^{p+\frac{n}{2}-1}(1-t)^s\,dt \\
&= \frac{\Gamma(\frac{n}{2}+s+1)}{\Gamma(\frac{n}{2})\Gamma(s+1)} \sum_p \frac{\Gamma(p+\frac{n}{2})\Gamma(s+1)}{\Gamma(p+s+\frac{n}{2}+1)} \|f_p\|_{\partial B^n}^2 \\
&= \sum_p \frac{(\frac{n}{2})_p}{(\frac{n}{2}+s+1)_p} \|f_p\|_{\partial B^n}^2, \qquad (64)
\end{aligned}$$

and, accordingly, the reproducing kernel of $\mathcal{H}_s$ is given by

$$K_s^{\rm harm}(x,y)=\sum_p \frac{(\frac n2+s+1)_p}{(\frac n2)_p} Z_p(x,y), \tag{65}$$

where $Z_p(x,y)$, the reproducing kernel of $\mathbf{H}^p$, is given by so-called zonal harmonics (expressible explicitly in terms of Gegenbauer polynomials); see Chapter 8 in [ABR] for the unweighted case, the weighted case being completely parallel.

The coefficients

$$C_p^{\rm harm}(s):=\frac{(\frac n2)_p}{(\frac n2+s+1)_p}, \qquad p\in\mathbf{N}, \tag{66}$$

extend to nonvanishing holomorphic functions of s on the entire $\mathbf{C}$, except for simple poles at $s=-\frac n2-1,\dots,-\frac n2-p$; accordingly, $K_s^{\rm harm}(x,y)$ extends to a holomorphic function of $s\in\mathbf{C}$. Due to the orthogonality of the spaces $\mathbf{H}^p$, the extended kernel — still denoted $K_s^{\rm harm}$ — will remain positive definite as long as $1/C_p^{\rm harm}(s)\ge 0$ $\forall p\in\mathbf{N}$, hence, precisely for $s\in[-\frac n2-1,+\infty)$. The last interval is thus the "harmonic Wallach set" of B^n. The norm in the corresponding reproducing kernel Hilbert spaces — still denoted by $\mathcal{H}_s$ — is still given by (64) for $s>-\frac n2-1$. For $s=-\frac n2-1$, (65) reduces to constant one, and the associated space thus consists only of the constants, with $\|\mathbf{1}\|=1$. As the "residue" at $s=-\frac n2-1$, we get the reproducing kernel

$$K_\square^{\rm harm}(x,y):=\lim_{s\searrow -\frac n2-1}\frac{K_s^{\rm harm}(x,y)-1}{s+\frac n2+1}. \tag{67}$$

The corresponding reproducing kernel Hilbert space $\mathcal{H}_\square$ consists of all harmonic functions on B^n for which

$$\|f\|^2_\square:=\sum_{p=0}^\infty C_p^\square\|f_p\|^2_{\partial B^n}<+\infty, \tag{68}$$

where

$$C_p^\square:=\lim_{s\searrow-\frac n2-1}(s+\tfrac n2+1)C_p^{\rm harm}(s)=p\frac{(\frac n2)_p}{p!}. \tag{69}$$

This can be viewed as the *harmonic Dirichlet space*. It is easily seen to coincide with the eponymous space studied by other authors, see e.g. [GKU] and the numerous references therein. The characterization of $\mathcal{H}_\square$ given in Theorem 19 below, however, seems not to appear in the literature (up to the authors' knowledge).

Remark 18. As in the holomorphic case, the limit $\lim_{s\searrow-\frac n2-1}\|f\|^2_s$ always exists for any $f\in\mathcal{H}_\square$ and coincides with $\|f\|^2_\square$. The proof is the same as for the holomorphic case. □

Finally, the following characterization of $\mathcal{H}_\square$ can be given along the same lines as in the preceding sections. Recall our notation

$$X_{jk}=x_j\partial_k-x_k\partial_j, \qquad j,k=1,\dots,n, \quad j\ne k,$$

for the tangential vector fields on $\mathbf{R}^n$, and $\mathcal{X}_m$, $m=1,\dots,n(n-1)$, for the collection of all the X_{jk} (in some fixed order). By a routine computation, one checks that

$$\sum_{j,k=1}^n X_{jk}^2=2\Delta_{\rm sph},$$

where Δ_{sph} is the spherical Laplacian on $\mathbf{R}^n$: for $x = r\zeta$ with $r > 0$ and $\zeta \in \partial B^n$,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathrm{sph}}.$$

The operator Δ_{sph} commutes with the action of the orthogonal group $O(n)$ of $\mathbf{R}^n$, hence it is automatically diagonalized by the Peter-Weyl decomposition (60): a simple computation reveals that

$$\Delta_{\mathrm{sph}}|\mathbf{H}^p = -p(p+n-2)I|\mathbf{H}^p \tag{70}$$

where I stands for the identity operator.

Theorem 19. *If f is harmonic on B^n, $n \geq 2$, then $f \in \mathcal{H}_\square$ if and only if*

$$\sum_{j_1,\dots,j_m=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_m} f\|^2_{2m-\frac{n}{2}-1} < +\infty.$$

for some (equivalently, any) integer $m > \frac{n}{4}$. Furthermore, the square roots of the left-hand sides are seminorms equivalent to $\|f\|_\square$.

Proof. Since the adjoint of X_{jk} in $L^2(\partial B^n, d\sigma)$ is just $-X_{jk}$, we have for any $g \in L^2(\partial B^n, d\sigma)$

$$\sum_{j=1}^{n(n-1)} \|\mathcal{X}_j g\|^2_{\partial B^n} = -\sum_{j,k=1}^{n} \langle X_{jk}^2 g, g\rangle_{\partial B^n} = -2\langle \Delta_{\mathrm{sph}} g, g\rangle_{\partial B^n},$$

so for $g = \sum_p g_p$, $g_p \in \mathbf{H}^p$, as in (60),

$$\sum_{j=1}^{n(n-1)} \|\mathcal{X}_j g\|^2_{\partial B^n} = \sum_p 2p(p+n-2)\|g_p\|^2_{\partial B^n}, \tag{71}$$

by (70). Iterating this procedure, we get

$$\sum_{j_1,\dots,j_m=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_m} g\|^2_{\partial B^n} = \sum_p [2p(p+n-2)]^m \|g_p\|^2_{\partial B^n}.$$

Applying this now to $g(\zeta) = f(r\zeta)$ where f is harmonic on B^n, we obtain

$$\sum_{j_1,\dots,j_m=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_m} f(r\cdot)\|^2_{\partial B^n} = \sum_p [2p(p+n-2)]^m r^{2p}\|f_p\|^2_{\partial B^n},$$

and, as in (64), for any $s > -1$,

$$\begin{aligned}
&\sum_{j_1,\dots,j_m=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_m} f\|^2_s \\
&\quad = \frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)} \int_0^1 \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \sum_{j_1,\dots,j_m=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_m} f(r\cdot)\|^2_{\partial B^n} (1-r^2)^s r^{n-1}\,dr \\
&\quad = \frac{\Gamma(\frac{n}{2}+s+1)}{\Gamma(\frac{n}{2})\Gamma(s+1)} \int_0^1 \sum_p [2p(p+n-2)]^m \|f_p\|^2_{\partial B^n} t^{p+\frac{n}{2}-1}(1-t)^s\,dt \\
&\quad = \sum_p [2p(p+n-2)]^m \frac{(\frac{n}{2})_p}{(\frac{n}{2}+s+1)_p}\|f_p\|^2_{\partial B^n}.
\end{aligned}$$

Now for all $p \geq 1$

$$[2p(p+n-2)]^m \frac{(\frac{n}{2})_p}{(\frac{n}{2}+s+1)_p} \asymp p^{2m-s-1} \asymp p^{2m-s-1-\frac{n}{2}} C_p^{\square},$$

whence

$$[2p(p+n-2)]^m \frac{(\frac{n}{2})_p}{(\frac{n}{2}+s+1)_p} \asymp C_p^{\square}$$

if $2m = s+1+\frac{n}{2}$ (for $p=0$, both sides vanish). This completes the proof. $\square$

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