THE DIXMIER TRACE OF HANKEL OPERATORS ON THE BERGMAN SPACE

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1. Introduction and Summary

Suppose f is a smooth function on the closed unit disk $\mathbf{D}$. Let H_f be the (big) Hankel operator with symbol function f acting on the Bergman space of the unit disk, $L^2_{\text{hol}}(\mathbf{D})$. We will show that $|H_f|$ has a finite Dixmier trace and that, with $\mathbf{T} = \partial\mathbf{D}$,

$$\text{Tr}_\omega(|H_f|) = \frac{1}{2\pi}\int_{\mathbf{T}} |\overline{\partial} f|\, d\theta. \tag{1.1}$$

More generally let $f_1, f_2, ..., f_k$ be additional smooth functions on the disk and let T_{f_i} be the Toeplitz operators with symbol functions f_i. Then $T = T_{f_1}\dots T_{f_k}|H_f|$ has a finite Dixmier trace given by

$$\text{Tr}_\omega(T) = \frac{1}{2\pi}\int_{\mathbf{T}} f_1 \dots f_k |\overline{\partial} f|\, d\theta. \tag{1.2}$$

In the next section we present background information. In the section after that we prove that (1.1) and (1.2) hold. The basic idea is to recast the issue as one about pseudodifferential operators on the circle and then use the relationship between the Dixmier trace of a pseudodifferential operator and the integral of the principal symbol of the operator. We also show that, under appropriate restrictions on f, the function $\text{Tr}(|H_f|^z)$ extends to a meromorphic function on the entire complex plane whose only singularities are simple poles at $z = 1, 0, -1, -2, \dots$. In Section 4 we consider the regularity that is necessary for (1.1); if f is harmonic and if either side of (1.1) is finite then so is the other and the two are equal. In the section after that we extend our results to Hankel operators on the Bergman space of finitely connected plane domains. In Section 6 we describe operators closely related to the Bergman space Hankel operator. The final section presents instances in which the right side of (1.1), or its analog for multiply connected domains, can be evaluated using considerations from function theory. In some cases that produces quantities which determine the conformal type of the domain.

2. Background

2.1. **Spaces and Operators.** Let $L^2(\mathbf{D})$ be the Lebesgue space of the disk with respect to the normalized measure $\pi^{-1} r dr d\theta$. Let P be the orthogonal projection

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onto the Bergman space, $L^2_{\text{hol}}(\mathbf{D})$, the subspace of holomorphic functions and let $\overline{P}_0$ be the projection onto conjugate holomorphic functions of mean zero.

For a symbol function f on $\mathbf{D}$ one commonly defines the Toeplitz, big Hankel, and small Hankel operators on the Bergman space by

$$T_f\phi = P(f\phi), \quad H_f\phi = (I-P)(f\phi), \quad h_f\phi = \overline{P}_0(f\phi) \qquad \phi \in L^2_{\text{hol}}(\mathbf{D}).$$

However it is convenient for us to extend these operators to all of $L^2(\mathbf{D})$ by setting them to zero on $L^2_{\text{hol}}(\mathbf{D})^\perp$. Thus we adopt the definitions

$$T_f := PfP, \qquad H_f := (I-P)fP, \qquad h_f := \overline{P}_0 fP. \tag{2.1}$$

(The use of $\overline{P}_0$ rather than $\overline{P}$ is to enhance the analogy with the situation in the Hardy space.)

Let $L^2(\mathbf{T})$ be the Lebesgue space of the circle with respect to the normalized measure $(2\pi)^{-1}d\theta$. Recall that the Fourier coefficients of $f \in L^2(\mathbf{T})$ are given by

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, e^{-ni\theta}\, d\theta.$$

The Hardy space, H^2, is the subspace of $L^2(\mathbf{T})$ consisting of those $f \in L^2(\mathbf{T})$ for which $\widehat{f}(-n) = 0$ for $n = 1, 2, ...$ We write S for the orthogonal (Szegö) projection of $L^2(\mathbf{T})$ to H^2; thus

$$\widehat{Su}(n) = \chi_+(n)\, \widehat{u}(n), \tag{2.2}$$

where χ_+ is the characteristic function of $\mathbf{Z}_{\geq 0}$. The Toeplitz and Hankel operators for the Hardy space, now for a symbol function f defined on the circle, are given by

$$T_f^H := SfS, \qquad H_f^H = h_f^H := (I-S)fS.$$

To prevent confusion, we will reserve the undecorated symbols T_f, H_f for the Bergman space operators given in (2.1).

2.2. **The Dixmier Trace.** Recall that if A is a compact operator acting on a Hilbert space then its sequence of singular values $\{s_n(A)\}_{n=1}^\infty$ is the sequence of eigenvalues of $|A| = (A^*A)^{1/2}$ arranged in nonincreasing order. In particular if $A \gg 0$ this will also be the sequence of eigenvalues of A in decreasing order. For $0 < p < \infty$ we say that A is in the Schatten ideal $\mathcal{S}_p$ if $\{s_n(A)\} \in l^p(\mathbf{Z}_{>0})$. If $A \gg 0$ is in $\mathcal{S}_1$, the trace class, then A has a finite trace and, in fact, $\operatorname{Tr}(A) = \sum s_n(A)$. If however we only know that

$$s_n(A) = O(n^{-1}) \text{ or that}$$
$$\sigma_n(A) := \sum_{k=1}^n s_n(A) = O(\log(1+n))$$

then A may have infinite trace. However in this case we may still try to compute its *Dixmier trace,* $\operatorname{Tr}_\omega(A)$. Informally $\operatorname{Tr}_\omega(A) = \lim_N \frac{1}{\log N}\sum_1^N s_n(A)$ and this will actually be true in the cases of interest to us. We begin with the definition. Select a continuous positive linear functional ω on $l^\infty(\mathbf{Z}_{>0})$ and denote its value on $a = (a_1, a_2, ...)$, by $\operatorname{Lim}_\omega(a_n)$. We require of this choice that $\operatorname{Lim}_\omega(a_n) = \lim a_n$ if the latter exists. We require further that ω be *scale invariant;* a technical requirement that is fundamental for the theory but will not be of further concern to us.

For a positive operator A with

$$\left(\frac{\sigma_n(A)}{\log(1+n)}\right) \in l^\infty \tag{2.3}$$

we define the Dixmier trace of A, $\mathrm{Tr}_\omega(A)$, by $\mathrm{Tr}_\omega(A) = \mathrm{Lim}_\omega\left(\frac{\sigma_n(A)}{\log(1+n)}\right)$. $\mathrm{Tr}_\omega(\cdot)$ is then extended by linearity to the full class of operators which satisfy (2.3). Although this definition does depend on ω the operators A we consider are *measurable,* that is, the value of $\mathrm{Tr}_\omega(A)$ is independent of the particular instance of Tr_ω considered. We refer to [9] and [8] for details and for discussion of the role of these functionals.

For $0 < p < \infty$ a Hankel operator with conjugate holomorphic symbol function acting on the Hardy space is in $\mathcal{S}_p$ if and only if its symbol function is in the diagonal Besov space $B^p(\mathbf{D})$ and the same is true for the small Hankel operator on the Bergman space. The result for the Hardy space is in [24]. Also, Theorem 8.9 there, together with the natural unitary map from the Hardy space to the Bergman space, gives the Bergman space case. A similar result holds for the big Hankel operator on the Bergman space for $p > 1$. However at $p = 1$ the story changes, if H_f is in the trace class then H_f is the zero operator [2]. On the other hand, if f is smooth then it is always true that $s_n(H_f) = O(n^{-1})$ [23]. Thus it is natural to consider $\mathrm{Tr}_\omega(|H_f|)$ and that is what we do here.

2.3. **Related Results.** A direct predecessor of this paper is the paper of Engliš, Guo, and Zhang [12]. A particular result there is that if H_f is the big Hankel operator acting on the Bergman space of the unit ball in $\mathbf{C}^d$, $d > 1$, and f is holomorphic then we have

Theorem 1.

$$\mathrm{Tr}_\omega(\left|H_{\bar{f}}\right|^{2d}) = \int_S \left(|\nabla f|^2 - |Rf|^2\right)^d d\sigma.$$

Here S is the boundary of the ball, $d\sigma$ is its normalized surface measure and R is radial differentiation.

In one dimension there is a rich relationship between the theory of Hankel operators and the geometric function theory. For instance we have the following:

Theorem 2. *Suppose ϕ is a holomorphic univalent map of the unit disk to a domain Ω of finite area;* $\mathrm{Area}(\Omega) < \infty$. *The Hankel operator $H_{\bar{\phi}}$ is in the Hilbert-Schmidt class $\mathcal{S}_2$ and*

$$\mathrm{Tr}(\left|H_{\bar{\phi}}\right|^2)^{1/2} = \mathrm{Area}(\Omega).$$

Our theorem leads to statements of a similar spirit.

Theorem 3. *Suppose ϕ is a holomorphic univalent map of the unit disk to a domain Ω which has a boundary of finite length,* $\mathrm{Length}(\partial\Omega) < \infty$. *Then*

$$\mathrm{Tr}_\omega(\left|H_{\bar{\phi}}\right|) = \mathrm{Length}(\partial\Omega).$$

Furthermore, if f is holomorphic on $\overline{\Omega}$ then

$$\mathrm{Tr}_\omega(T_{f\circ\phi}\left|H_{\bar{\phi}}\right|) = \int_{\partial\Omega} f(\zeta)\,|d\zeta|\,.$$

The second equality recalls the following result of Connes and Sullivan [9, Ch IV.3, Thms. 17, 26]. Suppose now that Ω is a bounded domain and that $\partial\Omega$ is the limit set of a quasi-Fuchsian group. Suppose that the Hausdorff dimension of $\partial\Omega$ is

$p > 1$ and that $d\Lambda_p$ is the p-dimensional Hausdorff measure on $\partial\Omega$. For the moment we consider operators on the *Hardy* space.

Theorem 4. *Suppose ϕ is a holomorphic univalent map of the unit disk to a domain Ω such as described above. There is a nonzero number c so that if f is holomorphic on $\overline{\Omega}$ then*

$$\operatorname{Tr}_\omega(T^H_{f\circ\phi}|H^H_{\bar\phi}|^p) = c\int_{\partial\Omega} f(\zeta)d\Lambda_p(\zeta).$$

3. Boutet de Monvel-Guillemin Theory for the Disc

Let $\mathbf{K}$ denote the Poisson extension operator, acting from functions on the unit circle into harmonic functions on the unit disc and let γ be its inverse, i.e. the operator of taking the (suitably interpreted) boundary values. The operator $\mathbf{K}$ has a well known description in terms of the Fourier coefficients: namely,

$$\mathbf{K}f(re^{i\theta}) \equiv (\mathbf{K}f)_r(e^{i\theta}) = \sum_{m\in\mathbf{Z}} \widehat{f}(m)\, r^{|m|}\, e^{mi\theta}. \tag{3.1}$$

Here and below we will denote by

$$F_r(e^{i\theta}) := F(re^{i\theta}), \qquad 0 \le r < 1,\, \theta \in \mathbf{R},$$

the restriction of F to the circle $r\mathbf{T}$.

Viewing $\mathbf{K}$ as a (bounded linear) operator from $L^2(\mathbf{T})$ into $L^2(\mathbf{D})$ its adjoint $\mathbf{K}^*$ is given by

$$\widehat{\mathbf{K}^*F}(n) = \int_0^1 r^{|n|}\,\widehat{F_r}(n)\,2r\,dr. \tag{3.2}$$

In particular,

$$\mathbf{K}^*\mathbf{K} = \Lambda,$$

where Λ is the Fourier multiplier

$$\widehat{\Lambda f}(n) = \frac{1}{|n|+1}\,\widehat{f}(n). \tag{3.3}$$

Setting

$$U := \mathbf{K}\Lambda^{-1/2}$$

it therefore follows that U is a unitary isomorphism of $L^2(\mathbf{T})$ onto the subspace $L^2_{\text{harm}}(\mathbf{D})$ of all harmonic functions in $L^2(\mathbf{D})$.

From the relations $\Lambda^{-1}\mathbf{K}^*\mathbf{K} = I = \gamma\mathbf{K}$ we see that

$$\gamma = \Lambda^{-1}\mathbf{K}^* \qquad \text{on} \operatorname{Ran}\mathbf{K}.$$

Also,

$$\mathbf{K}\Lambda^{-1}\mathbf{K}^* = P_{\text{harm}} \tag{3.4}$$

is the projection in $L^2(\mathbf{D})$ onto $L^2_{\text{harm}}(\mathbf{D})$. Indeed, $\mathbf{K}\Lambda^{-1}\mathbf{K}^*$ is obviously selfadjoint, is an idempotent, vanishes on $(\operatorname{Ran}\mathbf{K})^\perp$ (since $\mathbf{K}^*$ does), and acts as the identity on $\operatorname{Ran}\mathbf{K}$.

Recall that a pseudodifferential operator (ΨDO for short) on $\mathbf{R}$ is an operator of the form

$$Af(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{ix\xi}a(x,\xi)\,\widehat{f}(\xi)\,d\xi, \tag{3.5}$$

where (abusing the notation slightly, but there is no danger of confusion)

$$\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x)\, e^{-ix\xi}\, d\xi \tag{3.6}$$

denotes the Fourier transform of a function f on $\mathbf{R}$. Here a, called the "symbol" σ_A of A, is a function in $C^\infty(\mathbf{R}\times\mathbf{R})$; it is usually required to satisfy the estimates

$$|\partial_x^n \partial_\xi^l a(x,\xi)| \le c_{n,l}\ (1+|\xi|)^{m-l}, \qquad \forall n,l = 0,1,2,\dots,$$

for some $m \in \mathbf{R}$ — that is, to belong to Hörmander's class $S^m(\mathbf{R}\times\mathbf{R})$. The ΨDO (3.5) is called *classical* if a admits the asymptotic expansion

$$a(x,\xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x,\xi), \tag{3.7}$$

where $a_{m-j} \in S^{m-j}(\mathbf{R}\times\mathbf{R})$ is positive homogenous of degree $m-j$ in ξ for $|\xi|>1$, and "$\sim$" means that

$$a(x,\xi) - \sum_{j=0}^{N-1} a_{m-j}(x,\xi) \in S^{m-N}(\mathbf{R}\times\mathbf{R}), \qquad \forall N = 0,1,2,\dots.$$

We denote the vector space of all classical ΨDOs of order m (that is, with $a \in S^m(\mathbf{R}\times\mathbf{R})$) by $\mathbf{\Psi}^m(\mathbf{R})$. One calls a_m the "leading", or "principal", symbol of A. The "total symbol" $a(x,\xi)$ can be recovered from A by

$$a(x,\xi) = e^{-ix\xi} A(e^{i\,\cdot\,\xi})\Big|_{\,\cdot\,=x}.$$

If $A \in \mathbf{\Psi}^m(\mathbf{R})$ and $B \in \mathbf{\Psi}^n(\mathbf{R})$, then $AB \in \mathbf{\Psi}^{m+n}(\mathbf{R})$ and

$$\sigma_{AB}(x,\xi) \sim \sum_{j=0}^{\infty} \frac{(-i)^j}{j!}\, [\partial_\xi^j \sigma_A(x,\xi)]\, [\partial_x^j \sigma_B(x,\xi)]. \tag{3.8}$$

See e.g. Folland [13], [14] or Treves [31] for these properties of ΨDO.

It further turns out that ΨDOs also behave well under coordinate changes, making it possible to define a ΨDO on a compact manifold by declaring that it be of the form (3.5) when restricted to each coordinate chart; see again [13] or [31] for the details. In particular, we can define in this way classical ΨDOs of order m on the unit circle $\mathbf{T}$; we denote the vector space of all such operators by $\mathbf{\Psi}^m(\mathbf{T})$.

The above material on ΨDOs is, of course, very standard; it turns out that for the particular case of the circle $\mathbf{T}$ there is a much more convenient variant using the Fourier coefficients instead of the Fourier transform. Recall that the s-th order Sobolev space, $W^s(\mathbf{T}) \equiv W^s$, on the unit circle consists of all distributions u on $\mathbf{T}$ for which

$$\|u\|_s^2 := \sum_{n\in\mathbf{Z}} (1+|n|)^{2s}\, |\widehat{u}(n)|^2 < \infty.$$

The intersection of all W^s is $C^\infty(\mathbf{T})$, and the usual Frechet topology on $C^\infty(\mathbf{T})$ coincides with the one induced by the seminorms $\|\cdot\|_s$, $s\in\mathbf{R}$ (or $s \in \mathbf{Z}$). Any continuous operator $A : C^\infty(\mathbf{T}) \to C^\infty(\mathbf{T})$ thus extends to an operator from W^{s_1} into W^{s_2} for some s_1 and s_2; and since the Fourier series $u(e^{it}) = \sum_n \widehat{u}(n)\, e^{nit}$ of

$u \in W^{s_1}$ converges in W^{s_1}, it follows that

$$\begin{aligned} Au(e^{it}) &= \sum_{n\in\mathbf{Z}} \widehat{u}(n)\ Ae^{nit} \\ &= \sum_{n\in\mathbf{Z}} \sigma_A(e^{it},n)\ \widehat{u}(n)\ e^{nit} \end{aligned} \tag{3.9}$$

(convergence in W^{s_2}), where

$$\sigma_A(e^{it},n) := e^{-nit}Ae^{nit}. \tag{3.10}$$

One calls operators of the form (3.9) *periodic* (or *discrete*) *ΨDOs* (pΨDOs for short), with "periodic symbol" $a = \sigma(A) \in C^\infty(\mathbf{T}\times\mathbf{Z})$. The periodic analogue of the Hörmander class is the space $S^m(\mathbf{T}\times\mathbf{Z}) \equiv S^m$ of all periodic symbols a satisfying

$$|\partial_t^j \Delta_n^k a(e^{it},n)| \le c_{j,k}\ (1+|n|)^{m-k}, \qquad \forall j,k=0,1,2,\dots,$$

where Δ_n stands for the difference operator

$$\Delta_n a(e^{it},n) := a(e^{it},n+1) - a(e^{it},n).$$

One has also the obvious analogue of the asymptotic expansion (3.7) of a symbol, and of the notion of a classical ΨDO; we denote the vector space of all classical pΨDOs of order m by $\mathbf{\Psi}^m_{\text{per}}(\mathbf{T}) \equiv \mathbf{\Psi}^m_{\text{per}}$.

Periodic ΨDOs on $\mathbf{T}$ were studied by many authors, see e.g. Turunen and Vainikko [33] and the references therein. In particular, it was proved by McLean [19] that

$$\mathbf{\Psi}^m_{\text{per}}(\mathbf{T}) = \mathbf{\Psi}^m(\mathbf{T}),$$

i.e. that the "ordinary" and "periodic" ΨDOs on $\mathbf{T}$ coincide. See also Saranen and Wendland [28], Melo [22], and Turunen [32]. The symbol calculus of pΨDOs was worked out by Turunen and Vainikko [33], who proved also the "periodic" analogue of the product formula (3.8): namely, for $A \in \mathbf{\Psi}^m_{\text{per}}$ and $B \in \mathbf{\Psi}^n_{\text{per}}$, AB belongs to $\mathbf{\Psi}^{m+n}_{\text{per}}$ and

$$\sigma_{AB}(e^{it},n) \sim \sum_{j=0}^\infty \frac{1}{j!}\,[\Delta_n^j \sigma_A(e^{it},n)]\,[\partial_t^{(j)}\sigma_B(e^{it},n)]. \tag{3.11}$$

Here $\partial_t^{(j)}$ stands for the "shifted derivative"

$$\partial_t^{(j)} := \prod_{k=0}^{j-1} \Big(\frac{1}{i}\,\frac{\partial}{\partial t} - k\Big).$$

Note that, in particular, the operator Λ from (3.3) and the Szegö projection S in (2.2) are pΨDOs of order -1 and 0, respectively, with symbols

$$\begin{aligned} \sigma_\Lambda(e^{it},n) &= \frac{1}{|n|+1}, \\ \sigma_S(e^{it},n) &= \chi_+(n) \equiv \begin{cases} 1 & \text{for } n \ge 0, \\ 0 & \text{for } n < 0. \end{cases} \end{aligned}$$

After all these preparations, we can return to the Toeplitz and Hankel operators.

Our strategy will be to transfer the operators (2.1) on $L^2(\mathbf{D})$, via the isomorphism U, to operators on $L^2(\mathbf{T})$, which turn out to be of the form (3.9), i.e. periodic ΨDOs.

We claim, first of all, that

$$P\mathbf{K} = \mathbf{K}S. \tag{3.12}$$

Indeed, let $u \in L^2(\mathbf{T})$ and set $v = \gamma P\mathbf{K}u$. Then for all $w \in H^2$

$$\langle \mathbf{K}u, \mathbf{K}w\rangle = \langle P\mathbf{K}u, \mathbf{K}w\rangle = \langle \mathbf{K}\gamma P\mathbf{K}u, \mathbf{K}w\rangle = \langle \mathbf{K}v, \mathbf{K}w\rangle,$$

so

$$\langle \Lambda(u - v), w\rangle = 0.$$

It follows that $S\Lambda(u - v) = 0$. As, by (3.3) and (2.2),

$$S\Lambda = \Lambda S, \tag{3.13}$$

and Λ is invertible, we get $Su = Sv = v$. Thus $\mathbf{K}Su = \mathbf{K}v = P\mathbf{K}u$, proving the claim.

Combining (3.12) with (3.4) we see that

$$\begin{aligned} \gamma P &= \gamma P P_{\text{harm}} \\ &= \gamma P\mathbf{K}\Lambda^{-1}\mathbf{K}^* \\ &= \gamma \mathbf{K}S\Lambda^{-1}\mathbf{K}^*, \end{aligned}$$

i.e.

$$\gamma P = S\Lambda^{-1}\mathbf{K}^*.$$

Finally, let $f \in L^\infty(\mathbf{D})$. Then for any $u \in L^2(\mathbf{T})$

$$\begin{aligned} U^*T_f U u &= \Lambda^{-1/2}\mathbf{K}^* P f P \mathbf{K}\Lambda^{-1/2} u \\ &= \Lambda^{-1/2} S\mathbf{K}^* f \mathbf{K} S \Lambda^{-1/2} && \text{by (3.12)} \\ &= S\Lambda^{-1/2}\mathbf{K}^* f \mathbf{K}\Lambda^{-1/2} S && \text{by (3.13)}. \end{aligned}$$

That is,

$$U^*T_f U = SB_f S, \tag{3.14}$$

where

$$B_f = \Lambda^{-1/2}\mathbf{K}^* f \mathbf{K}\Lambda^{-1/2}.$$

In other words, upon transferring the Toeplitz operator T_f on $L^2(\mathbf{D})$, by means of the isomorphism U, to an operator on $L^2(\mathbf{T})$, it becomes basically a Hardy-space Toeplitz operator but with the multiplication by f replaced by the operator B_f above. One of the starting points of the theory of Boutet de Monvel and Guillemin [7] [6] [16] is the following.

Theorem 5. *For $f \in C^\infty(\overline{\mathbf{D}})$, B_f is a classical pΨDO of order ≤ 0. More precisely, if f vanishes at $\partial\mathbf{D}$ to order $k \ge 0$, then $B_f \in \mathbf{\Psi}^{-k}_{per}$. The beginning of the expansion of the symbol of B_f is*

(3.15)

$$\sigma_{B_f}(e^{it}, n) \sim f(e^{it}) + \frac{\partial_r f(e^{it})}{2(|n|+1)} + \frac{\partial_t^2 f(e^{it}) + 2\partial_r f(e^{it}) + 2\partial_r^2 f(e^{it}) - 2i\partial_t\partial_r f(e^{it})}{8(|n|+1)^2} + \dots$$

for $n > 0$; for $n < 0$, one replaces i by $-i$.

Note that the theorem implies that SB_fS also belongs to $\mathbf{\Psi}^{-k}_{\text{per}}$, and

$$\sigma_{SB_fS}(e^{it},n)=\sigma_{B_f}(e^{it},n)\chi_+(n).$$

In fact, for any pΨDO A we have by (3.11)

$$\sigma_{AS}\sim\sigma_{SA}\sim\sigma_S\sigma_A=\chi_+\sigma_A. \tag{3.16}$$

(For σ_{AS}, the right-hand side of (3.11) in fact coincides with $\sigma_A\sigma_S$; for σ_{SA}, it differs from $\sigma_S\sigma_A$ only at finitely many values of n.)

Before giving the proof of the theorem, let us list some corollaries for Hankel and little Hankel operators.

Corollary 1. *For $f\in C^\infty(\overline{\mathbf{D}})$, the operator $|H_f|^2=H_f^*H_f=T_{\overline{f}f}-T_{\overline{f}}T_f$, when transferred to $L^2(\mathbf{T})$ via the isomorphism U, becomes the pΨDO*

$$U^*|H_f|^2U=SR_fS \tag{3.17}$$

where

$$R_f=B_{\overline{f}f}-B_{\overline{f}}SB_f$$

is a pΨDO of order -2, with leading symbol

$$\frac{|\partial_rf(e^{it})-i\partial_tf(e^{it})|^2}{4(|n|+1)^2}=\frac{|\overline{\partial}f(e^{it})|^2}{(|n|+1)^2}$$

for $n>0$, and $|\partial f|^2/(|n|+1)^2$ for $n<0$.

Proof. By (3.14), we immediately get (3.17). The assertion about the symbol follows from the formula (3.15) and the product rule (3.11) by a routine computation. □

Corollary 2. *For $f\in C^\infty(\overline{\mathbf{D}})$, H_f belongs to the Dixmier class, and*

$$\text{Tr}_\omega(|H_f|)=\frac{1}{2\pi}\int_{\mathbf{T}}|\overline{\partial}f|d\theta.$$

Proof. By the classical result of Wodzicki [34], if T is a ΨDO of order $-n$ on a compact manifold of real dimension n, then T is in the Dixmier class and $\text{Tr}_\omega(T)$ equals the integral of the principal symbol of T over the unit cosphere bundle $|\xi|=1$. Now by Seeley's work [30] on powers of elliptic ΨDOs, McLean's result about the coincidence of the ordinary and periodic ΨDOs, and the preceding corollary, it follows that $U^*|H_f|U=(SR_fS)^{1/2}$ is a ΨDO on $\mathbf{T}$ of order -1 with leading symbol $|\overline{\partial}f(e^{it})|\chi_+(\xi)/|\xi|$. Taking $T=U^*|H_f|U$, the assertion follows. □

Corollary 3. *For any $f_1,\dots,f_k,f\in C^\infty(\overline{\mathbf{D}})$, $T=T_{f_1}\dots T_{f_k}|H_f|$ belongs to the Dixmier class and*

$$\text{Tr}_\omega(T)=\frac{1}{2\pi}\int_{\mathbf{T}}f_1\dots f_k|\overline{\partial}f|d\theta.$$

Proof. The first part is immediate from the preceding corollary since the Dixmier class is an ideal. Concerning the second part, observe that by (3.16),

$$\begin{aligned}U^*TU&=SB_{f_1}SB_{f_2}S\dots(SR_fS)^{1/2}\\&\sim SB_{f_1}B_{f_2}\dots B_{f_k}S(SR_fS)^{1/2}\end{aligned}$$

is a pΨDO of order -1 with leading symbol $f_1f_2\dots f_k|\overline{\partial}f|\chi_+(n)/(n+1)$. Appealing to Wodzicki's result as in the preceding corollary completes the proof. □

Corollary 4. *For $f \in C^\infty(\overline{\mathbf{D}})$, $U^*|h_f|U$ is a smoothing operator. Consequently, $|h_f|$ is in the Dixmier class and $\operatorname{Tr}_\omega(|h_f|) = 0$. Also, for any $f_1, \dots, f_k \in C^\infty(\overline{\mathbf{D}})$, $\operatorname{Tr}_\omega(T_{f_1} \dots T_{f_k}|h_f|) = 0$.*

Proof. Note that $h_f = (P_{\text{harm}} - P)fP$. Thus

$$h_f^* h_f = P\overline{f}P_{\text{harm}}fP - P\overline{f}PfP.$$

Using (3.4), (3.12), (3.14) and (3.13), this becomes

$$\begin{aligned} U^*|h_f|^2U &= \Lambda^{-1/2}\mathbf{K}^*P\overline{f}\mathbf{K}\Lambda^{-1}\mathbf{K}^*fP\mathbf{K}\Lambda^{-1/2} - (U^*T_{\overline{f}}U)(U^*T_fU) \\ &= \Lambda^{-1/2}S\mathbf{K}^*\overline{f}\mathbf{K}\Lambda^{-1}\mathbf{K}^*f\mathbf{K}S\Lambda^{-1/2} - SB_{\overline{f}}SB_fS \\ &= S\Lambda^{-1/2}\mathbf{K}^*\overline{f}\mathbf{K}\Lambda^{-1}\mathbf{K}^*f\mathbf{K}\Lambda^{-1/2}S - SB_{\overline{f}}SB_fS \\ &= SB_{\overline{f}}B_fS - SB_{\overline{f}}SB_fS \\ &= SB_{\overline{f}}[B_f, S]S. \end{aligned}$$

But by (3.16), this is ~ 0, proving the first part of the corollary.

For the second part, note that $U^*|h_f|^2U =: T \sim 0$ implies that $\Lambda^{-2}T\Lambda^{-2} \in \mathbf{\Psi}^{-\infty}_{\text{per}}$ is a bounded operator on $L^2(\mathbf{T})$; thus so is its square root $(\Lambda^{-2}T\Lambda^{-2})^{1/2}$ and, using polar decomposition, also $T^{1/2}\Lambda^{-2}$. Since Λ^2 is trace class, it follows that $T^{1/2} = U^*|h_f|U$ is trace class, and hence has vanishing Dixmier trace. The last part of the corollary also follows immediately, since trace class operators form an ideal. □

Let us now turn to the proof of Theorem 5. We begin with a preparatory lemma.

Lemma 1. *For $G \in C^\infty(\mathbf{T})$ and $M = 0, 1, 2, \dots$, the sum*

$$\sum_{|m| \le n} \int_0^1 e^{mi\theta} r^{2n+m} (1 - r^2)^M \, \widehat{G}(m) \, 2r \, dr$$

has the asymptotic expansion

$$\sum_{l=0}^\infty \frac{1}{(n+1)^{l+M+1}} \sum_{j=0}^l \left(\frac{i}{2} \frac{\partial}{\partial\theta}\right)^{l-j} G(e^{i\theta}) \, (-1)^j \binom{l+M}{l-j} c_M(M+j) \tag{3.18}$$

as $n \to +\infty$. Here $c_M(m)$ are the numbers defined recursively by

$$\begin{aligned} c_0(m) &= \delta_{m,0}, \\ c_{M+1}(m) &= \sum_{l=M}^{m-1} \binom{m}{l} c_M(l). \end{aligned} \tag{3.19}$$

One can show that

$$c_1(m) = 1, \qquad c_2(m) = 2^m - 2,$$

and, generally,

$$c_M(m) = \sum_{j=0}^M (-1)^{M-j} \binom{M}{j} j^m.$$

Proof of Lemma 1. Let us first consider the case of $M = 0$, i.e.

$$\sum_{|m|\le n} e^{mi\theta}\widehat{G}(m)\int_0^1 r^{2n+m}\,2r\,dr = \sum_{|m|\le n}\frac{e^{mi\theta}\widehat{G}(m)}{n+\frac{m}{2}+1} \equiv Q_n.$$

Using the summation formula for geometric progression,

$$\frac{n+1}{n+\frac{m}{2}+1} = \sum_{k=0}^{N-1}(-1)^k\Big(\frac{m/2}{n+1}\Big)^k + \frac{(-1)^N\Big(\frac{m/2}{n+1}\Big)^N}{1+\frac{m/2}{n+1}}$$

(where $N = 1, 2, 3, \dots$), we get

$$\begin{aligned} Q_n &= \sum_{k=0}^{N-1}\frac{(-1)^k}{(n+1)^{k+1}}\sum_{|m|\le n}\Big(\frac{m}{2}\Big)^k e^{mi\theta}\widehat{G}(m) \\ &\qquad + \sum_{|m|\le n} e^{mi\theta}\widehat{G}(m)\frac{(-m/2)^N}{(n+1)^{N+1}(n+\frac{m}{2}+1)} \\ &\equiv \sum_{k=0}^{N-1} Q_{n,k} + Q_{n,N}. \end{aligned} \tag{3.20}$$

By integrating by parts, for $m \ne 0$,

$$\begin{aligned} \widehat{G}(m)e^{mi\theta} &= \int_0^{2\pi} G(e^{it})\,e^{mi(\theta-t)}\,\frac{dt}{2\pi} \\ &= \int_0^{2\pi}\frac{e^{mi(\theta-t)}}{(mi)^N}\,\partial_t^N G(e^{it})\,\frac{dt}{2\pi}, \end{aligned} \tag{3.21}$$

which yields the estimate

$$\begin{aligned} |Q_{n,N}| &\le \frac{2n+1}{(n+1)^{N+1}}\,\|\partial_t^N G\|_\infty \cdot \frac{2^{-N}}{\frac{n}{2}+1} \\ &\le \frac{2^{2-N}\|\partial_t^N G\|_\infty}{(n+1)^{N+1}}, \end{aligned} \tag{3.22}$$

i.e. the last summand in (3.20) is $O((n+1)^{-N-1})$, uniformly in θ. On the other hand, by (3.21) again but with $k+j$ in the place of N ($j = 2, 3, \dots$),

$$|\widehat{G}(m)| \le \frac{\|\partial_t^{k+j}G\|_\infty}{m^{k+j}}, \tag{3.23}$$

so

$$\begin{aligned} \Big|\sum_{|m|>n}\widehat{G}(m)e^{mi\theta}\Big(\frac{-m}{2}\Big)^k\Big| &\le 2\sum_{m=n}^{\infty}\frac{\|\partial_t^{k+j}G\|_\infty}{m^j} \\ &\approx \|\partial_t^{k+j}G\|_\infty\, n^{1-j} = O(n^{-\infty}) \end{aligned}$$

since j can be taken arbitrary. Hence

$$
\begin{aligned}
Q_{n,k} + O(n^{-\infty}) &= \frac{(-1)^k}{(n+1)^{k+1}} \sum_{m\in\mathbf{Z}} \Big(\frac{m}{2}\Big)^k e^{mi\theta} \widehat{G}(m) \\
&= \frac{1}{(n+1)^{k+1}} \sum_{m\in\mathbf{Z}} \Big(-\frac{1}{2i}\frac{\partial}{\partial\theta}\Big)^k e^{mi\theta} \widehat{G}(m) \\
&= \frac{1}{(n+1)^{k+1}} \Big(-\frac{1}{2i}\frac{\partial}{\partial\theta}\Big)^k G(e^{i\theta}). \qquad (3.24)
\end{aligned}
$$

Combining (3.22) and (3.24) gives

$$
Q_n = \sum_{k=0}^{N-1} \frac{1}{(n+1)^{k+1}} \Big(\frac{i}{2}\,\partial_\theta\Big)^k G(e^{i\theta}) + O\Big(\frac{1}{(n+1)^{N+1}}\Big).
$$

Since N was arbitrary, this proves the lemma for $M = 0$.

For general M, note that

$$
\Delta_n r^{2n+m} = -(1-r^2)\, r^{2n+m}.
$$

Thus

$$
\begin{aligned}
&-\Delta_n\Big(\sum_{|m|\le n}\int_0^1 e^{mi\theta}\, r^{2n+m}\, \widehat{G}(m)\, 2r\, dr\Big) \\
&\qquad = \sum_{|m|\le n}\int_0^1 e^{mi\theta}\, (1-r^2)\, r^{2n+m}\, \widehat{G}(m)\, 2r\, dr \\
&\qquad\qquad - \sum_{|m|=n+1}\int_0^1 e^{mi\theta}\, r^{2n+2+m}\, \widehat{G}(m)\, 2r\, dr.
\end{aligned}
$$

By (3.23) again, the last term is $O(n^{-\infty})$. Repeating the same argument M times, we get

$$
\begin{aligned}
&\sum_{|m|\le n}\int_0^1 e^{mi\theta}\, r^{2n+m}\, (1-r^2)^M\, \widehat{G}(m)\, 2r\, dr + O(n^{-\infty}) \\
&\qquad = (-1)^M \Delta_n^M \Big(\int_0^1 e^{mi\theta}\, r^{2n+m}\, \widehat{G}(m)\, 2r\, dr\Big) \\
&\qquad = (-1)^M \Delta_n^M Q_n \\
&\qquad \approx \sum_{k=0}^{\infty} \Big(\frac{i}{2}\frac{\partial}{\partial\theta}\Big)^k G(e^{i\theta})\, (-1)^M \Delta_n^M \frac{1}{(n+1)^{k+1}}. \qquad (3.25)
\end{aligned}
$$

By Taylor's formula, we have for any $\nu \in \mathbf{C}$ and $|z| < 1$,

$$
(1-z)^{-\nu} = \sum_{j=0}^{\infty} \frac{(\nu)_j}{j!}\, z^j. \qquad (3.26)
$$

Here $(\nu)_j := \nu(\nu+1)\dots(\nu+j-1)$ is the Pochhammer symbol (raising factorial). Taking $z = -\frac{1}{n+1}$, we get

$$\begin{aligned}-\Delta_n \frac{1}{(n+1)^\nu} &= \frac{1}{(n+1)^\nu} - \frac{1}{(n+2)^\nu} \\ &= \frac{1}{(n+1)^\nu}\Big[1 - \Big(1+\frac{1}{n+1}\Big)^{-\nu}\Big] \\ &= -\sum_{j=1}^\infty \frac{(\nu)_j\,(-1)^j}{j!(n+1)^{\nu+j}},\end{aligned}$$

with the series converging for $n > 0$, and also as an asymptotic expansion as $n \to +\infty$. Iterating the last formula yields

$$\Delta_n^M \frac{1}{(n+1)^\nu} = \sum_{m=M}^\infty \frac{(-1)^m(\nu)_m}{m!(n+1)^{\nu+m}}\, c_M(m), \tag{3.27}$$

with the numbers $c_M(m)$ given by (3.19). Take $\nu = k+1$; since $(k+1)_m = (k+m)!/k!$, we can also write the formula as

$$\Delta_n^M \frac{1}{(n+1)^{k+1}} = \sum_{m=M}^\infty \binom{k+m}{m} (-1)^m \frac{c_M(m)}{(n+1)^{k+m+1}}.$$

Substituting this into (3.25) yields (3.18), completing the proof of the lemma. □

Proof of Theorem 5. That $\mathbf{K}^* f\mathbf{K}$ and, hence, $B_f = \Lambda^{-1/2}\mathbf{K}^* f\mathbf{K}\Lambda^{-1/2}$, is a pΨDO of order ≤ 0 with leading symbol $f|_\mathbf{T}$ is, of course, a fact from the theory of "Poisson" (like $\mathbf{K}$) and "trace" (like γ) operators initiated by Boutet de Monvel [5], combined with McLean's result [19] that $\mathbf{\Psi}^m = \mathbf{\Psi}^m_{\text{per}}$ for any $m \in \mathbf{R}$. For the disc, however, one can also proceed by a much simpler argument: namely, if $u \in C^\infty(\mathbf{T})$, so that $\widehat{u}(m) = O(m^{-\infty})$ by (3.23), it follows from (3.1) that $\mathbf{K}u \in C^\infty(\overline{\mathbf{D}})$, and $\mathbf{K}$ maps $C^\infty(\mathbf{T})$ into $C^\infty(\overline{\mathbf{D}})$ continuously. Similarly if $F \in C^\infty(\overline{\mathbf{D}})$, then by (3.2) and integration by parts in θ (as in (3.21)) it follows that $\widehat{\mathbf{K}^*F}(m) = O(m^{-\infty})$, or $\mathbf{K}^*F \in C^\infty(\mathbf{T})$, and $\mathbf{K}^*$ maps $C^\infty(\overline{\mathbf{D}})$ into $C^\infty(\mathbf{T})$ continuously. Hence for $f \in C^\infty(\overline{\mathbf{D}})$, $\mathbf{K}^* f\mathbf{K}$ maps $C^\infty(\mathbf{T})$ into itself continuously. By the remarks preceding (3.9), it is therefore a pΨDO with symbol $\sigma_{\mathbf{K}^*f\mathbf{K}} \in C^\infty(\mathbf{T}\times\mathbf{Z})$, establishing the claim.

It thus remains to show that $\sigma_{\mathbf{K}^*f\mathbf{K}}$ is classical, belongs to $S^{-k}(\mathbf{T}\times\mathbf{Z})$ if f vanishes at the boundary to order k, and has the asymptotic expansion as asserted.

By (3.10),

$$\sigma_{\mathbf{K}^*f\mathbf{K}}(e^{it}, n) = e^{-nit}\mathbf{K}^* f\mathbf{K} e^{nit}. \tag{3.28}$$

Since both $\mathbf{K}$ and $\mathbf{K}^*$ commute with complex conjugation, it follows that

$$\sigma_{\mathbf{K}^*f\mathbf{K}}(e^{it}, -n) = \overline{\sigma_{\mathbf{K}^*\overline{f}\mathbf{K}}(e^{it}, n)}.$$

Thus it is enough to consider $n \to +\infty$, i.e. to prove (3.15). So we will assume $n > 0$ from now on.

From (3.28), (3.1) and (3.2),

$$\begin{aligned}\sigma_{\mathbf{K}^* f \mathbf{K}}(e^{it}, n) &= e^{-itn} \sum_m e^{mit} \int_0^1 \int_0^{2\pi} r^{|m|}\, e^{-mi\theta}\, f(re^{i\theta})\, r^{|n|}\, e^{ni\theta}\, \frac{r\,dr\,d\theta}{\pi} \\ &= \sum_m e^{(m-n)it} \int_0^1 \int_0^{2\pi} r^{|m|+|n|}\, e^{(n-m)i\theta}\, f(re^{i\theta})\, \frac{r\,dr\,d\theta}{\pi} \\ &= \sum_m e^{mit} \int_0^1 \int_0^{2\pi} r^{|m+n|+|n|}\, f(re^{i\theta})\, e^{-mi\theta}\, \frac{r\,dr\,d\theta}{\pi}. \end{aligned} \tag{3.29}$$

We claim that the contribution from $|m| > n$ to the last sum is $O(n^{-\infty})$. Indeed, by (3.21), we have for any $k \geq 0$

$$\Big| \int_0^{2\pi} f(re^{i\theta})\, e^{-mi\theta}\, \frac{d\theta}{2\pi} \Big| \leq \frac{\|\partial_\theta^k f\|_\infty}{m^k}, \tag{3.30}$$

so

$$\begin{aligned} &\Big| \sum_{|m|>n} e^{mit} \int_0^1 \int_0^{2\pi} f(re^{i\theta})\, e^{-mi\theta}\, \frac{d\theta}{2\pi}\, r^{|m+n|+|n|}\, 2r\,dr \Big| \\ &\qquad \leq \sum_{|m|>n} \frac{\|\partial_\theta^k f\|_\infty}{m^k} \int_0^1 r^{|m+n|+|n|}\, 2r\,dr \\ &\qquad \leq \sum_{|m|>n} \frac{\|\partial_\theta^k f\|_\infty}{m^k} \cdot \frac{2}{n} \\ &\qquad \lesssim 2\|\partial_\theta^k f\|_\infty \cdot n^{-k} = O(n^{-\infty}), \end{aligned}$$

since k was arbitrary. (Here, as well as everywhere else in this paper, the various $O(n^{\cdots})$ terms are always understood to hold uniformly in t or θ.) Thus

$$\sigma_{\mathbf{K}^* f \mathbf{K}}(e^{it}, n) = O(n^{-\infty}) + \sum_{|m|\leq n} e^{mit} \int_0^1 \int_0^{2\pi} r^{2n+m}\, f(re^{i\theta})\, e^{-mi\theta}\, \frac{r\,dr\,d\theta}{\pi}. \tag{3.31}$$

Since $f \in C^\infty(\overline{\mathbf{D}})$, we can write, for any $N = 1, 2, 3, \dots$,

$$f(re^{i\theta}) = \sum_{j=0}^{N-1} (1-r^2)^j F_j(e^{i\theta}) + (1-r^2)^N G_N(re^{i\theta}), \tag{3.32}$$

with $F_j \in C^\infty(\mathbf{T})$ given by

$$F_j(e^{i\theta}) := \frac{(-1)^j}{j!} \frac{\partial^j}{\partial r^j} f(\sqrt{r}\, e^{i\theta}) \big|_{r=1} \tag{3.33}$$

and $G_N \in L^\infty(\mathbf{D})$. Substituting this into (3.31), the contribution from the term containing G_N can be estimated by

$$\begin{aligned}\Big|\sum_{|m|\le n} e^{mit} \int_0^1 \int_0^{2\pi} r^{2n+m}\,(1-r^2)^N\, G_N(re^{i\theta})\, e^{-mi\theta}\, \frac{r\,dr\,d\theta}{\pi}\Big| \\ \le \sum_{|m|\le n} \|G_N\|_\infty \int_0^1 r^{2n+m}\,(1-r^2)^N\, 2r\,dr \\ = \sum_{|m|\le n} \|G_N\|_\infty\, \frac{N!}{(n+\frac{m}{2}+1)\dots(n+\frac{m}{2}+N+1)} \\ \le (2n+1)\|G_N\|_\infty \frac{N!}{(\frac{n}{2}+1)^{N+1}} = O(n^{-N}).\end{aligned}$$

On the other hand, the contributions from the F_j,

$$\sum_{|m|\le n} e^{mit} \int_0^1 r^{2n+m}\,(1-r^2)^j\, \widehat{F_j}(m)\, 2r\,dr$$

are precisely of the form handled by the last lemma. Combining everything together, we thus obtain

$$\begin{aligned}\sigma_{\mathbf{K}^* f\mathbf{K}}(e^{it},n) = \sum_{M=0}^{N-1} \sum_{l=0}^{N-1-M} \frac{1}{(n+1)^{l+M+1}} \sum_{j=0}^{l} \Big(\frac{i}{2}\frac{\partial}{\partial\theta}\Big)^{l-j} F_M(e^{it}) \\ \cdot (-1)^j \binom{l+M}{l-j} c_M(M+j) + O\Big(\frac{1}{(n+1)^N}\Big).\end{aligned}$$

Since N was arbitrary, we see that $\sigma_{\mathbf{K}^* f\mathbf{K}}$ has the asymptotic expansion

$$\sigma_{\mathbf{K}^* f\mathbf{K}}(e^{it},n) = \sum_{\substack{M\ge 0,\\ 0\le j\le l}} \frac{(-1)^j c_M(M+j)}{(n+1)^{l+M+1}} \binom{l+M}{l-j} \Big(\frac{i}{2}\frac{\partial}{\partial\theta}\Big)^{l-j} F_M(e^{it}), \tag{3.34}$$

uniformly in t, as $n\to+\infty$.

Coming now back momentarily to (3.29), note that owing to the estimate (3.30) it is legitimate to differentiate (3.29) with respect to t term by term. Using again integration by parts, we thus get, for any $j=0,1,2,\dots$,

$$\begin{aligned}\partial_t^j \sigma_{\mathbf{K}^* f\mathbf{K}}(e^{it},n) &= \sum_m (mi)^j\, e^{mit} \int_0^1\int_0^{2\pi} r^{|m+n|+|n|}\, f(re^{i\theta})\, e^{-mi\theta}\, \frac{r\,dr\,d\theta}{\pi} \\ &= \sum_m e^{mit} \int_0^1\int_0^{2\pi} r^{|m+n|+|n|}\, f(re^{i\theta})\,(-\partial_\theta)^j e^{-mi\theta}\, \frac{r\,dr\,d\theta}{\pi} \\ &= \sum_m e^{mit} \int_0^1\int_0^{2\pi} r^{|m+n|+|n|}\, \partial_\theta^j f(re^{i\theta})\, e^{-mi\theta}\, \frac{r\,dr\,d\theta}{\pi} \\ &= \sigma_{\mathbf{K}^*(\partial_\theta^j f)\mathbf{K}}(e^{it},n).\end{aligned}$$

Consequently, by (3.34), $\partial_t^j \sigma_{\mathbf{K}^* f\mathbf{K}}$ has also an asymptotic expansion as $n\to+\infty$, and in fact it is the one obtained by applying ∂_t^j to (3.34) term by term.

Finally, as

$$\Delta_n r^{2n+m} = -(1-r^2)\, r^{2n+m}$$

for $n \geq 0$, it follows from (3.31) that $\Delta_n \sigma_{\mathbf{K}^* f \mathbf{K}}(e^{it}, n)$ is again given by (3.31) except that $f(re^{i\theta})$ is replaced by $-(1-r^2)f(re^{i\theta})$. On the level of (3.32) and, hence, (3.34), this amounts to a sign change combined with the shift $F_j \mapsto F_{j-1}$, which by (3.27) amounts in turn to applying Δ_n to the right-hand side of (3.34) term by term.

Combining the observations from the last two paragraphs, we thus see that we can apply $\Delta_n^k \partial_t^j$ to the right-hand side of (3.34) term by term, so that (3.34) is not only an asymptotic expansion (uniform in t) as $n \to +\infty$ in the sense of values, but even the asymptotic expansion (3.7) in the sense of symbols. Thus $\mathbf{K}^* f \mathbf{K} \in \mathbf{\Psi}^{-1}_{\mathrm{per}}$, and

$$\begin{aligned}\sigma_{\mathbf{K}^* f \mathbf{K}}(e^{it}, n) \sim \sum_{j,k,M \geq 0} &\frac{1}{(n+1)^{j+k+M+1}} \Big(\frac{i}{2}\,\partial_\theta\Big)^k F_M(e^{it}) \\ &\cdot (-1)^j \binom{j+k+M}{k} c_M(M+j),\end{aligned} \tag{3.35}$$

with F_M given by (3.33). The first three terms (i.e. $j+k+M \leq 2$) in the expansion are

$$\begin{aligned}\sigma_{\mathbf{K}^* f \mathbf{K}}(e^{it}, n) \sim \frac{f(e^{it})}{n+1} &+ \frac{i\partial_t f(e^{it}) - \partial_r f(e^{it})}{2(n+1)^2} \\ &+ \frac{-\partial_t^2 f(e^{it}) + \partial_r f(e^{it}) - 2i\partial_t\partial_r f(e^{it}) + \partial_r^2 f(e^{it})}{4(n+1)^3} + \dots .\end{aligned} \tag{3.36}$$

If f vanishes at $\partial\mathbf{D}$ to order l, then $F_0 = \cdots = F_{l-1} = 0$, so the summation in (3.35) is only over $M \geq l$. This means that $\mathbf{K}^* f \mathbf{K} \in \mathbf{\Psi}^{-l-1}_{\mathrm{per}}$.

Finally, using the fact that

$$\sigma_{\Lambda^{-1/2}}(e^{it}, n) = \sqrt{n+1}$$

and the product formula (3.11), the facts just established for $\mathbf{K}^* f \mathbf{K}$ are easily transferred into the ones about $\Lambda^{-1/2}\mathbf{K}^* f \mathbf{K}\Lambda^{-1/2} = B_f$. (Note that the differences $\Delta_n^j \sigma_{\Lambda^{-1/2}}$ can be handled using (3.27) with $\nu = -\frac{1}{2}$.) Thus, in particular, B_f always belongs to $\mathbf{\Psi}^0_{\mathrm{per}}$, belongs even to $\mathbf{\Psi}^{-k}_{\mathrm{per}}$ if f vanishes at $\partial\mathbf{D}$ to order k, and using (3.36) to evaluate the first terms in the expansion of σ_{B_f} yields (3.15). This completes the proof of Theorem 5. □

These ideas can be extended to show that $\mathrm{Tr}(|H_f|^z)$ is meromorphic with only simple poles. We will prove that but will not develop explicit formulas for the general residues.

Theorem 6. *Suppose $f \in C^\infty(\overline{\mathbf{D}})$ is such that $\overline{\partial} f$ does not vanish on $\mathbf{T}$. Then the function $\zeta(|H_f|, z) := \mathrm{Tr}(|H_f|^z)$ which is holomorphic in $\{z : \mathrm{Re}\, z > 1\}$ extends to a meromorphic function on the entire complex plane $\mathbf{C}$, whose only singularities are simple poles at $z = 1, 0, -1, -2, \dots,$ and*

$$\mathrm{Res}_{z=1}\, \zeta(|H_f|, z) = \mathrm{Tr}_\omega(|H_f|) = \int_{\mathbf{T}} |\overline{\partial} f| d\theta.$$

Proof. We will use standard facts on complex powers A^z and zeta functions $\zeta(A, z) = \mathrm{Tr}(A^z)$ of positive elliptic ΨDO/s A, cf. e.g. Shubin [29].

We have seen that

$$U^*|H_f|U = SQ_fS$$

for a ΨDO/ Q_f on $\mathbf{T}$ of order -1 with asymptotic expansion

$$Q_f = |\overline{\partial} f|\Lambda + \sum_{j=2}^{\infty} g_j \Lambda^j,$$

with some $g_j \in C^\infty(\mathbf{T})$. If $\overline{\partial} f$ does not vanish on $\mathbf{T}$, then Q_f is elliptic, and thus by the standard theory of Seeley has complex powers

$$Q_f^z = |\overline{\partial} f|^z \Lambda^z + \sum_{j=1}^{\infty} g_{j,z} \Lambda^{z+j}, \qquad z \in \mathbf{C},$$

with some $g_{j,z} \in C^\infty(\mathbf{T})$ depending holomorphically on $z \in \mathbf{C}$. For uniformity of notation we also set $g_{0,z} := |\overline{\partial} f|^z$. Since

$$\operatorname{Tr}(Sg\Lambda^z S) = \sum_{j\geq 0} \langle g\Lambda^z \zeta^j, \zeta^j \rangle = \sum_{j\geq 0} \int_{\mathbf{T}} \frac{g(\zeta)}{(j+1)^z}\, d\zeta = \zeta(z) \int_{\mathbf{T}} g d\theta$$

(implying, in particular, that $\operatorname{Tr}(A\Lambda^z)$ is finite and holomorphic in $\{z : \operatorname{Re} z > 1\}$ for any ΨDO A on $\mathbf{T}$ of order ≤ 0), we obtain

$$\begin{aligned} \operatorname{Tr}(|H_f|^z) &= \operatorname{Tr}(SQ_f^z S) \\ &= \sum_{j=0}^{N-1} \zeta(z+j) \int_{\mathbf{T}} g_{j,z} + (\text{a function holomorphic on } \operatorname{Re} z > 1-N), \end{aligned}$$

for any $N = 1, 2, 3, \ldots$. Since $\zeta(z)$ extends to be holomorphic on $\mathbf{C} \setminus \{1\}$ and has a simple pole at $z = 1$ with residue 1, the theorem follows. □

In principle, the use of periodic ΨDOs can be circumvented by passing from the disc to the upper half-plane $\mathbf{U} = \{x + yi \in \mathbf{C} : y > 0\}$. The Cayley transform

$$\mathcal{C}(z) = \frac{z-i}{z+i}$$

is a biholomorphism of $\mathbf{U}$ onto $\mathbf{D}$, and the weighted composition operator

$$U_{\mathcal{C}} : f \mapsto (f \circ \mathcal{C}) \cdot \mathcal{C}'$$

is a unitary isomorphism of $L^2(\mathbf{D})$ onto $L^2(\mathbf{U})$, as well as of the corresponding Bergman subspaces $L^2_{\text{hol}}(\mathbf{D})$ onto $L^2_{\text{hol}}(\mathbf{U})$. A Toeplitz operator T_f, $f \in L^\infty(\mathbf{D})$, on $\mathbf{D}$ corresponds under this isomorphism to the Toeplitz operator $T_{f\circ\mathcal{C}}$ on $L^2(\mathbf{U})$, and similarly for Hankel operators. The role of the Fourier coefficients is taken over by the Fourier transform (3.6), and the formula for the Poisson operator becomes

$$\mathbf{K}f(x+yi) \equiv (\mathbf{K}f)_y(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi - y|\xi|}\, \widehat{f}(\xi)\, d\xi,$$

where we are now denoting by $F_y(x) := F(x+yi)$, $x \in \mathbf{R}$, $y > 0$, the restriction of a function F on $\mathbf{U}$ to the line $\mathbf{R} + yi$. The adjoint of $\mathbf{K}$ is given by

$$\widehat{\mathbf{K}^* F}(\xi) = \int_0^\infty e^{-y|\xi|}\, \widehat{F_y}(\xi)\, dy$$

and $\mathbf{K}^*\mathbf{K} = \Lambda$ is again a Fourier multiplier

$$\widehat{\Lambda f}(\xi) = \frac{1}{2|\xi|}\, \widehat{f}(\xi).$$

Also, for any $f \in C^\infty(\overline{\mathbf{D}})$, the function $g = f \circ \mathcal{C}$ satisfies the estimates

$$|\partial_x^m \partial_y^n g(x+yi)| \le \frac{c_{m,n}}{(|x|+y+1)^{m+n+2}},$$

which serve as a substitute for the estimates (3.30). Using all this, the proof of Theorem 5 carries over with minor modifications also to the half-plane setting. However, the serious trouble that arises is that now the operator Λ is no longer bounded on $L^2(\mathbf{R})$ (and, in particular, $\mathbf{K}$ is also only densely defined and unbounded as an operator from $L^2(\mathbf{R})$ into $L^2(\mathbf{U})$). This has the effect that the various ΨDOs like $\mathbf{K}^* f \mathbf{K}$, B_f, etc., have symbols with singularities of the form $|\xi|^{-m}$ at the origin. Although this technical difficulty can probably be circumvented, it seems much simpler to use the periodic ΨDOs instead.

Another difficulty with the half-plane approach is that the little Hankel operators on $\mathbf{D}$ are not mapped into the ones on $\mathbf{U}$ by the Cayley isomorphism $U_{\mathcal{C}}$ (the reason being that $U_{\mathcal{C}}$ maps holomorphic functions into holomorphic functions, but not conjugate-holomorphic into conjugate-holomorphic); thus for h_f we need to work in $\mathbf{D}$ directly. (For the same reason, however, our Corollary 4 cannot be transferred to little Hankel operators on $\mathbf{U}$.)

We also remark that in higher dimensions (i.e. for the disc replaced by a bounded strictly pseudoconvex domain Ω in $\mathbf{C}^n$ with smooth boundary), the Boutet de Monvel-Guillemin theory gets much more complicated. The main differences against the one-dimensional case are that S itself is no longer a ΨDO on $\partial\Omega$; the operators S and Λ need no longer commute; likewise, $P\mathbf{K} \ne \mathbf{K}S$ in general; and $[B_f, S]$ need not be smoothing. (In fact, one of the cornerstones of the theory is the result that there exists a ΨDO Q such that $SQS = 0$ and $[B_f + Q, S] \sim 0$.) For the case of the unit ball $\mathbf{B}^n$ of $\mathbf{C}^n$, an analysis similar to ours has recently been done by Zhang, Guo and one of the authors [12], using instead of the Boutet de Monvel-Guillemin theory a related technique due to Howe [18]. It turns out that for $\mathbf{B}^n$ with $n \ge 2$, $B_{f\bar{f}} - B_{\bar{f}} B_f$ is of order -1 not -2 (its leading symbol being $\|\nabla f\|^2 - |\mathcal{R} f|^2$ — where $\mathcal{R}$ stands for the radial derivative — which vanishes if $n = 1$); hence, it is enough to evaluate one less term in the asymptotic expansions like (3.35) and (3.15), thus paradoxically making the case $n \ge 2$ easier than the case $n = 1$ of the unit disc. In principle, it should not be difficult to obtain also our results by Howe's method ("pseudo-Toeplitz operators" on the Fock space), and it would be no less interesting to have explicit formulas like (3.15) also for some higher-dimensional situations, e.g. for the unit ball $\mathbf{B}^n$.

4. Optimal regularity for harmonic symbols

Theorem 5 was proved under the *a priori* assumption that f is smooth. In general we do not know how much that requirement can be relaxed; however if f is harmonic we can give a precise statement. First note that such f is uniquely decomposable as $f = f_1 + \bar{f}_2$ with both f_i holomorphic and $f_1(0) = 0$. Also $\bar\partial f = \overline{f_2'}$ and $H_f = H_{\bar{f}_2}$. Hence we can restrict attention to $H_{\bar{f}}$ for holomorphic f.

A holomorphic function g is said to be in the Hardy space H^1 if

$$\|g\|_{H^1} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})| d\theta < \infty.$$

We set $IH^1 = \{f : f' \in H^1\}$. We need notation for two sequence spaces slightly larger than $l^1(\mathbf{Z}_{>0})$. For any sequence $\{s_i\}$ of numbers with limit zero let$\{s_i^*\}$ be the sequence $\{|s_n|\}$ arranged in nonincreasing order. We will say that a sequence $\{s_n\}_{n>0}$ is in *weak* l^1, $\{s_n\} \in l^1_w$, if $s^*_n = O(n^{-1})$. The l^1_w quasinorm of such a sequence is $\sup ns^*_n$ We say that the sequence is in $l^{1,\infty}$ if $\sum_{k=1}^n s^*_n = O(\log(1+n))$. The $l^{1,\infty}$ norm of such a sequence is $\sup(\log(1+n))^{-1}\sum_{k=1}^n s^*_n$. We then have the proper inclusions $l^1 \subset l^1_w \subset l^{1,\infty}$. (A WARNING ABOUT NOTATION: The notation just introduced is in line with that used in the literature on the Dixmier trace but at odds with notation sometimes used for Lorentz sequence spaces, for instance in [27].)

The local oscillation of the symbol function is closely related to the singular values of Hankel operators. When the symbol function is smooth the needed oscillation information is captured by the normalized derivative and it is sufficient to consider those quantities on an appropriately thick discrete set. Pick and fix $r > 0$ and $M, \varepsilon > 0$ with M very large and ε very small. Select a set of points $Z = \{z_i\}$ in the disk so that the hyperbolic balls centered at z_i and of radius εr, $\{B(z_i, \varepsilon r)\}$, are disjoint and that the expanded balls $\{B(z_i, Mr)\}$ cover the disk with bounded overlap; i.e. $\sum \chi_{B(z_i,Mr)}$ is bounded. For given holomorphic g we define the oscillation numbers, $Osc(g(z_i))$ by

$$Osc(g(z_i)) = \sup\left\{(1-|z_i|^2)\,|g'(z)| : z \in B(z_i, Mr)\right\}.$$

We will prove the following regularity result. Earlier work in this direction was done by Li and Russo in [20].

Theorem 7. *Suppose f is a holomorphic function on the disk and select a choice of Tr_ω. The following are equivalent:*

(1) *f is in IH^1.*
(2) *The numbers $\{Osc(f'(z_i))\}$ are in the sequence space l^1_w.*
(3) *The numbers $\{Osc(f'(z_i))\}$ are in the sequence space $l^{1,\infty}$.*
(4) *$Tr_\omega(|H_{\bar f}|) < \infty$.*
(5)

$$Tr_\omega(|H_{\bar f}|) = \frac{1}{2\pi}\int_{\mathbf{T}} |f'|d\theta < \infty. \tag{4.1}$$

Furthermore the inclusions in (2) and (3) do not depend on the particular choice of Z and when (4) or (5) hold for one choice for Tr_ω they hold for every choice. In particular $|H_{\bar f}|$ is measurable.

Finally, the quantities in (4.1) are comparable to both the l^1_w quasinorm and the $l^{1,\infty}$ norm of the sequence $\{Osc(f'(z_i))\}$.

Proof. The sequence space inclusion shows that (2) implies (3) and it is automatic that (5) implies (4). We will show that (1) implies (2), (3) implies (4), (4) implies (1), and finally that (1) is equivalent to (5). The equivalence of the norms and quasinorm are implicit in the proof.

(1) implies (2): It is proved as Theorem C of [27] that if f is in the Besov space B^1 then (2) holds. However that proof starts by noting that $B^1 \subset IH^1$ and then gives a direct argument that condition (1) implies condition (2).

(3) implies (4): In Theorem 4' of [21] Luecking gives conditions on general functions f which ensure that $|H_{\bar f}|$ is in the Schatten ideal $\mathcal{S}_p$. He also notes on page 262 that his proof actually gives more. In particular it shows that if the parameters

used in constructing $\{z_i\}$ are chosen appropriately then there is a $c>0$ so that for all n, $s_n(H_{\bar{f}}) \leq c(b(z_i))^*(n)$. Hence $|H_{\bar{f}}|$ is in the domain of Tr_ω for any ω.

(4) implies (1): We again use the ideas and some of the computations in [21]. It will be convenient to be more specific about the choice of Z. We do that in two steps. Pick and fix a, $1<a<2$ and K large. On the circle $\{z:|z|=1-a^{-n}\}$ distribute Ka^n points, uniformly spaced. Enumerate $\tilde{Z}$ so that points closer to the origin have lower indices. This $\tilde{Z}$ satisfies the covering and separation conditions described earlier. Hence, the linear map $\tilde{T}$ of an abstract Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_i\}$ into the Bergman space which takes e_i to the normalized reproducing kernel at z_i; $\tilde{T}(e_i) = (1-|z_i|^2)(1-\bar{z}_i z)^{-2}$, is bounded [35]. Furthermore the operator norm is bounded by a number that depends only on the separation constants of $\tilde{Z}$. Pick and fix the symbol function $\bar{f}$. We now adjust $\tilde{Z}$ to a new set Z. The point $z_i \in \tilde{Z}$ is on a circle centered at the origin. On that circle it sits in an arc connecting its two nearest (on that circle) neighbors. Let z_i^* be the point on that arc where $|f'|$ is largest. Set $Z=\{z_i^*\}$. This new set will have essentially the same covering data as $\tilde{Z}$, that is, large balls centered at the z_i^* will cover the disk and there will be an upper bound on the depth of the covering. We define T analogously to $\tilde{T}$ but now using the set Z instead of $\tilde{Z}$. We now study $H_{\bar{f}}T$.

Luecking also constructs an additional auxiliary operator S from $\mathcal{H}$ to the Bergman space. With his construction on page 264 of [21] Luecking obtains the estimate that, for some R

$$\inf\left\{\frac{1}{|B(z_i,R)|}\int_{B(z_i,R)}|f-h|^2 : h\in \mathrm{Hol}\left(B(z_i,2R)\right)\right\}^{1/2} \leq C\left|\left\langle S^*H_{\bar{f}}Te_i,e_i\right\rangle\right|$$

Straightforward estimates shows that this gives

$$Osc(f(z_i)) = (1-|z_i|^2)\,|f'(z_i)| \leq C\left|\left\langle S^*H_{\bar{f}}Te_i,e_i\right\rangle\right|. \tag{4.2}$$

We now sum this over all the Ka^n indices which give points on the same circle as z_i. Setting $K'=Ka/(a-1)$ we find

$$\begin{aligned}\sum^{Ka^n}|f'(z_i)|\left(1-|z_i|^2\right) &\leq C\sum^{Ka^n}\left|\left\langle S^*H_{\bar{f}}Te_i,e_i\right\rangle\right| \\ &\leq C\sum^{Ka^n}s_i\left(S^*H_{\bar{f}}T\right) \\ &\leq C\sigma_{K'a^n}\left(S^*H_{\bar{f}}T\right) \\ &\leq C\sigma_{K'a^n}\left(H_{\bar{f}}\right).\end{aligned}$$

On the other hand the sum on the left is, up to a constant factor, an upper Riemann sum for $\int|f'|$ on that circle; hence

$$I(|z_i|) := \int_{|z|=|z_i|}|f'(z)|\,|dz| \leq C\sigma_{K'a^n}\left(H_{\bar{f}}\right).$$

Now pick and fix a large number M and we repeat this analysis on the circles of radius $1-a^{-(n+1)}, ..., 1-a^{-(n+M)}$. The number of points involved is now $\approx a^{n+M}$. Recall that because f is holomorphic $I(r)$ is an increasing function of r. Combining these facts we have

$$MI(|z_i|) \leq C\sigma_J\left(H_{\bar{f}}\right) \text{ with } J\approx a^{n+M}.$$

Dividing by $\log J$ we have

$$\frac{M}{M+n} I\left(|z_i|\right) \leq C \frac{1}{\log J} \sigma_J\left(H_{\bar{f}}\right).$$

Letting $M \to \infty$ we obtain

$$I\left(|z_i|\right) \leq C \underline{\lim}_{J\to\infty} \frac{1}{\log J} \sigma_J\left(H_{\bar{f}}\right).$$

We know $f \in IH^1$ if and only if the left hand side is bounded and thus if the right hand side is bounded. This completes the proof.

(1) is equivalent to (5): We already have the equivalence of the first four conditions and (5) certainly implies (4). To finish we show (1) through (4) imply (5). For $0 < r < 1$ define f_r by $f_r(z) = f(rz)$. By Theorem 5 we know that

$$Tr_\omega\left(\left|H_{\bar{f}_r}\right|\right) = \left\|(f_r)'\right\|_{H^1}.$$

We know that as $r \to 1$ the right hand side converges to $\|f'\|_{H^1}$. Set $g_r = \bar{f}_r - \bar{f}$. We have $H_{\bar{f}_r} - H_{\bar{f}} = H_{g_r}$. Thus

$$\begin{aligned}
\overline{\lim}_r \left|Tr_\omega\left(\left|H_{\bar{f}_r}\right|\right) - Tr_\omega\left(\left|H_{\bar{f}}\right|\right)\right| &\leq C\ \overline{\lim}_r Tr_\omega\left(|H_{g_r}|\right) \\
&\leq C\ \overline{\lim}_r\ \overline{\lim}_N \frac{1}{\log N} \sigma_N\left(H_{g_r}\right) \\
&\leq C\ \overline{\lim}_r\ \overline{\lim}_N \frac{1}{\log N} \sum^N Osc(g_r'(z_i)) \\
&\leq C\ \overline{\lim}_r\ \overline{\lim}_N \left\|Osc(g_r'(z_i))\right\|_{l^1_w} \\
&\leq C\ \overline{\lim}_r\ \left\|g_r'\right\|_{H^1} \\
&= 0.
\end{aligned}$$

Here the passage from the second line to the third uses the fact that (3) implies (1) and the norm and quasinorm equivalences. The passage from the fourth to the fifth uses the fact that condition (1) implies condition (2).

The fact that the first two conditions do not depend on the choice of Z is based on standard estimates such as can be found in [35]. The proof did not use any particulars related to the choice of Tr_ω and hence it holds for any choice. The right hand side of (4.1) does not involve the choice of Tr_ω and hence all choices give the same value. □

There are two places in the proof where aspects of holomorphy play a role. First, the equivalence of (1) and (2) is the statement that a certain potential space (defined by integrability of a derivative) coincides with a Besov type space (defined by global control of local oscillation). Such occurrences are unusual when the spaces are not Hilbert spaces. This is discussed in the Appendix of [10] where it is shown that the space of functions in d dimensions with one derivative in L^d coincides with a weak type Besov space with index d. It is noted there that the result fails for $d = 1$, their proof only yielding the conclusion that the boundary values of f have bounded variation and hence that f' is a finite measure. However in our context we have the additional hypothesis of holomorphy and hence can appeal to the F. and M. Riesz theorem to see that the measure is absolutely continuous giving a direct proof that (2) implies (1).

Second, the passage from (2) to (3) follows from the obvious sequence space inclusion. However we eventually obtain that (3) implies (2). At its heart that result is based on the fact which we used in proving that (4) implies (1): the integral means $\int \left|f'(re^{i\theta})\right| d\theta$ are an increasing function of r, a fact proved using considerations of subharmonicity.

5. Other Bergman Spaces

Once we have Theorem 5 we can obtain similar results for Hankel operators on Bergman spaces of multiply connected domains.

Let Ω be a bounded domain in the plane with boundary consisting of finitely many smooth disjoint curves $\{\Gamma_i\}_{i=1}^n$. Let dA be Lebesgue area measure. (An easy calculation shows that normalizing dA to, say, total mass one doesn't affect the singular values of Hankel operators.) Let $d\gamma$ be arclength rescaled on each boundary component to give the component unit mass; $d\gamma = \sum (\text{length}\,(\Gamma_i))^{-1} \chi_{\Gamma_i} |dz|$. The Bergman space of such a domain, $L^2_{\text{hol}}(\Omega)$, is the closed subspace of $L^2(\Omega) = L^2(\Omega, dA)$ consisting of holomorphic functions. For convenience in this section we will write $\mathcal{B}(\Omega)$ for $L^2_{\text{hol}}(\Omega)$. We write P for the orthogonal projection of $L^2(\Omega, dxdy)$ to $\mathcal{B}(\Omega)$.

For a function $f \in C^\infty(\bar{\Omega})$ we define the Hankel operator with symbol f, H_f^Ω, as a linear operator from $L^2(\Omega)$ to $L^2(\Omega)$ given by

$$H_f^\Omega g = (I - P) f P g.$$

Theorem 8. *For any choice of Dixmier trace for operators on $L^2(\Omega)$ we have*

$$Tr_\omega\left(\left|H_f^\Omega\right|\right) = \int_{\partial\Omega} |\bar{\partial} f| d\gamma. \tag{5.1}$$

Proof. Suppose first that $n = 1$, i.e. that Ω is simply connected. Let $\phi : \mathbf{D} \to \Omega$ be a univalent holomorphic map of $\mathbf{D}$ onto Ω. Define $U = U_\phi : L^2(\Omega, dA) \to L^2(\mathbf{D})$ by $U(g(z)) = g(\phi(z))\phi'(z)$. The following facts are straightforward: U is unitary, U maps the subspace $\mathcal{B}(\Omega)$ into and onto the subspace $\mathcal{B}(\mathbf{D})$, and, with M_g denoting the operator of multiplication by g, $UM_f = M_{f\circ\phi}U$. Using these facts it is immediate that $UH_f^\Omega = H_{f\circ\phi}^{\mathbf{D}}U$. Recall also that $\bar{\partial}(f \circ \phi)(z) = \bar{\partial} f(\phi(z))\,\bar{\phi}'(z)$. Hence the case $n = 1$ of the theorem follows from (1.1).

We just noted that the hypotheses and conclusion transform well under a biholomorphic change of variable. Hence, without loss of generality, we can, and do, suppose that all Γ_j are circles, with Γ_2 the unit circle and all other Γ_j, $j \neq 2$, contained in the unit disc $\mathbf{D}$.

Let Ω_j, $j = 1, \ldots, n$, be the component of $\mathbf{C} \setminus \Gamma_j$ which contains Ω, and let $\mathcal{B}_j$ be the subspace of $\mathcal{B}$ consisting of all the functions which extend to be holomorphic in Ω_j, and which vanish at ∞ if Ω_j is unbounded (i.e. for $j \neq 2$). It is not hard to see that $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \cdots + \mathcal{B}_n$, a non-orthogonal direct sum decomposition. We denote the associated (oblique) projections from $\mathcal{B}$ onto $\mathcal{B}_j$ by Q_j.

We now consider the case of $n = 2$. This is for notational convenience; the details for $n > 2$ are straightforward extensions and we will omit them.

For compact operators A and B we will write $A \approx B$ if the singular values satisfy $s_n(A - B) = O(c^n)$ for some c, $0 < c < 1$. This is enough to ensure that $\text{Tr}_\omega(|A|) = \text{Tr}_\omega(|B|)$. (In fact, $s_n(A - B) = O(n^{-2})$ would do; cf. [15], Lemma II.4.2, and [9], Ch. IV.3, Lemma 9 on p. 320 (with $\alpha = \frac{1}{2}$).) To prove the theorem we will replace H^Ω_f by a sequence of simpler operators all related through $\approx$.

For $i=1,2$ let f_i be smooth functions on Ω that are supported in small disjoint neighborhoods of Γ_i and which agree with f in those neighborhoods. We will verify first that there is no loss replacing f by f_1+f_2:

Claim 1. $H^\Omega_f \approx H^\Omega_{f_1+f_2} = H^\Omega_{f_1} + H^\Omega_{f_2}$.

We then show that for each summand there is no loss in restricting the operator to functions which are large near that boundary component:

Claim 2. $H^\Omega_{f_1} \approx H^\Omega_{f_1} Q_1$*;* $H^\Omega_{f_2} \approx H^\Omega_{f_2} Q_2$

To analyze $|H^\Omega_{f_1}Q_1 + H^\Omega_{f_2}Q_2|$ we verify

Claim 3.

$$\begin{aligned}(H^\Omega_{f_1}Q_1 + H^\Omega_{f_2}Q_2)^*(H^\Omega_{f_1}Q_1 + H^\Omega_{f_2}Q_2) &\approx (H^\Omega_{f_1}Q_1)^* H^\Omega_{f_1}Q_1 + (H^\Omega_{f_2}Q_2)^* H^\Omega_{f_2}Q_2 \\ &= (H^\Omega_{f_1}Q_1 \oplus H^\Omega_{f_2}Q_2)^*(H^\Omega_{f_1}Q_1 \oplus H^\Omega_{f_2}Q_2).\end{aligned}$$

This will ensure us that $|H^\Omega_{f_1}Q_1 + H^\Omega_{f_2}Q_2| \approx |H^\Omega_{f_1}Q_1| \oplus |H^\Omega_{f_2}Q_2|$ and hence that $\mathrm{Tr}_\omega(|H^\Omega_{f_1}Q_1 + H^\Omega_{f_2}Q_2|) = \mathrm{Tr}_\omega(|H^\Omega_{f_1}Q_1|) + \mathrm{Tr}_\omega(|H^\Omega_{f_2}Q_2|)$. The two summands on the right are handled similarly; we just look at the second. The operator $H^\Omega_{f_2}Q_2$ maps $L^2(\Omega)$ into itself. We extend it to $L^2(\Omega_2)$ by writing $L^2(\Omega_2) = L^2(\Omega) \oplus L^2(\Omega_2 \setminus \Omega)$, identifying $L^2(\Omega)$ with $L^2(\Omega) \oplus \{0\}$. We then extend $H^\Omega_{f_2}Q_2$ as $H^\Omega_{f_2}Q_2 \oplus 0$.

Let $H^{\mathbf{D}}_{\widetilde{f}_2} = (I - P_{\mathbf{D}})\widetilde{f}_2 P_{\mathbf{D}}$ be the Hankel operator on the Bergman space of the unit disc with symbol $\widetilde{f}_2$, the extension of f_2 to $\mathbf{D}$ by zero; also a map of $L^2(\Omega_2)$ into itself. We will show

Claim 4. $H^\Omega_{f_2}Q_2 \oplus 0 \approx H^{\mathbf{D}}_{\widetilde{f}_2}$.

The earlier claims reduced the issue to considering $H^\Omega_{f_2}Q_2$. With this final claim we complete the proof because

$$\begin{aligned}\mathrm{Tr}_\omega(|H^\Omega_{f_2}Q_2|) &= \mathrm{Tr}_\omega(|H^\Omega_{f_2}Q_2 \oplus 0|) \\ &= \mathrm{Tr}_\omega(|H^{\mathbf{D}}_{\widetilde{f}_2}|) \\ &= \int_{\Gamma_2} |\overline{\partial} f_2| \, d\gamma \\ &= \int_{\Gamma_2} |\overline{\partial} f| \, d\gamma.\end{aligned}$$

The evaluation of Tr_ω used the case $n=1$ of the theorem.

We now proceed to the claims.

(Claim 1.) Just note that $H^\Omega_f - H^\Omega_{f_1+f_2} = H^\Omega_g$ with g supported on a compact subset of Ω. For such g, the next proposition implies that $H^\Omega_g \approx 0$.

Proposition 1. *Let Ω be a domain in $\mathbf{C}$, g a bounded function supported on a compact subset of Ω, and let $M^\Omega_g = gP$ be the restriction to $\mathcal{B}(\Omega)$ of the operator $f \mapsto gf$ on $L^2(\Omega)$ of multiplication by g. Then $M^\Omega_g \approx 0$.*

Proof. For $x \in \Omega$, let D_x and d_x denote the discs with center x and radii $\mathrm{dist}(x, \partial\Omega)$ and $\frac{1}{2}\mathrm{dist}(x, \partial\Omega)$, respectively. There exist finitely many $d_{x_j} \equiv d_j$, $j = 1, \dots, m$, that cover the support of g; and we can decompose g as a sum $g = \sum_{j=1}^m g_j$ with g_j supported in d_j. (For instance, take for g_j the restriction of g to $d_j \setminus \bigcup_{k=1}^{j-1} d_k$.) Then $M_g^\Omega = \sum_j M_{g_j}^\Omega = \sum_j \iota_j M_{g_j}^{\Omega_j} r_j$, where $\iota_j : L^2(D_j) \to L^2(\Omega)$ and $r_j : \mathcal{B}(\Omega) \to \mathcal{B}(D_j)$ are the inclusion and the restriction maps from Ω to $D_j \equiv D_{x_j}$, respectively. Since i_j and r_j are bounded (in fact — even contractive), it is enough to prove that $M_{g_j}^{\Omega_j} \approx 0$.

We have thus reduced to the situation when $\Omega = \mathbf{D}$ and g is a bounded function supported in $\{z : |z| < \frac{1}{2}\} \equiv \frac{1}{2}\mathbf{D}$. Clearly, we may also assume that $\|g\|_\infty \leq 1$. Let χ denote the indicator function of $\frac{1}{2}\mathbf{D}$. Since (we will drop the superscripts $\mathbf{D}$ in the rest of this proof) $M_g^* M_g = T_{|g|^2} \leq T_\chi = M_\chi^* M_\chi$, whence $s_n(M_g) \leq s_n(M_\chi)$ for all n, it is in fact enough to deal with $g = \chi$. However, an easy calculation using the fact that the monomials z^n are an orthogonal basis of eigenvectors for $M_\chi^* M_\chi = T_\chi$ shows that $s_n(M_\chi) = 2^{-n-1}$. This completes the proof. □

The argument works, without modifications, for an arbitrary domain Ω in $\mathbb{C}^n$.

(Claim 2.) We need to show that $H_{f_1}^\Omega Q_2 \approx 0$, or $(I-P) f_1 Q_2 P \approx 0$. We claim that even

$$f_1 Q_2 P \approx 0. \tag{5.2}$$

To see this, let $\rho : \mathcal{B}(\mathbf{D}) \to \mathcal{B}_2$ be the restriction map, and $\iota = \rho^{-1} : \mathcal{B}_2 \to \mathcal{B}(\mathbf{D})$ the inclusion of $\mathcal{B}_2$ into $\mathcal{B}(\mathbf{D})$. Then $f_1 Q_2 P = f_1 \rho \iota Q_2 P$ so it is enough to show that $f_1 \rho \approx 0$. However, for any $\phi, \psi \in \mathcal{B}(\mathbf{D})$,

$$\langle f_1 \rho \phi, f_1 \rho \psi \rangle_\Omega = \int_\Omega |f_1|^2 \phi \overline{\psi} = \langle T_{|\widetilde{f}_1|^2}^{\mathbf{D}} \phi, \psi \rangle_{\mathbf{D}}, \tag{5.3}$$

where $\widetilde{f}_1$, the extension by zero of f_1 to $\mathbf{D}$, is compactly supported in $\mathbf{D}$. Consequently, $(f_1\rho)^*(f_1\rho) = T_{|\widetilde{f}_1|^2}^{\mathbf{D}} \approx 0$ by the preceding proposition. Thus $f_1 \rho \approx 0$.

(Claim 3.) We need to show that $(H_{f_1}^\Omega Q_1)^*(H_{f_2}^\Omega Q_2) \approx 0$ and also that a similar result holds with the indices interchanged. The two are similar and we will just look at the first. This is slightly more delicate than the previous claims, and will require some particulars of the Bergman kernels.

We have

$$\begin{aligned}(H_{f_1}^\Omega Q_1)^*(H_{f_2}^\Omega Q_2) &= Q_1^* P \overline{f}_1 (I-P) f_2 Q_2 P \\ &= -Q_1^* P \overline{f}_1 P f_2 Q_2 P.\end{aligned}$$

From (5.2) we have $\overline{f}_1 P \approx \overline{f}_1 Q_1 P$, and similarly, taking adjoints,

$$P f_2 \approx Q_2^* P f_2. \tag{5.4}$$

Thus we can continue with

$$(H_{f_1}^\Omega Q_1)^*(H_{f_2}^\Omega Q_2) \approx -Q_1^* P \overline{f}_1 Q_1 Q_2^* P f_2 Q_2 P.$$

We claim that we even have

$$Q_1 Q_2^* \approx 0. \tag{5.5}$$

Indeed, for any orthonormal basis $\{e_j\}_{j\geq 0}$ of $\mathcal{B}(\Omega)$, $Q_1 Q_2^* = \sum_{j\geq 0} \langle \cdot\,, Q_2 e_j \rangle Q_1 e_j$ is an integral operator on Ω with integral kernel

$$k(x,y) := Q_{1,x} \overline{Q}_{2,y} \sum_{j\geq 0} e_j(x) \overline{e_j(y)} = Q_{1,x} \overline{Q}_{2,y} K_\Omega(x,y),$$

K_Ω being the Bergman kernel of Ω, where the subscripts x, y indicate the variable to which the operator Q_j applies, and $\overline{Q}_2\overline{f} := \overline{Q_2 f}$. Since $\partial\Omega$ consists of circles and, hence, is real-analytic, it is a result of Bell [3] that K_Ω extends to a holomorphic function of $x, \overline{y}$ for (x, y) in a neighborhood of the closure of $\Omega \times \Omega$ minus the boundary diagonal; thus, in particular, to $x \in \Omega \cup$ (a neighborhood of Γ_1) and $y \in \Omega \cup$ (a neighborhood of Γ_2). By the next lemma, it follows that $k(x, y)$ is actually holomorphic in $x, \overline{y}$ for x in a neighborhood of $\overline{\Omega}_1$ and y in a neighborhood of $\overline{\Omega}_2 (= \overline{\mathbf{D}})$; and the next proposition then implies that $Q_1 Q_2^* \approx 0$.

Lemma 2. *If $f \in \mathcal{B}(\Omega)$ extends to be holomorphic in a neighborhood of Γ_j, then so does $Q_j f$; that is, $Q_j f$ extends to be holomorphic in a neighborhood of $\overline{\Omega}_j$.*

Proof. We give the proof for $j = 2$. By Cauchy's formula, $Q_2 f$ is given by

$$Q_2 f(z) = \frac{1}{2\pi i} \oint_{r\mathbf{T}} \frac{f(\xi)}{\xi - z}\, d\xi,$$

for any r, $|z| < r < 1$, the value of the integral being independent of the choice of such r. If f is even holomorphic on $|z| < 1 + \delta$, then we can even take any r with $|z| < r < 1 + \delta$, showing that $Q_2 f$ likewise extends to be holomorphic in $|z| < 1 + \delta$. □

Proposition 2. *Let T be an integral operator on a bounded domain Ω,*

$$Tf(x) = \int_\Omega k(x, y) f(y)\, dy,$$

whose integral kernel $k(x, y)$ belongs to the complex conjugate of $\mathcal{B}(\Omega)$ for each fixed x, and is holomorphic on $\Omega_0 \supset \overline{\Omega}$ for each fixed y. Then $T \approx 0$.

Proof. Let $\Omega_{1/2}$ be a domain containing $\overline{\Omega}$ but such that its closure is contained in Ω_0. Morera's and Fubini's theorems imply that the integral

$$\int_\Omega k(x, y) \overline{k(z, y)}\, dy$$

is holomorphic (hence — continuous) in $x, \overline{z}$ on $\Omega_0 \times \Omega_0$; taking $x = z$ it follows, in particular, that

$$\|k(x, \cdot)\|_{L^2(\Omega)} \le C \qquad \forall x \in \Omega_{1/2}$$

for some finite C. Straightforward estimates then show that the operator

$$\widetilde{T} f(x) := \int_\Omega k(x, y) f(y)\, dy$$

is bounded from $\mathcal{B}(\Omega)$ into $\mathcal{B}(\Omega_{1/2})$ (with norm not exceeding $C|\Omega|^{1/2}$). Now $T = \tau \widetilde{T}$ where τ is the restriction map $\tau : \mathcal{B}(\Omega_{1/2}) \to \mathcal{B}(\Omega)$; by the same argument as in (5.3), it follows from Proposition 1 that $\tau^* \tau = T^{\Omega_{1/2}}_{\chi_\Omega} \approx 0$. Hence $\tau \approx 0$ and $T \approx 0$. □

(Claim 4.) To cope with the various identifications, we denote by $R : L^2(\mathbf{D}) \to L^2(\Omega)$ the restriction map, and by $E = R^* : L^2(\Omega) \to L^2(\mathbf{D})$ the map of prolonging by zero. We are thus claiming that $EH^\Omega_{f_2} Q_2 P R \approx H^{\mathbf{D}}_{\widetilde{f}_2}$, where $\widetilde{f}_2 = E f_2$.

We have

$$\begin{aligned} EH^{\Omega}_{f_2}Q_2PR - H^{\mathbf{D}}_{\widetilde{f}_2} &= E(I-P)f_2Q_2PR - (I-P_{\mathbf{D}})\widetilde{f}_2P_{\mathbf{D}} \\ &= E(I-P)f_2Q_2PR - (I-P_{\mathbf{D}})Ef_2RP_{\mathbf{D}} \\ &= E(I-P)f_2(Q_2PR - RP_{\mathbf{D}}) + [E(I-P)-(I-P_{\mathbf{D}})E]f_2RP_{\mathbf{D}} \\ &= E(I-P)f_2(Q_2PR - RP_{\mathbf{D}}) - (EP - P_{\mathbf{D}}E)f_2RP_{\mathbf{D}} \\ &= E(I-P)f_2(Q_2PR - RP_{\mathbf{D}}) - (EQ_2P - P_{\mathbf{D}}E)f_2RP_{\mathbf{D}} - EQ_1Pf_2RP_{\mathbf{D}}. \end{aligned}$$

Thus it is enough to show that

$$Q_2PR - RP_{\mathbf{D}} \approx 0, \tag{5.6}$$

$$EQ_2P - P_{\mathbf{D}}E \approx 0, \tag{5.7}$$

$$Q_1Pf_2 \approx 0. \tag{5.8}$$

From (5.4) we have $Q_1Pf_2 \approx Q_1Q_2^*Pf_2$ and thus (5.8) is immediate from (5.5). For (5.6), observe that for $F \in L^2(\mathbf{D})$ and $x \in \Omega$,

$$\begin{aligned} (Q_2PR - RP_{\mathbf{D}})F(x) &= \int_{\Omega} Q_{2,x}K_{\Omega}(x,y)F(y)\,dy - \int_{\mathbf{D}} K_{\mathbf{D}}(x,y)F(y)\,dy \\ &= \int_{\Omega} [Q_{2,x}K_{\Omega}(x,y) - K_{\mathbf{D}}(x,y)]F(y)\,dy - \int_{\mathbf{D}\setminus\Omega} K_{\mathbf{D}}(x,y)F(y)\,dy. \end{aligned}$$

The second summand is just $RT^{\mathbf{D}}_{\chi_{\mathbf{D}\setminus\Omega}}F(x)$, and $T^{\mathbf{D}}_{\chi_{\mathbf{D}\setminus\Omega}} \approx 0$ by Proposition 1. The first summand vanishes if $RF \perp \mathcal{B}(\Omega)$, while on $\mathcal{B}(\Omega)$ it acts as integral operator with kernel $Q_{2,x}K_{\Omega}(x,y) - K_{\mathbf{D}}(x,y) = Q_{2,x}[K_{\Omega}(x,y) - K_{\mathbf{D}}(x,y)]$. From Theorem 23.4 of [4] we know that the difference $K_{\Omega}(x,y) - K_{\mathbf{D}}(x,y)$ extends to be holomorphic in $x, \overline{y}$ in a neighborhood of $\Gamma_2 = \partial\mathbf{D}$; by Lemma 2 we thus conclude that $Q_{2,x}[K_{\Omega}(x,y) - K_{\mathbf{D}}(x,y)]$ is in fact holomorphic for x in a neighborhood of $\overline{\mathbf{D}}$ and y in $\Omega \cup$ (a neighborhood of Γ_2). By Proposition 2, the corresponding integral operator is ≈ 0, thus proving (5.6).

With (5.6) in hand, it follows that

$$\begin{aligned} 0 \approx E(Q_2PR - RP_{\mathbf{D}})E &= EQ_2P - ERP_{\mathbf{D}}E \\ &= EQ_2P - P_{\mathbf{D}}E + (I - ER)P_{\mathbf{D}}E. \end{aligned}$$

As $(I - ER)P_{\mathbf{D}} = \chi_{\mathbf{D}\setminus\Omega}P_{\mathbf{D}} \approx 0$ by Proposition 1, (5.7) follows. □

6. Related Operators

In an effort to understand better the difference between the singular value behavior of the small and big Hankel operators the authors of [25] studied operators built from multiplication followed by projection onto subspaces of $L^2(\mathbf{D})$ which sit between $\overline{L^2_{\text{hol}}(\mathbf{D})}$ and $L^2_{\text{hol}}(\mathbf{D})^{\perp}$. Set $\bar{D} = \bar{z}\bar{\partial}$ and note that the Bergman space, which we will denote by A_0 in this section, is the closure of the smooth functions in the kernel of $\bar{D}$. Let $A_1 = \ker(\bar{D}^2)$, set $A^1 = A_1 \ominus A_0$ and let P_{A^1} be the orthogonal projection onto A^1.

Pick and fix a smooth holomorphic symbol function $b = \sum b_n z^n$ and recall that $H_{\bar{b}} = P^{\perp}\bar{b}P$. We will compare this with the *intermediate Hankel operator* $K_{\bar{b}} = P_{A^1}\bar{b}P$; that is $K_{\bar{b}}(f) = P_{A^1}\left(\bar{b}f\right)$. The difference $K_{\bar{b}} - H_{\bar{b}}$ is, in the notation of [25], the operator $-H^1_b$. Theorem 5 of [25] states that for b in the Besov space

B^1 the operator H_b^1 will be in the trace class. A trace class perturbation of an operator A will not change $Tr_\omega(|A|)$. Thus we have the following corollary.

Theorem 9. *For any choice of* Tr_ω, $Tr_\omega\left(|K_{\bar b}|\right) = Tr_\omega\left(|H_{\bar b}|\right)$.

Theorem 1 in [25] gives orthonormal bases for both A_0 and A^1. An orthornomal basis for A_0 is given by the functions

$$e_{0,n} = \sqrt{n+1} z^n \;\; n = 0, 1, 2, ...$$

and for A^1 by the functions

$$e_{1,n} = \sqrt{n+1}\left(2\left(n+1\right)\log r + 1\right) z^n \;\; n = 0, 1, 2, ...$$

Using these bases we can compute the matrix of $K_{\bar b}$; the matrix $M_{\bar b} = (\beta_{ij})$ with $\bar\beta_{i,j} = \langle K_{\bar b}(e_{0,i}), e_{1,j}\rangle$.

Proposition 3. *The entries of the matrix* $M_{\bar b}$ *are given by*

$$\begin{aligned}\beta_{i,j} &= \frac{\left(j+1\right)^{1/2}\left(i-j\right)\bar b_{i-j}}{\left(i+1\right)^{3/2}} \text{ if } i \geq j \\ &= 0 \text{ otherwise.}\end{aligned}$$

Proof. The $\beta_{i,j}$ depend conjugate linearly on b so it suffices to do the computation for the monomial $b = z^N$. We have

$$\begin{aligned}\beta_{i,j} &= \langle K_{\bar b}(e_{0,i}), e_{i,j}\rangle = \left\langle P_{A^1}\bar b e_{0,i}, e_{i,j}\right\rangle \\ &= \left\langle \bar b e_{0,i}, e_{i,j}\right\rangle = \langle e_{0,i}, b e_{i,j}\rangle \\ &= \left\langle e_{0,i}, z^N e_{i,j}\right\rangle.\end{aligned}$$

Computing the inner product by first doing the θ integration shows that this quantity is zero unless $i = j + N$. In that case the remaining integral is

$$\begin{aligned}\beta_{i,j} &= \frac{1}{\pi} 2\pi\sqrt{i+1}\sqrt{j+1}\int_0^1 r^{2i+1}\left(2\left(j+1\right)\log r + 1\right) dr \\ &= 2\sqrt{i+1}\sqrt{j+1}\left(\frac{1}{2i+2} - \frac{2\left(j+1\right)}{\left(2i+2\right)^2}\right) \\ &= \left(j+1\right)^{1/2}\left(i+1\right)^{-3/2}\left(i-j\right)\end{aligned}$$

which gives the required result. □

We can regard $M_{\bar b}$ as the matrix of an operator on the Hardy spaces with respect to the monomial basis and give that operator a function theoretic description. Recall the operator Λ introduced in (3.3) and for any real α let Λ^α be the generalized differentiation or integration operator on $L^2(\mathbf{T})$ defined through

$$\Lambda^\alpha\left(e^{in\theta}\right) = \left(1 + |n|\right)^{-\alpha} e^{in\theta}.$$

Recall that T_f^H denotes the Toeplitz operator on the Hardy space with symbol f. It is a straightforward computation that $M_{\bar b}$ is the matrix of the operator $S_{\bar b} = \Lambda^{-1/2} T^H_{\overline{zb'}} \Lambda^{3/2}$ and thus we have $Tr_\omega(|S_{\bar b}|) = \int |b'|$.

In fact we could have applied the ideas of Section 3 directly to $S_{\bar b}$ and then used the results of [25] to pass the results of that analysis back to Bergman space Hankel operators and obtained (1.1). That would have the advantage of staying in the Hardy space where the computations are a bit simpler but would use the results of [25] which are more computational than conceptual. Also, it is not clear that approach could also yield (1.2).

7. Computations

In some cases evaluating the integrals in (1.1) or (5.1) is straightforward. For instance if $f(z) = \bar{z}$ then

$$Tr_\omega\left(\left|H_{\bar{z}}^{\Omega}\right|\right) = \text{number of components of } \partial\Omega.$$

Also, on the disk if $f = \bar{g}$ and g' is a finite Blaschke product, or any inner function, then $Tr_\omega(|H_f|) = 1$.

For some f it is possible to use the Cauchy-Riemann equations to give a geometric or function theoretic interpretations to the values of the integrals in (1.1) or (5.1). Suppose Γ is a real analytic simple closed curve bounding the bounded domain Ω. Suppose Γ' is another simple closed curve which is inside Ω (and which we think of as being near and roughly parallel Γ). Denote by Ω' the subdomain of Ω bounded by Γ and Γ'. If dh is a harmonic differential on a domain in the plane we denote by $*dh$ the harmonic differential which is conjugate to dh (see Ch. II of [1]).

Proposition 4. (1) *Suppose $f(z)$ is continuous on $\bar{\Omega}'$ and holomorphic on Ω'. Suppose further that $|f(z)| \equiv 1$ on Γ and that for some $c > 0$, $c \leq |f(z)| \leq 1$. It follows that*

$$\int_\Gamma \left|\bar{\partial}\bar{f}\right| |dz| = \int_\Gamma d\arg(f) = \int_\Gamma *d\log|f|. \tag{7.1}$$

(2) *Suppose h is continuous on $\bar{\Omega}'$, harmonic and negative on Ω', and $h \equiv 0$ on Γ then*

$$\int_\Gamma \left|\bar{\partial}e^h\right| |dz| = \frac{1}{2}\int_\Gamma *dh. \tag{7.2}$$

(3) *If, instead, Γ were the* inner *boundary then analogous statements hold with a negative sign inserted on the right hand side. The variation in which $|f|$ or h has a minimum on Γ also introduces a negative sign*

Proof. We know f has constant modulus on Γ and that Γ is real analytic hence we can use the reflection principle to extend f to be holomorphic in a small neighborhood of Γ. Pick and fix $x \in \Gamma$ and U a small simply connected neighborhood of x. Let $\arg f(z)$ be a choice of argument which is harmonic in U and thus $\log f(z) = \log|f(x)| + i\arg f(z)$ is holomorphic there. In U the integrand is

$$\left|\bar{\partial}\bar{f}\right| = |\partial f| = |f\partial\log f| = |\partial\log f| = |\partial\log|f| + \partial\arg f|.$$

The integration is along Γ and $\log|f|$ is constant on Γ thus the integrand simplifies to $|\partial\arg f|$. Now note that the directional derivative of $\log|f|$ in the direction of the outward normal to $\bar{\Omega}$ at x is positive. Hence, by the Cauchy-Riemann equations the directional derivative of $\arg f$ in the direction of the positively oriented tangent to Γ at x is positive. Thus we can drop the absolute value and find that

$$\int_{\Gamma\cap U} \left|\bar{\partial}\bar{f}\right| |dz| = \int_{\Gamma\cap U} d\arg(f).$$

Piecing together these local results gives (7.1).

We could obtain the second statement by working directly with the fact that h is harmonic. Alternatively note that, locally, $e^h = f\bar{f}$ for a holomorphic function f

with $\log|f| = \frac{1}{2}h$. Hence $\left|\bar\partial e^h\right| = \left|\bar\partial f\bar f\right| = \left|f\bar\partial f\right| = \left|\bar\partial f\right|$ and the desired conclusion follows from the first statement.

The third statement is straightforward. □

On the disk it is immediate from (1.1) that $Tr_\omega\left(|H_{\bar z^n}|\right) = n$. Using the proposition we see that the same conclusion holds if z^n is replaced by any Blaschke product with n factors.

Suppose now that Ω is bounded by n smooth curves. Pick $\zeta \in \Omega$ and consider the holomorphic function $g(z)$ which solves the following extremal problem:

$$\text{maximize } \operatorname{Re} g'(\zeta) \text{ subject to } g(\zeta) = 0 \text{ and } \sup|g(z)| \le 1.$$

This function, often called the Ahlfors function, represents Ω as an $n-$sheeted cover of the disk with the boundary going to the boundary [3]. In particular we can use (5.1) and (7.1) to conclude that $Tr_\omega\left(|H^\Omega_{\overline{g}}|\right) = n$. (Recall that the boundary measure $d\gamma$ in (5.1) is built from *normalized* arc length measures.)

We now consider symbol functions of the form $g(z) = \phi(z)e^{h(z)}$ where h is real valued and harmonic and ϕ is a localizing function. We suppose ϕ is smooth, is identically one on a neighborhood of one boundary component, say Γ_1, and identically zero on neighborhoods of the other boundary components. In that case, by (5.1) we have

$$Tr_\omega\left(\left|H^\Omega_g\right|\right) = \int_{\partial\Omega}|\bar\partial g|d\gamma = \int_{\Gamma_1}|\bar\partial g|d\gamma = \int_{\Gamma_1}|\bar\partial e^h|d\gamma. \tag{7.3}$$

In some situations we can use the previous proposition to continue the computation.

First we consider double connected domains. Select r, $0 < r < 1$, and let $\Omega = \Omega(r)$ be the ring domain with outer boundary Γ_1 the circle centered at the origin with radius 1 and with inner boundary Γ_r the concentric circle with radius r. Let $h(z)$ be the harmonic function on Ω with boundary values 0 on Γ_1 and -1 on Γ_r. Pick the smooth function ϕ which is one near Γ_1 and 0 near Γ_r. Again setting $g = \phi e^h$ and combining (7.2) with the previous equality we find

$$Tr_\omega\left(\left|H^\Omega_g\right|\right) = \frac{1}{2}\frac{1}{2\pi}\int_{|z|=1} *dh. \tag{7.4}$$

We know that h must be of the form $A + B\log|z|$ for some real A, B and we find

$$h(z) = 1 + \frac{2}{\log r}\log|z| = \operatorname{Re}\left(1 + \frac{2}{\log r}\log z\right).$$

Hence

$$*dh(z) = d\operatorname{Im}\left(1 + \frac{2}{\log r}\log z\right) = \frac{2}{\log r}d\arg z = \frac{2}{\log r}d\theta.$$

We combine this with the earlier computation and conclude that

$$Tr_\omega\left(\left|H^\Omega_g\right|\right) = \frac{1}{\log r}.$$

If $\tilde\Omega$ is any other doubly connected domain we can do the same analysis. That is, we can let $\tilde h$ be the harmonic function which is 0 on the outer boundary and -1 on the inner boundary. Then construct $\tilde g$ by localizing $\exp\tilde h$ to a neighborhood of the outer boundary and consider $T_\omega(|H^{\tilde\Omega}_{\tilde g}|)$. There is a unique $\bar r$ so that $\tilde\Omega$ is conformally equivalent to $\Omega(\bar r)$ and we can choose the conformal map to take one outer boundary to the other outer boundary. We noted earlier that $Tr_\omega(|H^{\tilde\Omega}_{\cdot}|)$ behaves well under conformal maps. Also, the conformal map takes harmonic functions to harmonic functions. Combining these facts we find.

$$T_\omega(|H_{\tilde{g}}^{\tilde{\Omega}}|) = \frac{1}{\log \tilde{r}}.$$

In particular the trace is determined by the conformal type of $\tilde{\Omega}$. Conversely the conformal type of a doubly connected domain is completely determined by the parameter r. We conclude, with the natural interpretation of $\breve{\Omega}$ and $\breve{g}$, that

$$\tilde{\Omega} \text{ and } \breve{\Omega} \text{ are conformally equivalent iff } Tr_\omega(|H_{\tilde{g}}^{\tilde{\Omega}}|) = Tr_\omega(|H_{\breve{g}}^{\breve{\Omega}}|).$$

This analysis extends to multiply connected domains. Suppose Ω is a domain bounded by n real analytic curves $\Gamma_1, ..., \Gamma_n$. Select a Γ_i and consider the associated *harmonic measure,* that is, the harmonic function h_i with boundary values 1 on Γ_i and 0 on the other components. For each index j, where $i = j$ is allowed, set $h_{ij} = \phi_j h_i$ where ϕ_j is smooth, one near Γ_j and zero in neighborhoods of the other boundary components. The straightforward extension of the previous argument gives

$$\alpha_{ij} := Tr_\omega(|H^\Omega_{\exp h_{ij}}|) = \pm \int_{\Gamma_j} *dh_i.$$

As before, the numbers $\{\alpha_{i,j}\}$ are conformal invariants of Ω. Also, they again determine the conformal structure of the domain, but now only up to reflection. That last fact is Proposition 4.10 in [26].

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