Let $T$ be a bounded linear operator on a separable Hilbert space $H$. A well-known result of Sz.-Nagy and Foias says that the following two assertions are equivalent:

(a) $I - TT^* \geq 0$ (i.e. $T$ is a contraction) and $T^n \to 0$ in the strong operator topology;

(b) $T^*$ is unitarily equivalent to the restriction of a backward shift of infinite multiplicity to an invariant subspace.

In this talk we describe a generalization of this result for commuting $n$-tuples $T = (T_1, \ldots, T_n)$ of operators on $H$.

Let $\Omega$ be a domain in $\mathbb{C}^n$ and $\mathcal{H}$ a Hilbert space of analytic functions on $\Omega$ which satisfies the following properties:

(1) $\mathcal{H}$ is invariant under the operators $Z_j$ of multiplications by the coordinate functions ($j = 1, \ldots, n$).

(2) The evaluation functionals are continuous on $\mathcal{H}$. Consequently, there exists a reproducing kernel $K(z, w)$ for $\mathcal{H}$.

(3) $\mathcal{H}$ contains all polynomials and they are dense in it.

(4) The function $1/K(z, w)$ is a polynomial (in $z$ and $w$).

It is well known that for any orthonormal basis $\{\psi_k\}$ of $\mathcal{H}$, the reproducing kernel is given by

$$K(z, w) = \sum_k \psi_k(z)\overline{\psi_k(w)}.$$

In view of (3), by applying the Gramm-Schmidt orthogonalization process, we may construct a basis such that all $\psi_k$ are polynomials (and, conversely, any polynomial is a linear combination of a finite number of the $\psi_k$). We fix such a basis from now on. For each $m$, set

$$f_m(z, w) = \sum_{k \geq m} \frac{\psi_k(z)\overline{\psi_k(w)}}{K(z, w)}.$$

Then $f_0(z, w) \equiv 1$ on $\Omega \times \Omega$. By virtue of (4) and our choice of the basis, the difference $f_0 - f_m$ is a polynomial in $z, w$, for each $m$; thus $f_m$ themselves are, in fact, polynomials.
For any polynomial \( p(z, w) \) of \( z, w \in \mathbb{C}^n \), define
\[
p(T, T^*) = \sum_{\alpha, \beta} p_{\alpha, \beta} T^\alpha T^{*\beta} \quad \text{if} \quad p(z, w) = \sum_{\alpha, \beta} z^\alpha w^\beta.
\]
(Up to the order of \( T \) and \( T^* \), this coincides with the “hereditary calculus” of Agler [A1].) Hence, in particular, \( f_m(T, T^*) \) are defined for any commuting operator tuple \( T \), and \( f_0(T, T^*) = I \). Our main result is the following.

**Theorem.** Let \( \mathcal{H}, \psi_k, f_m \) be as above and let \( T \) be a commuting \( n \)-tuple of operators on a separable Hilbert space \( H \). Denote by \( M_z \) the operator \( n \)-tuple of \( z \otimes I \) on the Hilbert space tensor product \( \mathcal{H} \otimes H \). Then the following are equivalent:

(a) \( \frac{1}{n} \langle T, T^* \rangle \geq 0 \) and \( \langle f_m(T, T^*)h, h \rangle \to 0 \ \forall h \in H \).

(b) \( T^* \) is unitarily equivalent to a restriction of \( M_z^* \) to an invariant subspace.

**Example.** Let \( n = 1, \Omega = \mathbb{D} \), the unit disc, and \( \mathcal{H} = H^2 \), the Hardy space. Then \( K(z, w) = 1/(1 - zw) \), so \( \frac{1}{n}(T, T^*) = I - TT^* \), and using the standard orthonormal basis \( \psi_k(z) = z^k \) we get \( f_m(T, T^*) = T^m T^* \). Further, the operator \( M_z \) on the tensor product \( H \otimes H \) is just the forward shift of infinite multiplicity (its wandering subspace is \( I \otimes H \)). Thus we recover the result of Nagy-Foias mentioned in the beginning. \( \square \)

Other results covered by the last Theorem include the regular dilations of commuting \( n \)-tuples of operators (for \( \mathcal{H} \) the Hardy space on the polydisc \( \mathbb{D}^n \)), the \( k \)-hypercontractions of Agler [A2] (for \( \mathcal{H} \) the Bergman space on \( \mathbb{D} \) with respect to the weight \( (1 - |z|^2)^{k-2} \)), and the “spherical hypercontractions” of Müller and Vasilescu [MV] (for \( \mathcal{H} \) a certain weighted Bergman space on the unit ball of \( \mathbb{C}^n \)).

**Sketch of the proof.** (a) \( \implies \) (b). Denote \( D_T = \frac{1}{n}(T, T^*)^{1/2} \). Define an operator \( V : H \to \mathcal{H} \otimes H \) by
\[
Vh = \sum_k \psi_k(z) \otimes D_T \psi_k(T)^* h.
\]

We claim that \( V \) is well-defined (i.e. the sum converges) and is, in fact, an isometry satisfying \( VT^* = M_z^* V \). This clearly establishes (b).

To see that \( V \) is well-defined and an isometry, observe that for any \( j < m \) and \( h \in H \)
\[
\| \sum_{j \leq k < m} \psi_k(z) \otimes D_T \psi_k(T)^* h \|^2 = \sum_{j \leq k < m} \| D_T \psi_k(T)^* h \|^2
\]
\[
= \sum_{j \leq k < m} \langle \psi_k(T) D_T^2 \psi_k(T)^* h, h \rangle
\]
\[
= \langle (f_j - f_m)(T, T^*) h, h \rangle.
\]

As \( \langle f_m(T, T^*)h, h \rangle \to 0 \) by hypothesis, it follows that the partial sums of the right-hand side of (*) form a Cauchy sequence, and letting \( j = 0 \) and \( m \to \infty \) shows that \( \|Vh\|^2 = \langle f_0(T, T^*)h, h \rangle = \|h\|^2 \), i.e. \( V \) is an isometry.
To prove that $VT^* = M^*_z V$, observe that $\forall h, h' \in H$ and any $k$

$$\langle Vh, \psi_k \otimes h' \rangle = \langle D_T \psi_k(T)^* h, h' \rangle = \langle h, \psi_k(T) D_T h' \rangle,$$

so, by virtue of our choice of the basis $\psi_k$,

$$(Vh, f \otimes h') = \langle h, f(T) D_T h' \rangle$$

for any polynomial $f$. Thus

$$\langle VT^*_h, \psi_k \otimes h' \rangle = \langle T^*_j h, \psi_k(T) D_T h' \rangle = \langle T^*_j h, z_j \psi_k(T) D_T h' \rangle = \langle M^*_z Vh, \psi_k \otimes h' \rangle,$$

and the assertion follows.

(b) $\Rightarrow$ (a). Let $U : H \to H \otimes H$ be an isometry such that $UT^* = M^*_z U$. Then a simple calculation shows that for any polynomial $p(z, w)$ in $z$ and $w$,

$$p(T, T^*) = U^* p(M_z, M^*_z) U = U^* (p(Z, Z^*) \otimes I) U$$

(here, as before, $Z$ stands for the operator tuple of multiplication by the coordinate functions on $H$). Thus it suffices to show that $\frac{1}{K} (Z, Z^*) \geq 0$ and $\langle f_m(Z, Z^*) h, h \rangle \to 0 \forall h \in H$.

To see the former, recall that for any $w \in \Omega$

$$Z^*_j K_w = \overline{w}_j K_w$$

where $K_w(z) := K(z, w)$. It follows that for any polynomial $p(z, \overline{w})$ and $x, y \in \Omega$,

$$\langle p(Z, Z^*) K_y, K_x \rangle = p(x, \overline{y}) \langle K_y, K_x \rangle = p(x, \overline{y}) K(x, y),$$

so, in particular, $\langle \frac{1}{K} (Z, Z^*) K_y, K_x \rangle = 1 \forall x, y \in \Omega$. As also $\langle K_y, 1 \rangle \langle 1, K_x \rangle = \overline{1(y)} 1(x) = 1 \forall x, y \in \Omega$, it follows that

$$\frac{1}{K} (Z, Z^*) = \langle \cdot, 1 \rangle 1$$

which is a positive operator.

For the latter assertion, observe that

$$(f_0 - f_m)(Z, Z^*) h = \sum_{0 \leq k < m} \psi_k(Z) \frac{1}{K} (Z, Z^*) \psi_k(Z)^* h$$

$$= \sum_{0 \leq k < m} \psi_k(Z) (\langle \cdot, 1 \rangle 1) \psi_k(Z)^* h \quad \text{by (**)}$$

$$= \sum_{0 \leq k < m} \langle \psi_k(Z)^* h, 1 \rangle \psi_k$$

$$= \sum_{0 \leq k < m} \langle h, \psi_k \rangle \psi_k,$$
and as $\psi_k$ is an orthonormal basis and $f_0 \equiv 1$, it follows that
\[(f_0 - f_m)(Z, Z^*)h = h - f_m(Z, Z^*)h \to h \quad \text{as } m \to \infty,\]
i.e. even $f_m(Z, Z^*)h \to 0 \ \forall h \in H$. This completes the proof. \qed

We remark that in view of the boundedness of $V$, the formula (†) defines a (non-multiplicative) “functional calculus” $g \mapsto g(T)$ for functions $g$ on $\Omega$ of the form $g(z) = K(z, z)^{-1/2}f(z)$, $f \in \mathcal{H}$ (defining $g(T)h (= “f(T)D_T h”) := V^*(f \otimes h)$ one has $\|g(T)\| \leq \|V\|||f||$).

An example of function spaces $\mathcal{H}$ satisfying the axioms (1) – (4) are, for instance, various weighted Bergman and Hardy spaces on bounded symmetric domains in $\mathbb{C}^n$ (matrix balls etc.).

Under the additional hypothesis that the Taylor spectrum $\sigma(T) \subset \Omega$, it turns out that the condition $\langle f_m(T, T^*)h, h \rangle \to 0$ can be omitted, and the axioms (3) and (4) for $\mathcal{H}$ replaced by the weaker requirement that $K(z, w) \neq 0$ on $\Omega \times \Omega$. For this and further details we refer to the joint work [AEM].

**References**


