BEREZIN TRANSFORM ON THE HARMONIC FOCK SPACE

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Abstract. We show that the Berezin transform associated to the harmonic Fock (Segal-Bargmann) space on $\mathbf{C}^n$ has an asymptotic expansion analogously as in the holomorphic case. The proof involves a computation of the reproducing kernel, which turns out to be given by one of Horn's hypergeometric functions of two variables, and an ad hoc determination of the asymptotic behaviour of the resulting integrals, to which the ordinary stationary phase method is not directly applicable.

1. Introduction

Let $\mathcal{F}_h$ be the Segal-Bargmann (or Fock) space of all entire functions on $\mathbf{C}^n$ square-integrable with respect to the Gaussian

$$d\mu_h(z) := \frac{1}{(\pi h)^n} e^{-|z|^2/h}\, dz, \qquad h > 0,$$

dz being the Lebesgue volume measure on $\mathbf{C}^n$. It is known that

$$K_h(x,y) = e^{\langle x,y\rangle/h}$$

is the reproducing kernel for $\mathcal{F}_h$; thus

$$f(x) = \int_{\mathbf{C}^n} f(y) K_h(x,y)\, d\mu_h(y) = \langle f, K_{h,x}\rangle, \qquad K_{h,x} := K_h(\cdot, x),$$

for all $f \in \mathcal{F}_h$ and $x \in \mathbf{C}^n$. (See, for instance, Berger and Coburn [2] or Folland [17].) Recall that for $f \in L^\infty(\mathbf{C}^n)$, the Berezin transform $B_h f$ of f is the function on $\mathbf{C}^n$ defined by

$$B_h f(x) = \frac{\langle f K_{h,x}, K_{h,x}\rangle}{\langle K_{h,x}, K_{h,x}\rangle} = K_h(x,x)^{-1} \int_{\mathbf{C}^n} f(y) |K_h(x,y)|^2\, d\mu_h(y).$$

Explicitly,

$$B_h f(x) = \frac{1}{(\pi h)^n} \int_{\mathbf{C}^n} f(y)\, e^{-|x-y|^2/h}\, dy = (e^{h\Delta/4} f)(x),$$

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i.e. B_h is the heat solution operator $e^{t\Delta}$ at the time $t = h/4$. It follows that as $h \searrow 0$, there is an asymptotic expansion

$$B_h f(x) \approx f(x) + \frac{h\Delta f(x)}{4} + \frac{h^2 \Delta^2 f(x)}{2!4^2} + \dots,$$

whenever $f \in L^\infty(\mathbf{C}^n)$ is smooth in a neighbourhood of x.

It turns out that this kind of situation prevails in much greater generality. Namely, consider a strictly plurisubharmonic real-valued smooth function Φ on a domain Ω in $\mathbf{C}^n$. Then $g_{i\bar{j}} = \partial^2\Phi/\partial z_i \partial \overline{z}_j$ defines a Kähler metric on Ω, with the associated volume element $d\mu(z) = \det[g_{i\bar{j}}]\, dz$. For any $h > 0$, we then have, in particular, the weighted Bergman spaces $L^2_{\text{hol}}(\Omega, e^{-\Phi/h}\, d\mu) =: L^2_{\text{hol},h}$ of all holomorphic functions in $L^2(\Omega, e^{-\Phi/h}\, d\mu) =: L^2_h$, and the corresponding reproducing kernels $K_h(x, y)$ and Berezin transforms $B_h f$. Furthermore, for $f \in L^\infty(\Omega)$ one has the Toeplitz operator $T^{(h)}_f$ with symbol f, namely, the operator on $L^2_{\text{hol},h}$ defined by $T^{(h)}_f \phi = P_h(f\phi)$, where $P_h : L^2_h \to L^2_{\text{hol},h}$ is the orthogonal projection. Now, assume that $\Omega \subset \mathbf{C}^n$ is smoothly bounded and strictly pseudoconvex, and $e^{-\Phi}$ is a defining function for Ω;[1] or that Ω is a bounded symmetric domain in $\mathbf{C}^n$ and e^Φ is the (unweighted) Bergman kernel of Ω; or that $\Omega = \mathbf{C}^n$ and $\Phi(z) = |z|^2$. Then as $h \searrow 0$, there are asymptotic expansions ([11], [5], [4], [7], [6])

$$K_h(x, x) \approx e^{\Phi(x)/h}\, h^{-n} \sum_{j=0}^\infty h^j\, b_j(x); \tag{1}$$

$$B_h f \approx \sum_{j=0}^\infty h^j\, Q_j f; \qquad \text{and} \tag{2}$$

$$T^{(h)}_f T^{(h)}_g \approx \sum_{j=0}^\infty h^j\, T^{(h)}_{C_j(f,g)} \qquad \text{(in operator norm)}, \tag{3}$$

for some functions $b_j \in C^\infty(\Omega)$, with $b_0 = \mathbf{1}$; some differential operators Q_j, with $Q_0 = I$ and Q_1 the Laplace-Beltrami operator with respect to the metric $g_{i\bar{j}}$; and some bidifferential operators C_j, where $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = \frac{i}{2\pi}\{f, g\}$ (the Poisson bracket of f and g with respect to the metric $g_{i\bar{j}}$).

The formulas (1)–(3) have an elegant application to *quantization on Kähler manifolds*. Recall that the traditional problem of quantization consists in looking for a map $f \mapsto Q_f$ from $C^\infty(\Omega)$ into operators on some (fixed) Hilbert space which is linear, conjugation-preserving, $Q_{\mathbf{1}} = I$, and as the Planck constant $h \searrow 0$,

$$[Q_f, Q_g] \approx \frac{ih}{2\pi} Q_{\{f,g\}}. \tag{4}$$

(The spectrum of Q_f is then interpreted as the possible outcomes of measuring the observable f in an experiment; and (4) amounts to a correct semiclassical limit.) The formula (3) implies that (4) holds for $Q_f = T^{(h)}_f$, the Toeplitz operators on the Bergman spaces $L^2_{\text{hol},h}$. This is the so-called *Berezin-Toeplitz quantization.*

[1] Recall that $\rho \in C^\infty(\overline{\Omega})$ is called a *defining function* for Ω if $\rho > 0$ on Ω, and $\rho = 0$, $\|\nabla\rho\| \neq 0$ on $\partial\Omega$.

From the point of view of these applications, the weighted Bergman spaces $L^2_{\text{hol},h}$ have an obvious disadvantage in that their very definition requires a holomorphic structure (hence, in particular, they can make sense only on complex manifolds). On the other hand, the other ingredients — the Toeplitz operators and the Berezin transform — make sense not only for L^2_{hol}, but for any subspace of L^2 with reproducing kernel. Hence it is of interest to investigate whether any such spaces other than weighted Bergman spaces can be used for quantization.

One such candidate, namely, the pluriharmonic Bergman spaces L^2_{ph}, consisting of all functions f in L^2 for which $\partial^2 f/\partial z_j \partial \overline{z}_k = 0 \ \forall j,k$, has recently been investigated in [12] and [13]. Unfortunately, it turned out that the analogue of (4),

$$\frac{1}{h}[T^{(h)}_f, T^{(h)}_g] \approx \frac{i}{2\pi} T^{(h)}_{\{f,g\}} \qquad \text{as } h \searrow 0,$$

in general fails, even for the unit disc $\Omega = \mathbf{D} \subset \mathbf{C}$ with the hyperbolic metric (given by $\Phi(z) = \lg \frac{1}{1-|z|^2}$). On the other hand, the analogues of (1) and (2) turned out to remain in force e.g. for the pluriharmonic Bergman spaces on bounded symmetric domains and the pluriharmonic Fock (or Segal-Bargmann) space on $\mathbf{C}^n$ (basically because the pluriharmonic Bergman kernels are then just the real parts of the ordinary holomorphic ones).

The aim of the present paper is to show that an analogue of the asymptotic expansion (2) for the Berezin transform prevails also in the case of the *harmonic*, rather than pluriharmonic, Segal-Bargmann (Fock) space on $\mathbf{C}^n \cong \mathbf{R}^{2n}$; that is, for the space

$$\mathcal{H}_h := \{f \in L^2(\mathbf{R}^{2n}, d\mu_h) : \Delta f = 0\} \tag{5}$$

of all harmonic functions in $L^2(\mathbf{R}^{2n}, d\mu_h)$, $n > 1$. (For $n = 1$, the harmonic functions coincide with the pluriharmonic ones, and thus this case is already covered by the above results for L^2_{ph}.) Let $\mathcal{R}$ denote the radial derivative

$$\mathcal{R} := \sum_{j=1}^{n} \Big(z_j \frac{\partial}{\partial z_j} + \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \Big) = \sum_{j=1}^{m} \Big(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \Big), \qquad z_j = x_j + i y_j,$$

on $\mathbf{R}^{2n} \cong \mathbf{C}^n$. Our main result is the following.

Theorem 1. *There exist linear differential operators $R_0, R_1, R_2, \dots$ on $\mathbf{R}^{2n} \setminus \{0\}$, of the form*

$$R_j = \sum_{\substack{k,l \geq 0 \\ k+2l \leq 2j}} \rho_{jkl} |y|^{2l-2j} \mathcal{R}^k \Delta^l \tag{6}$$

with some constants ρ_{jkl} (depending only on n), such that for any $y \neq 0$ and for any $f \in L^\infty(\mathbf{R}^{2n})$ smooth in a neighbourhood of y, the harmonic Berezin transform B^{harm}_h associated to the spaces (5) has the asymptotic expansion

$$B^{\text{harm}}_h f(y) \approx \sum_{j=0}^{\infty} h^j R_j f(y) \qquad \text{as } h \searrow 0. \tag{7}$$

Furthermore, $R_0 = I$, the identity operator, and

$$R_1 = \frac{\Delta}{4(2n-1)} + \frac{(4n-3)(n-1)}{2(2n-1)|y|^2}\,\mathcal{R} + \frac{n-1}{2(2n-1)|y|^2}\,\mathcal{R}^2. \tag{8}$$

Finally,

$$B_h^{\text{harm}} f(0) \approx \sum_{j=0}^{\infty} h^j \frac{\Delta^j f(0)}{j!4^j} \qquad \text{as } h \searrow 0 \tag{9}$$

for any $f \in L^\infty(\mathbf{R}^{2n})$ smooth in a neighbourhood of the origin.

Note that (6) does not reduce to (9) when $y = 0$ (in fact, the operator R_1 is even singular there), thus the asymptotic behaviour of B_h^{harm} has a discontinuity at $y = 0$; apparently, this is a kind of Stokes phenomenon.

The known proofs of (1)–(3) for the strictly-pseudoconvex case rely on microlocal analysis and employ a trick, going back to Forelli and Rudin [18], of expressing the Cartesian direct sum of the spaces $L^2_{\text{hol},h}$, $h = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$, as the (unweighted) Bergman space on a certain "disc bundle" domain over Ω [5] [11]. For the case of bounded symmetric domains or $\mathbf{C}^n$, the proofs rely on the homogeneous nature of the domain and invariance considerations [4] [6] or use the standard machinery of pseudodifferential operators [7]; in disguise, these were also the kind of methods used in [13]. For the harmonic Bergman spaces treated in this paper, however, none of these approaches seems to apply, and a completely different ad hoc argument must be used.

The paper is organized as follows. In Section 2, we compute the reproducing kernels H_h of the spaces $\mathcal{H}_h$; it turns out that they are given by an expression involving a certain hypergeometric function of two variables. A contour integral representation in combination with (essentially) a variant of the stationary phase method is used in Section 3 to get the asymptotic behaviour of B_h^{harm} and prove Theorem 1. Analogues of the formula (1) for the asymptotic behaviour as $h \searrow 0$ of the reproducing kernels H_h are established in Section 4. Some concluding remarks and open problems are collected in the final Section 5.

We remark that the harmonic Bergman spaces (5) make perfect sense also on any $\mathbf{R}^m$, $m = 1, 2, 3, \dots$, instead of $\mathbf{C}^n \cong \mathbf{R}^{2n}$ (the Gaussian measure $d\mu_h$ being then given, of course, by $d\mu_h(x) = (\pi h)^{-m/2} e^{-|x|^2/h}\,dx$). Though at the moment we are unable to prove Theorem 1 also for odd m, most of the results in Section 2 hold in this generality, and are therefore stated in that way.

Since the holomorphic Berezin transform B_h will not already appear in the rest of this paper[2], we will drop the superscript $^{\text{harm}}$ in B_h^{harm} from now on.

2. Harmonic Fock kernels

Let Ω be an arbitrary domain in $\mathbf{R}^n$, $n \geq 1$, and w a positive continuous weight function on Ω. The harmonic Bergman space $L^2_{\text{harm}}(\Omega, w)$ consists of all harmonic functions in $L^2(\Omega, w)$:

$$L^2_{\text{harm}}(\Omega, w) := \{f \in L^2(\Omega, w) : \Delta f = 0\}.$$

[2] With a sole exception in §5.6.

By the mean value property of harmonic functions, for each $x \in \Omega$, the evaluation functional $f \mapsto f(x)$ is continuous on L^2_{harm}, and thus is given by the inner product with a uniquely determined element, H_x, of $L^2_{\text{harm}}(\Omega, w)$:

$$f(x) = \langle f, H_x \rangle = \int_\Omega f(y)\,\overline{H_x(y)}\,w(y)\,dy$$

for all $f \in L^2_{\text{harm}}(\Omega, w)$. The function

$$H(x,y) := H_y(x) = \langle H_y, H_x \rangle = \overline{H_x(y)}$$

on $\Omega \times \Omega$ is called the (weighted) *harmonic Bergman kernel* of Ω with respect to the weight w. Since the complex conjugate $\overline{f}$ is harmonic whenever f is, we have

$$\overline{\langle f, H_y \rangle} = \overline{f(y)} = \langle \overline{f}, H_y \rangle = \overline{\langle f, \overline{H_y} \rangle},$$

implying that $H_y = \overline{H_y}$, i.e. H is real-valued:

$$H(x,y) \in \mathbf{R}, \qquad H(x,y) = H(y,x).$$

Explicit formulas for $H(x,y)$ for Ω the upper half-space $\mathbf{H}^n = \{x \in \mathbf{R}^n : x_n > 0\}$, with the weights $w(x) = x_n^\alpha$, $\alpha > -1$, or for the unit ball $\mathbf{B}^n = \{x \in \mathbf{R}^n : |x| < 1\}$, with weights $w(x) = (1 - |x|^2)^\alpha$, $\alpha > -1$, have been computed in many places, see e.g. Coifman and Rochberg [9], Jevtic and Pavlovic [19], Miao [22], or the book by Axler, Bourdon and Ramey [1]. For $\Omega = \mathbf{B}^n$ and $w = \mathbf{1}$ (i.e. the unweighted situation), the kernel is given by

$$H(x,y) = \frac{(n-4)|x|^4|y|^4 + (8\langle x,y\rangle - 2n - 4)|x|^2|y|^2 + n}{n\tau_n(1 - 2\langle x,y\rangle + |x|^2|y|^2)^{n/2+1}},$$

where τ_n is the Euclidean volume of $\mathbf{B}^n$. For the weighted case with the standard weights $w(x) = (1 - |x|^2)^\alpha$ on $\mathbf{B}^n$, things already get much more complicated, in particular there seems to be no simple explicit formula for H for general α, even integer. Even less is known for domains more general than $\mathbf{B}^n$ or $\mathbf{H}^n$.

In the rest of this section, we derive a formula for the reproducing kernel H_h of the harmonic Fock space

$$\mathcal{H}_h = L^2_{\text{harm}}(\mathbf{R}^n, d\mu_h), \qquad d\mu_h(x) = (\pi h)^{-n/2} e^{-|x|^2/h}\,dx, \quad h > 0,\ n \geq 3.$$

(The normalizing factor $(\pi h)^{-n/2}$ is inserted to make $d\mu_h$ of total mass 1, i.e. a probability measure. For the cases of $n = 1, 2$, see §5.1.)

We begin by recalling some facts on spherical harmonics; see e.g. [1] for more details and proofs. Let $\mathbf{S}^{n-1} = \partial\mathbf{B}^n$ denote the unit sphere in $\mathbf{R}^n$, and $\mathcal{Y}_k$ the space of all harmonic polynomials on $\mathbf{R}^n$ which are homogeneous of degree k. We equip $\mathcal{Y}_k$ with the norm and inner product coming from $L^2(\mathbf{S}^{n-1})$:

$$\langle f, g \rangle_{\mathcal{Y}_k} := \int_{\mathbf{S}^{n-1}} f(\zeta)\,\overline{g(\zeta)}\,d\sigma(\zeta),$$

where $d\sigma$ stands for the normalized surface measure on $\mathbf{S}^{n-1}$. Since $\mathcal{Y}_k$ is finite dimensional, the evaluation functional at any $y \in \mathbf{R}^n$ is automatically continuous on it, and thus $\mathcal{Y}_k$ possesses a reproducing kernel $Y_k(x,y)$. Explicitly, Y_k — whose restriction to $\mathbf{S}^{n-1}$ is usually known as the *zonal spherical harmonic* — is given by $Y_0 = 1$ and [1, Theorem 5.38]

$$Y_k(x,y) = |x|^k|y|^k\Big(\frac{n}{2}+k-1\Big)\sum_{j=0}^{[k/2]}\frac{(-1)^j 2^{k-2j}(\frac{n}{2})_{k-j-1}}{j!(k-2j)!}\Big(\frac{\langle x,y\rangle}{|x||y|}\Big)^{k-2j}, \qquad x,y\in\mathbf{R}^n,$$

for $k>0$, where $[\frac{k}{2}]$ means the greatest integer not exceeding $\frac{k}{2}$, and $(a)_k$ stands for the Pochhammer symbol (raising factorial)

$$(a)_k := a(a+1)(a+2)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

The harmonic polynomials are dense in $\mathcal{H}_h$, hence each $f\in\mathcal{H}_h$ has the homogeneous decomposition

$$f = \sum_{k=0}^\infty f_k \qquad \text{(convergence in } H_h\text{)}, \tag{10}$$

with $f_k\in\mathcal{Y}_k$. Furthermore, the spaces $\mathcal{Y}_k$ and $\mathcal{Y}_j$ are orthogonal for $j\neq k$, both in $L^2(\mathbf{S}^{n-1})$ and in $\mathcal{H}_h$; thus the decomposition (10) is, in fact, orthogonal.

Recall that the hypergeometric function of two variables Φ_2 from Horn's list [3, §5.7.1] is defined as

$$\Phi_2\Big(\begin{matrix}a,b\\c\end{matrix}\Big| z,w\Big) = \sum_{j,k=0}^\infty \frac{(a)_j(b)_k}{(c)_{j+k}j!k!}z^jw^k.$$

The series converges for all $z,w\in\mathbf{C}$ and defines an entire function on $\mathbf{C}^2$.

Proposition 2. *The harmonic Fock kernel H_h is given by*

$$H_h(x,y) = \Phi_2\Big(\begin{matrix}\frac{n}{2}-1,\frac{n}{2}-1\\ \frac{n}{2}-1\end{matrix}\Big|\frac{t_1+it_2}{h},\frac{t_1-it_2}{h}\Big), \tag{11}$$

where

$$t_1 = \langle x,y\rangle, \qquad t_2 = \sqrt{|x|^2|y|^2-\langle x,y\rangle^2}.$$

Proof. Observe that for any $f_k\in\mathcal{Y}_k$, the norms of f_k in $L^2(\mathbf{S}^{n-1})$ and in $\mathcal{H}_h$ are related by

$$\|f_k\|^2_{\mathcal{H}_h} = c_k\|f_k\|^2_{\mathbf{S}^{n-1}}, \qquad c_k := (\tfrac{n}{2})_k h^k. \tag{12}$$

Indeed, by the homogeneity of f_k and integration in polar coordinates

$$\begin{aligned}\|f_k\|^2_{\mathcal{H}_h} &= (\pi h)^{-n/2}\int_{\mathbf{R}^n}|f_k(x)|^2\, e^{-|x|^2/h}\,dx\\ &= (\pi h)^{-n/2}\int_0^\infty\int_{\mathbf{S}^{n-1}}|f_k(r\zeta)|^2\, e^{-r^2/h}\, r^{n-1}\,dr\,d\zeta\\ &= (\pi h)^{-n/2}\int_0^\infty\int_{\mathbf{S}^{n-1}} r^{2k}|f_k(\zeta)|^2\, e^{-r^2/h}\, r^{n-1}\,dr\,d\zeta\\ &= \operatorname{vol}(\mathbf{S}^{n-1})\,\|f_k\|^2_{\mathbf{S}^{n-1}}\,(\pi h)^{-n/2}\int_0^\infty r^{2k}e^{-r^2/h}\,r^{n-1}\,dr\\ &= c_k\|f_k\|^2_{\mathbf{S}^{n-1}},\end{aligned}$$

since the last integral equals $\frac{1}{2}h^{k+\frac{n}{2}}\Gamma(k+\frac{n}{2})$, while

$$\mathrm{vol}(\mathbf{S}^{n-1}) =: \sigma_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}.$$

Of course, the proportionality of norms (12) implies that the same holds also for the corresponding inner products. Using the decomposition (10), we thus have

$$\begin{aligned}\langle f, H_{h,x}\rangle_{\mathcal{H}_h} &= f(x) = \sum_{k=0}^{\infty} f_k(x) = \sum_{k=0}^{\infty}\langle f_k, Y_{k,x}\rangle_{\mathbf{S}^{n-1}} \\ &= \sum_{k=0}^{\infty}\frac{\langle f_k, Y_{k,x}\rangle_{\mathcal{H}_h}}{c_k} = \Big\langle f\ , \sum_{k=0}^{\infty}\frac{Y_{k,x}}{c_k}\Big\rangle_{\mathcal{H}_h}.\end{aligned}$$

Consequently,

$$H_h(x,y) = \sum_{k=0}^{\infty}\frac{Y_k(x,y)}{c_k} = \sum_{k=0}^{\infty}\frac{Y_k(x,y)}{h^k(\frac{n}{2})_k}. \tag{13}$$

Recall that the k-th Gegenbauer polynomial C_k^ν with parameter ν is given by [3, §10.9 (18)]

$$C_k^\nu(z) = \sum_{j=0}^{[k/2]}\frac{(-1)^j(\nu)_{k-j}}{j!(k-2j)!}\,(2z)^{k-2j}.$$

Thus, for all $k \geq 0$,

$$Y_k(x,y) = |x|^k|y|^k\,\frac{\frac{n}{2}+k-1}{\frac{n}{2}-1}\,C_k^{\frac{n}{2}-1}\Big(\frac{\langle x,y\rangle}{|x||y|}\Big).$$

Introduce the complex number

$$z = \frac{\langle x,y\rangle + i\sqrt{|x|^2|y|^2-\langle x,y\rangle^2}}{h},$$

so that

$$z = re^{i\theta} \qquad \text{with } r = \frac{|x||y|}{h},\ \cos\theta = \frac{\langle x,y\rangle}{|x||y|}.$$

By [3, §10.9 (17)],

$$C_k^\nu(\cos\theta) = \sum_{j=0}^{k}\frac{(\nu)_j(\nu)_{k-j}}{j!(k-j)!}\,e^{-i(k-2j)\theta}.$$

Consequently,

$$\begin{aligned}\frac{Y_k(x,y)}{h^k(\frac{n}{2})_k} &= \frac{|x|^k|y|^k}{h^k(\frac{n}{2}-1)_k}\,C_k^{\frac{n}{2}-1}\Big(\frac{\langle x,y\rangle}{|x||y|}\Big) \\ &= \frac{r^k}{(\frac{n}{2}-1)_k}\sum_{j=0}^{k}\frac{(\frac{n}{2}-1)_j(\frac{n}{2}-1)_{k-j}}{j!(k-j)!}\,e^{-(k-2j)i\theta}.\end{aligned}$$

Inserting this into (13) and switching from the summation variable k to $l = k - j$, we get

$$\begin{aligned} H_h(x,y) &= \sum_{k=0}^{\infty} \frac{r^k}{(\frac{n}{2}-1)_k} \sum_{j=0}^{k} \frac{(\frac{n}{2}-1)_j(\frac{n}{2}-1)_{k-j}}{j!(k-j)!}\, e^{-(k-2j)i\theta} \\ &= \sum_{j,l=0}^{\infty} \frac{r^{j+l}}{(\frac{n}{2}-1)_{j+l}} \frac{(\frac{n}{2}-1)_j(\frac{n}{2}-1)_l}{j!l!}\, e^{(j-l)i\theta} \\ &= \Phi_2\Big(\begin{matrix} \frac{n}{2}-1, \frac{n}{2}-1 \\ \frac{n}{2}-1 \end{matrix} \Big| re^{i\theta}, re^{-i\theta} \Big), \end{aligned}$$

completing the proof. $\square$

According to the general definition, the Berezin transform associated to the harmonic Fock space $\mathcal{H}_h$ is defined as

$$B_h f(x) = \frac{(\pi h)^{-n/2}}{H_h(x,x)} \int_{\mathbf{R}^n} f(y)\, |H_h(x,y)|^2\, e^{-|y|^2/h}\, dy \tag{14}$$

(the modulus signs around $H_h(x,y)$ being, in fact, superfluous in view of the real-valuedness of H_h). We want to know its asymptotic expansion as $h \searrow 0$. To get that, it would clearly be convenient to know the behaviour of $H_h(x,y)$ as $h \searrow 0$.

For $x = y$, (11) becomes

$$H_h(x,x) = \sum_{j,k=0}^{\infty} \frac{(\frac{n}{2}-1)_j(\frac{n}{2}-1)_k}{(\frac{n}{2}-1)_{j+k}} \frac{|x|^{2(j+k)}}{j!k!h^{j+k}}.$$

Using the familiar "binomial formula" for Pochhammer symbols,

$$\sum_{j=0}^{m} \frac{(\nu)_j(\mu)_{m-j}}{j!(m-j)!} = \frac{(\nu+\mu)_m}{m!}$$

(which is easily proved from the Taylor expansion

$$(1-z)^{-\nu} = \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!}\, z^k$$

by comparing the coefficients at like powers of z on both sides of the equality $(1-z)^{-\nu}(1-z)^{-\mu} = (1-z)^{-\nu-\mu}$), this becomes

$$H_h(x,x) = \sum_{m=0}^{\infty} \frac{(n-2)_m}{(\frac{n}{2}-1)_m} \frac{|x|^{2m}}{m!h^m} = {}_1F_1\Big(\begin{matrix} n-2 \\ \frac{n}{2}-1 \end{matrix} \Big| \frac{|x|^2}{h} \Big),$$

the ordinary confluent hypergeometric function ${}_1F_1$. Using [3, §6.13.1 (3)] we thus get the asymptotic expansion as $h \searrow 0$

$$H_h(x,x) \approx \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(n-2)}\, e^{|x|^2/h} \frac{|x|^{n-2}}{h^{\frac{n}{2}-1}} \sum_{j=0}^{\infty} \frac{(1-\frac{n}{2})_j(3-n)_j}{j!} \frac{h^j}{|x|^{2j}} \tag{15}$$

for $x \neq 0$, while $H_h(x,x) = 1\ \forall h$ for $x = 0$.

Unfortunately, for $x \neq y$ no analogous asymptotic formula seems to be available in the literature. We conclude this section by deriving a contour integral representation for $H_h(x,y)$ which will serve as a substitute.

The standard integral representation for Φ_2,

$$\Phi_2\Big(\begin{matrix} a,b \\ c \end{matrix}\Big| z,w\Big) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b)} \iint_{\substack{u,v\ge 0,\\ u+v\le 1}} u^{a-1}v^{b-1}(1-u-v)^{c-a-b-1}\, e^{uz+vw}\, du\, dv$$

is valid only for $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$, $\operatorname{Re}(c-a-b) > 0$, and thus is of no use in our case when $a = b = c = \frac{n}{2} - 1$. We use the standard workaround, see e.g. [23, §4.4.1].

Consider the complex plane cut along the real axis from $-\infty$ to 1, and let γ denote the contour going from 0 to $1-\epsilon$ along the "upper" edge of the cut, then around 1 clockwise, and then back from $1-\epsilon$ to 0 along the "lower" edge of the cut.

Proposition 3. *For any $z, w \in \mathbf{C}$ and $\alpha, \beta \in \mathbf{C}$ with $\operatorname{Re}\alpha > 0$,*

$$\oint_\gamma a^{2\alpha-1}(a-1)^{\beta-2\alpha-1}\int_{-1}^{1}(1-t^2)^{\alpha-1}e^{\frac{1+t}{2}az+\frac{1-t}{2}aw}\,dt\,da = \frac{-\pi i 4^\alpha \Gamma(\alpha)^2}{\Gamma(\beta)\Gamma(1+2\alpha-\beta)}\,\Phi_2\Big(\begin{matrix} \alpha,\alpha \\ \beta \end{matrix}\Big| z,w\Big). \tag{16}$$

Here $a^{2\alpha-1}$ and $(a-1)^{\beta-2\alpha-1}$ are the principal branches of the powers on $\mathbf{C}\setminus(-\infty,1]$.

Proof. In terms of the entire function

$$\phi_\alpha(z) := \int_{-1}^{1}(1-t^2)^{\alpha-1}\, e^{zt}\, dt,$$

the left-hand side of (16) can be written as

$$\oint_\gamma a^{2\alpha-1}(a-1)^{\beta-2\alpha-1}e^{\frac{az+aw}{2}}\phi_\alpha\big(\tfrac{az-aw}{2}\big)\, da.$$

By Cauchy's theorem, the value of the last integral is independent of ϵ. Furthermore, for the integral over the middle piece $|a-1| = \epsilon$ of γ we have the straightforward estimate

$$\Big|\oint_{|a-1|=\epsilon} a^{2\alpha-1}(a-1)^{\beta-2\alpha-1}\int_{-1}^{1}(1-t^2)^{\alpha-1}e^{\frac{1+t}{2}az+\frac{1-t}{2}aw}\,dt\,da\Big| \le 2\pi\epsilon\cdot\epsilon^{\operatorname{Re}(\beta-2\alpha-1)}\cdot C_\epsilon,$$

where $C_\epsilon = \sup_{|a-1|=\epsilon}|a^{2\alpha-1}e^{\frac{az+aw}{2}}\phi_\alpha(\frac{az-aw}{2})|\cdot e^{\pi|\operatorname{Im}(\beta-2\alpha)|}$ tends to a finite limit as $\epsilon \searrow 0$. It follows that for $\operatorname{Re}(\beta-2\alpha) > 0$, the integral tends to zero as $\epsilon \searrow 0$.

On the other hand, the integral along the upper edge of the cut then tends to

$$\begin{aligned}
&\int_0^1 a^{2\alpha-1}(1-a)^{\beta-2\alpha-1}e^{(\beta-2\alpha-1)\pi i}e^{\frac{az+aw}{2}}\phi_\alpha(\tfrac{az-aw}{2})\,da \\
&\qquad= e^{(\beta-2\alpha-1)\pi i}\int_0^1 a^{2\alpha-1}(1-a)^{\beta-2\alpha-1}\int_{-1}^1 (1-t^2)^{\alpha-1}e^{\frac{1+t}{2}az+\frac{1-t}{2}aw}\,dt\,da \\
&\qquad= 2^{2\alpha-1}e^{(\beta-2\alpha-1)\pi i}\int_0^1\int_{-1}^1 \Big(\frac{1-t^2}{4}a^2\Big)^{\alpha-1}(1-a)^{\beta-2\alpha-1}e^{\frac{1+t}{2}az+\frac{1-t}{2}aw}\,\frac{a}{2}\,dt\,da,
\end{aligned}$$

which upon the change of variables $\frac{1+t}{2}a=u$, $\frac{1-t}{2}a=v$ becomes

$$\begin{aligned}
&2^{2\alpha-1}e^{(\beta-2\alpha-1)\pi i}\iint\limits_{\substack{u+v\le 1\\ u,v\ge 0}}(uv)^{\alpha-1}(1-u-v)^{\beta-2\alpha-1}e^{uz+vw}\,du\,dv \\
&\qquad= 2^{2\alpha-1}e^{(\beta-2\alpha-1)\pi i}\sum_{j,k=0}^\infty \frac{z^j}{j!}\frac{w^k}{k!}\iint\limits_{\substack{u+v\le 1\\ u,v\ge 0}} u^{j+\alpha-1}v^{k+\alpha-1}(1-u-v)^{\beta-2\alpha-1}\,du\,dv \\
&\qquad= 2^{2\alpha-1}e^{(\beta-2\alpha-1)\pi i}\sum_{j,k=0}^\infty \frac{z^j}{j!}\frac{w^k}{k!}\frac{\Gamma(j+\alpha)\Gamma(k+\alpha)\Gamma(\beta-2\alpha)}{\Gamma(j+k+\beta)} \\
&\qquad= 2^{2\alpha-1}e^{(\beta-2\alpha-1)\pi i}\frac{\Gamma(\alpha)^2\Gamma(\beta-2\alpha)}{\Gamma(\beta)}\,\Phi_2\Big(\begin{matrix}\alpha,\alpha\\ \beta\end{matrix}\Big|z,w\Big).
\end{aligned}$$

Similarly, for the integral along the lower edge we get the same expression, only with $e^{(\beta-2\alpha-1)\pi i}$ replaced by $-e^{-(\beta-2\alpha-1)\pi i}$. Thus for $\mathrm{Re}(\beta-2\alpha)>0$, the left-hand side of (16) equals

$$2^{2\alpha}i\sin(\beta-2\alpha-1)\pi\;\frac{\Gamma(\alpha)^2\Gamma(\beta-2\alpha)}{\Gamma(\beta)}\Phi_2\Big(\begin{matrix}\alpha,\alpha\\ \beta\end{matrix}\Big|z,w\Big).$$

By the functional equation for the Gamma function,

$$\sin(\beta-2\alpha-1)\pi\,\Gamma(\beta-2\alpha)=\frac{-\pi}{\Gamma(1-\beta+2\alpha)},$$

and (16) thus follows, for $\mathrm{Re}\,\beta>2\,\mathrm{Re}\,\alpha$. Since, for α fixed, both sides of (16) are entire functions of β, they must in fact coincide for all $\beta\in\mathbf{C}$, completing the proof. □

Taking $\beta=\alpha$, we obtain the following corollary.

Corollary 4. *For any $z,w\in\mathbf{C}$ and $\alpha\in\mathbf{C}$ with $\mathrm{Re}\,\alpha>0$,*

$$\Phi_2\Big(\begin{matrix}\alpha,\alpha\\ \alpha\end{matrix}\Big|z,w\Big)=\frac{i\alpha}{4^\alpha}\oint_\gamma a^{2\alpha-1}(a-1)^{-\alpha-1}\int_{-1}^1(1-t^2)^{\alpha-1}e^{\frac{1+t}{2}az+\frac{1-t}{2}aw}\,dt\,da.$$

In particular, for $\alpha=\frac{n}{2}-1$ we get from (11)

$$H_h(x,y)=\frac{i}{\pi}\frac{n-2}{2^{n-1}}\oint_\gamma a^{n-3}(a-1)^{-\frac{n}{2}}\int_{-1}^1(1-t^2)^{\frac{n}{2}-2}e^{a\frac{\langle x,y\rangle+itV(x,y)}{h}}\,dt\,da, \tag{17}$$

where we have set, for the sake of brevity,

$$V(x,y):=\sqrt{|x|^2|y|^2-\langle x,y\rangle^2}.$$

3. Proof of Theorem 1

From now on, we will only consider the case of $n \geq 3$ even:

$$n = 2N+2, \qquad N = 1, 2, 3, \dots .$$

In that case the function $(a-1)^{-n/2}$ is single-valued, so the contour integral in (17) can be evaluated explicitly using the residue theorem:

$$\begin{aligned}
\oint_\gamma a^{n-3}(a-1)^{-n/2} e^{aw}\, da &= \int_{-C(1,\epsilon)} a^{2N-1}(a-1)^{-N-1} e^{aw}\, da \\
&= -2\pi i \operatorname{Res}_{a=1} \frac{a^{2N-1} e^{aw}}{(a-1)^{N+1}} \\
&= \frac{2\pi}{i} \frac{1}{N!} \frac{\partial^N}{\partial a^N} [a^{2N-1} e^{aw}]_{a=1}.
\end{aligned}$$

Substituting $\frac{\langle x,y\rangle + itV}{h}$ for w, we thus have by (17)

$$\begin{aligned}
H_h(x,y) &= \frac{2N}{2^{2N+1}} \int_{-1}^{1} (1-t^2)^{N-1} \frac{2}{N!} \frac{\partial^N}{\partial a^N} \Big[a^{2N-1} e^{a \frac{\langle x,y\rangle + itV}{h}} \Big]_{a=1} dt \\
&= \frac{2^{1-2N}}{(N-1)!} \frac{\partial^N}{\partial a^N} \Big[a^{2N-1} \int_{-1}^{1} (1-t^2)^{N-1}\, e^{a \frac{\langle x,y\rangle + itV}{h}}\, dt \Big]_{a=1},
\end{aligned} \tag{18}$$

where, for the sake of brevity, we write just V for $V(x,y)$. Integrating by parts $N-1$ times yields

$$\begin{aligned}
H_h(x,y) &= \frac{2^{1-2N}}{(N-1)!} \frac{\partial^N}{\partial a^N} \Big[a^{2N-1} \int_{-1}^{1} G_N(t) \Big(\frac{h}{iVa}\Big)^{N-1} e^{a \frac{\langle x,y\rangle + itV}{h}}\, dt \Big]_{a=1} \\
&= \frac{2^{1-2N}}{(N-1)!} \Big(\frac{h}{iV}\Big)^{N-1} \int_{-1}^{1} G_N(t) \frac{\partial^N}{\partial a^N} \Big[a^N e^{a \frac{\langle x,y\rangle + itV}{h}} \Big]_{a=1} dt,
\end{aligned}$$

where

$$G_N(t) := (-1)^{N-1} \frac{\partial^{N-1}}{\partial t^{N-1}} (1-t^2)^{N-1}.$$

Finally, by the Leibniz rule

$$\frac{\partial^N}{\partial a^N} \Big[a^N e^{aw} \Big]_{a=1} = \sum_{j=0}^{N} \binom{N}{j} \frac{N!}{j!} w^j e^w,$$

hence

$$\begin{aligned}
H_h(x,y) = \frac{2^{1-2N}}{(N-1)!} \Big(\frac{h}{iV}\Big)^{N-1} &\sum_{j=0}^{N} \binom{N}{j} \frac{N!}{j!} \\
&\int_{-1}^{1} G_N(t) \Big(\frac{\langle x,y\rangle + itV}{h}\Big)^j e^{\frac{\langle x,y\rangle + itV}{h}}\, dt.
\end{aligned} \tag{19}$$

After these preparations, we are ready to prove the main result of this paper.

Proof of Theorem 1. For $y = 0$, it is immediate from (11) that $H_h(x, 0) = 1$ for all x and h, whence

$$B_h f(0) = (\pi h)^{-N-1} \int_{\mathbf{C}^{N+1}} f(y) e^{-|y|^2/h} \, dy = (e^{h\Delta/4} f)(0).$$

By the standard asymptotic expansion of Gaussian integrals [16],

$$B_h f(0) \approx \sum_{j=0}^{\infty} \frac{h^j}{j! 4^j} \Delta^j f(0),$$

proving (9).

Throughout the rest of the proof, we thus assume that $y \neq 0$. We need to prove (7), (6) and the formulas

$$R_0 = I, \qquad R_1 = \frac{\Delta}{4(2N+1)} + \frac{(4N+1)N}{2(2N+1)|y|^2} \mathcal{R} + \frac{N}{2(2N+1)|y|^2} \mathcal{R}^2. \tag{20}$$

(The latter is just (8) in terms of N). For greater clarity, the proof will be broken into a series of steps.

Step 1. *It is enough to show that* (7) *holds for all points y of the form*

$$y = (Y, 0, 0, \dots, 0), \qquad Y > 0, \tag{21}$$

and that R_j are of the form

$$R_j = \sum_{\substack{k,l \geq 0 \\ k+2l \leq 2j}} r_{jkl}(|y|) \, \mathcal{R}^k \Delta^l \tag{22}$$

for some functions r_{jkl} on $\mathbf{R}_+$, with R_0 and R_1 given by (20).

Indeed, let U be an arbitrary orthogonal transformation[3] of $\mathbf{R}^n \cong \mathbf{C}^{N+1}$. Since $d\mu_h(Ux) = d\mu_h(x)$, the composition map

$$f \longmapsto f \circ U \tag{23}$$

is unitary on $L^2(\mathbf{R}^n, d\mu_h)$; as it also maps harmonic functions into harmonic functions, and its inverse $f \mapsto f \circ U^{-1}$ enjoys the same properties, (23) is in fact unitary also on $\mathcal{H}_h$. Now for any $g \in \mathcal{H}_h$,

$$\langle g, H_{Uy} \rangle = g(Uy) = (g \circ U)(y) = \langle g \circ U, H_y \rangle = \langle g, H_y \circ U^{-1} \rangle,$$

so $H_{Uy} = H_y \circ U^{-1}$, or $H(x, y) = H(Ux, Uy)$. From the definition (14) of the harmonic Berezin transform, it therefore follows that

$$B_h(f \circ U) = (B_h f) \circ U, \tag{24}$$

[3]The argument in this paragraph in fact holds for $\mathbf{R}^n$ with any n (i.e. not necessarily even); that is why we decided to include orthogonal transformations, even though unitary transformations of $\mathbf{C}^{N+1}$ would be enough for the purpose at hand.

i.e. B_h is invariant under the orthogonal transformations U. On the other hand, the Laplace operator Δ as well as the radial derivative

$$\mathcal{R} = \sum_{j=1}^{n} y_j \frac{\partial}{\partial y_j}, \qquad y \in \mathbf{R}^n,$$

are clearly also invariant under orthogonal transformations of $\mathbf{R}^n$, while the quantity $|y|^2$ is preserved by them. Hence, any linear differential operator L which is a polynomial in $\mathcal{R}$ and Δ with coefficients depending only on $|y|$,

$$Lf(y) = \sum_{k,l}^{\text{finite}} a_{kl}(|y|)\, \mathcal{R}^k \Delta^l,$$

is likewise invariant under U:

$$L(f \circ U) = (Lf) \circ U.$$

In particular, this applies to the operators R_j in (22). By (24), the validity of (7) for f at y is therefore equivalent to its validity for $f \circ U$ at $U^{-1}y$. Since any given y can be mapped by a suitable U into a point of the form (21), with $Y = |y|$, it is indeed enough to prove (7) only for points y of the latter form.

It remains to show that if (7) holds with R_j as in (22), then in fact $r_{jkl}(|y|) = \rho_{jkl}|y|^{2l-2j}$, so that we actually have (6). Observe that for any $t > 0$,

$$H_h(x, y) = H_h(tx, t^{-1}y).$$

Denoting by δ_t the dilation operator

$$\delta_t f(x) := f(tx), \qquad x \in \mathbf{R}^n,$$

it follows easily from (14) that

$$\delta_t B_h = B_{h/t^2} \delta_t.$$

Consequently,

$$\delta_t R_j = t^{-2j} R_j \delta_t.$$

Since $\Delta\delta_t = t^2\delta_t\Delta$ and $\delta_t\mathcal{R} = \mathcal{R}\delta_t$, it follows that

$$r_{jkl}(t|y|) = t^{2l-2j} r_{jkl}(y),$$

so, indeed, $r_{jkl}(|y|) = \rho_{jkl}|y|^{2l-2j}$ with $\rho_{jkl} = r_{jkl}(1)$, proving Step 1.

STEP 2. *It is enough to show that there exist functions a_{jkm} on $\mathbf{R}_+$ such that, for any $y = (Y, 0, 0, \dots, 0)$, $Y > 0$, and $f \in L^\infty(\mathbf{R}^n)$ smooth in a neighbourhood of y,*

$$B_h f(y) \approx \sum_{j=0}^{\infty} h^j \widetilde{R}_j f(y) \qquad \text{as } h \searrow 0, \tag{25}$$

where

$$\widetilde{R}_j f(y)=\sum_{\substack{k,l\ge 0\\ k+2l\le 2j}}\widetilde{r}_{jkl}(Y)\frac{\partial^k}{\partial y_1^k}\Delta^l f(y), \tag{26}$$

and, in particular,

$$\widetilde{R}_0=I,\qquad \widetilde{R}_1=\frac{\Delta}{4(2N+1)}+\frac{N}{Y}\frac{\partial}{\partial y_1}+\frac{N}{2(2N+1)}\frac{\partial^2}{\partial y_1^2}. \tag{27}$$

Indeed, a straightforward induction argument reveals that

$$(|y|^{-1}\mathcal{R})^k=|y|^{-k}\mathcal{R}(\mathcal{R}-1)\dots(\mathcal{R}-k+1)=:|y|^{-k}p_k(\mathcal{R})$$

(where we are taking the liberty to write just $\mathcal{R}-j$ instead of the more correct $\mathcal{R}-jI$). Evaluating both sides at $y=(Y,0,\dots,0)$ gives

$$\frac{\partial^k f}{\partial y_1^k}(Y,0,\dots,0)=Y^{-k}p_k(\mathcal{R})f(Y,0,\dots,0),$$

whence, for any such y,

$$\begin{aligned}\widetilde{R}_j f(y)&=\sum_{k+2l\le 2j}\widetilde{r}_{jkl}(|y|)\,|y|^{-k}p_k(\mathcal{R})\Delta^l f(y)\\ &=\sum_{k+2l\le 2j} r_{jkl}(|y|)\;\mathcal{R}^k\Delta^l f(y),\end{aligned}$$

with $r_{jkl}(|y|)$ equal to the coefficient at z^k in the polynomial $\sum_{m=k}^{2j-2l}\frac{\widetilde{r}_{jml}(|y|)}{|y|^m}p_m(z)$. However, the last right-hand side is of the form (22), and for $j=0,1$ the formulas (27) translate exactly into (20); thus the assertion follows by Step 1. This completes the proof of Step 2.

For the rest of the proof of Theorem 1, we thus assume that y is of the form (21) with $Y>0$. We introduce the notation

$$x=(r,X),\qquad r\in\mathbf{R},\ X\in\mathbf{R}^{n-1},$$

and, further,

$$X=\rho\zeta,\qquad \rho\ge 0,\ \zeta\in\mathbf{S}^{n-2}\equiv\mathbf{S}^{2N}.$$

Then

$$\langle x,y\rangle=rY,\qquad V(x,y)=\rho Y,$$

while the Gaussian measure $d\mu_h(x)$ takes the form

$$d\mu_h(x)=\frac{e^{-(r^2+\rho^2)/h}}{(\pi h)^{N+1}}\,dr\,\rho^{2N}\,d\rho\;\sigma_{2N}\,d\sigma(\zeta).$$

Thus (19) becomes

$$H_h(x,y) = N2^{1-2N}\Big(\frac{h}{i\rho Y}\Big)^{N-1}\sum_{j=0}^{N}\frac{\binom{N}{j}}{j!}\int_{-1}^{1} G_N(t)\Big(\frac{rY+it\rho Y}{h}\Big)^j e^{\frac{rY+it\rho Y}{h}}\,dt,$$

and inserting this into (14), we arrive at the huge formula

$$\begin{aligned} e^{-Y^2/h}H_h(y,y)B_hf(y) = \frac{N^2 4^{1-2N}}{(\pi h)^{N+1}}\sum_{j,k=0}^{N}\frac{\binom{N}{j}}{j!}\frac{\binom{N}{k}}{k!}\Big(\frac{h^2}{-Y^2}\Big)^{N-1}\int_{r\in\mathbf{R}}\int_{\rho>0} \\ \int_{\zeta\in\mathbf{S}^{2N}}\int_{-1}^{1}\int_{-1}^{1} G_N(t)G_N(u)\Big(\frac{rY+it\rho Y}{h}\Big)^j\Big(\frac{rY+iu\rho Y}{h}\Big)^k \\ e^{\frac{-(r-Y)^2+i(t+u)\rho Y-\rho^2}{h}}\, f(r,\rho\zeta)\,du\,dt\,\sigma_{2N}\,d\sigma(\zeta)\,\rho^2\,d\rho\,dr. \end{aligned} \tag{28}$$

By (15), the product $e^{-Y^2/h}H_h(y,y)$ has an asymptotic expansion in increasing powers of h as $h\searrow 0$. In fact, in our current case of even $n=2N+2$, using the transformation rule for the confluent hypergeometric function [3, §6.3 (7)]

$${}_1F_1\Big(\begin{matrix}a\\c\end{matrix}\Big|x\Big) = e^x\,{}_1F_1\Big(\begin{matrix}c-a\\c\end{matrix}\Big|-x\Big),$$

we can rewrite this expansion in a more convenient form

$$\begin{aligned} e^{-Y^2/h}H_h(y,y) &= {}_1F_1\Big(\begin{matrix}1-\frac{n}{2}\\ \frac{n}{2}-1\end{matrix}\Big|-\frac{Y^2}{h}\Big) = {}_1F_1\Big(\begin{matrix}-N\\N\end{matrix}\Big|-\frac{Y^2}{h}\Big) \\ &= \sum_{j=0}^{N}\frac{(-N)_j}{j!(N)_j}\Big(-\frac{Y^2}{h}\Big)^j =: \sum_{j=0}^{N}\frac{c_j}{h^j}. \end{aligned} \tag{29}$$

Note that $c_N = Y^{2N}/(N)_N \neq 0$.

Suppose now that we can show that (28) also has an asymptotic expansion of the form

$$e^{-Y^2/h}H_h(y,y)B_hf(y) \approx \sum_{j=0}^{\infty} b_j h^{j-N} \qquad \text{as } h\searrow 0, \tag{30}$$

with b_j given by an expression of the form (26). By (29) and elementary power series manipulations, it will then follow that

$$B_hf(y) \approx \sum_{j=0}^{\infty} r_j h^j \qquad \text{as } h\searrow 0,$$

with r_j given recursively by

$$r_l = \frac{b_l - \sum_{j=1}^{\min(N,l)} c_{N-j}r_{l-j}}{c_N}, \qquad l=0,1,2,\dots. \tag{31}$$

Hence, r_j will also be of the form (26), and the proof of (25) — and, hence, of Theorem 1 — will be complete (except for the proof of the formulas (27) for R_0 and R_1, whose proofs we postpone for a moment). Let us thus prove that the right-hand side of (28) has the asymptotic expansion (30); we do this in the next two steps.

STEP 3. *The right-hand side of* (28) *has as asymptotic expansion*

$$e^{-Y^2/h} H_h(y,y) B_h f(y) \approx \sum_{\delta=-3}^{\infty} h^{\frac{\delta}{2}-N} b_{\delta/2}, \tag{32}$$

where $b_{\delta/2}$ are given by expressions of the form (26) *but with $2j$ replaced by $\delta+1$, i.e.*

$$b_{\delta/2} = \sum_{\beta+2l\le\delta+1} \widetilde{r}_{\delta\beta l}(Y) \frac{\partial^\beta \Delta^l f(y)}{\partial r^\beta}, \tag{33}$$

and $b_{-3/2} = b_{-1} = 0$.

To see this, set

$$F(r,\rho) := \sigma_{2N} \int_{\mathbf{S}^{2N}} f(r,\rho\zeta)\, d\sigma(\zeta).$$

By hypothesis, f is smooth near y, i.e. near $(r,\rho\zeta) = (Y,0)$; thus by Taylor's formula, we have for any $m = 1,2,3,\dots,$

$$f(r,X) = \sum_{j+|\kappa|\le m} \frac{\partial^{j+|\kappa|} f}{\partial r^j \partial X^\kappa}(Y,0)\, \frac{(r-Y)^j X^\kappa}{j!\kappa!} + O((r-Y)^2 + |X|^2)^{\frac{m+1}{2}}$$

(the summation is over all $j \ge 0$ and multiindices $\kappa \in \mathbf{N}^{n-1}$ satisfying $j + |\kappa| \le m$, with the usual multiindex notation). Integrating term by term and using the formula

$$\sigma_{2N} \int_{\mathbf{S}^{2N}} \zeta^\kappa \, d\sigma(\zeta) = \begin{cases} 0 & \text{if some entry of } \kappa \text{ is odd,} \\ \dfrac{2\prod_{j=1}^{n-1} \Gamma(\lambda_j + \frac12)}{\Gamma(|\lambda| + N + \frac12)} & \text{if } \kappa = 2\lambda \end{cases}$$

(note that taking $\lambda = 0$ gives a formula for σ_{2N}!), we see that

$$\begin{aligned} F(r,\rho) &= \sigma_{2N} \sum_{\substack{j+|\kappa|\le m \\ \kappa=2\lambda}} \frac{\partial^{j+|\kappa|} f}{\partial r^j \partial X^\kappa}(Y,0)\, \frac{(r-Y)^j \rho^{|\kappa|}}{j!\kappa!}\, \frac{(\frac12)_\lambda}{(N+\frac12)_{|\lambda|}} \\ &\qquad\qquad + O((r-Y)^2 + \rho^2)^{(m+1)/2} \\ &= \sigma_{2N} \sum_{j+2|\lambda|\le m} \frac{\partial^{j+2|\lambda|} f}{\partial r^j \partial X^{2\lambda}}(Y,0)\, \frac{(r-Y)^j \rho^{2|\lambda|}}{j!4^{|\lambda|}\lambda!(N+\frac12)_{|\lambda|}} \\ &\qquad\qquad + O((r-Y)^2 + \rho^2)^{(m+1)/2} \\ &= \sigma_{2N} \sum_{\substack{j+2k\le m \\ j,k\ge 0}} \frac{\partial^j \Delta'^k f}{\partial r^j}(Y,0)\, \frac{(r-Y)^j \rho^{2k}}{j!k!4^k(N+\frac12)_k} \\ &\qquad\qquad + O((r-Y)^2 + \rho^2)^{(m+1)/2}; \end{aligned}$$

that is, as $(r,\rho)\to(Y,0)$,

$$F(r,\rho)\approx\sigma_{2N}\sum_{\beta,\gamma\ge0} f_{\beta\gamma}\cdot(r-Y)^\beta\rho^{2\gamma} \tag{34}$$

with

$$f_{\beta\gamma}:=\frac{1}{\beta!\gamma!4^\gamma(N+\frac12)_\gamma}\,\frac{\partial^\beta\Delta'^\gamma f}{\partial r^\beta}(Y,0). \tag{35}$$

Here we have used the doubling formula for the Gamma function

$$\frac{(\frac12)_\lambda}{(2\lambda)!}=\frac{1}{\lambda!2^{2|\lambda|}},$$

and the multinomial formula

$$\sum_{|\lambda|=k}\frac{k!}{\lambda!}\,\frac{\partial^{2k}}{\partial X^{2\lambda}}=\Delta'^k,$$

where Δ' denotes the Laplacian with respect to the X variable, i.e.

$$\Delta'=\Delta-\frac{\partial^2}{\partial r^2}. \tag{36}$$

Returning to (28), we claim that for each of the resulting integrals

$$\int_{r\in\mathbf{R}}\int_{\rho>0}\int_{-1}^1\int_{-1}^1 G_N(t)G_N(u)\Big(\frac{rY+it\rho Y}{h}\Big)^j\Big(\frac{rY+iu\rho Y}{h}\Big)^k \\ e^{\frac{-(r-Y)^2+i(t+u)\rho Y-\rho^2}{h}}\,F(r,\rho)\,\rho^2\,du\,dt\,d\rho\,dr, \tag{37}$$

we can obtain its asymptotic expansion simply by substituting for $F(r,\rho)$ the right-hand side of (34), and integrating term by term.

Indeed, assume that, for some m,

$$|F(r,\rho)|\le C\cdot R^q,\qquad R:=\sqrt{(r-Y)^2+\rho^2}.$$

Then a brute force estimate shows that (37) is dominated by

$$C'\int_{r\in\mathbf{R}}\int_{\rho>0}\Big(\frac{R+Y}{h}\Big)^{j+k}R^{2N+q}e^{-R^2/h}\,dr\,d\rho. \tag{38}$$

As

$$\int_{r\in\mathbf{R}}\int_{\rho>0}R^\nu e^{-R^2/h}\,dr\,d\rho=\pi\int_0^\infty R^{\nu+1}e^{-R^2/h}\,dR=\pi h^{\frac{\nu}{2}+1}\Gamma(\nu+2),$$

the integral (38) is $O(h^{N+1-j-k+\frac q2})=O(h^{\frac q2+1-N})$ as $h\searrow0$. Consequently, replacing $F(r,\rho)$ in (37) by the partial sum $\beta+\gamma\le q-1$ of the right-hand side of (34), with $q \geq 2(m+N-1)$, produces an error of order $O(h^m)$. As m can be taken arbitrarily large, the claim from the end of the preceding paragraph follows.

It thus remains to show that (28) has an asymptotic expansion of the form (30) when $F(r,\rho)$ is a polynomial in r and ρ^2, which we will thus assume from now on.

Note that the function G_N has parity $(-1)^{N-1}$, i.e.

$$G_N(-t) = (-1)^{N-1} G_N(t), \tag{39}$$

while $F(r,\rho)$ is clearly an even function of ρ. Consequently, the integrand in (37) remains unchanged if t,u,ρ are simultaneously replaced by $-t,-u,-\rho$. Instead of $(\rho,t,u) \in \mathbf{R}_+ \times (-1,1) \times (-1,1)$, we can therefore integrate over the domain

$$\rho \in \mathbf{R},\ t,u \in (-1,1),\ u+t \geq 0.$$

Since G_N is also a polynomial, we are thus reduced to obtaining the asymptotic expansion as $h \searrow 0$ of the integral

$$\int_{r\in\mathbf{R}} \iint_{\substack{u,t\in(-1,1)\\ u+t\geq 0}} \int_{\rho\in\mathbf{R}} p(r,t,u,\rho)\, e^{-\frac{(r-Y)^2+i(t+u)\rho Y-\rho^2}{h}}\, d\rho\, dt\, du\, dr, \tag{40}$$

for a polynomial $p(r,t,u,\rho)$ in the indicated variables.

We next shift the ρ integration in (40) into the complex plane, namely, from $\mathbf{R}$ to the parallel line $\mathbf{R} + iY\frac{u+t}{2}$. This makes sense since p is a polynomial (hence is defined also for complex ρ), and a routine check shows that the value of the integral remains unchanged (owing to the fast decay of the exponential and Cauchy's theorem). Thus (40) equals

$$\int_{r\in\mathbf{R}} \int_{\rho\in\mathbf{R}} \iint_{\substack{u,t\in(-1,1)\\ u+t\geq 0}} p(r,t,u,\rho+iY\tfrac{u+t}{2})\ e^{-\frac{(r-Y)^2+\rho^2+(\frac{u+t}{2}Y)^2}{h}}\, du\, dt\, d\rho\, dr.$$

Making the change of variable $u = v+w$, $t = v-w$, this becomes

$$\begin{aligned} 2\int_{r\in\mathbf{R}} \int_{\rho\in\mathbf{R}} \int_{v=0}^{1} \int_{w=-1+v}^{1-v} p(r,v-w,v+w,\rho+iYv)\, e^{-\frac{(r-Y)^2-\rho^2-Y^2v^2}{h}}\, dw\, dv\, d\rho\, dr \\ =: \iint_{r,\rho\in\mathbf{R}} \int_{v=0}^{1} P(r,v,\rho+iYv)\, e^{-\frac{(r-Y)^2-\rho^2-Y^2v^2}{h}}\, dv\, d\rho\, dr, \end{aligned}$$

where

$$P(r,v,z) := 2\int_{-1+v}^{1-v} p(r,v-w,v+w,z)\, dw$$

is again a polynomial. Standard estimate now shows that extending the v integration from $(0,1)$ to $(0,+\infty)$ introduces an error which is $O(e^{-Y^2/2h}) = O(h^\infty)$; thus we may instead work with the integral

$$\iint_{r,\rho\in\mathbf{R}} \int_{v=0}^{\infty} P(r,v,\rho+iYv)\, e^{-\frac{(r-Y)^2-\rho^2-Y^2v^2}{h}}\, dv\, d\rho\, dr.$$

However, this is already a standard Gaussian integral, whose asymptotic behaviour as $h \searrow 0$ is easy to compute (see e.g. Fedoryuk [16], Copson [8], de Bruijn [10], Evgrafov [15], etc.): namely, changing the variables r, ρ, v to $Y + r\sqrt{h}, \rho\sqrt{h}$ and $v\sqrt{h}/Y$, respectively, and using the identities

$$\int_0^\infty v^k\, e^{-v^2}\, dv = \frac{1}{2}\Gamma(\tfrac{k+1}{2})h^{\frac{k+1}{2}},$$

$$\int_{\mathbf{R}} t^{2j+1}\, e^{-t^2}\, dt = 0, \qquad \int_{\mathbf{R}} t^{2j}\, e^{-t^2}\, dt = \Gamma(j+\tfrac{1}{2})h^{j+\frac{1}{2}} = \frac{(2j)!\Gamma(\frac{1}{2})}{j!4^j}\, h^{j+\frac{1}{2}},$$

we obtain

$$\begin{aligned}\iint_{r,\rho\in\mathbf{R}} \int_{v=0}^\infty & P(r,v,\rho+iYv)\, e^{-\frac{(r-Y)^2-\rho^2-Y^2v^2}{h}}\, dv\, d\rho\, dr \\ \approx & \sum_{j,k,l=0}^\infty \frac{\Gamma(\frac{1}{2})}{j!4^j}\, h^{j+\frac{1}{2}}\, \frac{\Gamma(\frac{k+1}{2})}{k!2Y^{k+1}}\, h^{\frac{k+1}{2}}\, \frac{\Gamma(\frac{1}{2})}{l!4^l}\, h^{l+\frac{1}{2}} \\ & \cdot \frac{\partial^{2l}}{\partial r^{2l}} \frac{\partial^{2j}}{\partial \rho^{2j}} \frac{\partial^k}{\partial v^k} P(r,v,\rho+iYv)\Big|_{r=Y,\rho=v=0}\end{aligned}$$

(the sum on the right-hand side is in fact finite, since P is a polynomial).

Putting everything together, we thus arrive at the following asymptotic expansion, as $h \searrow 0$, of the integral (28):

$$\begin{aligned} e^{-Y^2/h} H_h(y,y) B_h f(y) \approx & \; 2\sigma_{2N}\, \frac{N^2 4^{1-2N}}{(\pi h)^{N+1}} \Big(\frac{-h^2}{Y^2}\Big)^{N-1} \sum_{j,k=0}^N \frac{\binom{N}{j}}{j!} \frac{\binom{N}{k}}{k!} \\ & \sum_{l=0}^\infty \frac{\Gamma(\frac{1}{2})}{l!4^l}\, h^{l+\frac{1}{2}} \frac{\partial^{2l}}{\partial r^{2l}}\Big|_{r=Y} \sum_{p=0}^\infty \frac{\Gamma(\frac{1}{2})}{p!4^p}\, h^{p+\frac{1}{2}} \frac{\partial^{2p}}{\partial \rho^{2p}}\Big|_{\rho=0} \\ & \sum_{q=0}^\infty \frac{\Gamma(\frac{q+1}{2})}{q!Y^{q+1}}\, h^{\frac{q+1}{2}} \frac{\partial^q}{\partial v^q}\Big|_{v=0} \\ & Y^{j+k} \int_{-1+v}^{1-v} G_N(v-w)G_N(v+w) \Big(\frac{r+i(v-w)(\rho+iYv)}{h}\Big)^j \\ & \Big(\frac{r+i(v+w)(\rho+iYv)}{h}\Big)^k\, dw \\ & \cdot (\rho+iYv)^2 \sum_{\beta,\gamma=0}^\infty f_{\beta\gamma}\, (\rho+iYv)^{2\gamma}(r-Y)^\beta, \end{aligned} \tag{41}$$

with $f_{\beta\gamma}$ given by (35). Note that the power of h in the generic term on the right-hand side is

$$h^{N+l-\frac{3}{2}+p+\frac{q}{2}-j-k} = h^{-N+(N-j)+(N-k)+l+p+\frac{q}{2}-\frac{3}{2}}.$$

Setting $\delta = 2(N-j) + 2(N-k) + 2l + 2p + q - 3$ and denoting by $b_{\delta/2}$ the corresponding coefficient at $h^{-N+\frac{\delta}{2}}$ in (41), we see that (32) holds with $b_{\delta/2}$ given by an expression of the form

$$\sum_{\substack{l,p,q,j,k,\beta,\gamma \\ 2(N-j)+2(N-k)+2l+2p+q-3=\delta}} c_{lpqjk\beta\gamma}(Y)\; f_{\beta\gamma}. \tag{42}$$

Since the last three lines in (41) are a polynomial of degree at least β in $(r-Y)$ and at least $2\gamma+2$ in (ρ,v), only the $f_{\beta\gamma}$ with

$$\beta \le 2l \quad \text{and} \quad 2\gamma+2 \le 2p+q \tag{43}$$

really occur, i.e.

$$\beta+2\gamma \le 2l+2p+q-2 \le \delta+1.$$

Thus, first of all, the sum (42) is finite, and $b_{\delta/2}$ is of the form (33) by (35) and (36); and, second, $\delta+1\ge 0$, whence $b_{-3/2}=b_{-1}=0$. This completes the proof of Step 3.

STEP 4. *Only the integer powers of h in (32) really appear, i.e. $b_{\delta/2}=0$ for δ odd.*

By linearity, it is enough to prove this for real-valued f. Since the integral kernel in the formula (14) defining the harmonic Berezin transform is real, $B_h f$ is then also real-valued, so in (41) we can replace the right-hand side by its complex conjugate, which amounts — since all the variables occurring there are real — to replacing the six occurrences of i by $-i$. Next, one can use $-r$ as the variable instead of r, i.e. replace in (41) $\frac{\partial}{\partial r}$ by $-\frac{\partial}{\partial r}$ (which has no effect since $2l$ is even), the two occurrences of rY after $G_N(v+w)$ by $-rY$, and the $(r-Y)^\beta$ at the end by $(-1)^\beta(r+Y)^\beta$. Finally, from (24) we have $(B_h f)(y)=(B_h\tilde f)(-y)$, where $\tilde f(x):=f(-x)$; as also $e^{-|y|^2/h}H_h(y,y)=e^{-|-y|^2/h}H_h(-y,-y)$, the right-hand side of (41) remains unchanged if we replace Y by $-Y$ and f by $\tilde f$, i.e. $f_{\beta\gamma}$ by $(-1)^\beta f_{\beta\gamma}$. However, upon making all these changes (i.e. $i\mapsto -i$, $r\mapsto -r$, $Y\mapsto -Y$ and $f_{\beta\gamma}\mapsto(-1)^\beta f_{\beta\gamma}$), the right-hand side of (41) assumes back its original form, with only one exception — the term Y^{q+1} (in the denominator before $\frac{\partial^q}{\partial v^q}$) gets replaced by $(-1)^{q+1}Y^{q+1}$. Since we know the two expressions to be equal, it follows that the summand on the right-hand side must in fact vanish if q is even, that is, if $\delta=2(N-j+N-k+l+p)+q-3$ is odd. This completes the proof of Step 4.

Restricting δ to be even — that is, q to be odd — in (41), we thus get

$$e^{-Y^2/h}H_h(y,y)B_hf(y)\approx\sum_{\delta=0}^\infty b_\delta\,h^{-N+\delta},$$

where (replacing the q in (41) by $2q+1$)

$$\begin{aligned} b_\delta = 2\sigma_{2N}\,\frac{N^2 4^{1-2N}}{\pi^{N+1}}\Big(\frac{-1}{Y^2}\Big)^{N-1} &\sum_{\substack{l,p,q,j,k,\beta,\gamma\ge0,\ j,k\le N\\ l+p+q+(N-j)+(N-k)=\delta+1\\ \beta\le 2l,\ \gamma+1\le p+q}} \frac{\binom{N}{j}}{j!}\,\frac{\binom{N}{k}}{k!} \\ &\frac{\Gamma(\frac12)}{l!4^l}\,\frac{\partial^{2l}}{\partial r^{2l}}\Big|_{r=Y}\frac{\Gamma(\frac12)}{p!4^p}\,\frac{\partial^{2p}}{\partial\rho^{2p}}\Big|_{\rho=0}\frac{q!}{(2q+1)!Y^{2q+2}}\,\frac{\partial^{2q+1}}{\partial v^{2q+1}}\Big|_{v=0} \\ &Y^{j+k}\int_{-1+v}^{1-v}G_N(v-w)G_N(v+w)\Big(\frac{r+i(v-w)(\rho+iYv)}{h}\Big)^j \\ &\qquad\Big(\frac{r+i(v+w)(\rho+iYv)}{h}\Big)^k\,dw \\ &\cdot\ f_{\beta\gamma}\cdot(\rho+iYv)^{2\gamma+2}(r-Y)^\beta. \end{aligned}\tag{44}$$

The restrictions on β and γ in the sum come from (43), and ensure that $\beta+2\gamma \le 2\delta$; thus b_j is of the form (26). Consequently, (30) holds, and, hence, (25), and, by (31), also (26). By Step 2, to complete the proof of Theorem 1 it only remains to prove the formulas (27) for $\widetilde{R}_0$ and $\widetilde{R}_1$; this is the content of the last two steps.

STEP 5. $\widetilde{R}_0 = I$.

To show this, we need to compute b_0. For brevity, denote

$$(\rho+iYv)^2 F(r,\rho+iYv) = \sum_{\beta,\gamma} f_{\beta\gamma}(r-Y)^\beta(\rho+iYv)^{2\gamma+2} =: G(\rho+iYv, r-Y),$$

$$\int_{-1+v}^{1-v} G_N(v-w)G_N(v+w)\,(rY+i(v-w)zY)^j\,(rY+i(v+w)zY)^k\,dw =: Q_{N-j,N-k}(r,v,z),$$

so that (44) becomes (upon supplying the value for σ_{2N})

$$\begin{aligned} b_\delta = \frac{N!2N4^{1-N}(-1)^{N-1}}{(2N-1)!Y^{2N-2}} &\sum_{l+p+q+(N-j)+(N-k)=\delta+1} \frac{\binom{N}{j}}{j!}\frac{\binom{N}{k}}{k!}\frac{1}{l!4^l}\frac{\partial^{2l}}{\partial r^{2l}}\Big|_{r=Y} \\ &\frac{1}{p!4^p}\frac{\partial^{2p}}{\partial\rho^{2p}}\Big|_{\rho=0}\frac{q!}{(2q+1)!Y^{2q+2}}\frac{\partial^{2q+1}}{\partial v^{2q+1}}\Big|_{v=0} \\ &G(\rho+iYv,r-Y)Q_{N-j,N-k}(r,v,\rho+iYv). \end{aligned} \tag{45}$$

Since $p+q \ge \gamma+1 \ge 1$ in (44), the only nonzero terms for $\delta = 0$ in the last sum occur for $(l, N-j, N-k, p, q) = (0,0,0,1,0)$ and $(0,0,0,0,1)$. The sum (45) for $\delta = 0$ therefore equals

$$\begin{aligned} \frac{1}{N!^2}\Big[\frac{1}{4Y^2}\frac{\partial^3}{\partial\rho^2\partial v} + \frac{1}{6Y^4}\frac{\partial^3}{\partial v^3}\Big]_{(r,\rho,v)=(Y,0,0)} \\ G(\rho+iYv,r-Y)Q_{00}(r,v,\rho+iYv). \end{aligned} \tag{46}$$

Note that, by the definition of G,

$$\begin{aligned} &\frac{\partial^l}{\partial r^l}\frac{\partial^p}{\partial\rho^p}\frac{\partial^q}{\partial v^q}G(\rho+iYv,r-Y)\Big|_{r=Y,\rho=v=0} \\ &\qquad = \begin{cases} 0, & \text{if } p+q \text{ is odd or } 0, \\ l!(\frac{p+q}{2})!(iY)^q f_{l,\frac{p+q-2}{2}}, & \text{for } p+q \text{ even} \ge 2. \end{cases} \end{aligned} \tag{47}$$

Using (47) and the Leibniz rule, we have at $(r,\rho,v) = (Y,0,0)$,

$$\begin{aligned} \frac{\partial^3}{\partial\rho^2\partial v}G\,Q_{00} &= \frac{\partial^2 G}{\partial\rho^2}\frac{\partial Q_{00}}{\partial v} + 2\frac{\partial^2 G}{\partial\rho\partial v}\frac{\partial Q_{00}}{\partial\rho} = \Big(\frac{\partial Q_{00}}{\partial v} + 2iY\frac{\partial Q_{00}}{\partial\rho}\Big) f_{00}, \\ \frac{\partial^3}{\partial v^3}G\,Q_{00} &= 3\frac{\partial^2 G}{\partial v^2}\frac{\partial Q_{00}}{\partial v} = -3Y^2\frac{\partial Q_{00}}{\partial v}\,f_{00}, \end{aligned}$$

hence (46) equals

$$\frac{f_{00}}{4Y^2N!^2}\Big(-\frac{\partial Q_{00}}{\partial v}+2iY\frac{\partial Q_{00}}{\partial\rho}\Big)\Big|_{(r,\rho,v)=(Y,0,0)}=\frac{f_{00}}{4Y^2N!}(iY\partial_3Q_{00}-\partial_2Q_{00}),$$

where we have taken the liberty to omit the arguments $(\rho+iYv,r-Y)$ and $(r,v,\rho+iYv)$ of G and Q_{00}, respectively, and also introduced the shorthand ∂_jQ_{00}, $j=1,2,3$, to mean the derivative of Q_{00} with respect to the j-th variable evaluated at $(r,v,z)=(Y,0,0)$. Now by the definition of Q_{00},

$$\begin{aligned}(iY\partial_3-\partial_2)Q_{00}&=\Big(iY\frac{\partial}{\partial z}-\frac{\partial}{\partial v}\Big)\Big[(1-v)\int_{-1}^1\prod_{\epsilon=\pm1}G_N(v+(1-v)\epsilon\tau)\\&\qquad (rY+i(v+(1-v)\epsilon\tau)zY)^N\,d\tau\Big]\Big|_{(r,v,z)=(Y,0,0)}\\&=iY\int_{-1}^1G_N(\tau)G_N(-\tau)\frac{\partial}{\partial z}(Y^2+i\tau zY)^N(Y^2-i\tau zY)^N\big|_{z=0}\,d\tau\\&\qquad+\int_{-1}^1G_N(\tau)G_N(-\tau)Y^{4N}\,d\tau\\&\qquad-\int_{-1}^1\frac{\partial}{\partial v}[G_N(v+(1-v)\tau)G_N(v-(1-v)\tau)]_{v=0}\,Y^{4N}\,d\tau\\&=Y^{4N}\int_{-1}^1[G_N(\tau)G_N(-\tau)-(1-\tau)G_N'(\tau)G_N(-\tau)\\&\qquad\qquad-(1+\tau)G_N(\tau)G_N'(-\tau)]\,d\tau.\end{aligned}$$

In view of (39), we can continue with

$$\begin{aligned}&=(-1)^{N-1}Y^{4N}\int_{-1}^1[G_N(\tau)^2+2\tau G_N'(\tau)G_N(\tau)]\,d\tau\\&=(-1)^{N-1}Y^{4N}\int_{-1}^1[\tau G_N(\tau)^2]'\,d\tau\\&=2(-1)^{N-1}Y^{4N}G_N(1)^2.\end{aligned}$$

On the other hand, from

$$G_N(t)=(-1)^{N-1}[(1-t^2)^{N-1}]^{(N-1)}=[(t-1)^{N-1}(t+1)^{N-1}]^{(N-1)}$$

and the Leibniz rule, we have

$$G_N^{(k)}(t)=\sum_{j=0}^{N-1+k}\binom{N-1+k}{j}[(t-1)^{N-1}]^{(j)}[(t+1)^{N-1}]^{(N-1+k-j)},$$

whence

$$G_N^{(k)}(1)=\binom{N+k-1}{k}(N-1)!\frac{(N-1)!}{(N-1-k)!}2^{N-1-k}. \tag{48}$$

Thus, in particular,

$$G_N(1) = (N-1)!2^{N-1}.$$

Putting everything together, we thus get

$$\begin{aligned} b_0 &= \frac{N!2N4^{1-N}(-1)^{N-1}}{(2N-1)!Y^{2N-2}} \, \frac{f_{00}}{4Y^2N!^2} \cdot 2(-1)^{N-1}Y^{4N}(N-1)!^2 4^{N-1} \\ &= \frac{(N-1)!}{(2N-1)!} Y^{2N} f_{00} \\ &= c_N f_{00}, \end{aligned}$$

with c_N from (29). Thus by (31)

$$r_0 = \frac{b_0}{c_N} = f_{00} = F(Y,0) = f(y).$$

So, indeed, $\widetilde{R}_0 f(y) = f(y)$, or $\widetilde{R}_0 = I$, proving Step 5.

STEP 6. *$\widetilde{R}_1$ is given by the formula in* (27).

Again, we need to compute b_1 and then r_1. This time, i.e. for $\delta = 1$, we obtain nonzero contributions in (45) from nine terms:

$$l=0,\ j=k=N, \qquad p=0,1,2, \quad q=2-p;$$

$$\left.\begin{array}{l} l=1,\ j=k=N \\ l=0,\ j=N-1,\ k=N \\ l=0,\ j=N,\ k=N-1 \end{array}\right\} \quad p=0,1, \quad q=1-p.$$

The corresponding sum in (45) becomes

$$\begin{aligned} &\frac{1}{N!^2}\Big[\frac{2}{5!Y^6}\frac{\partial^5}{\partial v^5} + \frac{1}{4\cdot 3!Y^4}\frac{\partial^5}{\partial\rho^2\partial v^3} + \frac{1}{4^2 2!Y^2}\frac{\partial^5}{\partial\rho^4\partial v}\Big] \\ &\qquad\qquad [G(\rho+iYv,0)Q_{00}(Y,v,\rho+iYv)]_{v=\rho=0} \\ &+ \frac{1}{N!^2}\Big[\frac{1}{3!Y^4}\frac{\partial^3}{\partial v^3} + \frac{1}{4Y^2}\frac{\partial^3}{\partial\rho^2\partial v}\Big] \\ &\qquad \Big[\frac{1}{4}\frac{\partial^2}{\partial r^2}\Big(G(\rho+iYv, r-Y)Q_{00}(r,v,\rho+iYv)\Big)_{r=Y} \\ &\qquad\qquad + N^2 G(\rho+iYv,0)(Q_{01}+Q_{10})(Y,v,\rho+iYv)\Big]_{\rho=v=0}. \end{aligned}$$

Computations a good deal more extensive than, but otherwise completely analogous to, those for the case $\delta = 0$ show that this equals

$$\begin{aligned} &\frac{(-1)^{N-1}Y^{4N-4}}{4N!^2}\Big[\Big(8G_N(1)G_N''(1) - 4G_N'(1)^2 + (6N^2-4N)G_N(1)^2\Big) f_{00} \\ &\qquad + Y^2 G_N(1)^2 f_{01} + 2NYG_N(1)^2 f_{10} + Y^2 G_N(1)^2 f_{20}\Big]. \end{aligned}$$

Hence, supplying the constant in front of the sum in (45) and using (48),

$$b_1 = \frac{(N-1)!Y^{2N-2}}{(2N-1)!2}[2N(2N-1)f_{00} + Y^2 f_{01} + 2NYf_{10} + Y^2 f_{20}].$$

Inserting this into (31) gives

$$r_1 = \frac{b_1 - c_{N-1} r_0}{c_N} = \frac{f_{01} + f_{20}}{2} + \frac{N}{Y} f_{10}.$$

Finally, supplying the values (35) for f_{01}, f_{10} and f_{20}, and recalling (36), we get

$$\widetilde{R}_1 = \frac{\Delta}{4(2N+1)} + \frac{N}{2(2N+1)} \frac{\partial^2}{\partial y_1^2} + \frac{N}{Y} \frac{\partial}{\partial y_1},$$

which is the second formula in (27), as claimed.

This completes the proof of Step 6, and, hence, of Theorem 1. $\square$

4. Harmonic Fock kernels on $\mathbf{C}^m$, $m > 1$

In this section we establish some more explicit formulas for the kernels $H_h(x, y)$. Though they do not seem to be of any use e.g. from the point of view of possible simplification of the proofs in the preceding section, we believe them to be of interest on their own merit. Besides, they not only better reveal the nature of these kernels, but also make it possible to describe their asymptotics as $h \searrow 0$.

Throughout this section, we again consider only the case of $\mathbf{R}^n$ with $n > 2$ even, setting as before

$$n = 2N + 2, \qquad N = 1, 2, 3, \dots,$$

so that $\mathbf{R}^n \cong \mathbf{C}^{N+1}$. For $x, y \in \mathbf{R}^n$, we also keep the previous notation

$$V \equiv V(x, y) = \sqrt{|x|^2 |y|^2 - \langle x, y \rangle^2}.$$

Furthermore, denote

$$E(z) = \frac{e^{iz} - e^{-iz}}{iz} = 2 \frac{\sin z}{z};$$

this is an entire function of $z \in \mathbf{C}$.

Our first formula expresses H_h as a finite sum of terms involving the function E and its derivatives.

Proposition 5. *For any $x, y \in \mathbf{R}^{2N+2}$ and $h > 0$,*

$$\begin{aligned} H_h(x, y) = \frac{N}{2^{2N-1}} \sum_{\substack{j,l \geq 0 \\ j+l \leq N}} \frac{(2N-1)!}{l! j! (N-l-j)! (N-1+l+j)!} \Big(\frac{\langle x, y \rangle}{h} \Big)^l e^{\langle x, y \rangle / h} \\ \sum_{k=0}^{N-1} \binom{N-1}{k} \Big(\frac{V}{h} \Big)^j E^{(2k+j)} \Big(\frac{V}{h} \Big) \\ = \frac{N}{2^{2N-1}} \sum_{j=0}^{N} \frac{\binom{2N-1}{N-j}}{j!} \; {}_1F_1 \Big(\begin{matrix} j-N \\ j+N \end{matrix} \Big| - \frac{\langle x, y \rangle}{h} \Big) e^{\langle x, y \rangle / h} \\ \sum_{k=0}^{N-1} \binom{N-1}{k} \Big(\frac{V}{h} \Big)^j E^{(2k+j)} \Big(\frac{V}{h} \Big). \end{aligned}$$

Proof. By the binomial theorem,

$$\int_{-1}^{1}(1-t^2)^{N-1}e^{izt}\,dt=\sum_{k=0}^{N-1}\binom{N-1}{k}\frac{\partial^{2k}}{\partial z^{2k}}\int_{-1}^{1}e^{izt}\,dt$$
$$=\sum_{k=0}^{N-1}\binom{N-1}{k}E^{(2k)}(z).$$

Thus by (18) and the Leibniz rule

$$\begin{aligned}H_h(x,y)&=\frac{2^{1-2N}}{(N-1)!}\frac{\partial^N}{\partial a^N}\Big[a^{2N-1}e^{a\frac{\langle x,y\rangle}{h}}\int_{-1}^{1}(1-t^2)^{N-1}\,e^{a\frac{itV}{h}}\,dt\Big]_{a=1}\\&=\frac{2^{1-2N}}{(N-1)!}\frac{\partial^N}{\partial a^N}\Big[a^{2N-1}e^{a\frac{\langle x,y\rangle}{h}}\sum_{k=0}^{N-1}\binom{N-1}{k}E^{(2k)}\Big(\frac{aV}{h}\Big)\Big]_{a=1}\\&=\frac{2^{1-2N}}{(N-1)!}\sum_{\substack{j,l\ge0\\ j+l\le N}}\frac{N!}{j!l!(N-l-j)!}\frac{\partial^{N-l-j}}{\partial a^{N-l-j}}a^{2N-1}\cdot\frac{\partial^l}{\partial a^l}e^{a\frac{\langle x,y\rangle}{h}}\\&\qquad\cdot\frac{\partial^j}{\partial a^j}\sum_{k=0}^{N-1}\binom{N-1}{k}E^{(2k)}\Big(\frac{aV}{h}\Big)\Big|_{a=1},\end{aligned}$$

yielding the first formula. The second formula follows upon summing over l. □

Using the elementary relations

$$E^{(m)}(z)=2\sum_{j=0}^{[\frac{m}{2}]}\frac{(-1)^{m+j}m!}{(2j)!}\frac{\sin z}{z^{m-2j+1}}-2\sum_{j=0}^{[\frac{m-1}{2}]}\frac{(-1)^{m+j}m!}{(2j+1)!}\frac{\cos z}{z^{m-2j}},$$

one can get the asymptotic behaviour of $H_h(x,y)$ as x,y are fixed and $h\searrow 0$. (Unfortunately, it seems not to be of much direct use for the proof in the preceding section, since it is not uniform in x.)

Our second formula for the kernel is obtained upon taking

$$X=\frac{\langle x,y\rangle+iV(x,y)}{h},\quad Y=\frac{\langle x,y\rangle-iV(x,y)}{h}$$

in the following proposition.

Proposition 6. *For any $N=1,2,\dots$ and $X,Y\in\mathbf{C}$,*

$$\Phi_2\Big(\begin{matrix}N,N\\N\end{matrix}\Big|X,Y\Big)=\frac{1}{(N-1)!}\Big(\frac{\partial^2}{\partial X\partial Y}\Big)^{N-1}\frac{XY^{N-1}e^X-YX^{N-1}e^Y}{X-Y}$$

(*with the usual interpretation of the right-hand side for $X=Y$*).

Proof. We have

$$\frac{XY^{N-1}e^X-YX^{N-1}e^Y}{X-Y}=\sum_{m=0}^{\infty}\frac{X^{m+1}Y^{N-1}-Y^{m+1}X^{N-1}}{m!(X-Y)}.$$

The m-th summand is a homogeneous polynomial in X, Y of degree $m+N-1$; thus for $m \le N-2$ it is annihilated by $\dfrac{\partial^{2N-2}}{\partial X^{N-1}\partial Y^{N-1}}$. For $m \ge N-1$, it equals

$$\frac{X^{N-1}Y^{N-1}}{m!}\,\frac{X^{m-N+2}-Y^{m-N+2}}{X-Y}=\frac{X^{N-1}Y^{N-1}}{m!}\sum_{j+k=m-N+1}X^jY^k.$$

Applying $\partial^{2N-2}/\partial X^{N-1}\partial Y^{N-1}$ and restoring the summation over m, we arrive at

$$\begin{aligned}&\frac{\partial^{2N-2}}{\partial X^{N-1}\partial Y^{N-1}}\frac{XY^{N-1}e^X-YX^{N-1}e^Y}{X-Y}\\&\quad=\sum_{j,k=0}^{\infty}\frac{1}{(j+k+N-1)!}\,\frac{(N-1+j)!}{j!}X^j\,\frac{(N-1+k)!}{k!}Y^k\\&\quad=\frac{(N-1)!^2}{(N-1)!}\,\Phi_2\Big(\begin{matrix}N,N\\N\end{matrix}\Big|X,Y\Big).\quad\Box\end{aligned}$$

Corollary 7. *For $N>1$, $H_h(x,y)$ equals*

$$\begin{aligned}\frac{4^{1-N}}{(N-1)!}\Big(\frac{\partial^2}{\partial A^2}+\frac{\partial^2}{\partial B^2}\Big)^{N-1}&\Big[e^A\,\frac{\sin B}{B}\,(A^2+B^2)\sum_{l=0}^{[\frac{N-2}{2}]}\binom{N-2}{2l}A^{N-2-2l}(-B^2)^l\\&+e^A\,(A^2+B^2)\cos B\sum_{l=0}^{[\frac{N-3}{2}]}\binom{N-2}{2l+1}A^{N-3-2l}(-B^2)^l\Big]\end{aligned}$$

evaluated at $A=\frac{\langle x,y\rangle}{h}$, $B=\frac{V(x,y)}{h}$.

For $N=1$, one has to replace the expression by

$$Ae^A\,\frac{\sin B}{B}+e^A\,\cos B.$$

Proof. By routine manipulation from Proposition 6, upon passing to the variables A, B from the variables $X=A+iB$, $Y=A-iB$. $\Box$

Note that the expression in the square brackets has the form

$$e^AA^N\frac{\sin B}{B}+e^AP_N(A,B)\sin B+e^AQ_N(A,B)\cos B$$

for some polynomials P_N, Q_N of two variables. Performing the differentiation we thus see that $H_h(x,y)$ equals

$$p_N(A,B)\sin B+q_N(A,B)\cos B+\sum_{k=0}^{N-1}r_{Nk}(A)e^A\Big(\frac{\sin B}{B}\Big)^{(2k)},$$

evaluated at $A=\frac{\langle x,y\rangle}{h}$, $B=\frac{V(x,y)}{h}$, with some polynomials p_N, q_N, r_{Nk} in the indicated variables. From this, one can again read off the asymptotic behaviour of $H_h(x,y)$ as $h\searrow 0$ for x,y fixed.

Note that the last expression for $H_h(x,y)$ differs from the one appearing in Proposition 5, which is of the form

$$\sum_{j,k,l} c_{jkl} A^l B^j E^{(j+2k)}(B),$$

evaluated at the same A, B, with the appropriate constants c_{jkl} (e.g. the highest derivative of E that appears is not $2N-2$ but $3N-2$); the equality of these two expressions is definitely not apparent.

Remark 8. Yet another formula for $H_h(x,y)$ can be obtained upon observing that

$$\Phi_2\Big(\begin{matrix} a, b \\ c \end{matrix}\Big| x, y\Big) = \lim_{\epsilon\to 0} F_1(\tfrac{1}{\epsilon}, a, b, c, \epsilon x, \epsilon y),$$

where F_1 is the first Horn hypergeometric function [3, §5.7 (6)]

$$F_1(\alpha, a, b, c, x, y) := \sum_{j,k=0}^{\infty} \frac{(\alpha)_{j+k}(a)_j(b)_k}{(c)_{j+k} j! k!} x^j y^k.$$

From the transformation formula for F_1 [3, §5.11 (3)]

$$F_1(\alpha, a, b, c, x, y) = (1-y)^{-\alpha} F_1(\alpha, a, c-a-b, c, \tfrac{y-x}{y-1}, \tfrac{y}{y-1})$$

we thus obtain

$$\begin{aligned}\Phi_2\Big(\begin{matrix} a, b \\ c \end{matrix}\Big| x, y\Big) &= e^y \Phi_2\Big(\begin{matrix} a, c-a-b \\ c \end{matrix}\Big| x-y, -y\Big) \\ &= e^x \Phi_2\Big(\begin{matrix} c-a-b, b \\ c \end{matrix}\Big| -x, y-x\Big).\end{aligned}$$

Hence, in particular,

$$\Phi_2\Big(\begin{matrix} a, a \\ a \end{matrix}\Big| x, y\Big) = e^x \Phi_2\Big(\begin{matrix} -a, a \\ a \end{matrix}\Big| -x, y-x\Big) = e^y \Phi_2\Big(\begin{matrix} a, -a \\ a \end{matrix}\Big| x-y, -y\Big),$$

and for $N = 1, 2, 3, \dots$,

$$\begin{aligned}\Phi_2\Big(\begin{matrix} N, N \\ N \end{matrix}\Big| x, y\Big) &= e^x \Phi_2\Big(\begin{matrix} -N, N \\ N \end{matrix}\Big| -x, y-x\Big) \\ &= e^x \sum_{j=0}^{N} \frac{(-N)_j}{(N)_j} \frac{(-x)^j}{j!} \, {}_1F_1\Big(\begin{matrix} N \\ N+j \end{matrix}\Big| y-x\Big).\end{aligned}$$

Replacing x, y by $\frac{\langle x,y\rangle + iV(x,y)}{h}$ and $\frac{\langle x,y\rangle - iV(x,y)}{h}$, respectively, we obtain a formula for $H_h(x,y)$ in terms of finitely many single-variable confluent hypergeometric functions ${}_1F_1$. Again, using known facts about the asymptotic expansion as $|z| \to +\infty$ of ${}_1F_1\Big(\begin{matrix} a \\ c \end{matrix}\Big| z\Big)$ [3, §6.13], one can get from here once more the asymptotic behavior as $h \searrow 0$ of $H_h(x,y)$ for fixed x, y (which should of course coincide with the ones obtained from Propositions 5 and 6, though we have not tried to check this).

5. Concluding remarks

5.1 Low dimensions. We have assumed throughout that $n \geq 3$. For $n = 2$, the harmonic functions on $\mathbf{R}^2 \cong \mathbf{C}$ coincide with the pluriharmonic ones, and thus the results of [13] apply (the harmonic kernel is just twice the real part of the ordinary holomorphic Bergman kernel minus one, etc.). For $n = 1$, $\mathcal{H}_h$ consists just of functions of the form $f(x) = ax + b$, with $a, b \in \mathbf{R}$, with reproducing kernel

$$H_h(x,y) = 1 + \frac{2xy}{h}, \qquad x, y \in \mathbf{R}.$$

Thus

$$\begin{aligned}\Big(1 + \frac{2y^2}{h}\Big) B_h f(y) &= \frac{1}{\sqrt{\pi h}} \int_{\mathbf{R}} f(x) \Big(1 + \frac{2xy}{h}\Big)^2 e^{-x^2/h}\, dx \\ &\approx \sum_{j=0}^{\infty} \frac{h^j}{j! 4^j} \Big[f^{(2j)}(0) + \frac{4y}{h} (xf)^{(2j)}(0) + \frac{4y^2}{h^2} (x^2 f)^{(2j)}(0) \Big]\end{aligned}$$

is always a quadratic polynomial in y, and its behaviour as $h \searrow 0$ is determined by the jet of f at the origin (rather than at y); in particular, $B_h f(y) \to f(0)$, so B_h is not even an approximate identity as $h \searrow 0$.

5.2 Different proof? As has already been noted at several places in Section 4, our proof of Theorem 1 in fact bypasses the asymptotics of the kernels $H_h(x,y)$ as $h \searrow 0$ (working instead directly with the whole integral (14)). Having a proof building on the asymptotic formulas from Section 4 might be of interest from several respects.

5.3 Toeplitz operators. Liu [20] shows that for $n = 2$, the Toeplitz operators $T_f^{(h)}$ on the harmonic Bergman space of the unit ball $\mathbf{B}^n$ in $\mathbf{R}^n$ satisfy

$$\|T_f^{(h)}\| \to \|f\|_\infty \qquad \text{as } h \searrow 0 \tag{49}$$

for any bounded continuous f; the same is shown to hold also for $n > 2$, provided f is in addition radial (i.e. $f(x)$ depends only on $|x|$). The proof actually goes via showing that $B_h f \to f$ pointwise. (The assertion then follows since $\|T_f^{(h)}\| \geq |B_h f(x)|$ for each x.) Our Theorem 1 thus implies that (49) remains in force also for the harmonic Fock (Segal-Bargmann) spaces on $\mathbf{R}^n$, for any even $n \geq 3$. (More precisely, Theorem 1 implies this for any bounded smooth f; to get it for general bounded continuous f, one can approximate f by bounded smooth functions in the uniform norm and use the fact that $\|T_f^{(h)}\| \leq \|f\|_\infty$.)

5.4 Higher order terms. Computer aided calculations lead to the following formula for the operator R_2 in (7):

$$\begin{aligned}R_2 = {}& \frac{3\Delta^2}{32(4n^2-1)} + \frac{n-1}{8(4n^2-1)|y|^2} \mathcal{R}^2 \Delta + \frac{(4n+1)(n-1)}{8(4n^2-1)|y|^2} \mathcal{R}\Delta \\ &+ \frac{(n-1)(2n-3)}{4(2n-1)|y|^2} \Delta + \frac{n(n-1)}{8(4n^2-1)|y|^4} \mathcal{R}^4 + \frac{(n-1)(4n^2-5n-2)}{4(4n^2-1)|y|^4} \mathcal{R}^3 \\ &+ \frac{(n-1)(16n^3-56n^2+27n+24)}{8(4n^2-1)|y|^4} \mathcal{R}^2 - \frac{(40n^3-72n^2+n+22)(n-1)}{4(4n^2-1)|y|^4} \mathcal{R}.\end{aligned}$$

It is absolutely unclear to the author what the formula for general R_j, $j \geq 3$, might be.

5.5 Jack polynomials. The sum

$$\sum_{j=0}^{k} \frac{(\nu)_j (\nu)_{k-j}}{j!(k-j)!} \, z^j w^{k-j} = \frac{\nu^k}{k!} \, J_{(k)}^{(1/\nu)}(z,w)$$

coincides, up to the constant factor $\frac{\nu^k}{k!}$, with the *Jack symmetric polynomial* $J_{(k)}^{(1/\nu)}$, with parameter $1/\nu$ and corresponding to the *signature* (or *partition*) (k), of two variables z, w; see MacDonald [21], p. 378 and Example 1 on p. 383. Consequently, the hypergeometric function Φ_2 with equal parameters can be written as

$$\Phi_2\left(\begin{matrix} a, a \\ a \end{matrix} \middle| z, w \right) = \sum_{k=0}^{\infty} \frac{a^k}{k!(a)_k} \, J_{(k)}^{(1/a)}(z,w). \tag{50}$$

In view of Proposition 2, we thus obtain an expansion of the harmonic Fock kernels $H_h(x,y)$ in terms of Jack polynomials of two variables with parameter $\frac{2}{n-2}$.

Jack polynomials play an important role in several fields of mathematics, like representation theory, statistics, combinatorics, and also in analysis on bounded symmetric domains in $\mathbf{C}^n$. In the latter, the Jack polynomials of two variables and with parameter $\frac{2}{n-2}$ correspond to an important series of bounded symmetric domains of rank 2, known as *Lie spheres*; it seems extremely intriguing to understand if there is any reason for their occurrence in the above context, what might be the connection between harmonic Fock space and analysis on Lie spheres, and why only single-entry partitions (k) appear in the expansion (50). See e.g. [14] and the references therein for more information on Jack polynomials in the analysis on rank 2 bounded symmetric domains in $\mathbf{C}^n$.

5.6 Translations. On the holomorphic Fock space $\mathcal{F}_h$ on $\mathbf{C}^n$, the translations $\tau_a : z \mapsto z + a$, $z, a \in \mathbf{C}^n$, induce the Weyl operators

$$W_a f(z) := e^{-\frac{\langle z,a \rangle}{h} - \frac{|a|^2}{2h}} f(z+a), \tag{51}$$

which are unitary on $L^2(\mathbf{C}^n, d\mu_h)$ as well as on $\mathcal{F}_h$. This is extremely helpful in many situations, for instance, the existence of W_a is responsible for the fact that the holomorphic Berezin transform commutes with translations:

$$B_h(f \circ \tau_a) = (B_h f) \circ \tau_a, \tag{52}$$

so that it is enough to prove asymptotic expansions like (7) only at the origin. For the harmonic Fock space $\mathcal{H}_h$, no operators like (51) exist, and (52) fails.

5.7 Open problems. Of course, the greatest deficiency of our method is that we are unable to treat the case of odd $n \geq 3$. The problem is that the da integral in (17) then cannot be explicitly evaluated. Proceeding by simply inserting it into (14) produces an integral whose asymptotic behaviour we were unable to determine (it is of the form

$$\int F(x) e^{-S(x)/h} \, dx$$

where the phase function S has a unique critical point, but a degenerate one).

Another problem is to extend our results to harmonic Bergman spaces on the unit ball of $\mathbf{R}^n$, or even to all real bounded symmetric domains. Of course, the ultimate generalization would be to the weighted harmonic Bergman spaces

$$A_h(\Omega) = L^2_{\text{harm}}(\Omega, r^{1/h}), \qquad h > 0,$$

on any smoothly bounded domain $\Omega \subset \mathbf{R}^n$, with $r \asymp \operatorname{dist}(\,\cdot\,, \partial\Omega)$; however, this seems to be completely out of reach at present.

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