$Q_p$-SPACES: GENERALIZATIONS TO BOUNDED
SYMMETRIC DOMAINS

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ABSTRACT. $Q_p$ spaces on the unit disc were introduced, and their basic properties established, in 1995 by Aulaskari, Xiao and Zhao. Later some of these results were extended also to the unit ball or even to strictly pseudoconvex domains in the complex $n$-space. We briefly review the theory of bounded symmetric domains, of which the disc and the ball are the simplest examples, and then discuss the $Q_p$ spaces in this setting. It turns out that some new phenomena appear, most notably concerning the relationships of these spaces to the various kinds of Bloch spaces on symmetric domains.

1. INTRODUCTION

The $Q_p$ spaces on the unit disc $D$ were introduced in 1995 by Aulaskari, Xiao and Zhao [AXZ] by

$$f \in Q_p \iff \sup_{a \in D} \int_D |f'(z)|^2 g(z, a)^p \, dz < \infty,$$


the square root of the right-hand side being, by definition, the (semi)norm in $Q_p$. Here $g(z, a)$ stands for the Green function

$$g(z, a) = \log \left| \frac{z - a}{1 - \bar{a} \pi z} \right|,$$

and $dz$ denotes the Lebesgue area measure. It is not difficult to see that one gets the same spaces, with equivalent seminorms, upon replacing the Green function by the function $\log |\frac{z - a}{1 - \bar{a} \pi z}|$ which has (for each fixed $a$) the same boundary behaviour:

$$f \in Q_p \iff \sup_{a \in D} \int_D |f'(z)|^2 \left( 1 - \left| \frac{z - a}{1 - \bar{a} \pi z} \right|^2 \right)^p \, dz < \infty. \quad (1)$$

We will adhere to this latter definition throughout the sequel.

The most notable feature of the $Q_p$ spaces is that they are Möbius invariant. Indeed, any Möbius map (i.e. a biholomorphic self-map of $D$) is of the form $\phi(z) =$
\( \epsilon \frac{a - z}{1 - \overline{a}z} \) with \(|\epsilon| = 1\) and \(a \in \mathbf{D}\). Thus the right-hand side of (1) can be rewritten as

\[
\sup_{\phi \in \text{Aut}(\mathbf{D})} \int_\mathbf{D} |f'(z)|^2 (1 - |\phi(z)|^2)^p \, dz
\]

\[
= \sup_{\phi \in \text{Aut}(\mathbf{D})} \int_\mathbf{D} \Delta |f|^2(z) (1 - |\phi(z)|^2)^p \, dz
\]

\[
= \sup_{\phi \in \text{Aut}(\mathbf{D})} \int_\mathbf{D} (\tilde{\Delta}|f|^2(z) (1 - |\phi(z)|^2)^p \, d\mu(z)
\]

\[
= \sup_{\phi \in \text{Aut}(\mathbf{D})} \int_\mathbf{D} \tilde{\Delta}|f \circ \phi(z)|^2 (1 - |z|^2)^p \, d\mu(z),
\]

where \(\tilde{\Delta} = (1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \overline{z}}\) and \(d\mu(z) = \frac{dz}{(1 - |z|^2)^{d/2}}\) are the invariant Laplacian and the invariant measure on \(\mathbf{D}\), respectively. From the last formula it is apparent that \(f \in Q_p\) implies \(f \circ \phi \in Q_p\) and \(f\) and \(f \circ \phi\) have the same norm in \(Q_p\), for all \(\phi \in \text{Aut}(\mathbf{D})\).

It was shown in [AXZ] that

\[
p > 1 \implies Q_p = \mathcal{B}, \quad \text{the Bloch space},
\]

\[
p = 1 \implies Q_p = \text{BMOA},
\]

\[
0 \leq p_1 < p_2 \leq 1 \implies Q_{p_1} \subsetneq Q_{p_2},
\]

\[
p = 0 \implies Q_p = \mathcal{D}, \quad \text{the Dirichlet space},
\]

\[
p < 0 \implies Q_p = \{\text{const}\}.
\]

Thus the \(Q_p\) spaces provide a whole range of Möbius-invariant function spaces on \(\mathbf{D}\) lying strictly between the Dirichlet space on the one hand, and \(\text{BMOA}\) and the Bloch space on the other.

The \(Q_p\) spaces subsequently attracted a lot of attention; see e.g. the book by Xiao [X] and the references therein. They were generalized to the unit ball \(\mathbf{B}^d \subset \mathbf{C}^d\) in 1998 by Ouyang, Yang and Zhao [OYZ]:

\[
f \in Q_p \iff \sup_{a \in \mathbf{B}^d} \int_{\mathbf{B}^d} \tilde{\Delta}|f(z)|^2 G(z, a)^p \, d\mu(z) < \infty
\]

\[
\iff \sup_{\phi \in \text{Aut}(\mathbf{B}^d)} \int_{\mathbf{B}^d} \tilde{\Delta}|f \circ \phi|^2 G(z, 0)^p \, d\mu(z) < \infty,
\]

where \(\tilde{\Delta}, d\mu\) and \(G(z, a)\) denote the invariant Laplacian, the invariant measure and the Green function of \(\tilde{\Delta}\) on \(\mathbf{B}^d\), respectively. Again, these spaces are Möbius invariant, and

\[
p \geq \frac{d}{d - 1}, \quad \implies Q_p = \{\text{const}\},
\]

\[
1 < p < \frac{d}{d - 1} \implies Q_p = \mathcal{B}(\mathbf{B}^d), \quad \text{the Bloch space},
\]

\[
p = 1 \implies Q_p = \text{BMOA}(\mathbf{B}^d),
\]

\[
\frac{d - 1}{d} < p_1 < p_2 \leq 1 \implies Q_{p_1} \subsetneq Q_{p_2},
\]

\[
p \leq \frac{d - 1}{d} \implies Q_p = \{\text{const}\}.
\]
The cut-off at \( p = \frac{d}{d+1} \) turns out to due to the pole of \( G(z, a) \) at \( z = a \), and disappears if we replace (as we did for the disc) the Green function by \( (1 - \frac{z-a}{1-(a,z)})^d \), i.e. upon setting

\[
    f \in Q_p \iff \sup_{\phi \in \text{Aut}(\mathbb{B}^d)} \int_{\mathbb{B}^d} \Delta |f \circ \phi|^2(z) (1 - \|z\|^2)^p d\mu(z) < \infty.
\]

Then \( Q_p = \mathcal{B}(\mathbb{B}^d) \) \( \forall p > 1 \), while the other cases remain unchanged. (We again stick to this latter definition in the sequel.)

Note that, in contrast to the disc, for \( d > 1 \) the Dirichlet space does not turn up as one of the \( Q_p \)'s, though in all other cases the situation is the same as for \( D \).

Other generalizations include \( Q_p \) spaces on smoothly bounded strictly pseudo-convex domains \([AC]\) or the \( F(p, q, s) \) spaces of Rätyyä and Zhao \([R],[Z]\). In this talk, we will consider generalization in another direction, suggested by the appearance of the invariant Laplacians \( \Delta \), the invariant measures \( d\mu \), and the invariance of the spaces under Möbius maps — namely, the generalization to bounded symmetric domains.

### 2. Bounded symmetric domains

Recall that a bounded domain \( \Omega \subset \mathbb{C}^d \) is called symmetric if \( \forall x \in \Omega \) there exists \( s_x \in \text{Aut}(\Omega) \) such that \( s_x \circ s_x = \text{id} \) and \( x \) is an isolated fixed-point of \( s_x \). One calls \( s_x \) the geodesic symmetry at \( x \). The motivating example behind this is, of course, the complex n-space \( \Omega = \mathbb{C}^n \) with \( s_x(z) = 2x - z \) (except that this is not a bounded domain). Another example is the unit disc \( D \) with \( s_0(z) = -z \) and \( s_x = \phi_x \circ s_0 \circ \phi_x \), where \( \phi_x(z) = \frac{x - z}{1 - \overline{x}z} \) is the geodesic symmetry interchanging 0 and \( x \). A more general example is the unit ball \( I_{r \times R} \) of \( r \times R \) complex matrices \( (R,r \geq 1) \), again with \( s_0(z) = -z \) and \( s_x = \phi_x \circ s_0 \circ \phi_x \), the geodesic symmetry \( \phi_x \) interchanging 0 and \( x \) being now given by

\[
    s_x(z) = (I_{r} - xx^*)^{-1/2} (x - z)(I_{R} - x^*z)^{-1} (I_{R} - x^*x)^{1/2}.
\]

Note that this includes the unit ball \( \mathbb{B}^d \subset \mathbb{C}^d \) as the special case \( I_{1d} \).

It turns out that symmetry implies homogeneity: the group \( \text{Aut}(\Omega) := G \) acts transitively on \( \Omega \). (In fact, already the symmetries \( s_x \) do.) It is a semisimple Lie group.

A bounded symmetric domain is called irreducible if it is not biholomorphic to a Cartesian product of two other bounded symmetric domains.

Irreducible bounded symmetric domains were completely classified by E. Cartan. There are four infinite series of such domains plus two exceptional domains in \( \mathbb{C}^{16} \) and \( \mathbb{C}^{27} \):

<table>
<thead>
<tr>
<th>Domain</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_{rR} )</td>
<td>( Z \in \mathbb{C}^{r \times R} ): ( |Z|_{\mathbb{C}^r} &lt; 1 ) ( \forall ) ( R \geq r \geq 1 )</td>
</tr>
<tr>
<td>( II_r )</td>
<td>( Z \in I_{rr}, Z = Z^t ) ( r \geq 2 )</td>
</tr>
<tr>
<td>( III_m )</td>
<td>( Z \in I_{mm}, Z = -Z^t ) ( m \geq 5 )</td>
</tr>
<tr>
<td>( IV_n )</td>
<td>( Z \in \mathbb{C}^{n \times 1} ): (</td>
</tr>
<tr>
<td>( V )</td>
<td>( Z \in \mathbb{O}^{1 \times 2} ): ( |Z| &lt; 1 )</td>
</tr>
<tr>
<td>( VI )</td>
<td>( Z \in \mathbb{O}^{3 \times 3} ): ( Z = Z^*, |Z| &lt; 1 )</td>
</tr>
</tbody>
</table>
The restrictions on \( R, r, m, n \) stem from a few isomorphisms in low dimensions:

\[
\begin{align*}
IV_1 & \cong III_2 \cong I_1, \\
IV_3 & \cong II_2, \\
IV_4 & \cong I_3, \\
IV_5 & \cong IV_3, \\
IV_6 & \cong I_{rR}, \\
II_1 & \cong III_2, \\
II_2 & \cong I \times D, \\
II_3 & \cong IV_3, \\
II_4 & \cong I^2, \\
III_3 & \cong I_1, \\
III_4 & \cong IV_6, \\
IV_7 & \cong I_{rR}. 
\end{align*}
\]

Up to biholomorphic equivalence, any irreducible bounded symmetric domain is uniquely determined by three integers, namely its rank \( r \) and its characteristic multiplicities \( a, b \).

<table>
<thead>
<tr>
<th>Domain</th>
<th>( r )</th>
<th>( a )</th>
<th>( b )</th>
<th>( d )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_{rR} ) ((r \leq R))</td>
<td>( r )</td>
<td>( 2 )</td>
<td>( R - r )</td>
<td>( rR )</td>
<td>( r + R )</td>
</tr>
<tr>
<td>( III_{r+\varepsilon} ), ( \varepsilon \in {0, 1} )</td>
<td>( r )</td>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( \frac{1}{2}r(r + 1) )</td>
<td>( r + 1 )</td>
</tr>
<tr>
<td>( IV_n )</td>
<td>( 2 )</td>
<td>( n - 2 )</td>
<td>( 0 )</td>
<td>( r(2n + 2\varepsilon - 1) )</td>
<td>( 4r + 2\varepsilon - 2 )</td>
</tr>
<tr>
<td>( V )</td>
<td>( 2 )</td>
<td>( 6 )</td>
<td>( 4 )</td>
<td>( 16 )</td>
<td>( 12 )</td>
</tr>
<tr>
<td>( VI )</td>
<td>( 3 )</td>
<td>( 8 )</td>
<td>( 0 )</td>
<td>( 27 )</td>
<td>( 18 )</td>
</tr>
</tbody>
</table>

The two other important quantities given in the table, the genus \( p \) and the dimension \( d \) are related to \( a, b \) and \( r \) by

\[
p = (r - 1)a + b + 2, \quad d = \frac{r(r - 1)}{2}a + rb + r.
\]

The domains with \( b = 0 \) are in some respects “simpler” than others and are called tube domains. Thus, for instance, \( I_{rR} \) is tube \( \iff \ r = R \).

The unit balls \( B^d = I_{1d} \) are the only bounded symmetric domains of rank 1, and the only bounded symmetric domains with smooth boundary.

The domains in the list above are called Cartan domains. Clearly, any Cartan domain is convex, contains the origin, and is circular with respect to it.

From now on, we will suppose (unless explicitly stated otherwise) that \( \Omega \) is a Cartan domain, and for each \( x \in \Omega \) we denote by \( \phi_x \) the (unique) geodesic symmetry interchanging 0 and \( x \). We further denote by \( K \) the stabilizer of the origin in \( G \),

\[
K := \{ k \in G : k0 = 0 \}.
\]

It is a consequence of one of Cartan’s theorems that any \( k \in K \) is automatically a unitary linear map on \( C^d \).

Note that from the definition of \( K \) it is immediate that any \( \phi \in G \) is of the form \( \phi = \phi_x k \), where \( k \in K \), \( x \in \Omega \). (In fact \( x = \phi(0) \).)

Having recalled the definition of bounded symmetric domains, we can turn to the \( Q_p \) spaces on them. We have seen in the Introduction that their definition involves three ingredients — namely, the invariant Laplacian \( \bar{\Delta} \), the invariant measure \( d\mu \), and powers of the function \( 1 - |z|^2 \) (or, for the ball, \( 1 - \|z\|^2 \)). Let us now clarify what are the counterparts of these on a general Cartan domain.

### 3. Invariant Differential Operators

A differential operator \( L \) on a Cartan domain \( \Omega \) is called invariant if

\[
L(f \circ \phi) = (Lf) \circ \phi \quad \forall \phi \in G = \text{Aut}(\Omega).
\]

It is well known that on the unit disc, invariant operators are precisely the polynomials of the invariant Laplacian \( \bar{\Delta} = (1 - |z|^2)^2 \Delta \). The same is true for \( B^d \). For a general bounded symmetric domain, the situation is more complicated: namely, the
algebra of all invariant differential operators consists of all polynomials in \( r \) commuting
differential operators \( \Delta_1, \ldots, \Delta_r \), of orders 2, 4, \ldots, \( 2r \), respectively, where
\( r \) is the rank. In particular, the monomials \( \Delta_1^{n_1} \cdots \Delta_r^{n_r} \) form a linear basis of all
invariant differential operators. However, often it is much more convenient to use
another basis, the construction of which we now describe.

For any invariant differential operator \( L \), let \( L_0 \) be the (non-invariant) linear
differential operator obtained upon freezing the coefficients of \( L \) at the origin, that is,
\( Lf(0) =: L_0 f(0) \). From the invariance of \( L \) it follows that
\[ k \in G, \ k0 = 0 \implies L_0( (f \circ k) ) = (L_0 f) \circ k \]
(i.e. \( L_0 \) is \( K \)-invariant) and
\[ Lf(z) = L_0( (f \circ \phi_z) )(0). \]  
(2)
Conversely, if \( L_0 \) is a \( K \)-invariant constant-coefficient differential operator, then the
recipe (2) clearly defines an invariant differential operator \( L \) on \( \Omega \). Thus there is a
1-to-1 correspondence between \((G-)\)invariant linear differential operators on \( \Omega \) and
\( K \)-invariant linear constant-coefficients differential operators on \( \mathbb{C}^d \).

Further, any constant-coefficient linear differential operator \( L_0 \) can be written
in the form \( L_0 = p(\partial, \bar{\partial}) \) for some polynomial \( p \) on \( \mathbb{C}^d \times \mathbb{C}^d \). It is not difficult to
see that such operator is \( K \)-invariant if and only if the polynomial \( p \) is \( K \)-invariant
in the sense that \( p(x, y) = p(kx, ky) \forall x, y \in \mathbb{C}^d \forall k \in K \).

Combining this with the observation in the preceding paragraph, we thus see
that invariant differential operators are in 1-to-1 correspondence with \( K \)-invariant
polynomials.

**Example.** Since \( K \) consists of unitary maps, the simplest \( K \)-invariant polynomial
(apart from the constants) is \( p(x, y) = \langle x, y \rangle \). The corresponding invariant
differential operator is
\[ Lf(a) = \Delta( (f \circ \phi_a) )(0). \]
This operator is called the invariant Laplacian of \( \Omega \); it coincides with the Laplace-
Beltrami operator with respect to the Bergman metric on \( \Omega \). Note that for \( f \)
holomorphic,
\[ L|f|^2(a) = \sum_{j=1}^d \left| \frac{\partial (f \circ \phi_a)(0)}{\partial z_j} \right|^2 = \| \partial (f \circ \phi_a)(0) \|^2 \]
is what we might call the invariant holomorphic gradient of \( f \).

In a moment, we will see that there exists a very natural basis for \( K \)-invariant
polynomials (which will thus yield the sought basis for invariant differential operators).
Prior to that, however, we review some facts about Bergman kernels on
Cartan domains.

4. Bergman spaces and kernels

The function \( h(x, y) := 1 - x\bar{y} \) on \( \mathbb{D} \) is noteworthy in a number of ways. First
of all, \( h^{-2} \) is, up to a constant factor, the Bergman kernel \( K(x, y) = \frac{1}{\pi(1-x\bar{y})^2} \).
Second, \( d\mu(z) = \frac{dz}{h(z, \bar{z})^2} \) is the invariant measure on \( \mathbb{D} \). Finally, for any \( \alpha > -1 \),
the Bergman kernel of $L^2_{\text{hol}}(D, h(z, z)^\alpha \, dz)$ is given by
\[ K_\alpha(x, y) = \frac{\text{const}}{h(x, y)^{\alpha+2}}. \]

The same properties are also possessed by the function $h(x, y) = 1 - \langle x, y \rangle$ on the ball, only in all three formulas 2 must be replaced by $d + 1$.

It is a notable fact that the same phenomenon persists for a general Cartan domain. Namely, the Bergman kernel of a Cartan domain has the form
\[ K(x, y) = \frac{1}{\text{vol } \Omega} h(x, y)^{-p}, \]
where $h(x, y)$ is an irreducible polynomial, analytic in $x$ and $y$, and such that $h(0, z) = h(z, 0) = 1 \geq h(z, z) \geq 0 \forall z \in \Omega$. The degree of $h$ is equal to the rank, $r$, and $p$ is the genus (this is always an integer $\geq 2$). Finally, the Bergman kernel of $L^2_{\text{hol}}(\Omega, h(z, z)^\alpha \, dz)$ equals
\[ K_\alpha(x, y) = \frac{\text{const}}{h(x, y)^{\alpha+p}}, \]
for any $\alpha > -1$, and
\[ d\mu(z) := \frac{dz}{h(z, z)^p} \]
is an invariant volume element on $\Omega$.

For the domains I and II in Cartan’s list, $h$ is given by $h(X, Y) = \det(I - XY^*)$; for domains of type III, the determinant gets replaced by the Pfaffian. Explicit formulas are known also for the types IV–VI.

Comparing all the facts above with the situations for the disc and the ball, we see that we should define the $Q_\nu$ spaces on a Cartan domain for any $\nu \in \mathbb{R}$ and any invariant differential operator $L$ as follows:
\[ f \in Q_{\nu,L} \iff \sup_{\phi \in G} \int_{\Omega} |L[f \circ \phi]|^2(z) \, h(z, z)^\nu \, d\mu(z) < \infty. \]
(3)

(Here we have started using the subscript $\nu$ instead of $p$ since the letter $p$ is already reserved for the genus.) Clearly, this reduces to the original definitions for $\Omega = D$ (or $B^d$) and $L$ the invariant Laplacian.

A small catch here is, however, that in order to have the square-root of the right-hand side for a seminorm, we need this right-hand side to be nonnegative for all holomorphic functions $f$. It is precisely at this point that the promised linear basis for invariant differential operators comes to the rescue; so let us exhibit it without further delay.

5. Peter–Weyl decomposition

Let $P$ denote the vector space of all (holomorphic) polynomials on $\mathbb{C}^d$. We endow $P$ with the Fock inner product
\[ \langle f, g \rangle_F := f(\partial) \, g^*(0), \quad \text{where} \quad g^*(z) := \overline{g(z)}, \]
\[ \quad = \pi^{-d} \int_{\mathbb{C}^d} f(z) \overline{g(z)} \, e^{-\|z\|^2} \, dz. \]
This makes $P$ into a pre-Hilbert space, and the action
\[ f \mapsto f \circ k, \quad k \in K, \]
is a unitary representation of $K$ on $\mathcal{P}$. It is a deep result of W. Schmidt that this representation has a multiplicity-free decomposition into irreducibles

$$\mathcal{P} = \sum_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}$$

where $\mathbf{m}$ ranges over all signatures, i.e. $r$-tuples $\mathbf{m} = (m_1, m_2, \ldots, m_r) \in \mathbb{Z}^r$ satisfying $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. Polynomials in $\mathcal{P}_{\mathbf{m}}$ are homogeneous of degree $|\mathbf{m}| := m_1 + m_2 + \cdots + m_r$; in particular, $\mathcal{P}_{(0)}$ are the constants and $\mathcal{P}_{(1)}$ the linear polynomials. Any holomorphic function thus has a decomposition $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$, $f_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}$, which refines the usual homogeneous expansion.

Since the spaces $\mathcal{P}_{\mathbf{m}}$ are finite dimensional, they automatically possess a reproducing kernel: there exist functions $K_{\mathbf{m}}(x, y)$ on $\mathbb{C}^d \times \mathbb{C}^d$ such that for each $f \in \mathcal{P}_{\mathbf{m}}$ and $y \in \mathbb{C}^d$, $f(y) = \langle f, K(\cdot, y) \rangle$. Explicitly, for any orthonormal basis $\{\psi_j\}_{j=1}^{\dim \mathcal{P}_{\mathbf{m}}}$ of $\mathcal{P}_{\mathbf{m}}$, $K_{\mathbf{m}}$ is given by

$$K_{\mathbf{m}}(x, y) = \sum_{j=1}^{\dim \mathcal{P}_{\mathbf{m}}} \psi_j(x)\overline{\psi_j(y)}. \quad (4)$$

It follows from the definition of the $\mathcal{P}_{\mathbf{m}}$ spaces that the kernels $K_{\mathbf{m}}(x, y)$ are $K$-invariant. By the discussion in the penultimate section, we therefore know that each $K_{\mathbf{m}}$ defines an invariant differential operator

$$\Delta_{\mathbf{m}} f(a) := K_{\mathbf{m}}(\partial, \partial)(f \circ \phi_a)(0), \quad a \in \Omega. \quad (5)$$

Further, one can show that $K_{\mathbf{m}}$ are actually a basis of all $K$-invariant polynomials, and, consequently, $\Delta_{\mathbf{m}}$ are a linear basis for invariant differential operators. Further, from (4) and (5) we see that for any $f$ holomorphic,

$$\Delta_{\mathbf{m}} |f|^2(a) = \sum_j |\psi_j(\partial)(f \circ \phi_a)(0)|^2 \geq 0.$$

What makes the basis $\Delta_{\mathbf{m}}$ important for our applications to the $Q_{\nu}$-spaces is the following converse to the last inequality.

**Theorem.** An invariant differential operator

$$L = \sum_{\mathbf{m}} l_{\mathbf{m}} \Delta_{\mathbf{m}}$$

satisfies $L|f|^2 \geq 0 \ \forall f$ holomorphic if and only if

$$l_{\mathbf{m}} \geq 0 \quad \forall \mathbf{m}.$$

### 6. Bloch spaces and $Q_{\nu}$ spaces on bounded symmetric domains

Thus we see that the invariant differential operators $L$ that can be used in (3) are precisely those which are linear combinations of $\Delta_{\mathbf{m}}$ with nonnegative coefficients. The most basic among such $L$ are evidently the operators $\Delta_{\mathbf{m}}$ themselves. We are thus lead to the following definitions.

**Definition.** For each signature $\mathbf{m}$, the $\mathbf{m}$-Bloch space is defined by

$$\mathcal{B}_{\mathbf{m}} = \{f \text{ holomorphic on } \Omega : \|\Delta_{\mathbf{m}} |f|^2\|_{\infty} < \infty \}.$$
We might call this the Arazy Bloch space and this reads (the unit disc, (1)). It is known that in that case the space \( B_{(1)} \) gives something new even for the unit disc.

Example. For \( m = (0) \) we have \( \Delta_m = I \), so \( B_{(0)} = H^\infty \). Also,
\[
Q_{\nu,(0)} = \begin{cases} H^\infty & \nu > p-1, \\ \{0\} & \nu \leq p-1. \end{cases}
\]

Example. For \( m = (1) \) we have \( \Delta_{(1)} = \tilde{\Delta} \), so that
\[
B_{(1)} = \{ f : \sup_a |\vartheta(f \circ \phi_a)(0)|^2 < \infty \}.
\]
This is known as the Timoney Bloch space \([T]\).

Example. Assume that \( \Omega \) is of tube type and \( s := d/r \in \mathbb{Z} \). Let \( m = (s, \ldots, s) \equiv (s^r) \). It is known that in that case the space \( \mathcal{P}_{(s^r)} = C N^s \) is one-dimensional (for the unit disc, \( N(z) = z \); for \( I_r, N(Z) = \det Z \)), the kernel \( K_m \) is given (up to a constant factor) by \( K_m(\vartheta, \tilde{\vartheta}) = N(\vartheta)^s N(\tilde{\vartheta})^s \), and the so-called Bol’s lemma says that for any \( f \) holomorphic, \( N(\vartheta)^s f(0) = \text{const} \cdot h(a)^s N(\tilde{\vartheta})^s f(a) \). (For the disc, this reads \( f(\circ \phi_a)(0) = -1(|a|^2 f'(a)) \).) Hence,
\[
\Delta_{(s^r)} f^2 = h^p |N(\vartheta)^s f|^2
\]
and
\[
B_{(s^r)} = \{ f \text{ holomorphic: } h^p |N(\vartheta)^s f|^2 \text{ is bounded} \}.
\]
We might call this the Arazy Bloch space \([A]\).

7. Example: the polydisc

For clarity, let us also see what is the situation for the bidisc \( \Omega = D^2 \). This is definitely NOT a Cartan domain (it is not irreducible), but in many respects it behaves like a Cartan domain with the rank, dimension and genus \( r = p = d = 2 \), multiplicities \( a = b = 0 \), and \( h(x, y) = (1 - x_1 y_1)(1 - x_2 y_2) \). Namely, the invariant measure is \( h(z, \bar{z})^{-2} \), the \( \Delta_j \) are symmetric polynomials in \( \Delta_1, \Delta_2 \), where \( \Delta_j := (1 - |z_j|^2)^2 \partial_j \bar{\partial}_j \); the Peter-Weyl spaces are given by \( \mathcal{P}_m = C z_1^{m_1} \bar{z}_2^{m_2} + C z_1^{m_2} \bar{z}_1^{m_1} \), for \( m = (m_1, m_2) \); and, up to a constant factor, \( K_m(x, y) = (x_1 y_1)^{m_1} (x_2 y_2)^{m_2} + (x_1 y_1)^{m_2} (x_2 y_2)^{m_1} \). It follows that
\[
\Delta_{(1,0)} = \tilde{\Delta}_1 + \Delta_2, \quad \Delta_{(1,1)} = \Delta_1 \Delta_2.
\]
Thus the Timoney Bloch space $B_{(1,0)}$ consists of all $f$ holomorphic on $D^2$ for which 
\[
(1 - |z_1|^2)^2 \left| \frac{\partial f}{\partial z_1} \right|^2 + (1 - |z_2|^2)^2 \left| \frac{\partial f}{\partial z_2} \right|^2 \text{ is bounded,}
\]
while the Arazy Bloch space $B_{(1,1)}$ consists of all $f$ holomorphic on $D^2$ for which 
\[
(1 - |z_1|^2)^2(1 - |z_2|^2)^2 \left| \frac{\partial^2 f}{\partial z_1 \partial z_2} \right|^2 \text{ is bounded.}
\]
We thus see that the Timoney Bloch space is contained in the Arazy Bloch space, and is a proper subset thereof: any holomorphic function of the form $f(z_1, z_2) = g(z_1)$ belongs to $B_{(1,0)}$, but does not belong to $B_{(1,1)}$ unless $g$ belongs to the Bloch space on the disc. We also see that the Timoney-Bloch norm vanishes precisely on the constants, while the Arazy-Bloch norm vanishes precisely on functions of the form $f(z_1) + g(z_2)$.

Similar situation can be seen to prevail for a general polydisc $D^n$: there are $n$ Bloch spaces, of which Timoney is the smallest, and Arazy the largest.

Remark. The big Hankel operator $H_f$ is compact $\iff f$ belongs to the Timoney Bloch space.

8. Composition series

The phenomenon that we have observed for the polydiscs is connected with the existence of the composition series. Let us explain this concept on the example of the unit disc $D$. There the following assertion holds.

Claim. Let $E$ be any topological vector space of holomorphic functions on $D$ which is M"obius invariant and on which the group of rotations acts strongly-continuously. Then either $E = \{0\}$, or $E = \{\text{constants}\}$, or $E$ contains all polynomials.

Proof. Let $E \ni f = \sum_{k=0}^{\infty} f_k z^k$; then by rotation invariance, 
\[
\int_0^{2\pi} \sum_{k=0}^{\infty} f_k z^k e^{-m \phi} \frac{d\phi}{2\pi} = f_m z^m \in E.
\] (7)

Thus if $f_m \neq 0$ for some $m$, then $z^m \in E$; hence, by invariance, $(\frac{a - z}{1 - \overline{a}z})^m \in E$ $\forall a \in D$. Taking this for the $f$ in (7) and 0 for the $m$ in (7), we get $a^m 1 \in E$; thus the constants are in $E$.

If even $f_m \neq 0$ for some $m \geq 1$, then, applying (7) to the same function again but this time taking 1 for the $m$ in (7) and noting that $(\frac{a - z}{1 - \overline{a}z})^m = a^m - m(1 - |a|^2)z + O(z^2)$, we see that $z \in E$; thus by invariance $a^{m-1} \frac{a - z}{1 - \overline{a}z} = a - (1 - |a|^2)z \sum_{j=1}^{\infty} \overline{a}_j z^j$ belongs to $E$, for any $a \in D$. Taking the last function for the $f$ in (7) shows that $z^j \in E \forall j$, i.e. all polynomials are in $E$. This completes the proof. \hfill $\Box$

The last theorem admits the following reformulation. Denote $\mathcal{M}_1 = \{\text{all holomorphic functions}\}$, $\mathcal{M}_0 = \{\text{constants}\}$, $\mathcal{M}_{-1} = \{0\}$. Then 
\[
E \setminus \mathcal{M}_j \neq \emptyset \implies \mathcal{P} \cap \mathcal{M}_j \subset E.
\]

It turns out that, in a sense, precisely the same thing holds for a general Cartan domain.

Namely, let 
\[
(x)_k := x(x+1) \ldots (x+k-1)
\]
denote the familiar Pochhammer symbol, and for a signature \( m = (m_1, m_2, \ldots, m_r) \), consider the function

\[
x \mapsto (x)_{m_1}(x - \frac{a}{2})_{m_2}(x - a)_{m_3} \cdots (x - \frac{r - 1}{2}a)_{m_r}, \quad x \in \mathbb{C}
\]

Let \( q(m) \) be the multiplicity of zero of this function at \( x = 0 \):

\[
q(m) := \text{card}\{j : m_j > j - \frac{1}{2}a \in \mathbb{Z}\}.
\]

Also denote by \( q \) the maximum possible value of \( q(m) \), i.e.

\[
q = \begin{cases} 
  r & \text{a even,} \\
  \lfloor \frac{r + 1}{2} \rfloor & \text{a odd.}
\end{cases}
\]

For \(-1 \leq j \leq q\), let

\[
\mathcal{M}_j = \{f = \sum_m f_m \text{ holomorphic} : f_m = 0 \text{ if } q(m) > j\}.
\]

Thus, in particular,

\[
\mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_q,
\]

\[
\mathcal{M}_{-1} = \{0\}, \quad \mathcal{M}_0 = \{\text{constants}\}, \quad \mathcal{M}_q = \{\text{all holomorphic}\}.
\]

The following deep result is due to Orsted, Faraut and Koranyi.

**Theorem.** (1) Each \( \mathcal{M}_j \) is \( G \)-invariant;

(2) for any \( G \)-invariant space \( E \) of holomorphic functions on which the action of \( K \) is strongly continuous,

\[
E \setminus \mathcal{M}_{j-1} \neq \emptyset \implies \mathcal{P} \cap \mathcal{M}_j \subset E.
\]

**Example.** For the bidisc \( D^2 \), \( q = 2 \) and

\[
\mathcal{M}_{-1} = \{0\}, \quad \mathcal{M}_0 = \{\text{constants}\}, \\
\mathcal{M}_1 = f(z_1) + g(z_2), \quad f, g \text{ holomorphic on } D, \\
\mathcal{M}_2 = \{\text{all holomorphic}\}.
\]

9. RESULTS

After all the preparations, we can finally state our results.

**Theorem 1.** If \( j < q(m) \), then the \( Q_{\nu,m} \)-norm vanishes on \( \mathcal{M}_j \); thus \( \mathcal{M}_j \) is contained in \( Q_{\nu,m} \) in a trivial way.

The same is true also for the Bloch space \( B_m \).

**Theorem 2.** If \( \nu > p - 1 \), then \( B_m \subset Q_{\nu,m} \) continuously.

**Theorem 3.** If \( q(m) \leq q(n) \), then \( Q_{\nu,m} \subset B_n \) continuously.

**Corollary.** \( \nu > p - 1 \implies Q_{\nu,m} = B_m \), with equivalent norms.

\[ q(m) \leq q(n) \implies B_m \subset B_n \text{ continuously.} \]

\[ q(m) = q(n) \implies B_m = B_n \text{ with equivalent norms.} \]

\[ q(m) = q(n), \nu > p - 1 \implies Q_{\nu,m} = Q_{\nu,n}, \text{ with equivalent norms.} \]

The last corollary exhausts the case \( \nu > p - 1 \) completely. What about \( \nu \leq p - 1 \)?

**Theorem 4.** If \( \nu < 0 \), then \( Q_{\nu,m} = \mathcal{M}_{q(m)-1} \).
Theorem 5. For \( r > 1 \) and \( m = (1, 0, \ldots, 0) \), that is, \( f \in Q_{\nu,(1)} \iff \sup_{a \in \Omega} \int_{\Omega} \tilde{\Delta} |f \circ \phi_a|^2 h^\nu \, d\mu, \) we have

\[
Q_{\nu,(1)} = \begin{cases} 
B_{(1)}, \text{ the Timoney Bloch space} & \nu > p - 1, \\
\{ \text{constants} \} & \nu \leq p - 1.
\end{cases}
\]

Note that the last theorem means that the situation for \( r > 1 \) differs radically from the one for \( r = 1 \) (disc, ball): there \( Q_{\nu} \) is nontrivial also for \( p - 2 < \nu \leq p - 1 \) (for the disc, even for \( p - 2 \leq \nu \leq p - 1 \)).

Theorem 6. For a tube domain with \( s = d \in \mathbb{Z} \), and \( m = (s^r) \),

\[
Q_{\nu,(s^r)} = \begin{cases} 
B_{(s^r)}, \text{ the Arazy Bloch space} & \nu > p - 1 \\
D & \nu = 0 \\
M_{q-1} & \nu < 0.
\end{cases}
\]

Here \( D \) is the Dirichlet space

\[
D = \{ f \text{ holomorphic: } N(\partial^*) f \in L^2(\Omega, dz) \}.
\]

At the moment, we do not know what are the spaces \( Q_{\nu,m} \) for \( \nu \) between 0 and \( p - 1 \) and \( |m| > 1 \) — for instance, whether they are properly increasing with \( \nu \). We can offer somewhat more complete information only for the polydisc:

Theorem 7. For the polydisc \( D^r \) (so that \( p = 2, q(m) = \# \{ j : m_j > 0 \} \), and \( q = r \)),

\[
q(m) < r \implies Q_{\nu,(m)} = \begin{cases} 
B_m & \nu > 1, \\
M_{q(m)-1} & \nu \leq 1;
\end{cases}
\]

\[
q(m) = r \implies Q_{\nu,(m)} = \begin{cases} 
\text{Arazy-Bloch} & \nu > 1, \\
D & \nu = 0, \\
M_{q(m)-1} & \nu < 0.
\end{cases}
\]

All this suggests the following conjecture about the nontriviality of the spaces \( Q_{\nu,m} \).

Conjecture. For tube domains with \( s = \frac{d}{r} \in \mathbb{Z} \) and \( q(m) = q \),

\[
M_{q(m)-1} \subsetneq Q_{\nu,m} \iff \nu \geq 0;
\]

in all other cases,

\[
M_{q(m)-1} \subsetneq Q_{\nu,m} \iff \nu > p - 1.
\]

10. Sketches of proofs

We proceed to give some hints about the proofs of the theorems. Details will appear in [AE].

Theorem 1. \( j < q(m) \implies M_j \subset Q_{\nu,m} \) and the norm vanishes. Similarly for \( B_m \).
Proof. Recall that
\[ f \in Q_{\nu, m} \iff \sup_a \int_{\Omega} \Delta_m |f|^2 (h \circ \phi_a)^\nu \, d\mu < \infty, \]
\[ f \in \mathcal{B}_m \iff \Delta_m |f|^2 \in L^\infty. \]

Now
\[ \Delta_m |f|^2(z) = K_m(\partial, \partial)|f \circ \phi_z|^2(0) \]
\[ = \sum_j \psi_j(\partial)\overline{\psi_j(\partial)}|f \circ \phi_z|^2(0) \]
\[ = \sum_j |\psi_j(\partial)(f \circ \phi_z)(0)|^2 \]
\[ = \sum_j |(f \circ \phi_z, \psi_j)_F|^2 \]
\[ = \|P_m(f \circ \phi_z)\|_F^2, \]
where \( P_m \) denotes the projection onto \( \mathcal{P}_m \).

Thus \( f \in \mathcal{M}_j \implies f \circ \phi_z \in \mathcal{M}_j \implies P_m(f \circ \phi_z) = 0 \implies \Delta_m |f|^2 = 0 \implies f \in \mathcal{B}_m \) and \( f \in Q_{\nu, m} \). \( \Box \)

**Theorem 2.** \( \nu > p - 1 \implies \mathcal{B}_m \subset Q_{\nu, m} \) continuously.

*Proof.* It is known that for \( \nu > p - 1 \), the measure \( h^\nu \, d\mu \) is finite. Thus \( \forall a \in \Omega, \)
\[ \int_{\Omega} (\Delta_m |f|^2) \circ \phi_a \, h^\nu \, d\mu \leq c_{\nu} \|\Delta_m |f|^2\|_\infty \]
\[ = c_{\nu} \|f\|_{\mathcal{B}_m} ^2. \]

**Theorem 3.** \( q(m) \leq q(n) \implies Q_{\nu, m} \subset \mathcal{B}_n \) continuously.

*Proof.* By the \( K \)-invariance of \( \Delta_m \) and \( h \), the integral
\[ \int_{\Omega} \Delta_m(fg) h^\nu \, d\mu \]
is a \( K \)-invariant bilinear form of \( f, g \in \mathcal{P} \). It follows from the Peter-Weyl decomposition of \( \mathcal{P} \) into the \( \mathcal{P}_m \) that any such bilinear functional must be of the form
\[ \sum_k c_{mk} \langle f_k, g_k \rangle_F, \]
for some coefficients \( c_{mk} \geq 0 \). Suppose we can show that
\[ c_{mn} > 0. \quad (8) \]

Since \( \Delta_n |f|^2(0) = \|P_n f\|_F^2 = \|f_n\|_F^2 \), it will follow that
\[ \Delta_n |f|^2(0) \leq \frac{1}{c_{mn}} \int_{\Omega} \Delta_m |f|^2 h^\nu \, d\mu. \]
Replacing \( f \) by \( f \circ \phi_a \), this becomes
\[ \Delta_n |f|^2(a) \leq \frac{1}{c_{mn}} \int_{\Omega} \Delta_m |f \circ \phi_a|^2 h^\nu \, d\mu. \]
Taking suprema over all \( a \in \Omega \) gives the assertion.
It remains to prove (8). But by the properties of the composition series,
\[ c_{mn} = 0 \iff \int_{\Omega} \Delta_m |f_n|^2 h^\nu \, d\mu = 0 \quad \forall f_n \in \mathcal{P}_n \]
\[ \iff \Delta_m |f_n|^2 (z) = 0 \quad \forall z \forall f_n \]
\[ \iff \|P_m(f_n \circ \phi_z)\|_F^2 = 0 \quad \forall z \forall f_n \]
\[ \iff P_m(f_n \circ \phi_z) = 0 \quad \forall z \forall f_n \]
\[ \iff P_m \mathcal{M}_{q(n)} = 0 \]
\[ \iff q(m) > q(n). \]

\( \square \)

**Theorem 4.** \( \nu < 0 \Rightarrow Q_{\nu,m} = \mathcal{M}_{q(m)-1}. \)

**Proof.** From the composition series we know that
\[ \mathcal{M}_{q(m)-1} \subseteq Q_{\nu,m} \Rightarrow \mathcal{P} \cap \mathcal{M}_{q(m)} \subset Q_{\nu,m} \]
\[ \Rightarrow \mathcal{P}_m \subset Q_{\nu,m} \]

\[ \Rightarrow \sup_a \int_{\Omega} \Delta_m |f|^2 (h \circ \phi_a)^\nu \, d\mu < \infty \quad \forall f \in \mathcal{P}_m. \]

Since \( K_m(z, z) = \sum_j |\psi_j(z)|^2 \) for any basis \( \{\psi_j\} \) of \( \mathcal{P}_m \), we can continue by
\[ \Rightarrow \sup_a \int_{\Omega} \Delta_m K_m \cdot (h \circ \phi_a)^\nu \, d\mu < \infty \]

where \( K_m = K_m(z, z) \). It can be shown that
\[ \exists m \gg 0 : \Delta_m K_m \geq c h^m. \]

Thus we can continue by
\[ \Rightarrow \sup_a \int_{\Omega} h^m (h \circ \phi_a)^\nu \, d\mu < \infty. \] (9)

Forelli-Rudin inequalities show that this happens iff \( \nu \geq 0. \)

\( \square \)

**Theorem 5.** For rank \( r > 1 \) and \( m = (1, 0, \ldots, 0) \),
\[ Q_{\nu,(1)} = \begin{cases} \mathcal{B}_{(1)}, \text{ the Timoney Bloch space} & \nu > p - 1, \\ \{\text{constants}\} & \nu \leq p - 1. \end{cases} \]

**Proof.** As above,
\[ \{\text{constants}\} \subseteq Q_{\nu,(1)} \iff \sup_a \int_{\Omega} \widetilde{\Delta} \|z\|^2 (h \circ \phi_a)^\nu \, d\mu < \infty. \]

The fact that
\[ \widetilde{\Delta} \|z\|^2 \approx \begin{cases} h^2 & \Omega = \mathcal{D}, \\ h & \Omega = \mathcal{B}^d, \\ 1 & r > 1 \end{cases} \]

and (9) again yield the conclusion. \( \square \)
Theorem 6. For a tube domain with $s = \frac{d}{r} \in \mathbb{Z}$, and $m = (s^r)$,

$$Q_{\nu,(s^r)} = \begin{cases} B_{(s^r)}, & \text{the Arazy Bloch space} \\ \mathcal{D} & \nu = 0 \\ \mathcal{M}_{q-1} & \nu < 0. \end{cases}$$

Proof. As mentioned before, in this case

$$\Delta_m = h^p N(\partial)^s N(\overline{\partial})^s$$

for a certain polynomial $N$ (the Jordan norm). Hence

$$f \in Q_{\nu,m} \iff \sup_a \int_{\Omega} |N(\partial)^s f|^2 (h \circ \phi_a)^\nu \, d\mu < \infty$$

$$\iff \sup_a \int_{\Omega} |N(\partial)^s f|^2 (h \circ \phi_a)^\nu \, dz < \infty.$$ 

Thus for $\nu \geq 0$, all polynomials belong to $Q_{\nu,m}$.

For $\nu = 0$, this coincides with the definition of the Dirichlet space.

The case $\nu < 0$ was settled by the previous theorem. □

Theorem 7. For the polydisc $D^r$ (so that $p = 2$, $q(m) = \#\{j : m_j > 0\}$, and $q = r$),

$$q(m) < r \implies Q_{\nu,(m)} = \begin{cases} B_{m} & \nu > 1, \\ \mathcal{M}_{q(m)-1} & \nu \leq 1; \end{cases}$$

$$q(m) = r \implies Q_{\nu,(m)} = \begin{cases} Arazy-Bloch & \nu > 1, \\ \mathcal{D} & \nu = 0, \\ \mathcal{M}_{q(m)-1} & \nu < 0. \end{cases}$$

Proof. Using explicit formulas for $K_m, \Delta_m$ etc. given in one of the preceding sections, this is easily reduced to explicit calculations on the disc. □

11. Open problems

We conclude the paper by a list of open problems.

(1) The first of them is, of course, to determine when $Q_{\nu,m}$ is nontrivial — we repeat here the conjecture stated above:

Conjecture. $Q_{\nu,m}$ is nontrivial iff

$$\nu \geq 0 \ (\text{for tube domain with } \frac{d}{r} \in \mathbb{Z} \text{ and } q(m) = q)$$

$$\nu > p - 1 \ (\text{otherwise}).$$

(2) If $q(m) = q(n)$, is $Q_{\nu,m} = Q_{\nu,n}$? (We have seen that this holds for the Bloch spaces, hence also for $\nu > p - 1$; the case of $\nu \leq p - 1$ remains unresolved.)

(3) If $\nu_1 < \nu_2$ and $Q_{\nu_1,m}, Q_{\nu_2,m}$ are nontrivial, is $Q_{\nu_1,m} \subset Q_{\nu_2,m}$? (For $D$, this was proved in [AXZ], and for $B^d$ in [AC].)
In principle, one can define $Q_{\nu,L}$ and $B_L$ for any invariant differential operator $L$, even when the right-hand side in (3) is not nonnegative, by

$$f \in B_L \iff L|f|^2 \text{ is bounded,}$$

$$f \in Q_{\nu,L} \iff \sup_a \int_{\Omega} |L \circ \phi_a|^2 h^\nu \, d\mu < \infty.$$  

If $L$ is such that $L|f|^2 \geq 0$ for all holomorphic $f$, i.e. if

$$L = \sum m l_m \Delta_m, \quad l_m \geq 0, \quad (10)$$

then it is easy to see that

$$Q_{\nu,L} = \bigcap_{m: l_m > 0} Q_{\nu,m}$$

and

$$B_L = B_m \quad \text{where } q(m) = \min\{q(k) : l_k > 0\}.$$  

What happens for operators $L$ not satisfying (10)?

For instance, does the space of all holomorphic $f$ on $D$ for which

$$\sup_{a \in D} \int_D |\Delta^2 f(\phi_a(z))|^2 (1 - |z|^2)^{\nu-2} \, dz < \infty$$

coincide with the Bloch space for $\nu > 1$?

**References**


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