UNIQUENESS OF SMOOTH RADIAL BALANCED METRICS ON THE DISC

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Abstract. We show that the usual Poincaré metric is the only radial balanced metric on the disc with not too wild boundary behaviour. Additionally, we identify explicitly all radial metrics with such boundary behaviour which satisfy the balanced condition as far as germs at the boundary are concerned. Related results for the annulus and the punctured disc are also established.

1. Introduction

Let Ω be a bounded domain in $\mathbf{C}^n$, $n \geq 1$, which we assume for simplicity to be contractible to a point, and let ϕ be a strictly plurisubharmonic function on Ω with the associated Kähler metric

$$g_{j\bar{k}}(z) = \frac{\partial^2 \phi(z)}{\partial z_j \partial \bar{z}_k}, \qquad j,k = 1,\dots,n, \quad z \in \Omega, \tag{1}$$

and volume element $g(z)\,dz$, where $g(z) = \det[g_{j\bar{k}}(z)]_{j,k=1}^n$ and dz denotes the Lebesgue measure on $\mathbf{C}^n$. In terms of the function $u := e^{-\phi}$, one has

$$\det[g_{j\bar{k}}] \equiv \det[\partial\overline{\partial}\phi] = \det[\partial\overline{\partial}\log\tfrac{1}{u}] = u^{-n-1} J[u],$$

where $J[u]$ is the Monge-Ampére determinant

$$J[u] := (-1)^n \det \begin{bmatrix} u & \partial u \\ \overline{\partial} u & \partial\overline{\partial} u \end{bmatrix}. \tag{2}$$

Setting, for brevity, $w := J[u] = u^{n+1} \det[\partial\overline{\partial}\log\frac{1}{u}] = e^{-(n+1)\phi}\det[\partial\overline{\partial}\phi]$, consider the weighted Bergman space $L^2_{\mathrm{hol}}(\Omega, w)$ of all holomorphic functions in $L^2(\Omega, w)$. It is well known that $L^2_{\mathrm{hol}}(\Omega, w)$ has bounded point evaluations and hence possesses a reproducing kernel $K_w(x,y)$, given in fact by $K_w(x,y) = \sum_j e_j(x)\overline{e_j(y)}$ for any orthonormal basis $\{e_j\}_j$ of $L^2_{\mathrm{hol}}(\Omega, w)$. The metric (1) — or, abusing terminology, the function u — is called *balanced* if

$$K_w(z,z) = \frac{c}{u(z)^{n+1}} \qquad \forall z \in \Omega \tag{3}$$

for some constant c. (One can check that this indeed depends only on the metric (1), not on its potential ϕ or, equivalently, on $u = e^{-\phi}$.)

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Example. Let $\Omega = \mathbf{D}$, the unit disc in $\mathbf{C}$, and $u(z) = 1 - |z|^2$. Then $w = J[u] = \mathbf{1}$ and the corresponding Bergman kernel is well known to be

$$K(x, y) = \frac{1}{\pi(1 - x\overline{y})^2}.$$

Thus (3) holds with $c = \frac{1}{\pi}$. $\square$

The notion extends in an obvious way from complex domains also to the more general setting of polarized Kähler manifolds. Namely, let Ω be a complex manifold of dimension n and ω a Kähler form on Ω such that the second cohomology class $[\omega]$ is integral. Then there exists a holomorphic Hermitian line bundle $\mathcal{L}$ over Ω with Kähler connection ∇ such that $\operatorname{curv} \nabla = \omega$. Let $\mathcal{L}^*$ be the dual bundle, and $L^2_{\text{hol}}(\mathcal{L}^*, \omega^n)$ the Bergman space of all square-integrable holomorphic sections of $\mathcal{L}^*$. For any orthonormal basis $\{s_j\}_j$ of this space, set

$$\epsilon(x) := \sum_j \|s_j(x)\|_x^2 \tag{4}$$

(where $\|\cdot\|_x$ denotes the fiber norm in $\mathcal{L}^*_x$); one can again show that $\epsilon(x)$ does not depend on the choice of the orthonormal basis $\{s_j\}_j$, and also does not really depend on the line bundle $\mathcal{L}$ but only on the Kähler form ω. The Kähler form ω (or the associated Kähler metric) is called *balanced* if

$$\epsilon \equiv \text{const.} \tag{5}$$

Of course, if Ω is contractible, then the bundle $\mathcal{L}$ is trivial, so fixing a trivialization its sections can be identified with functions on Ω, and under this identification the fiber norm of a function f at a point x is given by $h(x)|f(x)|^2$ for some positive smooth function h on Ω satisfying $\omega = i\partial\overline{\partial} \log h$. Setting $\phi := \frac{1}{n+1} \log h$, we recover the situation from the previous paragraph.

The function ϵ has appeared in the literature under different names. The earliest one was probably the *η-function* of Rawnsley [17] (later renamed to *ϵ-function* in [5]), defined for arbitrary Kähler manifolds; followed by the *distortion function* of Kempf [12] and Ji [11] for the special case of Abelian varieties, and of Zhang [19] for complex projective varieties. The metrics for which ϵ is constant were called *critical* in [19]; the term *balanced* was first used by Donaldson [7], who also established the existence of such metrics on any (compact) projective Kähler manifold with constant scalar curvature. Subsequent studies of the existence and uniqueness of balanced metrics in the compact case — where the balancedness condition (5) is actually tantamount to the existence of a "balanced" imbedding of Ω into the complex projective space $\mathbf{C}P^N$, $N = \dim L^2_{\text{hol}}(\mathcal{L}^*, \omega^n)$ — include Seyyedali [18], Li [13], and others; see also Phong and Sturm [16] for an overview.

However, much less seems to be known about the existence and uniqueness of balanced metrics in the noncompact setting of domains in $\mathbf{C}^n$. Apart from the example above for the disc, and the analogous situation for $u(z) = (1 - \|z\|^2)^\alpha$, $\alpha > \frac{n}{n+1}$, on the unit ball $\mathbf{B}^n$ of $\mathbf{C}^n$ (with $c = \frac{\Gamma(\alpha n+\alpha)}{\pi^n \alpha^n \Gamma(\alpha n+\alpha-n)}$), the only known examples of balanced metrics are the (appropriate multiples of the) Bergman metrics on bounded symmetric domains in $\mathbf{C}^n$, or, more generally, of invariant metrics on bounded homogeneous domains. The situation for Cartan-Hartogs domains is discussed in Loi and Zedda [15]. We will briefly recall some other examples in §4.3 below. Similarly, the Euclidean metric is balanced on $\mathbf{C}^n$ (with $\phi(z) = \|z\|^2$). As for uniqueness, it was shown in Cuccu and Loi [6] that $u = e^{-\phi}$ with $\phi(z) = \|z\|^2$ the potential for the Euclidean metric on $\mathbf{C}^n$ is, up to constant multiples, the only solution to $K_{u(z)^{n+1}}(z,z) = c/u(z)^{n+1}$ among the ϕ of the form $\phi(z) = \sum_{j=1}^n \phi_j(|z_j|^2)$, with some functions ϕ_j. In Greco and Loi [10], it was shown that $u(z) = 1 - |z|^2$ is the only solution to $K_{u(z)^3/(1-|z|^2)^2}(z,z) = c/u(z)^3$ of the form $u(z) = f(|z|^2)$ with f real analytic on the interval $(-\epsilon, 2+\epsilon)$ with some $\epsilon > 0$ and satisfying $f(1) = 0$, $f'(1) = -1$. Some remarks on Donaldson's original approach in the noncompact case appear in the final section of Arezzo and Loi [1]. In the negative direction, it was shown by Bommier, Youssfi and the present author [4] that on the Kepler manifold $\mathbf{K}^n := \{z \in \mathbf{C}^{n+1} \setminus \{0\} : \sum_{j=1}^{n+1} z_j^2 = 0\}$, balanced metric either does not exist, or exists but is not unique. The reason is, roughly speaking, that the isotropy subgroup of the group of all holomorphic automorphisms of $\mathbf{K}^n$ has a noncompact orbit on the tangent space; note that, however, this can never happen for a bounded domain $\Omega \subset \mathbf{C}^n$ in the place of $\mathbf{K}^n$, since the isotropy subgroup of a bounded domain is always compact.

It is a conjecture of the author's [9] that for $\Omega \subset \mathbf{C}^n$ bounded strictly pseudoconvex with smooth boundary and any $\alpha > \frac{n}{n+1}$, there exists a unique balanced metric on Ω with $u^{(n+1)/\alpha}$ vanishing precisely to the first order at the boundary $\partial\Omega$, i.e. $u(z) \asymp \operatorname{dist}(z, \partial\Omega)^{\alpha/(n+1)}$. In this paper, we consider this problem for $\alpha = n+1$ and u radial on the unit disc, i.e. $u(z) = f(|z|^2)$ for some $f \in C^\infty[0,1)$. Our strategy will be to look at the boundary behaviour of both sides of (3). More specifically, let us call a metric — or, abusing terminology, the function u — on a domain Ω *almost-balanced* if there exists a constant $c \neq 0$ such that

$$u(z)^{n+1} K_{J[u]}(z,z) - c \in C^\infty(\overline{\Omega}) \cap C_0^1(\overline{\Omega}), \tag{6}$$

the space of all functions smooth on the closure $\overline{\Omega}$ of Ω that vanish at $\partial\Omega$ together with their first derivatives (i.e. to second order). Our main results are the following.

Theorem 1. *Radial functions $u(z) = f(|z|^2)$ on the disc $\mathbf{D}$, with $f \in C^\infty(0,1]$ satisfying $f(1) = 0$, $f'(1) = -1$, that give rise to almost balanced metrics on $\mathbf{D}$ are precisely those of the form*

$$f(t) = t^a \frac{t^{-\sqrt{v}} - t^{\sqrt{v}}}{2\sqrt{v}} + h(t), \tag{7}$$

with $a, v \in \mathbf{R}$ and $h \in C^\infty(0,1]$ satisfying $h^{(k)}(1) = 0$ $\forall k$. Also, the constant c in (6) necessarily equals $\frac{1}{\pi}$.

Here for $v = 0$, (7) is to be interpreted as the limit $v \to 0$, i.e. $f(t) = -t^a \log t + h(t)$. Note also that the right-hand side of (7) remains unchanged upon replacing $\sqrt{v}$ by $-\sqrt{v}$, so there is no ambiguity connected with the choice of the square root.

Corollary 2. *If $u(z) = f(|z|^2)$ on the disc with $f(1) = 0$, $f'(1) = -1$ and f real-analytic near 1, then u is balanced if and only if $f(t) = 1 - t$.*

Finally, the hypothesis of the smoothness of u at the boundary in Theorem 1 can be weakened considerably: writing temporarily for brevity $r(z) := \operatorname{dist}(z, \partial\Omega)$, assume that $u \in C^\infty(\Omega)$ has an asymptotic expansion at $\partial\Omega$ of the form

$$u(z) \approx r(z) \sum_{k=0}^{\infty} \sum_{j=0}^{M_k} a_{kj}(z) r(z)^k (\log r(z))^j, \tag{8}$$

with some nonnegative integers M_k and functions $a_{kj} \in C^\infty(\overline{\Omega})$, where

$$M_0 = 0 \qquad \text{and} \qquad a_{00} = 1 \text{ on } \partial\Omega. \tag{9}$$

Here (8) means that u differs from the partial sum $\sum_{k=0}^{N-1}$ of the right-hand side by a function in $C^N(\overline{\Omega})$ all of whose partial derivatives up to order N vanish at $\partial\Omega$, for all $N = 0, 1, 2, \dots$. Note that $u \in C^\infty(\overline{\Omega})$ is equivalent to $M_k = 0\ \forall k$.

Theorem 3. *Assume that $u(z) = f(|z|^2)$ is a smooth radial function on the disc, with asymptotic expansion* (8) *satisfying* (9)*, that gives rise to an almost-balanced metric. Then $f \in C^\infty(0,1]$* (*and, hence, Theorem 1 applies*)*.*

The proofs of Theorem 1 and Corollary 2 are given in Section 2, and the proof of Theorem 3 in Section 3. The final section, Section 4, collects some concluding comments and remarks; among others, it is also shown there that there exist no complete radial balanced metrics on the disc or on the annulus that are real-analytic up to the exterior boundary.

2. The smooth case

Let quite generally $w(z) = w(|z|)$ be a radial weight on the unit disc $\mathbf{D}$. It is then standard (and easily shown by using polar coordinates) that the monomials form an orthogonal basis in $L^2_{\text{hol}}(\mathbf{D}, w)$, with norm squares

$$\|z^k\|^2 = \int_{\mathbf{D}} |z|^{2k} w(z)\, dz = \pi \int_0^1 t^k w(\sqrt{t})\, dt =: \pi c_k.$$

Furthermore, the reproducing kernel is given by

$$K_w(x, y) = \tfrac{1}{\pi} F(x\overline{y}), \qquad F(t) := \sum_{k=0}^{\infty} \frac{t^k}{c_k} \tag{10}$$

(with t^k/c_k interpreted as 0 if $c_k = +\infty$).

Proof of Theorem 1. Assume that $u(z) = f(|z|^2)$, with $f \in C^\infty(0,1]$ satisfying $f(1) = 0$, $f'(1) = -1$, gives rise to an almost-balanced metric. It will be convenient to switch to the variable

$$L := -\log |z|^2.$$

One then has, in the sense of (8),

$$u(z)^2 \approx L^2 \sum_{k=0}^{\infty} f_k L^k \tag{11}$$

where $f_k = \frac{1}{k!}\frac{d^k}{dL^k}\frac{u(e^{-L})^2}{L^2}|_{L=0}$ are some real numbers; in particular, $f_0 = 1$ (from $f'(1) = -1$). Taking logarithm gives

$$\log u \approx \log L + \sum_{m=1}^{\infty} \frac{f'_m}{2} L^m,$$

where

$$\begin{aligned} f'_m &= \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k} \sum_{\substack{j_1,\dots,j_k \geq 1 \\ j_1+\dots+j_k=m}} f_{j_1} \dots f_{j_k} \\ &= f_m + (\text{a polynomial in } f_1, \dots, f_{m-1}). \end{aligned} \tag{12}$$

Now (8) can be differentiated termwise any number of times; using the formulas

$$\frac{\partial^2 \log L}{\partial z \partial \overline{z}} = -\frac{e^L}{L^2}, \qquad \frac{\partial^2 L^m}{\partial z \partial \overline{z}} = m(m-1)L^{m-2}e^L, \tag{13}$$

it therefore follows that

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log \frac{1}{u(z)} \approx \frac{e^L}{L^2}\Big[1 - \sum_{m=1}^{\infty} \frac{m(m-1)f'_m}{2} L^m\Big],$$

and, using again (11),

$$\begin{aligned} w \equiv u^2 \partial\overline{\partial} \log \tfrac{1}{u} &= e^L \Big[1 - \sum_{m=1}^{\infty} \frac{m(m-1)f'_m}{2} L^m\Big]\Big[1 + \sum_{j=1}^{\infty} f_j L^j\Big] \\ &= e^L \sum_{n=0}^{\infty} f''_n L^n, \end{aligned} \tag{14}$$

where, by (12),

$$\begin{aligned} f''_n &= f_n + \sum_{m=1}^{n} \frac{m(1-m)f'_m}{2} f_{n-m} \\ &= (1 + \tfrac{n(1-n)}{2}) f_n + (\text{a polynomial in } f_1, \dots, f_{n-1}). \end{aligned}$$

Now by elementary computation

$$\int_0^1 t^{k-1} (\log \tfrac{1}{t})^n \, dt = \int_0^\infty L^n e^{-kL} \, dL = \frac{n!}{k^{n+1}}.$$

From (14) we thus get the asymptotic expansion for the moments c_k

$$c_k = \int_0^1 w(\sqrt{t}) t^k \, dt \approx \sum_{n=0}^{\infty} \frac{n! f''_n}{k^{n+1}}$$

as $k \to +\infty$, where "$\approx$" now means that the difference between c_k and the partial sum $\sum_{n=0}^{N-1}$ of the right-hand side is $O(k^{-N-1})$ as $k \to +\infty$, for each $N = 0, 1, 2, \dots$. Taking reciprocals yields

$$\frac{1}{c_k} \approx 1 \Big/ \frac{1}{k}\Big[1 + \sum_{n=1}^{\infty} \frac{n! f''_n}{k^n}\Big] = k\Big[1 + \sum_{j=1}^{\infty} \frac{f'''_j}{k^j}\Big], \tag{15}$$

with

$$\begin{aligned} f'''_j &= \sum_{l=1}^{j} (-1)^l \sum_{\substack{n_1,\dots,n_l \ge 1 \\ n_1+\dots+n_l=j}} \prod_{i=1}^{l} (n_i! f''_{n_i}) \\ &= j!(\tfrac{j(j-1)}{2} - 1) f_j + (\text{a polynomial in } f_1, \dots, f_{j-1}). \end{aligned} \tag{16}$$

In particular, by a short computation,

$$f'''_1 = -f_1, \qquad f'''_2 = 0. \tag{17}$$

Now, quite generally, whenever a holomorphic function $f(t) = \sum_{k=0}^{\infty} a_k t^k$ on the disc has Taylor coefficients satisfying $a_k = O(k^{-N-2})$ as $k \to +\infty$ ($N = 0, 1, 2, \dots$), then by a straightforward estimate $f \in C^N(\overline{\mathbf{D}})$. Recalling the definition of Lerch's transcendental function [2, §1.11]

$$\Phi(z, s, v) := \sum_{k=0}^{\infty} \frac{z^k}{(k+v)^s}, \qquad s \in \mathbf{C}, \quad v \neq 0, -1, -2, \dots,$$

we have $\sum_{k=1}^{\infty} \frac{t^k}{k^j} = t\Phi(t, j, 1)$, so for the function F in (10) we thus get from (15)

$$F(t) = \frac{t}{(1-t)^2} + \frac{t f'''_1}{1-t} + \sum_{j=2}^{N+2} f'''_j t\Phi(t, j-1, 1) + (\text{a function in } C^N(\overline{\mathbf{D}})),$$

for any $N = 0, 1, 2, \dots$. By Lerch's formula [2, 1.11(9)], for an integer $j \ge 2$,

$$t\Phi(t, j-1, 1) = \frac{(-1)^{j-1}}{(j-2)!} L^{j-2} \log L + {\sum_{k=0}^{\infty}}' \frac{(-1)^k}{k!} \zeta(j-1-k) L^k, \qquad L := \log \frac{1}{t}, \tag{18}$$

where the sum on the right-hand side converges for $|L| < 2\pi$, and the $\sum'$ means that in the term $k = j-2$, $\zeta(1)$ should be replaced by $\sum_{n=1}^{j-2} \frac{1}{n}$.[1] Altogether, employing also the fact that $\frac{t}{(1-t)^2} - \frac{1}{L^2}$ and $\frac{t}{1-t} - \frac{1}{L}$ are holomorphic on $\mathbf{C}$, we thus obtain

$$F(t) \approx \frac{1}{L^2} + \frac{f'''_1}{L} + \sum_{j=2}^{\infty} \Big(a_j + \frac{f'''_j (-1)^{j-1}}{(j-2)!} \log L\Big) L^{j-2}, \qquad L := \log \frac{1}{t}, \tag{19}$$

[1] This formula is stated in [2, 1.11(9)] for $j \ge 3$; however, in this form it holds also for $j = 2$, when $t\Phi(t, 1, 1) = \log \frac{1}{1-t}$.

in the sense of (8), with some coefficients a_j, $j=2,3,\dots$. Now, on the other hand, the almost-balanced condition $u^2K_w - c \in C^\infty(\overline{\mathbf{D}}) \cap C^1_0(\overline{\mathbf{D}})$ means that $f^2F = \pi c[1+\sum_{j=2}^\infty b_jL^j]$; that is, by (11) again,

$$\begin{aligned} F(t) &\approx \frac{\pi c}{f^2}\Big[1+\sum_{j=2}^\infty b_jL^j\Big] \\ &\approx \frac{\pi c}{L^2}\Big[1+\sum_{j=2}^\infty b_jL^j\Big]\Big/\Big[1+\sum_{k=1}^\infty f_kL^k\Big] \\ &= \frac{\pi c}{L^2}\Big[1+\sum_{k=1}^\infty \tilde f_kL^k\Big], \end{aligned} \tag{20}$$

with some coefficients $\tilde f_k$, where $\tilde f_1 = -f_1$. Comparing this with (19), we see that necessarily ($c=\frac{1}{\pi}$ and)

$$f_1''' = -f_1, \qquad f_j''' = 0 \quad \forall j \ge 2. \tag{21}$$

By (17), the conditions on f_1''' and f_2''' are always fulfilled, while by (16), for $j \ge 3$ the equation $f_j''' = 0$ is always uniquely solvable for f_j (if $f_0, f_1, \dots, f_{j-1}$ are already given) since $\frac{j(j-1)}{2} - 1 \neq 0$. Altogether, we thus see that for any chosen $f_1, f_2 \in \mathbf{R}$ (and $f_0 = 1$), there exist unique f_j, $j \ge 3$, such that (21) holds. In other words, there exist at most[2] two-parameter family of germs of f at $t=1$ for which $u(z) = f(|z|^2)$ can give rise to an almost-balanced metric.

Since (7) is a two-parameter family of germs of f at $t=1$, it is thus enough to show that any f as in (7) gives rise to an almost-balanced metric; this will complete the proof of the theorem.

As we have seen above, the term $h(t)$ gives a contribution of order $O(k^{-\infty})$ to c_k and of order $C^\infty(\overline{\mathbf{D}})$ to u and K, hence is negligible as far as the almost-balanced condition is concerned; so we can take $h \equiv 0$. By direct computation, one then gets ($t = |z|^2$)

$$\begin{aligned} \partial\overline\partial \log \tfrac{1}{u} &= \frac{4v}{t(t^{\sqrt v} - t^{-\sqrt v})^2}, \\ w(z) &= t^{2a-1}, \\ c_k = \int_0^1 t^k w(\sqrt t)\,dt &= \frac{1}{k+2a} \qquad (k+2a>0), \end{aligned} \tag{22}$$

and

$$\begin{aligned} F(t) &= \sum_{k\ge 0,\ k+2a>0} (k+2a)t^k \\ &= \frac{t}{(1-t)^2} + \frac{2a}{1-t} + \sum_{0\le k\le -2a} (k+2a)t^k \\ &= \frac{1}{(1-t)^2}\Big[1+(2a-1)(1-t)+(1-t)^2p(t)\Big] \end{aligned} \tag{23}$$

[2] In fact, "at most" can be dropped here, since from (20) and (19) it follows that, conversely, once (21) if fulfilled, one has $F \approx \frac{\pi c}{f^2}[1+\sum_{j=2}^\infty b_jL^j]$ with some b_j and thus f indeed gives rise to an almost-balanced metric.

with some polynomial $p(t)$. On the other hand, by the familiar formula

$$t^{-b} = (1-(1-t))^{-b} = \sum_{k=0}^\infty \frac{(b)_k}{k!}(1-t)^k, \qquad b\in\mathbf{C}, \quad |1-t|<1$$

(where $(b)_k = b(b+1)\dots(b+k-1)$ is the usual Pochhammer symbol), we get

$$\begin{aligned} f(t) &= \frac{t^{a-\sqrt{v}} - t^{a+\sqrt{v}}}{2\sqrt{v}} \\ &= \sum_{k=0}^\infty \frac{(-a+\sqrt{v})_k - (-a-\sqrt{v})_k}{k!2\sqrt{v}}(1-t)^k \\ &= (1-t) + (\tfrac12 - a)(1-t)^2 + \sum_{k=3}^\infty q_k(1-t)^k \end{aligned}$$

with some q_k. Hence finally

$$\begin{aligned} &\pi K_w(z,z)u(z)^2 = f(t)^2F(t) \\ &\quad = \Big[1+(\tfrac12-a)(1-t)+\sum_{k=2}^\infty q_{k+1}(1-t)^k\Big]^2\Big[1+(2a-1)(1-t)+(1-t)^2p(t)\Big] \\ &\quad = 1+(1-t)^2q(t) \end{aligned}$$

with some function q holomorphic on the disc $|t-1|<1$. Consequently, $u^2K_w - \frac1\pi$ is smooth up to $\partial\mathbf{D}$ (in fact, even extends to a holomorphic function on $|z|^2<2$) and vanishes to second order at $|z|=1$. This means that the almost-balanced condition (6) is satisfied, with $c=\frac1\pi$, completing the proof. $\square$

Note that for f as in (7) we get from (11)

$$f_1 = -2a, \qquad f_2 = 2a^2 + \frac{v}{3},$$

or $a=-\frac{f_1}{2}$, $v = 3f_2 - \frac32 f_1^2$; thus when (f_1,f_2) runs through all of $\mathbf{R}^2$, so does (a,v).

Proof of Corollary 2. If f is real-analytic near 1, then it follows from (7) that so is h, hence $h\equiv 0$ by Theorem 1. As we already saw in (22), the metric is then given by

$$g(z) = \frac{4v}{t(t^{\sqrt{v}} - t^{-\sqrt{v}})^2}, \qquad t=|z|^2.$$

If $v<0$, say $v=-s^2$ with $s>0$, this becomes

$$g(z) = \frac{s^2e^L}{\sin^2(sL)}, \qquad L = \log\frac{1}{|z|^2},$$

which has singularities at $|z|^2 = e^{-k\pi/s}$, $k=0,1,2,\dots$. Similarly, for $v\to 0$, the metric becomes $g(z) = 1/t(\log t)^2$, which has a singularity at the origin $t=0$. Thus to have g smooth on the disc, necessarily $v>0$, say $v=s^2$ with $s>0$. Then the function $u(z) = f(|z|^2) = \frac{|z|^{2a-2s}-|z|^{2a+2a}}{2s}$, being of the form $u=e^{-\phi}$ with ϕ a (smooth) potential for the Kähler metric, has to be smooth at $z = 0$; hence both $a - s$ and $a + s > a - s$ must be nonnegative integers, thus $2a = (a + s) + (a - s)$ is an integer ≥ 1. On the other hand, the balanced condition (with $c = \frac{1}{\pi}$) then says that the function F from (10) satisfies

$$F(t) = \frac{1}{f(t)^2} = \frac{4s^2 t^{2s-2a}}{(1 - t^{2s})^2} = 4s^2 \sum_{k=0}^{\infty} (k+1) t^{2sk+2s-2a}. \tag{24}$$

But we have seen that F is given by (23); it follows that necessarily $s = \frac{1}{2}$ (since the exponents of t increase by 1 in (23)). Comparing (24) and (23) then gives $\sum_{k=0}^{\infty} (k+1) t^{k+1-2a} = \sum_{k \geq 0,\ k+2a>0} (k + 2a) t^k$, or

$$\sum_{k=0}^{\infty} (k+1) t^{k+1} = \sum_{k \geq 0,\ k+2a>0} (k + 2a) t^{k+2a}. \tag{25}$$

Since $2a > 0$, looking at the lowest order terms shows that $1t^1 = 2at^{2a}$, or $a = \frac{1}{2}$. Hence $f(t) = 1 - t$ and $u(z) = 1 - |z|^2$, proving the corollary. $\square$

3. The general case

We will need the following refinement of Lerch's formula (18).

Lemma 4. *The series*

$$\sum_{k=1}^{\infty} \frac{t^k}{k^s} \Big(\log \frac{1}{k} \Big)^n = \Big(\frac{d}{ds} \Big)^n t\Phi(t, s, 1), \qquad n = 0, 1, 2, \ldots, \tag{26}$$

equals

(27)
$$\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \Gamma^{(n-j)}(1 - s) L^{s-1} (\log L)^j + \sum_{k=0}^{\infty} \zeta^{(n)}(s - k) \frac{(-1)^k}{k!} L^k,$$
$$|L| < 2\pi, \quad s \neq 1, 2, 3, \ldots, \quad L := \log \frac{1}{t}.$$

For $s = 1, 2, 3, \ldots$, the first sum on the right-hand side of the last formula has to be replaced by

$$\sum_{j=0}^{n} \binom{n}{j} c_{s,n-j} L^{s-1} (\log L)^j + \frac{(-1)^{s-1}}{(s-1)!} \Big[\gamma_n - \frac{(\log L)^{n+1}}{n+1} \Big] L^{s-1}, \tag{28}$$

while the term $k = s - 1$ in the second sum on the right-hand side of (27) *has to be omitted. Here $c_{s,j}$ and γ_j are certain constants* (*given explicitly below*).

Proof. For $s \neq 1, 2, 3, \ldots$, we have by [2, 1.11(8)]

$$t\Phi(t, s, 1) = \Gamma(1 - s) L^{s-1} + \sum_{k=0}^{\infty} \zeta(s - k) \frac{(-1)^k}{k!} L^k, \tag{29}$$
$$|L| < 2\pi, \quad L := \log \frac{1}{t}.$$

Applying $(d/ds)^n$ to both sides, the Leibniz rule gives (27).

To get the formula for $s = m$, where $m = 1, 2, 3, \dots$, we first take $s = m + \epsilon$ in (29), then apply $(d/ds)^n$, and finally let $\epsilon \to 0$. Clearly there is no problem with the terms $k \neq m - 1$ in (29); thus we only need to evaluate the limit

$$\lim_{\epsilon\to 0} \Big(\frac{d}{ds}\Big)^n \Big[\Gamma(1-m-\epsilon)L^{m-1+\epsilon} + \zeta(\epsilon+1)\frac{(-1)^{m-1}}{(m-1)!}L^{m-1}\Big]. \tag{30}$$

Recall that

$$\zeta(\epsilon+1) = \frac{1}{\epsilon} + \sum_{j=0}^{\infty} \frac{\gamma_j}{j!}\epsilon^j, \qquad \epsilon \in \mathbf{C}, \tag{31}$$

where γ_j are certain coefficients (the Stieltjes constants). Similarly, $\Gamma(s)$ has a simple pole at $s = 1 - m$ with residue $(-1)^{m-1}/(m-1)!$, so

$$\Gamma(1-m-\epsilon) = \frac{(-1)^m}{(m-1)!\epsilon} + \sum_{j=0}^{\infty} \frac{c_{m,j}}{j!}\epsilon^j, \qquad |\epsilon| < 1, \tag{32}$$

with some coefficients $c_{m,j}$. After pulling out the factor L^{m-1} from both terms in (30), we thus get

$$\begin{aligned}
&\Gamma(1-m-\epsilon)L^\epsilon + \zeta(\epsilon+1)\frac{(-1)^{m-1}}{(m-1)!} \\
&\quad = \Big(\frac{(-1)^m}{(m-1)!}\frac{L^\epsilon - 1}{\epsilon} + \frac{(-1)^m}{(m-1)!\epsilon} + \sum_{j=0}^{\infty}\frac{c_{m,j}}{j!}\epsilon^j L^\epsilon\Big) + \frac{(-1)^{m-1}}{(m-1)!}\Big(\frac{1}{\epsilon} + \sum_{j=0}^{\infty}\frac{\gamma_j}{j!}\epsilon^j\Big) \\
&\quad = \frac{(-1)^m}{(m-1)!}\sum_{j=0}^{\infty}\frac{(\log L)^{j+1}}{(j+1)!}\epsilon^j + \sum_{j=0}^{\infty}\frac{c_{m,j}}{j!}\epsilon^j L^\epsilon + \frac{(-1)^{m-1}}{(m-1)!}\sum_{j=0}^{\infty}\frac{\gamma_j}{j!}\epsilon^j.
\end{aligned}$$

By the Leibniz rule,

$$\begin{aligned}
&\Big(\frac{d}{d\epsilon}\Big)^n \Big[\frac{(-1)^m}{(m-1)!}\sum_{j=0}^{\infty}\frac{(\log L)^{j+1}}{(j+1)!}\epsilon^j + \sum_{j=0}^{\infty}\frac{c_{m,j}}{j!}\epsilon^j L^\epsilon + \frac{(-1)^{m-1}}{(m-1)!}\sum_{j=0}^{\infty}\frac{\gamma_j}{j!}\epsilon^j\Big] \\
&= \sum_{j=0}^{\infty}\frac{(-1)^{m-1}}{(m-1)!}\Big[\gamma_{j+n} - \frac{(\log L)^{j+n+1}}{j+n+1}\Big]\frac{\epsilon^j}{j!} + \sum_{j=0}^{\infty}\sum_{q=0}^{n}\binom{n}{q} c_{m,j+n-q}\frac{\epsilon^j}{j!}L^\epsilon(\log L)^q.
\end{aligned}$$

Letting $\epsilon \to 0$ and inserting back the factor L^{m-1}, (28) follows (with q, m in the place of j, s). $\square$

For later use, we also note that the simple formula

$$\int_0^\infty L^s e^{-kL}\,dL = \frac{\Gamma(s+1)}{k^{s+1}}, \qquad \operatorname{Re} s > -1, \quad k = 1, 2, 3, \dots,$$

yields upon applying $(d/ds)^n$ to both sides ($n = 0, 1, 2, \dots$)

$$\int_0^\infty L^s(\log L)^n e^{-kL}\,dL = \sum_{l=0}^{n}\binom{n}{l}\frac{\Gamma^{(n-l)}(s+1)}{k^{s+1}}\Big(\log\frac{1}{k}\Big)^l, \tag{33}$$

again by the Leibniz rule.

Proof of Theorem 3. Assume that $u(z) = f(|z|^2)$ is a smooth radial function on the disc, with the asymptotic expansion (8) satisfying (9), that gives rise to an almost balanced metric. Passing again from the variable $t = |z|^2$ to $L = \log \frac{1}{t}$, (8) and (9) become

$$f(t) \approx L \sum_{k=0}^{\infty} \sum_{j=0}^{M_k} a_{kj} L^k (\log L)^j, \qquad M_0 = 0, \quad a_{00} = 1. \tag{34}$$

We will show that $M_k = 0$ for all k, so that $f \in C^\infty(0, 1]$ as claimed.

Assume, to the contrary, that there is $N \geq 1$ such that $M_0 = M_1 = \cdots = M_{N-1} = 0$ but $M_N \geq 1$ with $a_{NM_N} \neq 0$. By (34) and the definition of (8), we have

$$u \approx L\Big[1 + p_N(L) + \sum_{j=1}^{M} a_j L^N (\log L)^j + O(L^{N+\delta})\Big], \tag{35}$$

with any $0 < \delta < 1$; here we started writing just M and a_j for M_N and a_{Nj}, respectively, and p_N stands for some polynomial (not necessarily the same one at each occurrence) of degree N without constant term. Furthermore, (35) can be differentiated termwise any number of times. Taking logarithm we get

$$\log \frac{1}{u} \approx \log \frac{1}{L} + p_N(L) - \sum_{j=1}^{M} a_j L^N (\log L)^j + O(L^{N+\delta}).$$

Using again the formulas (13) and also

$$\begin{aligned} \frac{\partial^2}{\partial z \partial \bar{z}} L^m (\log L)^j = L^{m-2} e^L [j(j-1)(\log L)^{j-2} &+ j(2m-1)(\log L)^{j-1} \\ &+ m(m-1)(\log L)^j], \end{aligned} \tag{36}$$

this implies

$$\begin{aligned} \partial\bar{\partial} \log \frac{1}{u} \approx \frac{e^L}{L^2} \Big[1 + p_N(L) - \sum_{j=1}^{M} a_j L^N \Big(& j(j-1)(\log L)^{j-2} + j(2N-1)(\log L)^{j-1} \\ & + N(N-1)(\log L)^j \Big) + O(L^{N+\delta})\Big], \end{aligned}$$

and, combining with (35),

$$\begin{aligned} w \equiv u^2 \partial\bar{\partial} \log \frac{1}{u} \approx e^L \Big[1 + p_N(L) - \sum_{j=1}^{M} a_j L^N \Big(& j(j-1)(\log L)^{j-2} \\ & + j(2N-1)(\log L)^{j-1} + N(N-1)(\log L)^j \Big) + O(L^{N+\delta}) \Big] \\ & \cdot \Big[1 + p_N(L) + 2 \sum_{j=1}^{M} a_j L^N (\log L)^j + O(L^{N+\delta}) \Big] \\ = e^L \Big[1 + p_N(L) + L^N \sum_{j=1}^{M} a_j \Big((2 - N^2 + N)(\log L)^j & - j(2N-1)(\log L)^{j-1} \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - j(j-1)(\log L)^{j-2}\Big) + O(L^{N+\delta})\Big] \\
&= e^L\Big[1 + p_N(L) + \sum_{j=1}^M A_j L^N (\log L)^j + O(L^{N+\delta})\Big], \\
&\qquad A_j := \Big((2-N^2+N)a_j - (j+1)(2N-1)a_{j+1} - (j+2)(j+1)a_{j+2}\Big),
\end{aligned}
$$

where we set $a_{M+1} = a_{M+2} := 0$. For the moments c_k we thus obtain, in view of (33),

$$
\begin{aligned}
c_k &= \int_0^1 t^k w(\sqrt{t})\,dt = \int_0^\infty e^{-(k+1)L} w\,dL \\
&\approx \frac{1}{k} + \frac{1}{k} p_N(\frac{1}{k}) + \sum_{j=1}^M A_j \sum_{l=0}^j \binom{j}{l} \frac{\Gamma^{(j-l)}(N+1)}{k^{N+1}} \Big(\log\frac{1}{k}\Big)^l + O\Big(\frac{1}{k^{N+1+\delta}}\Big).
\end{aligned}
$$

Taking reciprocal yields

$$
\frac{1}{c_k} \approx k\Big[1 + p_N(\frac{1}{k}) - \sum_{j=1}^M A_j \sum_{l=0}^j \binom{j}{l} \frac{\Gamma^{(j-l)}(N+1)}{k^N} \Big(\log\frac{1}{k}\Big)^l + O\Big(\frac{1}{k^{N+\delta}}\Big)\Big].
$$

It transpires that

$$
\begin{aligned}
F(t) = \sum_{k=0}^\infty \frac{t^k}{c_k} &= \frac{t}{(1-t)^2} + \frac{\beta_1 t}{1-t} + \sum_{r=2}^N \beta_r t\Phi(t, r-1, 1) \\
&\quad - \sum_{j=1}^M A_j \sum_{l=0}^j \binom{j}{l} \Gamma^{(j-l)}(N+1) \Big(\frac{d}{ds}\Big)^l t\Phi(t,s,1)\Big|_{s=N-1} + R(t),
\end{aligned} \tag{37}
$$

with some coefficients β_r, where the remainder term $R(t)$ is $O((1-t)^{\delta-1})$ if $N=1$, and belongs to $C^{N-2}(0,1]$ if $N \geq 2$.

If $N = 1$, then the first sum on right-hand side of (37) is absent, while the second sum equals, by (27),

$$
\sum_{j=1}^M A_j \sum_{l=0}^j \binom{j}{l} \Gamma^{(j-l)}(2) \sum_{r=1}^l \binom{l}{r} (-1)^{l-r} \Gamma^{(l-r)}(1) \frac{(\log L)^r}{L} + h(L) \tag{38}
$$

with some function $h(L)$ holomorphic on the disc $|L| < 2\pi$. Now

$$
\begin{aligned}
&\sum_{r \leq l \leq j} \binom{j}{l} \Gamma^{(j-l)}(2) \binom{l}{r} (-1)^{l-r} \Gamma^{(l-r)}(1) \\
&\qquad = \binom{j}{r} \Big(\frac{d}{ds}\Big)^{j-r} \Gamma(2+s)\Gamma(1-s)\Big|_{s=0} \\
&\qquad = \binom{j}{r} \Big(\frac{d}{ds}\Big)^{j-r} \frac{\pi s(s+1)}{\sin \pi s}\Big|_{s=0} =: \frac{j!}{r!} C_{j-r},
\end{aligned} \tag{39}
$$

so (38) equals

$$\sum_{j=1}^{M}\sum_{r=1}^{j} A_j \frac{r!}{j!} C_{j-r} \frac{(\log L)^r}{L} + h(L) =: \sum_{r=1}^{M} \tilde{A}_r \frac{(\log L)^r}{L} + h(L),$$

and, finally,

$$\begin{aligned} F(t) &= \frac{1}{L^2} + \frac{\beta_1}{L} - \sum_{r=1}^{M} \tilde{A}_r \frac{(\log L)^r}{L} + h(L) + O(L^{\delta-1}) \\ &= \frac{1}{L^2}\Big[1 + p_N(L) - \sum_{r=1}^{M} \tilde{A}_r L(\log L)^r + O(L^{1+\delta})\Big]. \end{aligned} \tag{40}$$

On the other hand, from the almost-balanced condition we have, as in the preceding section,

$$F \approx \frac{\pi c}{u^2}\Big[1 + \sum_{j=2}^{\infty} b_j L^j\Big],$$

whence by (35)

$$\begin{aligned} F &\approx \frac{\pi c}{L^2}\Big[1 + \sum_{j=2}^{\infty} b_j L^j\Big]\Big[1 + p_N(L) - 2\sum_{j=1}^{M} a_j L(\log L)^j + O(L^{1+\delta})\Big] \\ &= \frac{\pi c}{L^2}\Big[1 + p_N(L) - 2\sum_{j=1}^{M} a_j L(\log L)^j + O(L^{1+\delta})\Big]. \end{aligned}$$

Comparing this with (40), we thus see that we must have ($c = \frac{1}{\pi}$ and) $\tilde{A}_j = 2a_j$ for all $j = 1, \dots, M$. For $j = M-1$, the latter reads, since $C_0 = C_1 = 1$,

$$\begin{aligned} 2a_{M-1} = \tilde{A}_{M-1} &= A_M M C_1 + A_{M-1} C_0 \\ &= (2a_M)M + (2a_{M-1} - M a_M), \end{aligned}$$

or $Ma_M = 0$, contradicting the hypothesis that $M \geq 1$ and $a_M \neq 0$. Thus $N = 1$ cannot occur.

For $N \geq 2$, by (18) the first sum on the right-hand side of (37) equals $p(L)\log L + h(L)$, with some polynomial p of degree $N-2$ and some function h holomorphic in the disc $|L| < 2\pi$. The second sum on the right-hand side of (37) equals, by (27) and (28),

$$\begin{aligned} &\sum_{j=1}^{M} A_j \sum_{l=0}^{j} \binom{j}{l} \Gamma^{(j-l)}(N+1) \\ &\qquad \Big[\sum_{r=1}^{l} \binom{l}{r} c_{N-1,l-r}(\log L)^r - \frac{(-1)^{N-2}}{(N-2)!}\frac{(\log L)^{l+1}}{l+1}\Big] L^{N-2} + h(L) \\ &=: \sum_{r=1}^{M+1} \tilde{A}_r L^{N-2}(\log L)^r + h(L), \end{aligned}$$

with some $h(L)$ as above. Consequently,

$$\begin{aligned} F(t) &= \frac{1}{L^2} + \frac{\beta_1}{L} + p(L)\log L - \sum_{r=1}^{M+1} \tilde{A}_r L^{N-2} (\log L)^r + h(L) + R_{N-2} \\ &= \frac{1}{L^2}\Big[1 + p_N(L) + L^2 p(L)\log L - \sum_{r=1}^{M+1} \tilde{A}_r L^N (\log L)^r + R_N\Big], \end{aligned}$$

where R_k denotes a remainder term in $C^k[0,+\infty)$. On the other hand, from the almost-balanced condition we get as above

$$\begin{aligned} F &\approx \frac{\pi c}{L^2}\Big[1 + \sum_{j=2}^{\infty} b_j L^j\Big]\Big[1 + p_N(L) - 2\sum_{j=1}^{M} a_j L^N (\log L)^j + R_N\Big] \\ &= \frac{\pi c}{L^2}\Big[1 + p_N(L) - 2\sum_{j=1}^{M} a_j L^N (\log L)^j + R_N\Big]. \end{aligned}$$

Comparing the two formulas, we see that we must have ($c = \frac{1}{\pi}$ and) $\tilde{A}_j = 2a_j$ for all $j = 2, \dots, M+1$. For $j = M+1$, this yields $\tilde{A}_{M+1} = 0$, i.e. $A_M = 0$, that is,

$$(2 - N^2 + N)a_M = 0.$$

Since $a_M \neq 0$ by hypothesis, necessarily $0 = (2 - N^2 + N) = (N+1)(2-N)$, so $N = 2$. If $M \geq 2$, then for $j = M$ we get

$$\begin{aligned} 2a_M = \tilde{A}_M &= -\frac{N!}{M}\frac{(-1)^{N-2}}{(N-2)!}A_{M-1} - A_M \Gamma'(N+1)\frac{(-1)^{N-2}}{(N-2)!} + A_M N! c_{N-1,0} \\ &= -\frac{2}{M}A_{M-1} \qquad \text{since } A_M = 0 \text{ and } N = 2 \\ &= -\frac{2}{M}(0a_{M-1} - 3Ma_M) = 6a_M, \end{aligned}$$

or $a_M = 0$, again a contradiction. Thus we must have $M = 1$.

We conclude by doing the computations for the remaining case $N = 2$, $M = 1$ with a bit more care for the terms $p_N(L)$. Namely, (35) becomes (changing the notation a little)

$$u \approx L[1 + a_1 L + a_2 L^2 + bL^2 \log L + O(L^{2+\delta})],$$

with $b \neq 0$. Hence, in turn, by tedious but routine manipulations,

$$\begin{aligned} \log\frac{1}{u} &\approx \log\frac{1}{L} - a_1 L + (\tfrac{1}{2}a_1^2 - a_2)L^2 - bL^2\log L + O(L^{2+\delta}), \\ \partial\bar{\partial}\log\frac{1}{u} &\approx \frac{e^L}{L^2}\Big[1 + (a_1^2 - 2a_2 - 3b)L^2 - 2bL^2\log L + O(L^{2+\delta})\Big], \\ w &\approx e^L\Big[1 + 2a_1 L + (2a_1^2 - 3b)L^2 + O(L^{2+\delta})\Big], \\ c_k &\approx \frac{1}{k}\Big[1 + \frac{2a_1}{k} + \frac{4a_1^2 - 6b}{k^2} + O\Big(\frac{1}{k^{2+\delta}}\Big)\Big], \\ \frac{1}{c_k} &\approx k\Big[1 - \frac{2a_1}{k} + \frac{6b}{k^2} + O\Big(\frac{1}{k^{2+\delta}}\Big)\Big], \end{aligned}$$

and

$$\begin{aligned} F(t) &\approx \frac{t}{(1-t)^2} - \frac{2a_1 t}{1-t} + 6b \log \frac{1}{1-t} + R_0 \\ &= \frac{1}{L^2}\Big[1 - 2a_1 L + (a_1 - \tfrac{1}{12})L^2 - 6bL^2 \log L + R_2\Big], \end{aligned}$$

while, on the other hand,

$$\frac{\pi c}{u^2} \approx \frac{\pi c}{L^2}\Big[1 - 2a_1 L + (3a_1^2 - 2a_2)L^2 - 2bL^2 \log L + R_2\Big].$$

Consequently, ($c = \frac{1}{\pi}$ and) $6b = 2b$, or $b = 0$, a contradiction again. This completes the proof. □

4. Concluding remarks

4.1 Punctured disc. Quite generally, for a radial weight $w(z) = w(|z|)$ on the punctured disc $\mathbf{D}^* := \mathbf{D} \setminus \{0\}$, the Laurent monomials $\{z^k\}_{k \in \mathbf{Z}}$ again form an orthogonal basis in $L^2_{\mathrm{hol}}(\mathbf{D}^*, w)$, with norm squares

$$\|z^k\|^2 = \pi \int_0^1 t^k w(\sqrt{t})\, dt =: \pi c_k, \qquad k \in \mathbf{Z},$$

and the reproducing kernel is given by

$$K(x,y) = \frac{1}{\pi} F(x\overline{y}), \qquad F(t) := \sum_{k=-\infty}^{\infty} \frac{t^k}{c_k}, \qquad 0 < |t| < 1,$$

with t^k/c_k interpreted as zero if $c_k = +\infty$. By general properties of Laurent series, the principal part $\sum_{k=-\infty}^{-1}$ of $F(t)$ converges on all of $\mathbf{C}\setminus\{0\}$, in particular is smooth at $t = 1$; thus the boundary behaviour as $t \nearrow 1$ of $F(t)$ is completely determined by the regular part $\sum_{k=0}^{\infty}$ of F, whose analysis we carried out in the last two sections. We thus immediately obtain the following extensions of Theorems 1 and 3 from the disc to the case of $\mathbf{D}^*$.

Theorem 5. *Let $u(z) = f(|z|^2)$ be a radial function on $\mathbf{D}^*$, with $f \in C^\infty(0,1]$ satisfying $f(1) = 0$, $f'(1) = -1$, that gives rise to a metric almost-balanced at $|z| = 1$, i.e. there exists a constant $c \neq 0$ such that $u(z)^2 K_{J[u]}(z,z) - c$ is smooth up to $|z| = 1$ and vanishes to (at least) second order there. Then f is again of the form* (7) (*and $c = \frac{1}{\pi}$*).

Theorem 6. *Assume $u(z) = f(|z|^2)$ is a smooth radial function on $\mathbf{D}^*$, with asymptotic expansion* (8) *satisfying* (9) *as $|z| \nearrow 1$, that gives rise to a metric almost-balanced at $|z| = 1$ as in the preceding theorem. Then $f \in C^\infty(0,1]$* (*and, hence, the conclusion of the preceding theorem applies*).

We also have an analogue of Corollary 2.

Corollary 7. *There exists no complete radial balanced metric on* $\mathbf{D}^*$ *with* u *real-analytic at* $|z|=1$.

Proof. If u is real-analytic near $|z|=1$, then as before $h\equiv 0$ in (7), hence, as we saw in (22), the metric is given by

$$g(z)=\frac{4v}{t(t^{\sqrt{v}}-t^{-\sqrt{v}})^2},\qquad t=|z|^2;$$

also $w(z)=t^{2a-1}$ and

$$c_k=\int_0^1 t^k w\,dt=\frac{1}{k+2a}\qquad \text{for } k+2a>0.$$

Write $-2a=m+\delta$, with $m\in\mathbf{Z}$ and $0\le\delta<1$. Then $k+2a>0$ is equivalent to $k\ge m+1$, and

$$F(t)=\sum_{k+2a>0}\frac{t^k}{c_k}=\sum_{k=m+1}^{\infty}(k-m-\delta)t^k=t^{m+1}\Big(\frac{1}{(1-t)^2}-\frac{\delta}{1-t}\Big). \tag{41}$$

By the balanced condition, this should be equal to $\pi c/f^2$, where by (7)

$$\frac{1}{f(t)^2}=\frac{4vt^{m+\delta}}{(t^{\sqrt{v}}-t^{-\sqrt{v}})^2}. \tag{42}$$

If $v<0$, say $v=-s^2$ with $s>0$, we have seen that this has poles at $t=e^{-k\pi/s}$, $k=0,1,2,\dots$, whereas (41) is holomorphic on all of $\mathbf{D}^*$. For $v\to 0$, (42) becomes $\frac{1}{f(t)^2}=\frac{t^{m+\delta}}{(\log t)^2}$, so $F=\frac{\pi c}{f^2}$ means that $\frac{\pi c t^{\delta-1}}{(\log t)^2}=\frac{1}{(1-t)^2}-\frac{\delta}{1-t}$ is holomorphic at $t=0$, which is clearly impossible. Finally for $v>0$, say $v=s^2$ with $s>0$, (42) has power series expansion

$$\frac{1}{f^2}=\frac{4vt^{m+\delta+2s}}{(1-t^{2s})^2}=4s^2\sum_{k=0}^{\infty}(k+1)t^{2sk+m+\delta+2s}.$$

Comparing this with $F=\sum_{l=0}^{\infty}(l+1-\delta)t^{m+l+1}$ (by (41)), we see that $F=\frac{\pi c}{f^2}$ is equivalent to, after pulling out a factor of $t^{m+\delta+1}$ from both sides,

$$\sum_{l=0}^{\infty}(l+1-\delta)t^{l-\delta}=4s^2\pi c\sum_{k=0}^{\infty}(k+1)t^{2sk+2s-1}.$$

Since the powers of t increase by 1 on the left-hand side, necessarily $s=\frac12$; and since the powers of t are then integers on the right-hand side, necessarily $\delta=0$; and, finally, looking e.g. at the lowest order terms, $c=\frac{1}{\pi}$. So we conclude that the only radial balanced metrics on $\mathbf{D}^*$ with f real-analytic at $t=1$ arise from

$$f(t)=t^{-\frac{m+1}{2}}(1-t),\qquad m\in\mathbf{Z},$$

and are given by

$$g(z)=\frac{1}{(1-t)^2},\qquad t=|z|^2.$$

However, the last is just the ordinary Poincare metric on the disc, which is not complete on $\mathbf{D}^*$ (the origin is at finite distance from any point of $\mathbf{D}^*$). This completes the proof. □

The last corollary means that either there exists no complete radial balanced metric on $\mathbf{D}^*$, or if it exists, then its boundary behaviour at $|z|=1$ must be more complicated than in (8).

4.2 The annulus. All that has been said in §4.1 applies mutatis mutandis also to the case of the annulus

$$\mathbf{A} \equiv \mathbf{A}_\Lambda := \{z \in \mathbf{C} : \ e^{-\Lambda} < |z| < 1\}, \qquad \Lambda > 0.$$

We thus obtain the following.

Theorem 8. *Let $u(z) = f(|z|^2)$ be a radial function on $\mathbf{A}$, with $f \in C^\infty(e^{-2\Lambda}, 1]$ satisfying $f(1) = 0$, $f'(1) = -1$, that gives rise to a metric almost-balanced at $|z| = 1$, i.e. there exists a constant $c \neq 0$ such that $u(z)^2 K_{J[u]}(z,z) - c$ is smooth up to $|z| = 1$ and vanishes to (at least) second order there. Then f is again of the form (7) (and $c = \frac{1}{\pi}$).*

Proof. The only difference is that the moments c_k are now given by integrals over $(e^{-2\Lambda}, 1)$ instead of $(0,1)$. However, it is well known that the behaviour of such integrals as $k \to +\infty$ depends only on the behaviour of the weight function w at $t = 1$: namely, for any $0 < \delta_1 < \delta_2 < 1$,

$$\int_{1-\delta_2}^{1-\delta_1} t^k w \, dt \leq (\delta_2 - \delta_1)\,(1-\delta_1)^k \sup_{[1-\delta_2, 1-\delta_1]} w$$

is exponentially small as $k \to +\infty$, hence negligible compared to the negative powers of k which were used in the proofs in Sections 2 and 3. Hence all our conclusions from there remain in force, with the same proofs. □

Theorem 9. *Assume $u(z) = f(|z|^2)$ is a smooth radial function on $\mathbf{A}$, with asymptotic expansion (8) satisfying (9) as $|z| \nearrow 1$, that gives rise to a metric almost-balanced at $|z| = 1$ as in the preceding theorem. Then $f \in C^\infty(e^{-2\Lambda}, 1]$ (and, hence, the conclusion of the preceding theorem applies).*

Again, we also have an analogue of Corollaries 2 and 7.

Corollary 10. *There exists no complete radial balanced metric on $\mathbf{A}$ with u real-analytic at $|z| = 1$.*

Proof. As before, the real analyticity implies that

$$f(t) = t^a \frac{t^{-\sqrt{v}} - t^{\sqrt{v}}}{2\sqrt{v}}, \qquad e^{-2\Lambda} < t < 1,$$

and the metric is given by

$$g(z) = \frac{4v}{t(t^{\sqrt{v}} - t^{-\sqrt{v}})^2}, \qquad t = |z|^2 \in (e^{-2\Lambda}, 1).$$

If $v \geq 0$ then, as we have already seen, $g(z)$ actually extends smoothly to all $t \in (0,1)$, hence is not complete at the interior boundary $t = e^{-2\Lambda}$ (the points on the interior boundary circle of $\mathbf{A}$ have finite distance from any point of the annulus). Thus $v < 0$, say $v = -s^2$ with $s > 0$, and then the last formula becomes

$$g(z) = \frac{s^2 e^L}{\sin^2(sL)}, \qquad L := \log \frac{1}{|z|^2} \in (0, 2\Lambda).$$

In order for g to be (smooth and) complete on $\mathbf{A}$, this expression has to (be smooth on $(0,2\Lambda)$ and) blow up at $L=2\Lambda$; thus necessarily $2\Lambda s=\pi$, or $s=\pi/2\Lambda$. Furthermore,

$$\frac{1}{f^2}=\frac{s^2}{t^{2a}}\Big(\frac{2i}{t^{-is}-t^{is}}\Big)^2=\frac{s^2e^{-2aL}}{\sin^2(sL)}, \qquad L=\log\tfrac{1}{t}.$$

On the other hand, by the balanced condition $\frac{\pi c}{f^2}=F$, where $F(t)=\sum_{k\in\mathbf{Z}}\frac{t^k}{c_k}$ is holomorphic on $\mathbf{A}_{2\Lambda}$; in terms of the variable $L=\log\frac{1}{t}$, this means that $F(e^{-L})$ is holomorphic in the strip $0<\operatorname{Re}L<2\Lambda$ and has period $2\pi i$. Hence

$$\frac{e^{-2aL}}{\sin^2(sL)}=\frac{1}{s^2f^2}=\frac{1}{\pi cs^2}F(e^{-L})$$

should be holomorphic in the strip $0<\operatorname{Re}L<2\Lambda$, and have period $2\pi i$. Now taking $L=\Lambda\pm\pi i$, the periodicity condition implies $e^{-2\pi ia}=e^{2\pi ia}$, or $2a=:m$ is an integer. Thus

$$\sin^{-2}(sL)=\frac{e^{mL}}{\pi cs^2}F(e^{-L})$$

should also have period $2\pi i$; that is, $\sin^2$ should have period $2\pi i/s=4\Lambda i$, $\Lambda>0$, which is absurd. The proof is complete. $\square$

Using the flip $z\mapsto e^{-\Lambda}/z$, one can get also the analogous assertions concerning the behaviour as $|z|\searrow e^{-\Lambda}$ (instead of $|z|\nearrow 1$). We omit the details.

The last corollary again means that either there exists no complete radial balanced metric on $\mathbf{A}$, or if it exists, then its boundary behaviour at $|z|=1$ must be more complicated than in (8).

4.3 Examples of balanced metrics. Apart from the prime example of balanced metric, namely $g(z)=(1-|z|^2)^{-2}$ on the disc, corresponding to $u(z)=1-|z|^2$ (with the "balanced constant" $c=\frac{1}{\pi}$), we have already mentioned in the Introduction that other examples are $u(z)=(1-|z|^2)^\alpha$, $\alpha>\frac12$, on $\mathbf{D}$, as well as $u(z)=(1-\|z\|^2)^\alpha$, $\alpha>\frac{n}{n+1}$, on the unit ball $\mathbf{B}^n$ of $\mathbf{C}^n$. For an irreducible bounded symmetric domain Ω in $\mathbf{C}^n$, one gets balanced metrics by taking $u(z)=h(z,z)^\alpha$, with $h(x,y)$ the Jordan triple determinant of Ω and $\alpha>\frac{p-1}{n+1}$, where p is the genus of Ω. In all these cases, the metric is complete and the constant c can be expressed explicitly as a ratio of products of certain Gamma functions. More generally, any invariant metric on a bounded homogeneous domain in $\mathbf{C}^n$ is balanced as soon as $L^2_{\text{hol}}(\Omega,w)\neq\{0\}$; see Loi and Mossa [14]. Balanced metrics on so-called Cartan-Hartogs domains were discussed by Loi and Zedda [15]. The Euclidean metric $g(z)\equiv 1$ on $\mathbf{C}^n$ is balanced, with $u(z)=e^{-\|z\|^2}$ (and likewise $g(z)\equiv a$, $a>0$, with $u(z)=e^{-a\|z\|^2}$); the corresponding spaces L^2_{hol} are the familiar Segal-Bargmann-Fock spaces on $\mathbf{C}^n$. We have also seen in §4.1 that $u(z)=|z|^{-m}(1-|z|^2)$, $m\in\mathbf{Z}$, give rise to non-complete radial balanced metrics on the punctured disc $\mathbf{D}^*$.

A somewhat less known example is given by the function

$$u(z)=e^{-a/|z|^2}, \qquad a>0,$$

on the punctured plane $\mathbf{C}^*=\mathbf{C}\setminus\{0\}$. The corresponding metric is

$$g(z)=\frac{a}{|z|^4}, \qquad a>0,\ z\in\mathbf{C}^*;$$

this is complete at $z = 0$, however is not complete at $z = \infty$. The weight $w = J[u] = u^2 g$ equals $\frac{a}{t^2} e^{-2a/t}$ ($t = |z|^2$), the moments are $c_k = a^k 2^{k-1} \Gamma(1-k)$, $k \leq 0$, and

$$K_w(z, z) = \sum_{k \in \mathbf{Z}} \frac{|z|^{2k}}{\pi c_k} = \frac{2}{\pi} e^{2a/|z|^2} = \frac{2}{\pi u(z)^2},$$

so the metric is balanced with constant $c = 2/\pi$. Of course, using the inversion $z \mapsto 1/z$, this example is just a disguised case of the ordinary Euclidean metric $g \equiv a$ on $\mathbf{C}$, and the Segal-Bargmann-Fock spaces related to it.

4.4 Scalar curvature. Quite generally, for a bounded strictly pseudoconvex domain Ω in $\mathbf{C}^n$ with smooth boundary, denoting for a moment by ρ any smooth function on $\overline{\Omega}$ that is positive on Ω and comparable to $\operatorname{dist}(\cdot, \partial\Omega)$ near $\partial\Omega$ (i.e. a positively signed defining function for Ω), the condition that $u \approx \rho^\alpha$, $\alpha > 0$, can be seen to imply $g \approx \alpha^n \rho^{-n-1} J[\rho]$; and recalling the definition of the Ricci tensor $\operatorname{Ric} = \partial\overline{\partial} \log g$ and the scalar curvature $S = g^{\overline{k}j} \operatorname{Ric}_{j\overline{k}}$ (with $[g^{\overline{k}j}(z)]$ the inverse matrix of the metric tensor $[g_{j\overline{k}}(z)]$), it transpires that

$$S(z) \to \frac{n(n+1)}{\alpha} \qquad \text{as } z \to \partial\Omega;$$

see [8]. Thus the condition $u \asymp \rho^\alpha$, mentioned in the Introduction, bears on the behaviour at $\partial\Omega$ of the scalar curvature S; in particular, the case $\alpha = n + 1$ considered in this paper corresponds to $S \to n$. The reformulation in terms of scalar curvature has the advantage of making sense also when $\partial\Omega$ is no longer assumed to be smooth (and, hence, neither will be $\operatorname{dist}(\cdot, \partial\Omega)$); for instance, for the case of an irreducible bounded symmetric domain $\Omega \subset \mathbf{C}^n$ and $u(z) = h(z, z)^\alpha$ mentioned in §4.3, one can show that $S = pn/\alpha$ (p again denoting the genus of Ω).

4.5 Invariant balanced metrics on punctured plane. We conclude by one last observation concerning balanced metrics on the punctured plane $\mathbf{C}^* := \mathbf{C} \setminus \{0\}$: namely, there exists no balanced metric on $\mathbf{C}^*$ invariant under the homotheties $z \mapsto \delta z$, $\delta \in \mathbf{C}^*$. Indeed, if $\rho(z)\, dz$ is such a metric, then from $\rho(\delta z)\, d(\delta z) = \rho(z)\, dz$ we obtain $|\delta|^2 \rho(\delta z) = \rho(z)$; taking $z = 1$ it follows that $\rho(\delta) = \frac{\rho(1)}{|\delta|^2}$. Thus the metric is given by $g(z) = \frac{2a}{|z|^2}$, with some $a > 0$; this admits $\phi(z) = a(\log|z|^2)^2$ as a potential, whence $u(z) = e^{-a(\log|z|^2)^2}$. The balanced condition $K_{J[u]}(z, z) = c/u(z)^2$ would thus mean that the reproducing kernel of $L^2_{\text{hol}}(\mathbf{C}^*, J[u])$ is given by $K_{J[u]}(z, z) = ce^{2a(\log|z|^2)^2}$, whence by the well-known uniqueness principle [3, Proposition II.4.7]

$$K_{J[u]}(x, y) = ce^{2a(\log(x\overline{y}))^2}.$$

However, the right-hand side is not single-valued on $\mathbf{C}^* \times \mathbf{C}^*$, a contradiction. Thus there are no $\mathbf{C}^*$-invariant balanced metrics on $\mathbf{C}^*$, as asserted.

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