

RADIAL BALANCED METRICS ON THE UNIT BALL OF THE KEPLER MANIFOLD

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ABSTRACT. We show that there is no radial balanced metric on the unit ball of the Kepler manifold with not too wild boundary behavior. Additionally, we identify explicitly the weights corresponding to radial metrics with such boundary behavior which satisfy the balanced condition as far as germs at the boundary are concerned. Related results for Poincaré metrics are also established.

1. INTRODUCTION

Let Ω be a domain in \mathbf{C}^n , $n \geq 1$, or on an n -dimensional complex manifold, and Φ a strictly plurisubharmonic function on Ω with the associated Kähler form $\omega = \frac{i}{2} \partial \bar{\partial} \Phi$ and volume element $\wedge^n \omega$. The weighted Bergman space $L_{\text{hol}}^2(\Omega, e^{-(n+1)\Phi} \wedge^n \omega)$ of all holomorphic functions on Ω square integrable with respect to $e^{-(n+1)\Phi} \wedge^n \omega$ is well known to have bounded point evaluations and hence possesses a reproducing kernel $K_{e^{-(n+1)\Phi} \wedge^n \omega}(x, y)$ (a weighted Bergman kernel). The Kähler metric associated to ω — or, abusing terminology, the function $u := e^{-\Phi}$ — is called *balanced* if

$$(1) \quad K_{e^{-(n+1)\Phi} \wedge^n \omega}(z, z) = ce^{(n+1)\Phi(z)} \quad \forall z \in \Omega,$$

that is,

$$(2) \quad K_{u^{n+1} \wedge^n (\frac{i}{2} \partial \bar{\partial} \log \frac{1}{u})}(z, z) = \frac{c}{u(z)^{n+1}} \quad \forall z \in \Omega$$

for some constant c . (One can check that this condition indeed depends only on the Kähler form ω , not on its potential Φ or, equivalently, on $u = e^{-\Phi}$.)

The notion extends in an obvious way also to the more general setting of functions replaced by sections of line bundles: namely, if \mathcal{L} is a holomorphic Hermitian line bundle over Ω with Kähler connection ∇ such that $\text{curv } \nabla = \omega$, let $L_{\text{hol}}^2(\mathcal{L}^*, \wedge^n \omega)$ be the Bergman space of all square-integrable holomorphic sections of its dual bundle \mathcal{L}^* , and for any orthonormal basis $\{s_j\}$ of this space, set

$$(3) \quad \epsilon(x) := \sum_j \|s_j(x)\|_x^2.$$

One can again show that $\epsilon(x)$ does not depend on the choice of the orthonormal basis $\{s_j\}$ and also does not depend on the line bundle \mathcal{L} but only on the Kähler form ω . The Kähler form, or the associated Kähler metric, is called *balanced* if

$$(4) \quad \epsilon \equiv \text{const.}$$

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When \mathcal{L} is trivial, its sections can be identified with functions on Ω , and one recovers the situation from the previous paragraph.

The function ϵ and the condition (4) have appeared in the literature under different names, cf. Rawnsley [20], Cahen, Gutt and Rawnsley [5], Kempf [14] and Ji [13], or Zhang [22]; the term *balanced* was first used by Donaldson [7], who also established the existence of such metrics on any (compact) projective Kähler manifold with constant scalar curvature. Subsequent studies of the existence and uniqueness of balanced metrics in the compact case include Seyyedali [21], Li [15], and others; see also Phong and Sturm [19] for an overview.

However, despite the extensive studies of the compact case, much less seems to be known concerning existence and uniqueness of balanced metrics in the noncompact setting of domains in \mathbf{C}^n or on complex manifolds. Beside the simplest example, which is the Bergman metric on the unit ball \mathbf{B}^n of \mathbf{C}^n , corresponding to

$$(5) \quad u(z) = (1 - |z|^2)^\alpha, \quad \alpha > \frac{n}{n+1},$$

the only known examples of balanced metrics are the (appropriate multiples of the) Bergman metrics on bounded symmetric domains in \mathbf{C}^n , or, more generally, of invariant metrics on bounded homogeneous domains; and the flat (Euclidean) metric on \mathbf{C}^n (with $\Phi(z) = |z|^2$). Miscellaneous partial results concerning uniqueness and/or existence of balanced metrics on domains are due to Loi and Zedda [16], Cuccu and Loi [6], Greco and Loi [12], Arezzo and Loi [1], or the present authors [4]. It is a conjecture of the author's [8] that for $\Omega \subset \mathbf{C}^n$ bounded strictly pseudoconvex with smooth boundary and any $\alpha > n$, there exists a unique balanced metric on Ω with $u^{(n+1)/\alpha}$ vanishing precisely to the first order at the boundary $\partial\Omega$, i.e. $u(z) \asymp \text{dist}(z, \partial\Omega)^{\alpha/(n+1)}$. For $\alpha = n + 1$ and u radial on the unit disc $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, i.e. $u(z) = f(|z|^2)$ for some $f \in C^\infty[0, 1)$, this problem was considered in [10], where it was shown that among all f with sufficiently nice boundary behavior, the only one giving rise to a balanced metric is (5), i.e. $f(t) = 1 - t$.

In this paper, we consider the above problem, again with $\alpha = n + 1$, in the setting of the unit disc replaced by the unit ball $\mathbf{M} = \mathbf{M}^n := \{z \in \mathbf{H} : |z| < 1\}$ of the Kepler manifold

$$(6) \quad \mathbf{H} = \mathbf{H}^n := \{z \in \mathbf{C}^{n+1} : z \cdot z = 0, z \neq 0\}, \quad n \geq 2$$

(here and throughout, $z \cdot w := \sum_j z_j w_j$). The latter is an n -dimensional complex submanifold in \mathbf{C}^{n+1} , which can be identified as a symplectic manifold with the cotangent bundle (minus its zero section) of the unit sphere $\mathbf{S}^n \subset \mathbf{R}^{n+1}$. The origin is a removable singularity for \mathbf{H} , i.e. $\mathbf{H} \cup \{0\}$ is a normal complex analytic space, and in fact is the simplest example of Jordan-Kepler varieties [11] which generalize the classical determinantal varieties. If the conjecture in the previous paragraph is valid, any balanced metric on \mathbf{M} has to be rotation-invariant, hence we will again be looking for $u = e^{-\Phi}$ in the form $u(z) = f(|z|^2)$ for some $f \in C^\infty[0, 1)$ which vanishes precisely to the first order at 1, i.e. $f(1^-) = 0$ and $f'(1^-) \neq 0$; replacing f by a suitable multiple thereof, we may assume that $f'(1^-) = -1$. As in [10] for the disc, our strategy will be to look at the boundary behavior of both sides of (2). Our main results are the following; for simplicity they are formulated for the simplest case $n = 2$, but the methods carry over to general n .

Theorem 1. *Assume that $f \in C^\infty(0, 1]$ satisfies $f(1) = 0$, $f'(1) = -1$ and there is a constant $c \neq 0$ such that*

$$(7) \quad \frac{c}{u(z)^3} - K_{u^3 \wedge^2 (\frac{i}{2} \partial \bar{\partial} \log \frac{1}{u})}(z, z), \quad u(z) := f(|z|^2),$$

is smooth on \mathbf{M}^2 up to $|z| = 1$. Then

$$(8) \quad tf'(ff' + tff'' - tf'^2) = \phi_v + h,$$

where $h \in C^\infty(0, 1]$ satisfies $h^{(k)}(1) = 0 \forall k$, and

$$(9) \quad \begin{aligned} \phi_v(t) &= e^{L/4} \left(\cosh \frac{L\sqrt{v}}{4} - \frac{1}{\sqrt{v}} \sinh \frac{L\sqrt{v}}{4} \right), \quad L := \log \frac{1}{t}, \\ &= \frac{(1 + \sqrt{v})t^{(-1+\sqrt{v})/4} - (1 - \sqrt{v})t^{(-1-\sqrt{v})/4}}{2\sqrt{v}} \end{aligned}$$

for some $v \in \mathbf{R}$.

Here for $v = 0$, (9) is to be interpreted as the limit $v \rightarrow 0$, i.e. as $e^{L/4}(1 - \frac{L}{4}) = t^{-1/4}(1 + \frac{1}{4} \log t)$. Note also that the right-hand side of (9) remains unchanged upon replacing \sqrt{v} by $-\sqrt{v}$, so there is no ambiguity connected with the choice of the square root \sqrt{v} .

Finding balanced metrics on \mathbf{M} thus reduces to looking for solutions of the differential equation (8).

Corollary 2. *If $u(z) = f(|z|^2)$ on \mathbf{M}^2 with $f(1) = 0$, $f'(1) = -1$ and f real-analytic at 1, then the balanced condition (2) is never satisfied. That is, there exists no balanced u which would be real-analytic up to the boundary $|z| = 1$.*

The last corollary is in contrast with the situation for the disc, where $f(t) = 1 - t$ gives a balanced metric. We also remark that for the disc in [10], we were able to identify explicitly the functions satisfying (7); this we are unable to offer here, though some observations are presented in Section 5.2 below. This also explains why a different method is needed here than in [10]: we first identify the weights $u^3 \wedge^2 (\frac{i}{2} \partial \bar{\partial} \log \frac{1}{u})$, which actually turn out to be given by the formula (8), and then are able to draw conclusions even without explicit knowledge of the solutions.

An analogue of Theorem 1 can be proved also for the disc, yielding another (though not much simpler) proof of Theorem 1 in [10]. (We pause to note that in [10], the condition that $u(z)^{n+1} K_{u^{n+1} \wedge^n (\frac{i}{2} \partial \bar{\partial} \log \frac{1}{u})}(z, z) - c$ be smooth up to the boundary and vanish to second order there was used instead of the smoothness of (7) up to the boundary; our Theorem 1 remains in force also for this modification, with the same proof.)

Finally, similarly as for the disc, the hypothesis of the smoothness of f at $t = 1$ in Theorem 1 can be weakened considerably: writing temporarily for brevity $r(z) := \text{dist}(z, \partial\Omega)$, assume that $u \in C^\infty(\Omega)$ has an asymptotic expansion at $\partial\Omega$ of the form

$$(10) \quad u(z) \approx r(z) \sum_{k=0}^{\infty} \sum_{j=0}^{M_k} a_{kj}(z) r(z)^k (\log r(z))^j,$$

with some nonnegative integers M_k and functions $a_{kj} \in C^\infty(\bar{\Omega})$, where

$$(11) \quad M_0 = 0 \quad \text{and} \quad a_{00} = 1 \text{ on } \partial\Omega.$$

Here (10) means that u differs from the partial sum $\sum_{k=0}^{N-1}$ of the right-hand side by a function in $C^N(\bar{\Omega})$ all of whose partial derivatives up to order N vanish at $\partial\Omega$, for all $N = 0, 1, 2, \dots$. Note that $u \in C^\infty(\bar{\Omega})$ is equivalent to $M_k = 0 \forall k$.

Theorem 3. *Assume that $u(z) = f(|z|^2)$ is a smooth radial function on \mathbf{M}^2 , with asymptotic expansion (10) satisfying (11), for which (7) is smooth up to $|z| = 1$. Then $f \in C^\infty(0, 1]$ (and, hence, Theorem 1 applies).*

In conclusion, if there exists a radial balanced metric on \mathbf{M}^2 , then either it has more complicated behavior at the boundary than given by (10) and (11); or it involves a nonzero “flat” piece corresponding to the function h in (8), so that $f \in C^\infty \setminus C^\omega[0, 1]$. Observe that any nonzero h in (8) must have some kind of singularity at $t = 1$; an example of such function is $h(t) = e^{-1/\sqrt{1-t}}$.

We pause to remark that weight functions with the boundary singularity given by (10) and (11) occur naturally in the analysis on strictly pseudoconvex domains: allowing a power at r in front of the double sum in (10) (or taking the appropriate root of the weight function), examples of functions of the form (10) and (11) include the Bergman kernel on the diagonal, the Szegő kernel on the diagonal, the potential of the Poincaré metric (i.e. the solution of the Monge-Ampère equation, cf. Section 5.2 below), the Bergman invariant, and so forth [9]. The necessity for a balanced metric to have more complicated boundary behavior would therefore seem a bit surprising.

The proofs of Theorem 1 and Corollary 2 are given in Section 3, after recalling some preliminaries about the Kepler manifold in Section 2. The proof of Theorem 3 occupies Section 4. The final section, Section 5, collects some concluding comments and remarks concerning completeness of balanced metrics, an identification of a family of Poincaré (i.e. Kähler-Einstein) metrics on \mathbf{M} , as well as a correction of a small overlook in the proof of a theorem in [10].

Remark. Our terminology in this paper is perhaps a bit at odds with common usage due to the powers $n + 1$ in (1) and (2): namely, in most of the literature one calls a Kähler form ω balanced if the reproducing kernel with respect to the volume element $e^{-\Phi} \wedge^n \omega$ equals ce^Φ on the diagonal. Thus ω is balanced in the sense of (1) if and only if $(n + 1)\omega$ is balanced in the sense of the preceding sentence. Of course, our reason for this small deviation was to avoid having to keep track of the factor $(n + 1)$ all the time. \square

2. PRELIMINARIES

Recall that the Kepler manifold \mathbf{H} is the orbit of the vector $e = (1, i, 0, \dots, 0)$ under the $O(n + 1, \mathbf{C})$ -action on \mathbf{C}^{n+1} ; its unit ball \mathbf{M} as well as the outer boundary $\partial\mathbf{M} = \{z \in \mathbf{H} : |z| = 1\}$ of the latter are invariant under $O(n + 1, \mathbf{C}) \cap U(n + 1) = O(n + 1, \mathbf{R})$, and in fact $\partial\mathbf{M}$ is the orbit of e under $O(n + 1, \mathbf{R})$. In particular, there is a unique $O(n + 1, \mathbf{R})$ -invariant probability measure $d\mu$ on $\partial\mathbf{M}$, coming from the Haar measure on the (compact) group $O(n + 1, \mathbf{R})$. Explicitly, denoting

$$(12) \quad \alpha := (n + 1) \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_{n+1} \quad \text{on } z_j \neq 0$$

(this is, up to constant factor, the unique $SO(n + 1, \mathbf{C})$ -invariant holomorphic n -form on \mathbf{H} , see [18]) and defining a $(2n - 1)$ -form η on $\partial\mathbf{M}$ by

$$\eta(z)(V_1, \dots, V_{2n-1}) := \alpha(z) \wedge \overline{\alpha(z)}(z, V_1, \dots, V_{2n-1}), \quad V_1, \dots, V_{2n-1} \in T_z(\partial\mathbf{M}),$$

we then have $d\mu = |\eta|/|\eta|(\partial\mathbf{M})$, where $|\eta|$ denotes the measure induced by η on $\partial\mathbf{M}$. Note that \mathbf{M} , $\partial\mathbf{M}$ and $d\mu$ are also invariant under the complex rotations

$$z \mapsto e^{i\theta}z, \quad \theta \in \mathbf{R}.$$

Proposition 4. *For a function $u(z) = f(|z|^2)$ on \mathbf{M} ,*

$$u^{n+1} \bigwedge^n \left(\frac{i}{2} \partial \bar{\partial} \log \frac{1}{u} \right) = \frac{W[f]}{(n+1)^2} \frac{\alpha \wedge \bar{\alpha}}{(-1)^{n(n+1)/2} (2i)^n},$$

where α is given by (12) and

$$(13) \quad W[f] := (-1)^n t f'^{(n-1)} (f f' + t f f'' - t f'^2).$$

(Here and throughout, f, f', f'' and $W[f]$ are evaluated at $t = |z|^2$.)

Proof. Working e.g. in the local chart $z_{n+1} \neq 0$, we choose the local coordinates $z = (Z, z_{n+1})$ on \mathbf{H} , with $Z := (z_1, \dots, z_n)$ and $z_{n+1} = \pm \sqrt{-Z \cdot Z}$. Then $u(z) = f(|z|^2) = f(|Z|^2 + |Z \cdot Z|)$. By elementary linear algebra,

$$u^{n+1} \bigwedge^n \left(\frac{i}{2} \partial \bar{\partial} \log \frac{1}{u} \right) = J[u] \left(\frac{i}{2} \right)^n dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n,$$

where $J[u]$ is the Monge-Ampère determinant

$$J[u] = (-1)^n \det \begin{bmatrix} u & \partial u / \partial z_k \\ \partial u / \partial \bar{z}_j & \partial^2 u / \partial z_k \partial \bar{z}_j \end{bmatrix}_{j,k=1}^n.$$

In our case,

$$\frac{\partial u}{\partial \bar{z}_j} = f'(|Z|^2 + |Z \cdot Z|) d_j, \quad d_j := z_j + \frac{Z \cdot Z}{|Z \cdot Z|} \bar{z}_j,$$

and similarly $\frac{\partial u}{\partial z_k} = f'(|z|^2) \bar{d}_k$; while

$$\frac{\partial^2 u}{\partial z_k \partial \bar{z}_j} = d_j \bar{d}_k f''(|z|^2) + \left(\delta_{jk} + \frac{\bar{z}_j z_k}{|Z \cdot Z|} \right) f'(|z|^2).$$

Omitting the argument $|z|^2$ at f and its derivatives, we thus get

$$\begin{aligned} (-1)^n J[u] &= \det \begin{bmatrix} f & \bar{d}_k f' \\ d_j f' & d_j \bar{d}_k f'' + \left(\delta_{jk} + \frac{\bar{z}_j z_k}{|Z \cdot Z|} \right) f' \end{bmatrix} \\ &= \det \begin{bmatrix} f & 0 \\ d_j f' & d_j \bar{d}_k f'' + \left(\delta_{jk} + \frac{\bar{z}_j z_k}{|Z \cdot Z|} \right) f' - f'^2 \frac{d_j \bar{d}_k}{f} \end{bmatrix} \\ &= f \det \left[\left(f'' - \frac{f'^2}{f} \right) d_j \bar{d}_k + \left(\delta_{jk} + \frac{\bar{z}_j z_k}{|Z \cdot Z|} \right) f' \right]_{j,k=1}^n \\ &= f f'^n \det \left[\left(\frac{f''}{f'} - \frac{f'}{f} \right) d_j \bar{d}_k + \left(\delta_{jk} + \frac{\bar{z}_j z_k}{|Z \cdot Z|} \right) \right] \\ &= f f'^n \det \left[I + \frac{\langle \cdot, Z \rangle Z}{|Z \cdot Z|} + \left(\frac{f''}{f'} - \frac{f'}{f} \right) \langle \cdot, d \rangle d \right], \end{aligned}$$

where d is the vector $(d_1, \dots, d_n) = Z + \frac{Z \cdot Z}{|Z \cdot Z|} \bar{Z}$. Passing to a basis containing Z, d shows that the last determinant equals

$$\left(1 + \frac{|Z|^2}{|Z \cdot Z|} \right) \left(1 + \left(\frac{f''}{f'} - \frac{f'}{f} \right) |d|^2 \right) - \left(\frac{f''}{f'} - \frac{f'}{f} \right) \left| \left\langle \frac{Z}{|Z \cdot Z|^{1/2}}, d \right\rangle \right|^2$$

(which formula remains in force even if d, Z are linearly dependent). Since $|d|^2 = 2|Z|^2 + 2|Z \cdot Z| = 2|z|^2$ and

$$\left\langle \frac{Z}{|Z \cdot Z|^{1/2}}, d \right\rangle = \frac{|Z|^2 + |Z \cdot Z|}{|Z \cdot Z|^{1/2}} = \frac{|z|^2}{|Z \cdot Z|^{1/2}}$$

while $|Z \cdot Z| = |z_{n+1}|^2$, the determinant equals

$$\frac{|z|^2}{|Z \cdot Z|} \left(1 + 2|z|^2 \left(\frac{f''}{f'} - \frac{f'}{f} \right) \right) - \frac{|z|^4}{|Z \cdot Z|} \left(\frac{f''}{f'} - \frac{f'}{f} \right) = \frac{|z|^2}{|z_{n+1}|^2} \left(1 + |z|^2 \left(\frac{f''}{f'} - \frac{f'}{f} \right) \right).$$

Now by (12)

$$\frac{\alpha \wedge \bar{\alpha}}{(-1)^{n(n+1)/2} (2i)^n} = \frac{(n+1)^2}{|z_{n+1}|^2} \left(\frac{i}{2} \right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,$$

so, switching to the notation $|z|^2 =: t$,

$$J[u] \left(\frac{i}{2} \right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n = \frac{(-1)^n t f f'^n (1 + t \left(\frac{f''}{f'} - \frac{f'}{f} \right))}{(n+1)^2} \frac{\alpha \wedge \bar{\alpha}}{(-1)^{n(n+1)/2} (2i)^n},$$

proving the claim. \square

The measure

$$d\rho(z) := \frac{\alpha \wedge \bar{\alpha}}{(-1)^{n(n+1)/2} (2i)^n}$$

admits a handy description in ‘‘polar coordinates’’: namely, it was shown in [17, Lemma 2.1] that for any measurable function f on \mathbf{H} ,

$$\int_{\mathbf{H}} f(z) d\rho(z) = c_{\mathbf{M}} \int_0^\infty \int_{\partial \mathbf{M}} f(\sqrt{t}\zeta) t^{n-2} d\mu(\zeta) dt,$$

where

$$c_{\mathbf{M}} = (n-1) \int_{\mathbf{M}} d\rho.$$

As in Theorem 5 on page 273 in [4], it then follows, in particular, that for any nonnegative integrable function ϕ on $(0, 1)$, the reproducing kernel $K_{\phi, d\rho}(x, y)$ of the weighted Bergman space $L^2_{\text{hol}}(\mathbf{M}, \phi d\rho)$ on \mathbf{M} is given by

$$K_{\phi, d\rho}(x, y) = \frac{1}{c_{\mathbf{M}}} \sum_{k=0}^{\infty} \frac{N(k)}{c_{k+n-2}} (x \cdot \bar{y})^k,$$

where

$$N(k) := \binom{k+n-1}{n-1} + \binom{k+n-2}{n-1}$$

while

$$c_k := \int_0^1 t^k \phi(t) dt$$

are the moments of the function ϕ .

Corollary 5. For $u(z) = f(|z|^2)$ on \mathbf{M} ,

$$K_{u^{n+1} \wedge^n (\frac{1}{2} \partial \bar{\partial} \log \frac{1}{u})}(z, z) = \frac{(n+1)^2}{c_{\mathbf{M}}} F(|z|^2),$$

where

$$F(t) := \sum_{k=0}^{\infty} \frac{N(k)}{c_{k+n-2}} t^k,$$

with $N(k)$ and c_k as above, for $\phi = W[f]$ given by (13).

3. THE SMOOTH CASE

Specializing the last corollary to $n = 2$, we see in particular that

$$K_{u^3 \wedge^2 (\frac{1}{2} \partial \bar{\partial} \log \frac{1}{u})} (z, z) = \frac{9}{c_{\mathbf{M}}} F(|z|^2),$$

where

$$(14) \quad F(t) = \sum_{k=0}^{\infty} \frac{2k+1}{\int_0^1 t^k \phi(t) dt} t^k$$

with $\phi = W[f] = tf'(ff' + tff'' - tf'^2)$. The balanced condition therefore reads

$$(15) \quad F(t) = \frac{c}{f(t)^3} \quad \forall t \in (0, 1)$$

for some nonzero constant c (differing from the one in (2) by the factor $9/c_{\mathbf{M}}$), while the hypothesis of Theorem 1 is simply that

$$(16) \quad F - \frac{c}{f^3} \in C^\infty(0, 1]$$

with some nonzero constant c (the same one as in the preceding formula).

Proof of Theorem 1. The hypothesis on f implies that $\phi \in C^\infty(0, 1]$ and $\phi(1) = 1$. In terms of the variable $L := \log \frac{1}{t}$, we thus have

$$\phi(t) \approx \sum_{j=0}^{\infty} a_j L^j, \quad a_0 = 1,$$

where $a_j := \frac{1}{j!} \frac{d^j}{dL^j} \phi(e^{-L})|_{L=0}$ are some real numbers, and “ \approx ” means that $\phi - \sum_{j=0}^{N-1} a_j L^j$ vanishes to order at least N at $t = 1$, for each $N = 0, 1, 2, \dots$. Using the formula

$$\int_0^1 t^k L^j dt = \frac{j!}{(k+1)^{j+1}},$$

this implies that

$$c_k = \int_0^1 t^k \phi(t) dt \approx \sum_{j=0}^{\infty} \frac{j! a_j}{(k+1)^{j+1}},$$

where “ \approx ” now means that $c_k - \sum_{j=0}^{N-1} \frac{j! a_j}{(k+1)^{j+1}}$ is $O(k^{-N-1})$ as $k \rightarrow +\infty$, for each $N = 0, 1, 2, \dots$. Taking reciprocal gives

$$(17) \quad \frac{1}{c_k} \approx (k+1) \sum_{m=0}^{\infty} \frac{A_m}{(k+1)^m}, \quad A_0 = 1,$$

where for $m \geq 1$

$$(18) \quad \begin{aligned} A_m &= \sum_{n=1}^m (-1)^n \sum_{\substack{j_1, \dots, j_n \geq 1 \\ j_1 + \dots + j_n = m}} \prod_{j=1}^n (j! a_{j_i}) \\ &= -m! a_m + (\text{a polynomial in } a_1, \dots, a_{m-1}). \end{aligned}$$

Thus

$$\begin{aligned}
F(t) &= \sum_{k=0}^{\infty} \frac{2k+1}{c_k} t^k \approx \sum_{k=0}^{\infty} (2(k+1)^2 - (k+1)) \sum_{m=0}^{\infty} \frac{A_m}{(k+1)^m} \\
&= \sum_{m=0}^{\infty} A_m (2\Phi(t, m-2, 1) - \Phi(t, m-1, 1)) \\
&= \sum_{m=-2}^{\infty} (2A_{m+2} - A_{m+1}) \Phi(t, m, 1) \quad (A_{-1} := 0) \\
&= \frac{2(1+t)}{(1-t)^3} + \frac{2A_1 - 1}{(1-t)^2} + \frac{2A_2 - A_1}{1-t} \\
(19) \quad &+ \sum_{m=1}^{\infty} \frac{2A_{m+2} - A_{m+1}}{t} \left[\frac{(-1)^m}{(m-1)!} L^{m-1} \log L + h_m(L) \right],
\end{aligned}$$

where h_m is holomorphic in the disc $|L| < 2\pi$; here $\Phi(t, s, v)$ stands for the Lerch transcendental function [2, §1.11]

$$\Phi(z, s, v) := \sum_{k=0}^{\infty} \frac{z^k}{(k+v)^s}, \quad s \in \mathbf{C}, \quad v \neq 0, -1, -2, \dots,$$

and we have made use of Lerch's formula [2, 1.11(9)]

$$(20) \quad t\Phi(t, m, 1) = \frac{(-1)^m}{(m-1)!} L^{m-1} \log L + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta(m-k) L^k, \quad L := \log \frac{1}{t},$$

valid for an integer $m \geq 1$, where the sum on the right-hand side converges for $|L| < 2\pi$, and the \sum' means that in the term $k = m-1$, $\zeta(1)$ should be replaced by $\sum_{j=1}^{m-1} \frac{1}{j}$.

By hypothesis, $F(t) - \frac{c}{f(t)^3}$ is smooth up to $t = 1$. This is only possible if all the log-terms in (19) vanish, i.e. $2A_{m+2} - A_{m+1} = 0$ for all $m \geq 1$, or $A_m = 2^{2-m} A_2$ for all $m \geq 2$. Feeding this into (17) gives, after summing a geometric series,

$$(21) \quad \frac{1}{c_k} = (k+1) \left[1 + \frac{A_1}{k+1} + \frac{2A_2}{(k+1)(2k+1)} \right] + O(k^{-\infty})$$

(where " $O(k^{-\infty})$ " means " $O(k^{-N}) \forall N > 0$ "). Hence

$$\begin{aligned}
F(t) &= \sum_{k=0}^{\infty} \frac{2k+1}{c_k} t^k \\
&= \frac{4}{(1-t)^3} + \frac{2A_1 - 3}{(1-t)^2} + \frac{2A_2 - A_1}{1-t} + H(t) \\
&= \frac{4}{L^3} + \frac{2A_1 + 3}{L^2} + \frac{A_1 + 2A_2 + 1}{L} + H(e^{-L}),
\end{aligned}$$

where H denotes a function (possibly a different one at each occurrence) smooth up to $t = 1$. On the other hand, from $f(1) = 0$, $f'(1) = -1$ we have $f(t) = L + O(L^2)$; using the hypothesis $F - \frac{c}{f^3} \in C^\infty(0, 1]$ again, we see that necessarily $c = 4$ and

$$f = \left(\frac{c}{F - H} \right)^{1/3} = L - \frac{2A_1 + 3}{12} L^2 + \frac{4A_1^2 + 6A_1 + 3 - 12A_2}{72} L^3 + O(L^4),$$

which implies by a laborious but routine calculation,

$$\phi = tf'(ff' + tff'' - tf'^2) = 1 - \frac{2A_1}{3}L + \frac{4A_1^2 + A_1 - 6A_2}{12}L^2 + a_3L^3 + O(L^4)$$

(we will not need the value of a_3). Thus $a_1 = -\frac{2A_1}{3}$, while by (18) $a_1 = -A_1$. Hence necessarily $a_1 = A_1 = 0$, and only A_2 (or a_2) remains as a free parameter. We have thus obtained a one-parameter family of germs of f at $t = 1$ that can satisfy the condition (16).

We finish the proof by showing that the one-parameter family of germs of ϕ given by (9) is the one that leads to the coefficients $A_1 = 0$, $A_m = 2^{2-m}A_2$ for $m \geq 2$. Namely, by (9), the coefficients a_j for ϕ_v are given by

$$a_j = \frac{(1 + \sqrt{v})(1 - \sqrt{v})^j - (1 - \sqrt{v})(1 + \sqrt{v})^j}{4^j j! 2\sqrt{v}}.$$

Hence

$$\begin{aligned} c_k &\approx \sum_{j=0}^{\infty} \frac{j! a_j}{(k+1)^{j+1}} \\ &= \sum_{j=0}^{\infty} \frac{(1 + \sqrt{v})(1 - \sqrt{v})^j - (1 - \sqrt{v})(1 + \sqrt{v})^j}{4^j 2\sqrt{v}(k+1)^{j+1}} \\ &= \frac{1}{2(k+1)\sqrt{v}} \left[\frac{1 + \sqrt{v}}{1 - \frac{1-\sqrt{v}}{4(k+1)}} - \frac{1 - \sqrt{v}}{1 - \frac{1+\sqrt{v}}{4(k+1)}} \right] \\ &= \frac{16k + 8}{16k^2 + 24k + 9 - v}, \end{aligned}$$

or

$$\frac{1}{c_k} = \frac{2k^2 + 3k + \frac{9-v}{8}}{2k+1} + O(k^{-\infty}).$$

However, this is the same thing as (21) with $A_1 = 0$ and $A_2 = \frac{1-v}{16}$. This completes the proof. \square

Proof of Corollary 2. If f is real-analytic near 1, then so is $\phi = W[f]$ and clearly, by (9), also ϕ_v , hence it follows from (8) that so is h ; since $h^{(k)}(1) = 0 \forall k$, we must have $h \equiv 0$. Thus we have for some $c \neq 0$

$$(22) \quad \frac{c}{f(t)^3} = \sum_{k=0}^{\infty} \frac{(2k+1)t^k}{\int_0^1 t^k \phi(t) dt},$$

where

$$(23) \quad \phi = tf'(ff' + tff'' - tf'^2) = \phi_v.$$

We show this leads to a contradiction.

For $v < 0$, say $v = -s^2$ with $s > 0$, we have $\phi_v(t) = e^{L/4}(\cos \frac{Ls}{4} - \frac{1}{s} \sin \frac{Ls}{4})$ where $L := \log \frac{1}{t} > 0$; this changes sign as $t \searrow 0$, contradicting the fact that $\phi(|z|^2) d\rho(z) = u^3 \wedge^2 (\frac{i}{2} \partial \bar{\partial} \log \frac{1}{u})$ should be a nonnegative volume element on \mathbf{M} . Thus $v \geq 0$. In that case, the integral in (22) is finite if and only if $k - \frac{1}{4} - \frac{1}{4}|\sqrt{v}| > -1$. Write

$$\frac{|\sqrt{v}| - 3}{4} = m - 1 + \delta, \quad m \in \{0, 1, 2, \dots\}, \quad 0 \leq \delta < 1.$$

The integral is then finite precisely for $k \geq m$, and equals, by a small computation,

$$\int_0^1 t^k \phi_v(t) dt = \frac{2k+1}{(2k+2m+2\delta+1)(k+1-m-\delta)}.$$

For the right-hand side of (22) we thus get, again by a small computation,

$$\sum_{k=0}^{\infty} \frac{(2k+1)t^k}{\int_0^1 t^k \phi(t) dt} = \frac{t^m(1+3t+4m(1-t) - \delta(4m+2\delta-1)(1-t)^2)}{(1-t)^3}.$$

Taking reciprocals, we should thus have

$$f(t)^3 = c \frac{(1-t)^3}{t^m(1+3t+4m(1-t) - \delta(4m+2\delta-1)(1-t)^2)},$$

or

$$f(t) = c^{1/3} \frac{1-t}{\sqrt[3]{t^m(1+3t+4m(1-t) - \delta(4m+2\delta-1)(1-t)^2)}}.$$

The condition $f'(1) = -1$ implies that $c = 4$, so finally

$$(24) \quad f(t) = \frac{2^{2/3}t^{-m/3}(1-t)}{\sqrt[3]{1+3t+4m(1-t) - \delta(4m+2\delta-1)(1-t)^2}}.$$

This should now satisfy $tf'(ff' + tf f'' - tf'^2) = \phi_v$. From (24) we see that $f(t) = t^{-m/3}e^{h(t)}$ with h holomorphic at the origin. This yields

$$tf'(ff' + tf f'' - tf'^2) = -\frac{1}{3}e^{3h(t)}t^{-m}(m - 3th'(t))(h'(t) + th''(t)).$$

Thus $t^m \phi_v(t)$ should be holomorphic at $t = 0$. This is clearly not the case when $v = 0$ (since ϕ_0 contains $\log t$); and it also cannot be the case when $0 < |\sqrt{v}| \neq 1$, since then ϕ_v is a linear combination of two powers of t (with nonzero coefficients) whose exponents sum to $-\frac{1}{2}$. The only case left is $v = 1$, corresponding to $m = 0$, $\delta = \frac{1}{2}$; however, then (24) becomes

$$f(t) = \frac{2^{2/3}(1-t)}{\sqrt[3]{1+3t}},$$

and

$$tf'(ff' + tf f'' - tf'^2) = \frac{16t(1+t)(1+2t+5t^2)}{(1+3t)^4},$$

which clearly does not equal $\phi_1(t) \equiv 1$. This proves the corollary. \square

4. THE GENERAL CASE

We recall the following refinement of Lerch's formula (20), proved in [10].

Lemma 6. (Lemma 4 in [10]) *The series*

$$(25) \quad \sum_{k=1}^{\infty} \frac{t^k}{k^s} \left(\log \frac{1}{k} \right)^n = \left(\frac{d}{ds} \right)^n t\Phi(t, s, 1), \quad n = 0, 1, 2, \dots,$$

equals

$$(26) \quad \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \Gamma(n-j) (1-s)L^{s-1} (\log L)^j + \sum_{k=0}^{\infty} \zeta^{(n)}(s-k) \frac{(-1)^k}{k!} L^k,$$

$$|L| < 2\pi, \quad s \neq 1, 2, 3, \dots, \quad L := \log \frac{1}{t}.$$

For $s = 1, 2, 3, \dots$, the first sum on the right-hand side of the last formula has to be replaced by

$$(27) \quad \sum_{j=0}^n \binom{n}{j} c_{s,n-j} L^{s-1} (\log L)^j + \frac{(-1)^{s-1}}{(s-1)!} \left[\gamma_n - \frac{(\log L)^{n+1}}{n+1} \right] L^{s-1},$$

while the term $k = s - 1$ in the second sum on the right-hand side of (26) has to be omitted. Here $c_{s,j}$ and γ_j are certain constants (given explicitly below).

Here γ_j are the Stieltjes constants, i.e.

$$\zeta(1+z) = \frac{1}{z} + \sum_{j=1}^{\infty} \frac{\gamma_j}{j!} z^j, \quad z \in \mathbf{C},$$

while $c_{m,j}$ are, similarly, the coefficients of the Laurent expansion of the Gamma function,

$$(28) \quad \Gamma(1-m-z) = \frac{(-1)^m}{(m-1)!z} + \sum_{j=0}^{\infty} \frac{c_{m,j}}{j!} z^j, \quad |z| < 1, \quad m = 1, 2, 3, \dots$$

From the functional equation $\Gamma(z+1) = z\Gamma(z)$ for the Gamma function we get the recurrence relations

$$(29) \quad c_{m+1,j} = -\frac{c_{m,j} + j c_{m+1,j-1}}{m} \quad \text{for } j \geq 1, \quad c_{m+1,0} = \frac{(-1)^m}{m!m} - \frac{c_{m,0}}{m},$$

with

$$(30) \quad c_{1,j} = -\frac{c_{0,j+1}}{j+1}, \quad c_{0,j} := (-1)^j \Gamma^{(j)}(1).$$

For later use, we also note that the simple formula

$$\int_0^{\infty} L^s e^{-kL} dL = \frac{\Gamma(s+1)}{k^{s+1}}, \quad \operatorname{Re} s > -1, \quad k = 1, 2, 3, \dots,$$

yields upon applying $(d/ds)^n$ to both sides ($n = 0, 1, 2, \dots$)

$$(31) \quad \int_0^{\infty} L^s (\log L)^n e^{-kL} dL = \sum_{l=0}^n \binom{n}{l} \frac{\Gamma^{(n-l)}(s+1)}{k^{s+1}} \left(\log \frac{1}{k} \right)^l,$$

by the Leibniz rule.

Proof of Theorem 3. Assume that $u(z) = f(|z|^2)$ is a smooth radial function on \mathbf{M}^2 , with the asymptotic expansion (10) satisfying (11), which satisfies (16). Passing again from the variable $t = |z|^2$ to $L = \log \frac{1}{t}$, (10) and (11) become

$$(32) \quad f(t) \approx L \sum_{k=0}^{\infty} \sum_{j=0}^{M_k} a_{kj} L^k (\log \frac{1}{L})^j, \quad M_0 = 0, \quad a_{00} = 1.$$

We will show that $M_k = 0$ for all k , so that $f \in C^\infty(0, 1]$ as claimed.

Assume, to the contrary, that there is $N \geq 1$ such that $M_0 = M_1 = \dots = M_{N-1} = 0$ but $M_N \geq 1$ with $a_{NM_N} \neq 0$. By (32) and the definition of (10), we have

$$(33) \quad u \approx L \left[1 + p_N(L) + \sum_{j=1}^M a_j L^N (\log \frac{1}{L})^j + O(L^{N+\delta}) \right],$$

with any $0 < \delta < 1$; here we started writing just M and a_j for M_N and a_{Nj} , respectively, and p_N stands for some polynomial (not necessarily the same one at each occurrence) of degree N without constant term. Furthermore, (33) can be differentiated termwise any number of times. Viewing, for the duration of this proof, f temporarily as a function of L rather than of $t = e^{-L}$, the formula for $W[f] = \phi$ becomes just $\phi = e^L f'(f'^2 - f f'')$, and a routine computation gives

$$(34) \quad \phi = 1 + p_N(L) + \sum_{j=1}^M A_j L^N (\log \frac{1}{L})^j + O(L^{N+\delta}),$$

with

$$A_M = (3 - N)(N + 1)a_M.$$

For the moments $c_k = \int_0^1 t^k \phi(t) dt$ we thus obtain, in view of (31),

$$c_k \approx \frac{1}{k+1} + \frac{p_N(\frac{1}{k+1})}{k+1} + \sum_{j=1}^M \frac{A'_j}{(k+1)^{N+1}} (\log(k+1))^j + O\left(\frac{1}{(k+1)^{N+1+\delta}}\right),$$

with $A'_M = N!A_M$. Taking reciprocal and multiplying by $2k+1$ gives

$$\frac{2k+1}{c_k} \approx 2(k+1)^2 \left[1 + p_N\left(\frac{1}{k+1}\right) + \sum_{j=1}^M \frac{A''_j}{(k+1)^N} (\log(k+1))^j + O\left(\frac{1}{(k+1)^{N+1+\delta}}\right) \right]$$

with $A''_M = -A'_M$. It follows that

$$(35) \quad F(t) = \sum_{k=0}^{\infty} \frac{2k+1}{c_k} t^k = 2 \sum_{j=0}^N \beta_j \Phi(t, j-2, 1) + 2 \sum_{j=1}^M A''_j (-1)^j \Phi^{(j)}(t, N-2, 1) + R(t),$$

where $\Phi^{(j)}$ denotes the j -th derivative of Φ with respect to the second argument, β_j are some coefficients, and the remainder term $R(t)$ is $O((1-t)^{N-3+\delta})$ for $N = 1, 2$, and belongs to $C^{N-3}(0, 1]$ if $N \geq 3$.

If $N = 1$, (35) becomes

$$F = \frac{4}{L^3} \left[1 - 2a_M L (\log \frac{1}{L})^M + \dots \right]$$

where the dots stand for lower-order terms. On the other hand,

$$\frac{1}{f^3} = \frac{1}{L^3} \left[1 - 3a_M L (\log \frac{1}{L})^M + \dots \right].$$

Thus the condition $F - \frac{c}{f^3} \in C^\infty(0, 1]$ forces ($c = 4$ and) $a_M = 0$, a contradiction. Thus $N = 1$ cannot occur.

If $N = 2$, (35) becomes, after a bit more lengthy but completely routine computation (using (25)–(30)) which we omit,

$$F = \frac{4}{L^3} \left[1 + \beta'_1 L - 3a_M L^2 (\log \frac{1}{L})^M - (3a_{M-1} - \frac{5M}{2} a_M) L^2 (\log \frac{1}{L})^{M-1} + \dots \right],$$

while, by (33),

$$\frac{1}{f^3} = \frac{1}{L^3} \left[1 + \beta''_1 L - 3a_M L^2 (\log \frac{1}{L})^M - 3a_{M-1} L^2 (\log \frac{1}{L})^{M-1} + \dots \right]$$

(the dots again denote lower-order terms). Hence $F - \frac{c}{f^3} \in C^\infty(0, 1]$ again forces ($c = 4$ and) $\frac{5M}{2} a_M = 0$, contradicting the hypothesis that $M \geq 1$ and $a_M \neq 0$. Thus $N = 2$ cannot occur either.

If $N \geq 4$, (35) becomes, using Lemma 6,

$$F = \frac{4}{L^3} \left[1 + \sum_{j=1}^{N-1} \beta'_j L^j + \sum_{j=3}^{N-1} \frac{A''_j}{2} \frac{(-1)^j}{(j-3)!} L^j \log L - \frac{N(N+1)(N-1)(N-2)(N-3)}{2(M+1)} (-1)^N L^N \left(\log \frac{1}{L} \right)^{M+1} + \dots \right],$$

while, by (33),

$$\frac{1}{f^3} = \frac{1}{L^3} \left[1 + \sum_{j=1}^{N-1} \beta''_j L^j - 3a_M L^N \left(\log \frac{1}{L} \right)^M + \dots \right].$$

Thus $F - \frac{c}{f^3} \in C^\infty(0, 1]$ can only hold if ($c = 4$, $A''_j = 0$ for all $j = 1, \dots, N-1$, and) $N(N+1)(N-1)(N-2)(N-3) = 0$, i.e. $N \in \{1, 2, 3\}$, a contradiction again. Thus $N \geq 4$ is likewise not possible.

We are thus left with the case of $N = 3$; note that in that case $A_M = 0$ even though $a_M \neq 0$, so a bit more detailed analysis is needed. Computing A_{M-1} , (34) gives

$$\phi = 1 + p_2(L) + 4Ma_M L^3 \left(\log \frac{1}{L} \right)^{M-1} + \dots$$

provided $M > 1$. This gives, in turn,

$$c_k = \frac{1}{k+1} + \frac{p_2\left(\frac{1}{k+1}\right)}{k+1} + \frac{24Ma_M}{(k+1)^4} \left(\log(k+1) \right)^{M-1} + \dots,$$

$$\frac{2k+1}{c_k} = 2(k+1)^2 \left[1 + p_2\left(\frac{1}{k+1}\right) - \frac{24Ma_M}{(k+1)^3} \left(\log(k+1) \right)^{M-1} + \dots \right],$$

and

$$F = \frac{4}{L^3} \left[1 + p_2(L) - 12a_M L^3 \left(\log \frac{1}{L} \right)^M + \dots \right],$$

while by (34),

$$\frac{1}{f^3} = \frac{1}{L^3} \left[1 + p_2(L) - 3a_M L^3 \left(\log \frac{1}{L} \right)^M + \dots \right].$$

Hence the condition $F - \frac{c}{f^3} \in C^\infty(0, 1]$ forces ($c = 4$ and) $a_M = 0$, a contradiction as before.

This finally leaves us with the situation when $N = 3$ and $M = 1$, so that

$$f = L(1 + \alpha_1 L + \alpha_2 L^2 + aL^3 \log \frac{1}{L} + \dots), \quad a \neq 0,$$

yielding in turn

$$\phi = 1 + (4\alpha_1 + 1)L + (4\alpha_1 + 6\alpha_1^2 + 3\alpha_2 + \frac{1}{2})L^2 + (4a + 2\alpha_1 + 6\alpha_1^2 + 4\alpha_1^3 + 3\alpha_2 + 10\alpha_1\alpha_2 + \frac{1}{6})L^3 + \dots,$$

$$c_k = \frac{1}{k+1} + \frac{4\alpha_1 + 1}{(k+1)^2} + \frac{2(4\alpha_1 + 6\alpha_1^2 + 3\alpha_2 + \frac{1}{2})}{(k+1)^3} + \frac{6(4a + 2\alpha_1 + 6\alpha_1^2 + 4\alpha_1^3 + 3\alpha_2 + 10\alpha_1\alpha_2 + \frac{1}{6})}{(k+1)^4} + \dots,$$

$$\frac{2k+1}{c_k} = 2(k+1)^2 \left[1 - \frac{4\alpha_1 + \frac{3}{2}}{k+1} + \frac{\frac{1}{2} + 2\alpha_1 + 4\alpha_1^2 - 6\alpha_2}{(k+1)^2} + \frac{2\alpha_1^2 + 8\alpha_1^3 - 12\alpha_1\alpha_2 - 3\alpha_2 - 24a}{(k+1)^3} + \dots \right],$$

and

$$F = \frac{4}{L^3} \left[1 + \left(\frac{1}{4} - 2\alpha_1\right)L + (2\alpha_1^2 - \alpha_1 - 3\alpha_2)L^2 + (12a - \alpha_1^2 - 4\alpha_1^3 + \frac{3}{2}\alpha_2 + 6\alpha_1\alpha_2)L^3 \log L + \dots \right],$$

while

$$\frac{1}{f^3} = \frac{1}{L^3} \left[1 - 3\alpha_1 L + (6\alpha_1 - 3\alpha_2)L^2 + 3aL^3 \log L + \dots \right].$$

From $F - \frac{c}{f^3} \in C^\infty(0, 1]$ we thus get in turn, comparing the coefficients, $c = 4$, $\alpha_1 = -\frac{1}{4}$ and $a = 0$, contradicting once again the assumption that a is nonzero. This completes the proof. \square

5. CONCLUDING REMARKS

5.1. Completeness. An obvious distinction of \mathbf{M} from domains in \mathbf{C}^n is, of course, the presence of the (removable) singularity at $z = 0$. We pause to note that, in fact, there can exist no balanced metric on \mathbf{M} (radial or not, smooth up to $|z| = 1$ or not) that would be complete at the origin.

Theorem 7. *Let u be a solution to the equation (2). Then the balanced metric $\partial\bar{\partial} \log \frac{1}{u}$ is not complete at $z = 0$.*

Proof. Recall that in some local coordinate chart, the above mentioned metric is given explicitly by the coefficients

$$g_{j\bar{k}}(z) = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log \frac{1}{u(z)};$$

and the length of a differentiable curve $\psi : (0, 1) \rightarrow \mathbf{M}$ is then given by

$$(36) \quad \int_0^1 \sqrt{\sum_{j,k} g_{j\bar{k}}(\psi(x)) \psi'_j(x) \overline{\psi'_k(x)}} dx.$$

We apply this to the special case when $\psi(x) = xz$ is the segment joining the origin to some fixed point z of \mathbf{M} . The last sum then equals, by an elementary computation,

$$(37) \quad \frac{\partial^2}{\partial x \partial \bar{x}} \log \frac{1}{u(xz)}.$$

Now by the balanced condition (2), $\log c + (n+1) \log \frac{1}{u(z)} = \log K(z, z)$ where $K = K_{u^{n+1} \wedge^n (\frac{i}{2} \partial\bar{\partial} \log \frac{1}{u})}$ is the reproducing kernel. Thus, up to the (immaterial) constant factor $n+1$, (37) equals

$$\frac{\partial^2}{\partial x \partial \bar{x}} \log K(xz, xz).$$

In terms of any orthonormal basis $\{e_j\}$ of the corresponding Bergman space, the last kernel is given by $\sum_j |e_j(xz)|^2$. Note that since the origin is a removable singularity in \mathbf{H} (cf. [11, Theorem 2.4] even for the more general situation than our ordinary Kepler manifold), each e_j actually extends to a holomorphic function in some neighborhood of the origin in \mathbf{C}^{n+1} . Let $m \geq 0$ be the largest integer with the property that all the functions $x \mapsto e_j(xz)$, $j = 0, 1, 2, \dots$ (holomorphic in one complex variable x), vanish to order m at $x = 0$. We then obtain

$$K(xz, xz) = |x|^{2m} e^{F(x,x)}$$

with some $F(x, y)$ holomorphic in x, \bar{y} near $(x, y) = (0, 0)$. Consequently,

$$\frac{\partial^2}{\partial x \partial \bar{x}} \log K(xz, xz) = \frac{\partial^2}{\partial x \partial \bar{x}} F(x, x)$$

is (nonnegative and) continuous in a neighborhood of $x = 0$. Hence so will be its square root, and thus the integral (36) is finite. This means that there is a curve of finite length joining z to the origin, proving therefore that the metric is not complete at the origin. \square

The last argument in fact shows that no complete balanced metric can exist on a normal complex analytic space with singular locus of codimension ≥ 2 .

5.2. Poincaré metrics. Recall that, quite generally, for a Kähler metric $g_{j\bar{k}} = \partial_j \bar{\partial}_k \Phi$ given by a potential Φ , the volume element is given, in the local chart, by $g = \det[g_{j\bar{k}}] = e^{(n+1)\Phi} J[e^{-\Phi}]$, so that the Ricci tensor $\text{Ric}_{j\bar{k}} = \partial_j \bar{\partial}_k \log g$ satisfies

$$\text{Ric}_{j\bar{k}} = \partial_j \bar{\partial}_k J[u] + (n+1)g_{j\bar{k}} \quad (u = e^{-\Phi}).$$

The metrics with $\partial \bar{\partial} J[u] \equiv 0$ therefore satisfy $\text{Ric}_{j\bar{k}} = (n+1)g_{j\bar{k}}$, i.e. are *Kähler-Einstein* metrics with constant $(n+1)$. Those for which u in addition vanishes precisely to the first order at the boundary (i.e. $u = 0, \nabla u \neq 0$ on $\partial\Omega$) are usually called *Poincaré* metrics on Ω , cf. [3, Chapter 11]. We have seen in course of the proof of Proposition 4 that for $u(z) = f(|z|^2)$ on \mathbf{M} , $J[u](z) = W[f](|z|^2)/|z_{n+1}|^2$, with $W[f]$ given by (13). It follows, in particular, that functions f which are solutions to

$$(38) \quad (-1)^n t f'^{(n-1)}(f f' + t f f'' - t f'^2) \equiv c, \quad f(1) = 0, f'(1) \neq 0,$$

give rise to Poincaré metrics on \mathbf{M} . Replacing f by an appropriate multiple, we can assume that $f'(1) = -1$ and $c = 1$. Note that, in particular, for $n = 2$ (38) is precisely the equation (8) for $v = 1$ and $h \equiv 0$ (thus Poincaré metrics are among the “good candidates” for a balanced metric, in the radial situation). In this subsection, we want to discuss the differential equation (38) in more detail.

For simplicity, we treat again in detail only the simplest case of $n = 2$, so that the equation reads

$$W[f] \equiv t f'(f f' + t f f'' - t f'^2) = 1.$$

Differentiating the expression

$$\Psi(t) := -\frac{t}{f(t)^3} + \frac{t^2 f'(t)^2}{2f(t)^2} - \frac{t^3 f'(t)^3}{f(t)^3},$$

we see that

$$\Psi'(t) = \frac{f(t) - 3t f'(t)}{f(t)^4} (W[f](t) - 1).$$

Thus $W[f] \equiv 1$ implies $\Psi \equiv c$ for some constant c .

The equation $x^3 + \frac{1}{2}x^2 = a$ has for each $a \geq 0$ a unique nonnegative root $x =: \rho(a) \geq 0$. From $\Psi \equiv c$ we thus obtain

$$(39) \quad f'(t) = -\frac{f(t)}{t} \rho\left(c + \frac{t}{f(t)^3}\right)$$

as long as

$$(40) \quad c + \frac{t}{f^3} \geq 0.$$

The condition $f(1) = 0$ means that this will be fulfilled as $t \nearrow 1$, and by standard existence theorems (Peáno — note that since $\rho(a) \approx a^{1/3}$ when $a \rightarrow +\infty$, the function $(t, y) \mapsto -\frac{y}{t}\rho(c + ty^{-3})$ is continuous near $(t, y) = (1, 0)$) applied to (39) yield a unique solution $f(t)$ for $t \in (t_0, 1)$ with some $0 < t_0 < 1$. As long as $\rho > 0$, we will have $f' < 0$, so f will be decreasing and, hence, positive. When $c \geq 0$, it is possible to continue in this way down to $t = 0$ ((40) will still be fulfilled), and for $t \rightarrow 0+$ we will have $f'(t) \approx -\rho(c)f(t)/t$, or $f(t) \approx At^{-\rho(c)}$. When $c < 0$, the solution reaches for some $t_0 \in (0, 1)$ the situation when $c + \frac{t}{f(t)^3} = 0$, whence $f'(t_0) = 0 < f(t_0)$; from $W[f](t) = 0$ we then obtain $f''(t_0^+) = -\infty$, so f develops a singularity at $t = t_0$ and the solution terminates there (cf. also Remark 11 below).

Altogether, we thus arrive at a family $f_c(t)$, $c \geq 0$, of Poincaré metrics on \mathbf{M} , smooth up to the outer boundary $|z| = 1$.

Remark 8. The constant c appears in the Taylor expansion of f at 1, but only in the fourth derivative: namely, taking the Taylor expansion of $\Psi(t) - c = 0$ at $t = 1$ yields in turn

$$f(1) = 0, \quad f'(1) = -1, \quad f''(1) = \frac{1}{2}, \quad f'''(1) = -\frac{3}{4}, \quad f''''(1) = \frac{15+16c}{8}.$$

The condition on the solution f to reach as far as $t = 0$ thus is $f''''(1) \geq \frac{15}{8}$. \square

Remark 9. The only explicit solution we know is $f(t) = 2 - 2\sqrt{t}$, corresponding to $c = 0$. For general n , an explicit solution to (38) is $g(t) = \frac{n}{n-1}(1 - t^{(n-1)/n})$. \square

Remark 10. For $c \geq 0$, from $f_c \sim t^{-\rho(c)}$ as $t \searrow 0$ we see that the volume density $g = u^{-3}J[u] \sim t^{3\rho(c)}$ is nonvanishing at the origin only for $c = 0$ (and then we know explicitly that $f(t) = 2 - 2\sqrt{t}$, by the preceding remark). For $c > 0$, observing that ρ is C^∞ in a neighborhood of c and writing $f(t) = t^{-\rho(c)}h(t)$ with $h(0) \neq 0$, we get $\frac{tf'}{f} = \frac{th'}{h} - \rho(c)$ so that (39) becomes

$$\frac{th'}{h} = \rho(c) - \rho\left(c + \frac{t^{3\rho(c)+1}}{h^3}\right) \approx -\rho'(c)\frac{t^{3\rho(c)+1}}{h^3} = -\frac{t^{3\rho(c)+1}}{h^3\rho(c)(3\rho(c) + 1)},$$

since $\rho' = 1/(3\rho^2 + \rho)$. Solving for h gives

$$h \approx h(0) - \frac{t^{3\rho(c)+1}}{h(0)^2\rho(c)(3\rho(c) + 1)^2}.$$

Continuing in this fashion ultimately yields

$$f(t) = t^{-\rho(c)}Q(t^{3\rho(c)+1})$$

for some function $Q \in C^\infty[0, 1]$. This gives a complete description of the Poincaré metric corresponding to $c > 0$ in the neighborhood of the singularity at the origin.

Note that, by a similar argument as in the preceding subsection, for any $c \geq 0$ the corresponding Poincaré metric is incomplete at the origin $z = 0$. In fact, the integrand in (36) behaves for $x \searrow 0$ as $x^{3\rho(c)}$ for $c > 0$, and as $x^{-1/2}$ for $c = 0$, hence the integral is always finite. \square

Remark 11. For $c < 0$, we have seen that as t decreases from 1, we reach at some $t = t_0$ the situation when $t_0 > 0$, $f(t_0) > 0$ but $c + \frac{t_0}{f(t_0)^3} = 0$, whence $f'(t_0) = 0$ and $f''(t_0^+) = -\infty$. A similar analysis as in the preceding remark, starting from the observation that

$$\rho(x) \approx \sqrt{2x} - 2x + 5\sqrt{2}x^{3/2} - 32x^2 + \dots$$

is a smooth function of \sqrt{x} at the origin, shows that

$$f(t) = Q(\sqrt{t-t_0})$$

for some $Q \in C^\infty[0, \sqrt{1-t_0}]$, with

$$Q(0) = f(t_0), \quad Q'(0) = Q''(0) = 0 \quad \text{and} \quad Q'''(0) = \frac{4\sqrt{2}}{t_0\sqrt{f(t_0)}}.$$

In particular, as $t \searrow t_0$,

$$f'(t) \approx -\frac{\sqrt{2(t-t_0)}}{t_0\sqrt{f(t_0)}}, \quad f''(t) \approx -\frac{1}{t_0}\sqrt{\frac{t-t_0}{2f(t_0)}}.$$

In particular, the solution f cannot be continued in any way across $t = t_0$ — it would have to assume imaginary values for $t < t_0$. (The same is, of course, true for $c = 0$ and $t_0 = 0$, when $f(t) = 2 - 2\sqrt{t}$.) \square

5.3. Erratum. We conclude by giving a fix for a small overlook in the proof of Theorem 3 in [10]: the argument treating the case $N = 1$ after (42) there exhibits a contradiction by taking $j = M - 1$, where $j \in \{1, \dots, M\}$, $M \geq 1$. This is fine for $M \geq 2$, but makes no sense for $M = 1$.

To handle the overlooked case of $N = 1$, $M = 1$, we make the computations there in greater detail: namely, starting again with (as before, the dots always denote lower order terms)

$$(41) \quad f = L(1 + bL \log L + aL + \dots), \quad b \neq 0,$$

gives, in turn, in the notations of [10] (C is the Euler constant)

$$\begin{aligned} w &\equiv u^2 \partial \bar{\partial} \log \frac{1}{u} = 1 + 2bL \log L + (2a - b + 1)L + \dots, \\ c_k &= \frac{1}{k+1} - \frac{2b}{(k+1)^2} \log(k+1) + \frac{2a+b+1-2bC}{(k+1)^2} + \dots, \\ \frac{1}{c_k} &= (k+1) \left[1 - \frac{2b}{k+1} \log(k+1) + \frac{2Cb - 2a - b - 1}{(k+1)^2} + \dots \right], \\ F &= \frac{1}{L^2} \left[1 - 2bL \log L - (2a+b)L + \dots \right], \end{aligned}$$

and

$$f^2 F = 1 - bL + \dots$$

The condition that $f^2 F - c\pi$ is smooth up to $t = 1$ and vanishes to second order there thus implies that ($c = \frac{1}{\pi}$ and) $b = 0$, contradicting the hypothesis $b \neq 0$ in (41). This completes the proof.

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