Acta Mathematica Sinica, English Series
Sep., 201x, Vol. x, No. x, pp. 1–16
Published online: August 15, 201x
DOI: 0000000000000000
Http://www.ActaMath.com

Higher Laplace-Beltrami operators on bounded symmetric domains

Miroslav ENGLIŠ
Mathematics Institute, Žitná 25, 11567 Prague 1, Czech Republic
and Mathematics Institute, Na Rybníčku 1, 74601 Opava, Czech Republic
E-mail: *englis@math.cas.cz*

Hao XU
Center of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China
E-mail: *mathxuhao@gmail.com*

Abstract It was conjectured by the first author and Peetre that the higher Laplace-Beltrami operators generate the whole ring of invariant operators on bounded symmetric domains. We give a proof of the conjecture for domains of rank ≤ 6 by using a graph manipulation of Kähler curvature tensor. We also compute higher order terms in the asymptotic expansions of the Bergman kernels and the Berezin transform on bounded symmetric domain.

Keywords higher Laplace-Beltrami operators, bounded symmetric domains, Bergman Kernel

MR(2010) Subject Classification 32M15, 32A25

1 Introduction

Denote by $\mathscr{D}(\Omega)$ the algebra of all (biholomorphically) invariant differential operators on a bounded symmetric domain Ω of rank r. It is well known that $\mathscr{D}(\Omega)$ is a commutative algebra freely generated by r algebraically independent elements. It is an interesting problem to construct a set of generators explicitly. A survey of basic facts of bounded symmetric domains can be found in [1].

The higher Laplace-Beltrami operators were first introduced and studied in [16] and [9].

$$\bar{L}_m f = g^{a_1\bar{b}_1} \cdots g^{a_m\bar{b}_m} f_{/\bar{b}_1\cdots\bar{b}_m a_1\cdots a_m},$$

where $/$ in the subscript denotes covariant differentiation and $g^{a_i\bar{b}_i}$ are contravariant metric tensors. Throughout the paper, we will use the Einstein summation convention that any variable appearing in both upper and lower indices will be summed automatically. Sometimes we may omit $g^{a_1\bar{b}_1} \cdots g^{a_m\bar{b}_m}$ when it causes no confusion. In fact, they considered more general covariant Cauchy-Riemann operators twisted by a nontrivial vector bundles.

The first author and Peetre [9] conjectured that

Conjecture 1.1 (Engliš-Peetre [9]) *On any Hermitian symmetric space Ω, the operators L_m, $m \geq 0$ generate $\mathscr{D}(\Omega)$.*

Received x x, 201x, accepted x x, 201x
M.E. was supported by GA CR grant no. 201/12/G028 and by RVO funding for IC 67985840.

A stronger version is the following:

Conjecture 1.2 (Engliš) *On any Hermitian symmetric space Ω of rank $r \geq 1$, the operators L_{2m-1}, $1 \leq m \leq r$ generate $\mathscr{D}(\Omega)$.*

Conjecture 1.2 were proved by G. Zhang [25] for $r = 2$ and by the first author [7] for $r \leq 3$. For simple Hermitian symmetric spaces of any rank $r \geq 2$, Conjecture 1.2 has been proved by B. Schwarz [19]. In a recent work [17], Sahi and Zhang determined the eigenvalues of Shimura invariant differential operators. Since the higher Laplace-Beltrami operators L_m are sums of Shimura operators, it might be possible to prove Conjecture 1.2 using the Sahi-Zhang result.

The aim of this paper is to explore the algebraic relations of differential operators from the perspective of graph manipulations. As an application, we give a proof of Conjecture 1.2 for any bounded symmetric domains of rank $r \leq 6$ in §2 and indicate a possible general proof by our method. The results of §2 were obtained in Nov. 2012.

In §3, we compute the weight four term of the asymptotic expansion of Bergman kernels and study algebraic relations of curvature tensors, some of these results will be used in §4, where we compute the weight four and five terms of the asymptotic expansion of the Berezin transform. These results extend the previous work of the first author in [7].

2 Higher Laplace-Beltrami operators

First we fix notation and recall the work of [7, 9]. Let $\mathscr{L}$ be the set of all contravariant tensor fields obtained through (partial) contractions of curvature tensors,

$$T^{\beta_1 \cdots \beta_p} = g^{**} \cdots g^{**} R_{****/*\cdots*} \cdots R_{****/*\cdots*}.$$

Here β_i represents either a barred or unbarred index. There is an associated covariant differential operator $\mathrm{Op}(T)$ given by

$$\mathrm{Op}(T) f = T^{\beta_1 \cdots \beta_p} f_{\beta_1 \cdots \beta_p}. \tag{2.1}$$

At the center of normal coordinates, we do not distinguish between contravariant and covariant tensor fields. Denote by $\mathcal{O}p$ the algebra of all $\mathrm{Op}(T)$, $T \in \mathscr{L}$.

Let Ω be an irreducible bounded symmetric domain in $\mathbb{C}^N$ in its Harish-Chandra realization as a circular domain centered at the origin. The domain Ω is classified up to isomorphism by the rank r and the multiplicities a and b. Note that the dimension $N = r(1 + (r-1)a/2 + b)$. It is an important open problem of Yau [24] to characterize those Kähler manifolds that are covered by symmetric domains. Bounded symmetric domains are also natural arenas for the Berezin quantization [3].

Any bounded domain has a natural Kähler metric, the Bergman metric, which is invariant under biholomorphic mappings. Its curvature tensor is defined by (following the sign convention of [7])

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}k\bar{l}} - g^{m\bar{p}} g_{m\bar{j}\bar{l}} g_{i\bar{p}k}. \tag{2.2}$$

and satisfies (cf. [7, §5])

$$R_{i\bar{j}k\bar{l}/\alpha} = 0, \qquad R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = p \cdot g_{i\bar{j}}, \tag{2.3}$$

where $p = 2 + a(r-1) + b$ is the genus of Ω.

Recall the Ricci formula for a covariant tensor field T,

$$T_{\beta_1\ldots\beta_p/i\bar{j}} - T_{\beta_1\ldots\beta_p/\bar{j}i} = \sum_{k=1}^{p} R^{\gamma}_{\beta_k i\bar{j}} T_{\beta_1\ldots\beta_{k-1}\gamma\beta_{k+1}\ldots\beta_p}, \tag{2.4}$$

where $R^{\bar{k}}_{\bar{l}i\bar{j}} = g^{m\bar{k}} R_{m\bar{l}i\bar{j}}$, $R^{k}_{li\bar{j}} = -g^{k\bar{m}} R_{l\bar{m}i\bar{j}}$ and $R^{k}_{\bar{l}i\bar{j}} = R^{\bar{k}}_{li\bar{j}} = 0$.

The Kähler potential $\Phi(x)$ of the Bergman metric may be decomposed into a sum of homogeneous polynomials $\Phi(x) = \sum_{m=1}^{\infty} \Phi_m(x)$ with $\Phi_m(x)$ homogeneous of degree m in both x and $\bar{x}$.

Theorem 2.1 ([7, Prop. 7]) *On a bounded symmetric domain Ω of rank r, the algebra of covariant differential operators $\mathcal{O}p$ coincides with $\mathscr{D}(\Omega)$, which is freely generated by* $\mathrm{Op}(\Phi_1), \ldots, \mathrm{Op}(\Phi_r)$.

Let $\mathrm{Aut}_0(\Omega)$ be the identity component of the automorphism group of Ω and $K \subset \mathrm{Aut}_0(\Omega)$ the stabilizer subgroup of the origin. Under the action of K, the vector space $\mathcal{P}$ of all polynomials in $z \in \mathbb{C}^N$ equipped with the Fock inner product has a decomposition into irreducible subspaces $\mathcal{P} = \oplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}$, where $\mathbf{m}$ ranges over all *signatures*, i.e. r-tuples $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ satisfying $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. The reproducing kernel $K_{\mathbf{m}}(x, y)$ of $\mathcal{P}_{\mathbf{m}}$ are K-invariant polynomials of degree $m_1 + \cdots + m_r$, holomorphic in x and antiholomorphic in y and satisfy the Faraut-Koranyi formula [11]

$$h(x,y)^{-\nu} = \sum_{\mathbf{m}} (\nu)_{\mathbf{m}} K_{\mathbf{m}}(x,y), \quad \text{where } (\nu)_{\mathbf{m}} = \prod_{j=1}^{r} \frac{\Gamma\left(m_j + \nu - \frac{j-1}{2}a\right)}{\Gamma\left(\nu - \frac{j-1}{2}a\right)}, \tag{2.5}$$

for any $\nu \in \mathbb{C}$. Here $h(x, y)$ is the Jordan triple determinant satisfying $\Phi(x) = -\log h(x, x)$. Note that (2.5) encodes many algebraic relations among $K_{\mathbf{m}}(x, y)$.

As discussed in [2], $K_{\mathbf{m}}(x, y)$ defines an invariant differential operator

$$\Delta_{\mathbf{m}} f(x) := K_{\mathbf{m}}(\partial, \partial)(f \circ \phi_x)(0),$$

where ϕ_x the geodesic symmetry interchanging x and the origin. A proof of the following fundamental result can be found in [2, Prop. 2].

Theorem 2.2 *The polynomials $K_{\mathbf{m}}(x, y)$ form a basis of the space of all K-invariant sesqui-holomorphic polynomials on $\mathbb{C}^N \times \mathbb{C}^N$. Consequently, the operators $\Delta_{\mathbf{m}}$ form a basis for the vector space $\mathscr{D}(\Omega)$.*

The following explicit formula relating L_m and L_1 on the unit ball in $\mathbb{C}^N$ was due to Engliš and Peetre [9].

Theorem 2.3 ([9, Thm. 1.1]) *When Ω is the unit ball in $\mathbb{C}^N$, we have*

$$L_m = \prod_{j=1}^{m} (L_1 - (j-1)(j+N-1)). \tag{2.6}$$

Proof By (2.3) and the Ricci formula (2.4), we have

$$\begin{aligned} &L_m f - L_{m-1} L_1 f \\ &= \sum_{j=2}^{m} \left(f_{/\bar{b}_1\cdots\underline{\bar{b}_j a_1}\bar{b}_{j+1}\cdots\bar{b}_m a_2\cdots a_m} - f_{/\bar{b}_1\cdots\bar{b}_{j-1}\underline{a_1\bar{b}_j}\cdots\bar{b}_m a_2\cdots a_m}\right) \end{aligned}$$

$$= -\sum_{j=2}^{m}\left(R_{s\bar b_1 a_1\bar b_j} f_{/\bar s\bar b_2\cdots\bar b_{j-1}\bar b_{j+1}\cdots\bar b_m a_2\cdots a_m} + \sum_{i=2}^{j-1} R_{s\bar b_i a_1\bar b_j} f_{/\bar b_1\cdots\bar b_{i-1}\bar s\bar b_{i+1}\cdots\bar b_{j-1}\bar b_{j+1}\cdots\bar b_m a_2\cdots a_m}\right)$$
$$= -\sum_{j=2}^{m}\big((N+1)L_{m-1}f + 2(j-2)L_{m-1}f\big)$$
$$= -(m-1)(m+N-1)L_{m-1}f.$$

We get (2.6) immediately. Note that in the third equation, we used $R_{s\bar b_i a_1\bar b_j} = g_{s\bar b_i}g_{a_1\bar b_j} + g_{s\bar b_j}g_{a_1\bar b_i}$.

The rest of the section will be devoted to a proof of Engliš' conjecture for rank $r \le 6$.

Theorem 2.4 *Conjecture 1.2 holds for any bounded symmetric domains Ω of rank $r \le 6$.*

First we introduce some terminology. An *admissible graph* $G = (V, E)$ is defined to be a multidigraph (i.e. a directed graph with possible multiedges and loops) such that for each vertex $v \in V(G)$, both the indegree and outdegrees of v are no greater than 2 (i.e. $\deg^-(v) \le 2,\ \deg^+(v) \le 2$).

An *admissible tree* T is an admissible graph such that its underlying undirected graph is a simple tree (i.e. an oriented tree). Denote by $\mathscr{T}_k$ the set of admissible trees with k vertices. Obviously the k-vertex directed path $P_k \in \mathscr{T}_k$.

An admissible graph G canonically defines a covariant differential operator on bounded symmetric domains. This can be seen as follows: The completion $\overline{G}$ of G is a (unique) multidigraph with vertices $V(G) \cup \{\bullet\}$ and edges $E(G) \cup E'$, where E' consists of edges between $\bullet$ and $V(G)$ such that $\deg^-(v) = \deg^+(v) = 2$ for each $v \in V(G)$ in $\overline{G}$. Let $m = \deg^-(\bullet) = \deg^+(\bullet)$. We define a covariant differential operator L_G of order $2m$,

$$L_G f = \prod_{e = a_e b_e \in E(\overline{G})} g^{a_e \bar b_e} \prod_{v \in V(G)} R_{*\bar * * \bar *}\, f_{/b_1\cdots\bar b_m a_1\cdots a_m}, \tag{2.7}$$

where $***\,*$ denote the half-edges attached to v and $b_1\cdots\bar b_m a_1\cdots a_m$ denote all half-edges attached to $\bullet$. In particular, an admissible tree $T \in \mathscr{T}_k$ defines a covariant differential operator L_T of order $2k+2$. For simplicity, we will also use the graph G to denote L_G.

Lemma 2.5 *Let $k \ge 1$. Then*

$$L_{2k+1} - L_{2k}L_1 = \sum_{T \in \mathscr{T}_k} C_T L_T + Q_k, \tag{2.8}$$

where $(-1)^k C_T \ge 0$, $\forall T \in \mathscr{T}_k$ and $C_{P_k} \ne 0$ for directed paths, Q_k is a polynomial of $L_1, \ldots, L_{2k}$ and L_G of order $< 2k+2$ defined by connected admissible graphs G.

We also have

$$L_{2k} - L_{2k-1}L_1 = S_k, \tag{2.9}$$

where S_k is a polynomial of $L_1, \ldots, L_{2k-1}$ and L_G of order $< 2k+2$ defined by connected admissible graphs G.

Proof From the proof of Theorem 2.3, we have

$$L_m f = L_{m-1}L_1 f - (m-1)pL_{m-1}f \tag{2.10}$$

$$- \sum_{j=3}^{m} \sum_{i=2}^{j-1} R_{s\bar{b}_i a_1 \bar{b}_j} f_{/\bar{b}_2 \cdots \bar{b}_{i-1} \bar{b}_{i+1} \cdots \bar{b}_{j-1} \bar{b}_{j+1} \cdots \bar{b}_m \underline{\bar{b}_1 \bar{s}} a_2 \cdots a_{i-1} a_{i+1} \cdots a_{j-1} a_{j+1} \cdots a_m \underline{a_i a_j}}.$$

Next we move a_2 to the right adjacency of $\bar{b}_2$ using the Ricci formula. Repeat the process and note that if G is a disjoint union of connected subgraphs $G = G_1 \cup \cdots \cup G_n$, then

$$L_G = L_{G_1} \cdots L_{G_n} + \text{lower order operators},$$

we can write the summation in the right-hand side of (2.10) as a linear combination of $L_G L_1^j f$ with G connected and $j \geq 0$.

Note that each L_T appearing in the computation of $L_{2k+1} - L_{2k} L_1$ must have the same sign $(-1)^k$. We now check that in (2.8), $C_{P_k} \neq 0$.

$$\begin{aligned} L_{2k+1} f - L_{2k} L_1 f &= -R_{s\bar{b}_{2k} a_1 \bar{b}_{2k+1}} f_{/\bar{b}_1 \cdots \bar{b}_{2k-1} \bar{s} a_2 \cdots a_{2k+1}} + \cdots && (\text{switch } \bar{b}_{2k+1}, a_1) \\ &= R_{s\bar{b}_{2k} a_1 \bar{b}_{2k+1}} R_{t\bar{b}_{2k-1} a_2 \bar{s}} f_{/\bar{b}_1 \cdots \bar{b}_{2k-2} \bar{t} a_3 \cdots a_{2k+1}} + \cdots && (\text{switch } \bar{s}, a_2) \\ &= \cdots = (-1)^k L_{P_k} f + \cdots . \end{aligned}$$

The forms of (2.8) and (2.9) are not difficult to prove in view of the Ricci formula. We omit the details.

Remark 2.6 In [7], the first author computed

$$\begin{aligned} L_2 - L_1^2 &= -pL_1, \\ L_3 - L_2 L_1 &= -L(\circ) - 2pL_1^2 + 2p^2 L_1, \\ L_4 - L_3 L_1 &= -3pL_3 - 3L(\circ)L_1 + 6pL(\circ) + 3L(\circ \xrightarrow{2} \circ), \\ L_5 - L_4 L_1 &= 12L(\circ \xrightarrow{1} \circ) - 4pL_4 - 6L(\circ)L_1^2 + 30pL(\circ)L_1 + 12L(\circ \xrightarrow{2} \circ)L_1 \\ &\quad - 36p^2 L(\circ) - 30pL(\circ \xrightarrow{2} \circ) - 6L(\circ \xrightarrow{2} \circ \xrightarrow{2} \circ), \end{aligned}$$

where $L(\circ) f = R_{i\bar{j}k\bar{l}} f_{/\bar{j}\bar{l}ik} = 4\,\mathrm{Op}(\Phi_2) f$.

Remark 2.7 There are three trees in $\mathscr{T}_3$,

$$\circ \longrightarrow \circ \longrightarrow \circ \qquad \circ \longleftarrow \circ \longrightarrow \circ \qquad \circ \longrightarrow \circ \longleftarrow \circ$$

III_1 $\qquad$ III_2 $\qquad$ III_3

There are six trees in $\mathscr{T}_4$,

$$\circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \qquad \circ \longleftarrow \circ \longrightarrow \circ \longrightarrow \circ \qquad \circ \longrightarrow \circ \longleftarrow \circ \longrightarrow \circ \qquad \circ \longrightarrow \circ \longrightarrow \circ \longleftarrow \circ$$

IV_1 $\qquad$ IV_2 $\qquad$ IV_3 $\qquad$ IV_4

$$\begin{tikzcd} & \circ & \\ \circ \arrow[r] & \circ \arrow[u] \arrow[r] & \circ \end{tikzcd} \qquad \begin{tikzcd} & \circ \arrow[d] & \\ \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

IV_5 $\qquad$ IV_6

By a tedious computation following the procedure as the proof of (2.8), we get

$$\begin{aligned} L_7 - L_6 L_1 &= -180L(\mathrm{III}_1) - 90L(\mathrm{III}_2) - 180L(\mathrm{III}_3) + Q_3, \\ L_9 - L_8 L_1 &= 3360L(\mathrm{IV}_1) + 5040L(\mathrm{IV}_2) + 8400L(\mathrm{IV}_3) + 12320L(\mathrm{IV}_4) \\ &\quad + 1680L(\mathrm{IV}_5) + 2440L(\mathrm{IV}_6) + Q_4. \end{aligned}$$

Now we investigate relations among L_G for admissible graphs G. Let G be an arbitrary multidigraph. Consider the following two graphs,

$$G_1 = \begin{tikzcd} & G \\ \circ \arrow[r] & \circ \arrow[u, "i"'] \end{tikzcd} \qquad\qquad G_2 = \begin{tikzcd} & G \\ \circ & \circ \arrow[u, "i"'] \arrow[l] \end{tikzcd} \tag{2.11}$$

By $R_{i\bar{j}k\bar{l}/\alpha} = 0$ and the Ricci formula, we have

$$0 = R_{i\bar{j}k\bar{l}/p\bar{q}} - R_{i\bar{j}k\bar{l}/\bar{q}p} = -R_{s\bar{j}k\bar{l}}R_{i\bar{s}p\bar{q}} + R_{i\bar{s}k\bar{l}}R_{s\bar{j}p\bar{q}} - R_{i\bar{j}s\bar{l}}R_{k\bar{s}p\bar{q}} + R_{i\bar{j}k\bar{s}}R_{s\bar{l}p\bar{q}}, \tag{2.12}$$

which implies that $L_{G_1} = L_{G_2}$ (abbr. $G_1 = G_2$), or written graphically

$$\circ \longrightarrow \circ \xrightarrow{i} \;=\; \circ \longleftarrow \circ \xrightarrow{i} \tag{2.13}$$

Similarly, we have

$$\circ \longrightarrow \circ \xleftarrow{\bar{i}} \;=\; \circ \longleftarrow \circ \xleftarrow{\bar{i}} \tag{2.14}$$

More identities are collected in the following lemma.

Lemma 2.8 *We have*

$$\begin{tikzcd} & \circ & \\ {} \arrow[r, "\bar{j}"] & \circ \arrow[u] \arrow[r, "i"] & {} \end{tikzcd} = \begin{tikzcd} & \circ \arrow[d] & \\ {} \arrow[r, "\bar{j}"] & \circ \arrow[r, "i"] & {} \end{tikzcd} = 2 \begin{tikzcd} & \circ \arrow[r, "i"] & {} \\ {} \arrow[r, "\bar{j}"] & \circ \arrow[u] & \end{tikzcd} - \begin{tikzcd} & \circ \arrow[r, "i"] \arrow[d] & {} \\ {} \arrow[r, "\bar{j}"] & \circ & \end{tikzcd} \tag{2.15}$$

$$\begin{tikzcd} & \circ & \\ {} \arrow[r, "\bar{i}"] & \circ \arrow[u] & {} \arrow[l, "\bar{j}"'] \end{tikzcd} = \begin{tikzcd} & \circ & {} \arrow[l, "\bar{j}"'] \\ {} \arrow[r, "\bar{i}"] & \circ \arrow[u] & \end{tikzcd} = \begin{tikzcd} & \circ \arrow[d] & {} \arrow[l, "\bar{j}"'] \\ {} \arrow[r, "\bar{i}"] & \circ & \end{tikzcd} \tag{2.16}$$

$$\begin{tikzcd} & \circ \arrow[d] & \\ {} & \circ \arrow[l, "i"] \arrow[r, "j"] & {} \end{tikzcd} = \begin{tikzcd} & \circ \arrow[r, "j"] & {} \\ {} & \circ \arrow[u] \arrow[l, "i"] & \end{tikzcd} = \begin{tikzcd} & \circ \arrow[r, "j"] \arrow[d] & {} \\ {} & \circ \arrow[l, "i"] & \end{tikzcd} \tag{2.17}$$

Proof The first equation of (2.15) follows from (2.12). The second equation of (2.15) follows from

$$0 = R_{i\bar{n}k\bar{l}/m\bar{j}} - R_{i\bar{n}k\bar{l}/\bar{j}m} = -R_{s\bar{n}k\bar{l}}R_{i\bar{s}m\bar{j}} + R_{i\bar{s}k\bar{l}}R_{s\bar{n}m\bar{j}} - R_{i\bar{n}s\bar{l}}R_{k\bar{s}m\bar{j}} + R_{i\bar{n}k\bar{s}}R_{s\bar{l}m\bar{j}}.$$

The remaining identities can be proved similarly.

Lemma 2.9 *Let* $1 \le k \le 5$ *and* $T, T' \in \mathscr{T}_k$. *Then* $L_T = L_{T'}$.

Proof It is trivial when $k = 1$ or 2. Under the notation of Remark 2.7, $\mathrm{III}_1 = \mathrm{III}_2$ and $\mathrm{III}_1 = \mathrm{III}_3$ follow from (2.13) and (2.14) respectively.

It is also not difficult to see that (2.13) and (2.14) imply $\mathrm{IV}_1 = \mathrm{IV}_2$, $\mathrm{IV}_2 = \mathrm{IV}_3$, $\mathrm{IV}_3 = \mathrm{IV}_4$, (2.15) implies $\mathrm{IV}_5 = \mathrm{IV}_6$, and (2.17) implies $\mathrm{IV}_2 = \mathrm{IV}_5$.

The proof when $k = 5$ can be found in the appendix.

Remark 2.10 It would be very interesting to see whether the above lemma is true for higher k. Although it is probably too strong to be true, the validity of Lemma 2.9 for all k would imply Conjecture 1.2 and hence the Engliš-Peetre conjecture.

Recall that a *rooted tree* is an oriented tree with a special vertex, called the root, such that there is a unique directed path from the root to any vertex v, i.e. all edges point away from the root. Given a rooted tree T with i vertices, an *ordered decoration* of T is to attach $i+1$ outward external legs $a_1, \dots, a_{i+1}$ and $i+1$ inward external legs $b_1, \dots, b_{i+1}$ to vertices of T, such that (i) Each vertex of T has exactly two outward half-edges and two inward half-edges; (ii) b_1, b_2 are attached to the root; (iii) If uv is a directed edge of T and $b_{j_1} \in u$, $b_{j_2} \in v$, then $j_1 < j_2$.

Lemma 2.11 *Let $j \geq 2$. then*

$$\mathrm{Op}(\Phi_j) = \frac{1}{(2j)!} \sum_{T \in \mathscr{T}_{j-1}}^{\textit{rooted}} C_T L_T, \tag{2.18}$$

where T runs over all admissible rooted trees with $j-1$ vertices and $C_T > 0$ is equal to the number of ordered decorations of T.

Proof Since

$$\Phi_j(z) = \frac{1}{(2j)!} \sum_{a_i, b_i} g_{a_1 \bar{b}_1 \cdots a_j \bar{b}_j}(0) z_{a_1} \bar{z}_{b_1} \cdots z_{a_j} \bar{z}_{b_j},$$

(2.18) follows readily from [22, Thm. 4.4].

From Theorem 2.2 and Lemmas 2.5, 2.9 and 2.11, we can prove inductively that when $0 \leq k \leq 5$,

$$L_{2k+1} = c_k \,\mathrm{Op}(\Phi_{k+1}) + \text{a polynomial in } \mathrm{Op}(\Phi_1), \dots, \mathrm{Op}(\Phi_k),$$

where $c_k \neq 0$. So $L_1, L_3, \dots, L_{2k+1}$ generate $\mathscr{D}(\Omega)$ on domains of rank $\leq k+1$, which concludes the proof of Theorem 2.4.

3 Bergman kernel of bounded symmetric domains

Let $\Phi(z)$ be the Kähler potential of the Bergman metric on a bounded symmetric domain Ω. Consider the weighted Bergman space of all holomorphic function on Ω square-integrable with respect to the measure $e^{-\alpha\Phi} \frac{w_g^n}{n!}$, $\alpha > 0$. The reproducing kernel $K_\alpha(x, y)$ has an asymptotic expansion [3, 7, 8]

$$K_\alpha(x, y) \sim e^{\alpha\Phi(x,y)} \sum_{k=0}^{\infty} B_k(x, y) \alpha^{n-k}, \quad \alpha \to \infty. \tag{3.1}$$

These asymptotic coefficients have useful geometric implications on Hermitian symmetric spaces [12, 15]. The connection between Bergman kernel and heat kernel was studied in [4].

The coefficients B_j of the asymptotic expansion of the Bergman kernel of Ω satisfy $\sum_{j \geq 0} B_j z^j = \exp(\sum_{j \geq 1} k_j z^j)$, where k_j are given in terms of Bernoulli polynomials $\beta_j(x)$:

$$k_j = \frac{(-1)^{j+1}}{j(j+1)} \sum_{i=1}^{r} \left[\beta_{j+1}\left(-\frac{a}{2}(i-1)\right) - \beta_{j+1}\left(-\frac{N}{r} - \frac{a}{2}(i-1)\right) \right].$$

See [7, §5] for details. In particular,

$$B_4 = \frac{1}{24} k_1^4 + \frac{1}{2} k_1^2 k_2 + \frac{1}{2} k_2^2 + k_1 k_3 + k_4 \tag{3.2}$$

is a polynomial in $\mathbb{Q}[N, a, r, r^{-1}]$ and has 104 terms.

On the other hand, B_j can be computed by a recursive formalism developed by the first author [7] from the asymptotics of Laplace integrals, where B_j, $j \leq 3$ for Ω were explicitly computed. By applying an improved recursive formula [14], the following explicit closed formula of B_k was obtained in [21].

$$B_k(x) = \sum_G z(G) \cdot G = \sum_G \frac{(-1)^n \det(A - I)}{|\mathrm{Aut}(G)|} G, \tag{3.3}$$

where $G = G_1 \cup \cdots \cup G_n$ runs over stable (i.e. both the indegree and outdegree of each vertex are no less than 2) multidigraphs of weight k (i.e. $|E(G)| - |V(G)| = k$) such that each component G_i is strongly connected and A is the adjacency matrix of G. Note that vertices of G represent partial derivatives of metrics.

Below we derive an explicit formula of B_4 by using (3.3). Since all covariant derivatives of $R_{i\bar{j}k\bar{l}}$ vanish, we need only sum over balanced stable graphs. There are 82 weight 4 stable graphs, among which 48 are balanced (see [21, App. B]).

Table 1 in the appendix contains the 25 stable 4-vertex graphs of weight 4 (denoted by s_i, $1 \leq i \leq 25$), together with their coefficients $z(s_i)$ in B_4. Note that some s_i may be simplified into the following Weyl invariants (in the notation of [7])

$$\sigma_7 = R_{i\bar{j}k\bar{l}} R_{j\bar{i}m\bar{n}} R_{l\bar{k}n\bar{m}}, \quad \sigma_{15} = R_{i\bar{j}k\bar{l}} R_{j\bar{m}l\bar{n}} R_{m\bar{i}n\bar{k}}, \quad q = R_{i\bar{j}k\bar{l}} R_{j\bar{i}l\bar{k}} \tag{3.4}$$

using the Kähler-Einstein condition $R_{i\bar{j}} = p \cdot g_{i\bar{j}}$, where p is the genus of Ω.

Table 2 in the appendix contains the 23 stable balanced graphs of weight 4 and less than 4 vertices (denoted by t_i, $1 \leq i \leq 23$), together with their coefficients $z(t_i)$ in B_4. Note that t_i represent Weyl invariants in partial derivatives, i.e. each vertex represents a partial derivative of $g_{i\bar{j}}$ and each edge represents the contraction of a pair of indices.

For the one 1-vertex graph t_1 and seven 2-vertex graphs t_i, $2 \leq i \leq 8$, we may use [22, Thm. 4.4] to get their curvature tensor expressions

$$\begin{aligned}
D(t_1) =& 6s_2 + s_3 + 10s_5 + 15s_6 + 11s_7 + 20s_{11} + 7s_{12} + 8s_{13} \\
& + 8s_{14} + 44s_{15} + 20s_{18} + 14s_{19} + 14s_{20} + 2s_{22}, \\
D(t_2) =& 2s_2 + s_3 + 2s_5 + 2s_6 + 5s_7 + 4s_{11} + 2s_{12}, \\
D(t_3) =& s_2 + 4s_{11} + 4s_{18}, \\
D(t_4) =& 2s_5 + s_6 + s_7 + s_{12} + s_{14} + 4s_{15} + 3s_{19} + 4s_{20} + s_{22}, \\
D(t_5) =& s_5 + s_7 + 2s_{11} + s_{12} + 4s_{15}, \\
D(t_6) =& 7s_9 + s_{10} + 4s_{16} + 4s_{17} + 2s_{24}, \\
D(t_7) =& s_2 + s_5 + 3s_6 + 2s_{14} + 2s_{20}, \\
D(t_8) =& s_1 + 4s_4 + 4s_{21},
\end{aligned}$$

For the fifteen 3-vertex graphs t_i, $9 \leq i \leq 23$, we have

$$\begin{aligned}
D(t_9) &= 2s_6 + s_7, & D(t_{10}) &= 2s_9 + s_{10}, & D(t_{11}) &= s_1 + 2s_4, \\
D(t_{12}) &= s_2 + 2s_{11}, & D(t_{13}) &= s_5 + s_7 + s_{12}, & D(t_{14}) &= s_2 + 2s_6, \\
D(t_{15}) &= s_3 + 2s_7, & D(t_{16}) &= s_7 + 2s_{19}, & D(t_{17}) &= s_{11} + 2s_{15},
\end{aligned}$$

$$
\begin{aligned}
D(t_{18}) &= s_6 + s_{14} + s_{20}, & D(t_{19}) &= s_{12} + 2s_{20}, & D(t_{20}) &= s_9 + s_{17} + s_{24}, \\
D(t_{21}) &= s_4 + 2s_{21}, & D(t_{22}) &= s_5 + s_{20} + s_{22}, & D(t_{23}) &= s_8 + 2s_{23},
\end{aligned}
$$

By (3.3), we get

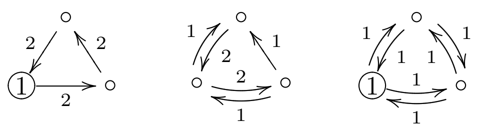
$$
\begin{aligned}
B_4 &= \sum_{i=1}^{23} z(t_i)D(t_i) + \sum_{i=1}^{25} z(s_i)s_i \\
&= \frac{1}{1152}q^2 - \frac{167}{360}s_2 - \frac{167}{960}s_3 - \frac{23}{72}p^2Nq + \frac{23}{60}p^2q + \frac{7}{40}p\sigma_{15} + \frac{1}{192}p^2N^2q - \frac{7}{24}pN\sigma_{15} \\
&\quad - \frac{2}{9}s_{11} + \frac{13}{120}s_{13} - \frac{1}{20}p^4N - \frac{4}{5}p\sigma_7 + \frac{7}{12}pN\sigma_7 + \frac{37}{36}s_{18} + \frac{1}{18}p^4N^2 - \frac{1}{48}p^4N^3 + \frac{1}{384}p^4N^4.
\end{aligned}
\tag{3.5}
$$

For further simplification, we need the following lemma.

Lemma 3.1 *Under the above notations, we have*

$$
2\sigma_7 - \sigma_{15} = pq, \qquad s_{11} = \frac{1}{2}(s_3 + p\sigma_{15}),
$$
$$
s_{13} = \frac{1}{4}s_3 - \frac{1}{2}s_2 + \frac{1}{4}p\sigma_{15} + p\sigma_7, \qquad s_{18} = \frac{1}{2}s_2 + \frac{1}{4}s_3 + \frac{1}{4}p\sigma_{15}.
$$

Proof $2\sigma_7 - \sigma_{15} = pq$ follows from [7, (5.8)]. Consider the following three graphs of weight 4, we will apply the Ricci formula (2.4) to the unique vertex with degree 6 in each of the following graphs:



(3.6)

From the first graph, we have

$$
0 = R_{h\bar{j}i\bar{k}}R_{j\bar{l}k\bar{m}}(R_{l\bar{h}m\bar{n}/n\bar{i}} - R_{l\bar{h}m\bar{n}/\bar{i}n}) = 2s_{11} - s_3 - s_7 = 2s_{11} - s_3 - p\sigma_{15}.
$$

From the second graph, we have

$$
0 = R_{h\bar{j}i\bar{k}}R_{k\bar{m}l\bar{n}}(R_{j\bar{h}m\bar{i}/n\bar{l}} - R_{j\bar{h}m\bar{i}/\bar{l}n}) = -2s_{18} + s_{11} + s_2.
$$

From the third graph, we have

$$
0 = R_{i\bar{h}k\bar{j}}R_{j\bar{k}m\bar{l}}(R_{h\bar{i}l\bar{n}/n\bar{m}} - R_{h\bar{i}l\bar{n}/\bar{m}n}) = s_{18} + s_{13} - s_{11} - s_{15} = s_{18} + s_{13} - s_{11} - p\sigma_7.
$$

They give the last three equations of the lemma.

By substituting equations of Lemma 3.1 into (3.5), we get an explicit formula of B_4 which is summarized in the following theorem.

Theorem 3.2 *Let Ω be an irreducible bounded symmetric domain in $\mathbb{C}^N$. Then*

$$
\begin{aligned}
B_4 = &\left(\frac{1}{384}N^4 - \frac{1}{48}N^3 + \frac{1}{18}N^2 - \frac{1}{20}N\right)p^4 + \frac{1}{192}p^2N^2q - \frac{1}{36}p^2Nq \\
&+ \frac{3}{80}p^2q + \frac{1}{1152}q^2 - \frac{1}{240}s_2 - \frac{1}{960}s_3 + \frac{1}{480}p\sigma_{15},
\end{aligned}
\tag{3.7}
$$

where p is the genus of Ω, *and* $q = R_{i\bar{j}k\bar{l}}R_{j\bar{i}l\bar{k}}$, $\sigma_{15} = R_{i\bar{j}k\bar{l}}R_{j\bar{m}l\bar{n}}R_{m\bar{i}n\bar{k}}$,

$$s_2 = R_{i\bar{j}h\bar{k}}R_{j\bar{i}l\bar{h}}R_{n\bar{m}t\bar{l}}R_{k\bar{n}m\bar{t}}, \quad s_3 = R_{i\bar{n}h\bar{t}}R_{j\bar{i}k\bar{h}}R_{l\bar{j}m\bar{k}}R_{n\bar{l}t\bar{m}}.$$

Remark 3.3 In fact, from our proof, the formula (3.7) holds for any Kähler metric in an open subset of $\mathbb{C}^N$ satisfying (2.3).

Let us check (3.7) when Ω is the unit ball in $\mathbb{C}^N$. In this case, we have

$$r = 1, \quad a = 2, \quad b = N-1, \quad p = N+1.$$

Since the Bergman metric of the unit ball has constant curvature, it is not difficult to get (cf. [7, 21])

$$q = 2N^2 + 2N, \quad \sigma_{15} = 4N^2 + 4N,$$
$$s_2 = 4N^3 + 8N^2 + 4N, \quad s_3 = 8N^2 + 8N.$$

From (3.7), we get

$$B_4 = \frac{1}{384}N^8 - \frac{1}{96}N^7 - \frac{1}{576}N^6 + \frac{1}{30}N^5 - \frac{5}{1152}N^4 - \frac{1}{32}N^3 + \frac{1}{288}N^2 + \frac{1}{120}N,$$

which agrees with that computed by (3.2).

Corollary 3.4 *Let* Ω *be an irreducible bounded symmetric domain in* $\mathbb{C}^N$. *Then*

$$\begin{aligned} 4s_2 + s_3 - 2p\sigma_{15} &= \frac{N(r-1)(r-2)a^4}{2} - \frac{3N(r-1)^2(N+r)a^3}{r} \\ &+ \frac{2N(r-1)(N+r)(2Nr - N + r^2 - 5r)a^2}{r^2} + \frac{12N(r-1)(N+r)^2 a}{r^2} + \frac{8N(N+2r)(N+r)}{r^2}. \end{aligned} \tag{3.8}$$

Proof The genus $p = 1 + \frac{N}{r} + \frac{(r-1)a}{2}$ and it was proved in [7, p.30] that

$$q = -\frac{(r-1)Na^2}{2} + \frac{(r-1)N(N+r)a}{r} + \frac{2N(N+r)}{r}.$$

So (3.8) follows readily from (3.2) and (3.7).

4 Berezin transform of bounded symmetric domains

On a bounded symmetric domain Ω, the *Berezin transform* is the integral operator

$$I_\alpha f(x) = \int_\Omega f(y) \frac{|K_\alpha(x,y)|^2}{K_\alpha(x,x)} e^{-\alpha\Phi(y)} \frac{w_g^n(y)}{n!}. \tag{4.1}$$

At any point for which $K_\alpha(x,x)$ invertible, the integral converges for any bounded measurable function f on Ω. Note that (3.1) implies that for any x, $K_\alpha(x,x) \neq 0$ if α is large enough.

The Berezin transform has an asymptotic expansion

$$I_\alpha f(x) = \sum_{k=0}^\infty Q_k f(x) \alpha^{-k}, \quad \alpha \to \infty, \tag{4.2}$$

where Q_k are linear differential operators and $Q_0 = Id$, $Q_1 = L_1$. It first appeared in the work of Berezin [3] in the quantization of Kähler manifolds. The convergence in various contexts has been extensively studied [5, 10, 13, 20], as well as relations to the star products [13, 18].

For a bounded symmetric domain Ω of rank r, is has been proved that $\mathscr{D}(\Omega)$ is freely generated by $Q_1, Q_3, \ldots, Q_{2r-1}$ [6, Thm. 1.1]. The first author [7] also proved a recursive formula of Q_k and computed Q_k, $k \leq 3$,

$$Q_2 = \frac{1}{2}L_1^2 + \frac{p}{2}L_1, \qquad Q_3 = \frac{1}{6}L_1^3 + \frac{p}{2}L_1^2 + \frac{p^2}{3}L_1 + \frac{1}{12}L(\circ). \tag{4.3}$$

The following closed formula of Q_k was proved in [22],

$$Q_k = \sum_{\Gamma=(V\cup\{\bullet\},E)}^{\text{strong}} \frac{\det(A(\Gamma_-) - I)}{|\mathrm{Aut}(\Gamma)|}\Gamma, \tag{4.4}$$

where Γ runs over all strongly connected graphs with a distinguished vertex $\bullet$ of weight k (i.e. $|E| - |V| = k$) and Γ_- is obtained from Γ by removing the distinguished vertex $\bullet$ from Γ. Note that vertices of Γ represent partial derivatives of metrics or the function. Effective methods of converting partial derivatives of metrics (functions) to covariant derivatives of curvature tensors (functions) on Kähler manifolds were developed in [22, 23], which made possible the computations of more terms of Q_k.

Theorem 4.1 *On a bounded symmetric domain Ω of genus p, we have*

$$Q_4 = -\frac{1}{12}L_3L_1 - \frac{1}{8}pL_3 + \frac{1}{8}L_1^4 + \frac{1}{8}pL_1^3 + \frac{1}{4}p^2L_1^2 + \frac{1}{2}p^3L_1, \tag{4.5}$$

$$\begin{aligned} Q_5 = {} & \frac{1}{720}L_5 - \frac{1}{144}L_4L_1 + \frac{1}{36}pL_4 - \frac{1}{36}L_3L_1^2 - \frac{19}{72}pL_3L_1 + \frac{1}{24}L_1^5 \\ & + \frac{5}{24}pL_1^4 - \frac{1}{4}p^2L_1^3 + \frac{2}{3}p^3L_1^2 + \frac{1}{3}p^4L_1 + \frac{1}{48}L(\circ \xrightarrow{2} \circ \xrightarrow{2} \circ). \end{aligned} \tag{4.6}$$

Proof By a lengthy calculation using (4.4) and the algorithm described in the proof of Lemma 2.5, we get

$$Q_4 = \frac{1}{24}L_4 + \frac{1}{2}pL_3 + \frac{3}{2}p^2L_2 + p^3L_1 + \frac{1}{4}L(\circ)L_1 + \frac{1}{4}pL(\circ) - \frac{1}{8}L(\circ \xrightarrow{2} \circ) + W, \tag{4.7}$$

where W is the sum of four differential operators of order 2,

$$W = \frac{3}{2}L\left[\begin{tikzcd} & \circ \arrow[dl, bend left, "1"] \arrow[dr, bend left, "1"] & \\ \circ \arrow[ur, bend left, "1"] \arrow[rr, "1"'] & & \circ \arrow[ul, bend left, "1"] \end{tikzcd}\right] - \frac{1}{12}L\left[\begin{tikzcd} & \circ \arrow[dr, "1"] & \\ \circ \arrow[ur, "1"] \arrow[rr, bend left, "1"] & & \circ \arrow[ll, bend left, "2"] \end{tikzcd}\right] - \frac{3}{4}\left[\begin{tikzcd} & \circ \arrow[dr, "2"] & \\ \circ \arrow[ur, "2"] & & \circ \arrow[ll, "1"] \end{tikzcd}\right] - \frac{2}{3}pL\left[\begin{tikzcd} \circ \arrow[r, bend left, "1"] & \circ \arrow[l, bend left, "2"] \end{tikzcd}\right].$$

Using the notation of (3.4) and Lemma 3.1, we get

$$W = \left(\frac{3}{2}\sigma_7 - \frac{1}{12}pq - \frac{3}{4}\sigma_{15} - \frac{2}{3}pq\right)L_1 = 0.$$

So (4.5) follows from the formulas of L_3 and L_4 in Remark 2.6.

The formula of Q_5 needs more work. Besides Lemma 3.1, we also need the following Lemma 4.2. Drastic simplifications occur in the computation. We omit the details.

Lemma 4.2 *We have the following equations among differential operators of order* 4.

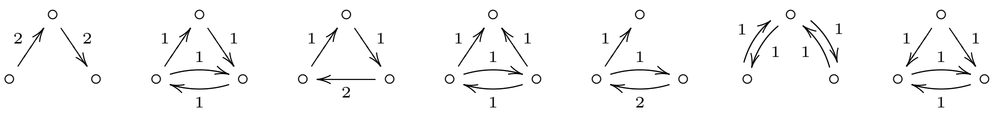
$$L(\circ \xrightarrow{2} \circ) = 2L\left[\begin{tikzcd} \circ \arrow[r, bend left, "1"] & \circ \arrow[l, bend left, "1"] \end{tikzcd}\right] - pL(\circ), \tag{4.8}$$

$$G_3 = G_4 = G_7 = \frac{1}{2}G_1 + \frac{1}{2}pL(\circ \xrightarrow{2} \circ), \tag{4.9}$$

$$2G_2 = G_3 + G_5, \tag{4.10}$$

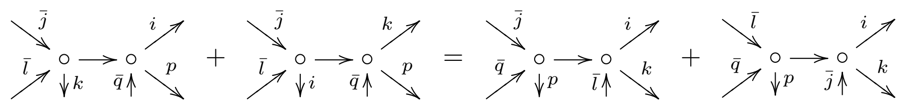
$$G_2 + G_6 = G_3 + pL\left[\circ \overset{1}{\underset{1}{\rightleftarrows}} \circ\right]. \tag{4.11}$$

where we use G to denote L_G and the corresponding graphs are



G_1 G_2 G_3 G_4 G_5 G_6 G_7

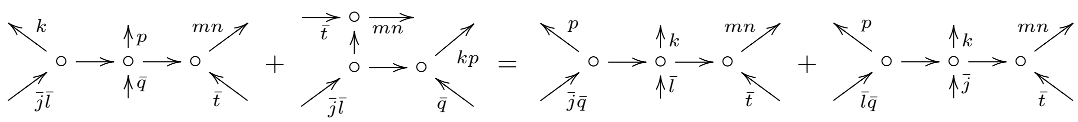
Proof Note that (2.12) may be written graphically as



(4.12)

Gluing the two external legs $p\bar{j}$ in the above equation, we get (4.8).

Attaching a new vertex to the external leg i in (4.12), we get



Gluing $m\bar{q}$ and $n\bar{j}$ gives $G_3 = G_4$. Gluing $m\bar{q}$ and $k\bar{t}$ gives $G_7 = G_3$. Gluing $k\bar{q}$ and $p\bar{t}$ gives $2G_7 = G_1 + pL(\circ \xrightarrow{2} \circ)$. Then (4.9) follows from these three equations. Finally, (4.10) follows from gluing $m\bar{j}$ and $n\bar{l}$. (4.11) follows from gluing $m\bar{q}$ and $p\bar{j}$.

In fact, (4.8) is equivalent to $L_3L_1 = L_1L_3$. When Ω is the unit ball in $\mathbb{C}^N$, (4.5) and (4.6) becomes

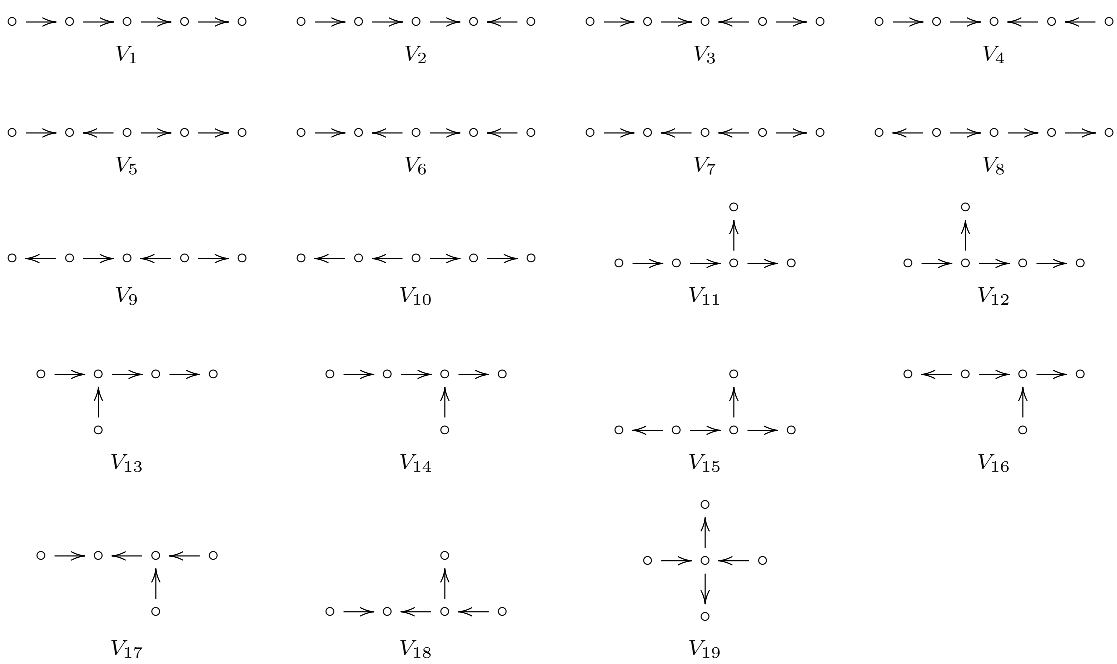
$$\begin{aligned}
Q_4 &= \frac{1}{24}[L_1^4 + (6N+10)L_1^3 + (11N^2+24N+13)L_1^2 + (6N^3+12N^2+6N)L_1], \\
Q_5 &= \frac{1}{120}[L_1^5 + (10N+20)L_1^4 + (35N^2+100N+73)L_1^3 \\
&\quad + (50N^3+146N^2+130N+30)L_1^2 + (24N^4+60N^3+40N^2-4)L_1],
\end{aligned}$$

which agree with that computed in [9, p. 53].

APPENDIX

1 Admissible trees with five vertices

In this appendix, we prove the case $k=5$ of Lemma 2.9. There are 19 admissible trees in $\mathscr{T}_5$.

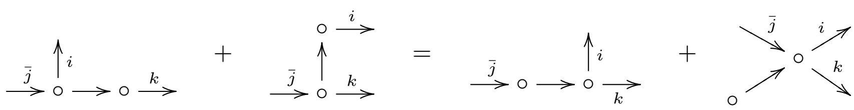


It is not difficult to verify that (2.13) and (2.14) imply $V_1 = V_2$, $V_1 = V_8$, $V_5 = V_{10}$, $V_2 = V_7$, $V_3 = V_4$, $V_3 = V_9$, $V_5 = V_6$, $V_{11} = V_{15}$, $V_{12} = V_{18}$, $V_{14} = V_{16}$, $V_{13} = V_{17}$.

Moreover, (2.15) implies $V_{12} = V_{13}$, $V_{11} = V_{14}$; (2.16) implies $V_{16} = V_7$, $V_{16} = V_3$; (2.17) implies $V_{12} = V_{10}$, $V_{12} = V_8$.

The above 17 equations imply that V_i, $1 \leq i \leq 18$ are all equal.

Lemma 1.3 *Under the convention of Lemma 2.8, we have*



(A1)

Proof The equation comes from switching p and $\bar{q}$ in $R_{i\bar{j}k\bar{l}/p\bar{q}}$ using the Ricci formula. The argument is similar to that of Lemma 3.1.

It is easy to see that (A1) implies $2V_{12} = V_{11} + V_{19}$. So we get $V_{19} = V_{11}$, which concludes the proof of the case $k = 5$ of Lemma 2.9.

Table 1 Stable 4-vertex graphs of weight 4

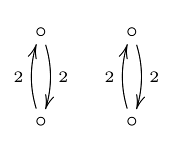
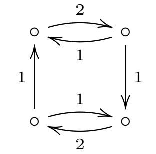
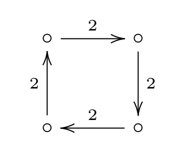
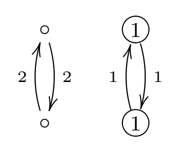
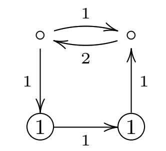
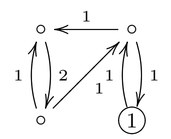
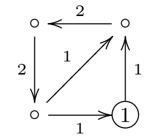
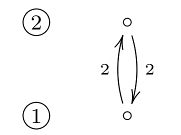
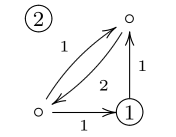
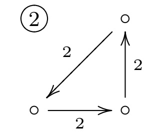
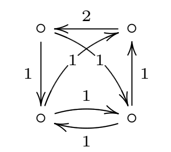
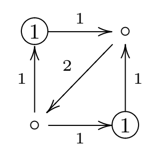
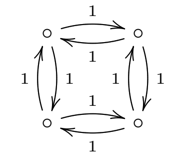
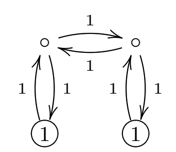
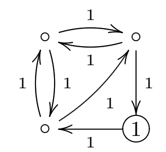
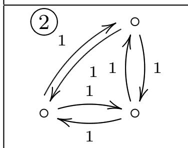
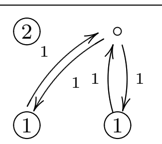
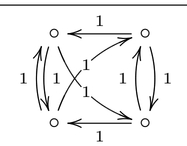
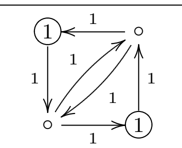
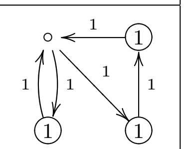
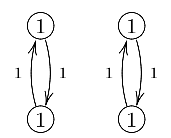
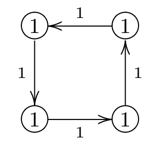
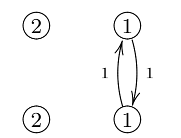
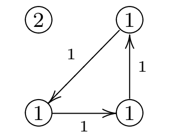
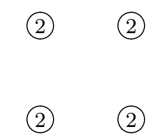
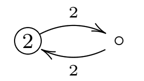
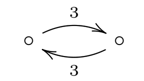
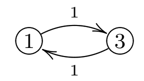
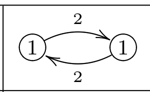
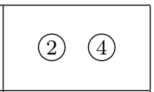
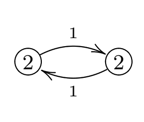
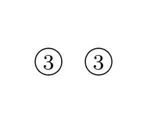
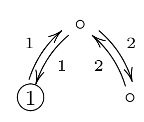
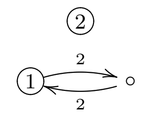
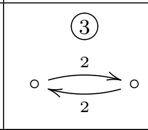
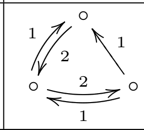
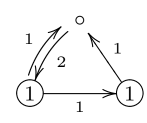
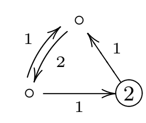
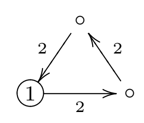
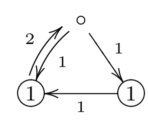
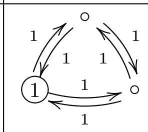
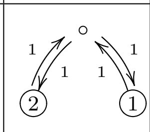
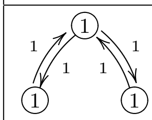
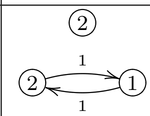
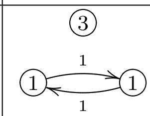
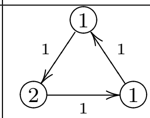
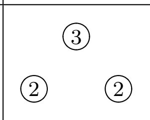
				
$s_1 = q^2$	s_2	s_3	$s_4 = p^2Nq$	$s_5 = p^2q$
9/128	3/8	15/64	3/16	1
				
$s_6 = p^2q$	$s_7 = p\sigma_{15}$	$s_8 = p^2N^2q$	$s_9 = p^2Nq$	$s_{10} = pN\sigma_{15}$
$-1/2$	1	3/64	1/2	7/48
				
s_{11}	$s_{12} = p^2q$	s_{13}	$s_{14} = p^4N$	$s_{15} = p\sigma_7$
2	0	3/8	$-1/2$	2
				
$s_{16} = pN\sigma_7$	$s_{17} = p^4N^2$	s_{18}	$s_{19} = p^2q$	$s_{20} = p^4N$
1/3	0	5/4	1/2	0
				
$s_{21} = p^4N^2$	$s_{22} = p^4N$	$s_{23} = p^4N^3$	$s_{24} = p^4N^2$	$s_{25} = p^4N^4$
1/8	1/4	1/16	1/6	1/384

Table 2 Stable balanced graphs of weight 4 and less than 4 vertices

					
t_1	t_2	t_3	t_4	t_5	t_6
$-1/30$	$5/8$	$1/9$	$1/6$	$1/2$	$1/16$
					
t_7	t_8	t_9	t_{10}	t_{11}	t_{12}
0	$1/18$	$-1/4$	$-1/2$	$-1/8$	$-7/4$
					
t_{13}	t_{14}	t_{15}	t_{16}	t_{17}	t_{18}
-1	$-1/4$	-1	-1	-2	$1/2$
					
t_{19}	t_{20}	t_{21}	t_{22}	t_{23}	
0	$-1/4$	$-1/6$	$-1/2$	$-1/24$	

Acknowledgements We thank the referees for very helpful comments and suggestions on the Conjecture 1.2.

References

[1] J. Arazy, *A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domainss*, Multivariable operator theory (R.E. Curto, R.G. Douglas, J.D. Pincus, N. Salinas, eds.), Contemporary Mathematics, vol. 185, Amer. Math. Soe., Providence, 1995, pp. 7–65.

[2] J. Arazy and M. Engliš, *Q_p-spaces on bounded symmetric domains*, J. Funct. Spaces Appl. **6** (2008), no. 3, 205–240.

[3] F. A. Berezin, *Quantization*, Math. USSR Izvest. **8** (1974), 1109–1163.

[4] X. Dai, K. Liu and X. Ma, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom. **72** (2006), no. 1, 1–41.

[5] M. Engliš, *Berezin quantization and reproducing kernels on complex domains*, Trans. Amer. Math. Soc. **348** (1996), 411–479.

[6] M. Engliš, *Invariant operators and the Berezin transform on Cartan domains*, Math. Nachr. **195** (1998), 61–75.

[7] M. Engliš, *The asymptotics of a Laplace integral on a Kähler manifold*, J. Reine Angew. Math. **528** (2000), 1–39.

[8] M. Engliš, *A Forelli-Rudin construction and asymptotics of weighted Bergman kernels*, J. Funct. Anal. **177** (2000), 257–281.

[9] M. Engliš and J. Peetre, *Covariant Cauchy-Riemann operators and higher Laplacians on Kähler manifolds*, J. Reine Angew. Math. **5478** (1996), 17–56.

[10] M. Engliš and J. Peetre, *On the correspondence principle for the quantized annulus*, Math. Scand. **78** (1996), 183–206.

[11] J. Faraut and A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), 64–89.

[12] Z. Feng and Z. Tu, *Balanced metrics on some Hartogs type domains over bounded symmetric domains*, Ann. Glob. Anal. Geom. **47** (2015), 305–333.

[13] A. Karabegov and M. Schlichenmaier, *Identification of Berezin-Toeplitz deformation quantization*, J. reine angew. Math. **540** (2001), 49–76.

[14] A. Loi, *The Tian-Yau-Zelditch asymptotic expansion for real analytic Kähler metrics*, Int. J. Geom. Methods in Modern Phys. **1** (2004), 253–263.

[15] A. Loi and M. Zuddas, *On the coefficients of TYZ expansion of locally Hermitian symmetric spaces*, Manuscripta Math. **148** (2015), 303–315.

[16] J. Peetre and. G. Zhang, *Harmonic analysis on the quantized Riemann sphere*, Internat. J. Math. Math. Sci. **16** (1993), no. 2, 225–243.

[17] S. Sahi and G. Zhang, *Positivity of Shimura operators*, arXiv:1606.05276.

[18] M. Schlichenmaier, *Berezin-Toeplitz quantization and naturally defined star products for K̈hler manifolds*, Anal. Math. Phys. (2018), 1–20.

[19] B. Schwarz, *Higher Laplacians on pseudo-Hermitian symmetric spaces*, arXiv:1410.3807.

[20] A. Unterberger and H. Upmeier, *The Berezin transform and invariant differential operators*, Comm. Math. Phys. **164** (1994), 563–597.

[21] H. Xu, *A closed formula for the asymptotic expansion of the Bergman kernel*, Comm. Math. Phys. **314** (2012), 555–585.

[22] H. Xu, *Bergman kernel and Kähler tensor calculus*, Pure Appl. Math. Q. (to appear).

[23] H. Xu and S.-T. Yau, *Trees and tensors on Kähler manifolds*, Ann. Global Anal. Geom. (to appear).

[24] S.-T. Yau, *Uniformization of geometric structures*, Proc. Symp. Pure Math. 48 (1988), 265–274.

[25] G. Zhang, *Invariant differential operators on Hermitian symmetric spaces and their eigenvalues*, Israel J. Math. **119** (2000), 157–185.