

Quantum Algorithms for Evaluating MIN-MAX Trees

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Abstract

We present a bounded-error quantum algorithm for evaluating MIN-MAX trees with $N^{\frac{1}{2}+o(1)}$ queries, where N is the size of the tree and where the allowable queries are comparisons of the form $[x_j < x_k]$. This is close to tight, since there is a known quantum lower bound of $\Omega(N^{\frac{1}{2}})$.

A MIN-MAX tree is a tree whose internal nodes are *minimum* and *maximum* gates, at alternating levels, and whose leaves are values from some underlying ordered set. The size N of such a tree is the number of its leaves, whose values are referred to as x_1, \dots, x_N . The value of a MIN-MAX tree is the value of its root, a function of x_1, \dots, x_N . In the *input value* query model, queries explicitly access the values of the leaves. In the *comparison* query model, the values of x_1, \dots, x_N are not directly accessible; rather, queries are comparisons of the form $[x_j < x_k]$. In this latter model, the appropriate output is any $j \in \{1, \dots, N\}$ such that x_j is the value of the tree.

Note that, when the ordered set is $\{0, 1\}$, a MIN-MAX tree reduces to an AND-OR tree. This implies that Barnum and Saks's lower bound of $\Omega(N^{\frac{1}{2}})$ [2] for the quantum query complexity of AND-OR trees applies to MIN-MAX trees.

Recent results initiated by Farhi *et al.* have shown that quantum algorithms can evaluate all AND-OR trees with order $N^{\frac{1}{2}+o(1)}$ queries [8, 5, 6, 1]. We show that these results carry over to MIN-MAX trees in both the input value model and the comparison model.

Let $W(N)$ be the query complexity for AND-OR trees of size N . We show that MIN-MAX trees can be evaluated with $O(W(N) \log(N))$ queries in both the input value model and the comparison model. Our algorithm combines the results on AND-OR trees in Refs. [1, 6] with the lemma below and Grover's search algorithm [9].

Lemma 1 *Let \mathcal{T} be a MIN-MAX tree with inputs x_1, x_2, \dots, x_N . Let \mathcal{T}^v be an AND-OR tree with identical structure to \mathcal{T} , but with AND and OR gates in place of MIN and MAX gates (respectively), and with the k^{th} input assigned to 1 if and only if $x_k \geq v$. Then $\text{value}(\mathcal{T}^v) = 1$ if and only if $\text{value}(\mathcal{T}) \geq v$.*

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Lemma 1 is easy to prove by induction. It implies that, if the underlying ordered set is a numerical range of size $N^{O(1)}$, then the tree can be evaluated in $\log(N)$ stages by a simple binary search. Each stage can be implemented with $O(W(N) \log \log(N))$ queries, which reflects the cost of evaluating an AND-OR tree amplified so that its error probability is $O(1/\log(N))$. The result is an $O(W(N) \log(N) \log \log(N))$ query algorithm.

A complication arises in performing such a binary search in the comparison model, where it is not possible to directly compute the midpoint of an interval like $[x_j, x_k]$. Problems can also arise in the input value model when the numerical range is too large: the binary search may not converge in a logarithmic number of steps. For this reason, we avoid the standard binary search approach where a midpoint is chosen as a pivot. Instead, we take a *random* input value among those that lie within a current interval as our pivot. What is noteworthy about this simple approach is that *it does not work efficiently in the classical case*: given an interval $[x_j, x_k]$, finding an interior point is as hard as searching, which can cost $\Omega(N)$ queries to do even once [3]. In the setting of *quantum* algorithms, we can utilize Grover's search algorithm [9, 4] which costs $O(\sqrt{N})$.

As an aside, we note that there is a classical reduction from MIN-MAX trees to AND-OR trees that yields an $O(N^{0.753})$ query algorithm for balanced MIN-MAX trees [10]. We can use that reduction with an $N^{\frac{1}{2}}$ query *quantum* algorithm for balanced AND-OR trees; however, the resulting algorithm for MIN-MAX costs $\Omega(N^{0.58})$. Our alternate approach yields exponent $\frac{1}{2} + o(1)$ and is not restricted to balanced trees.

What follows is a description of our algorithm with the analysis of its error. For convenience, let \perp and \top be such that $x_{\perp} < x_j$ and $x_{\top} > x_j$ for any $j \in \{1, \dots, N\}$ and let c be a constant.

QUANTUM MIN-MAX TREE EVALUATION

1. Let $\gamma \leftarrow \perp$ and $\delta \leftarrow \top$, and initialize the stack.
2. Repeat the following steps for $c \log(N)$ iterations, then go to Step 3:
 - (a) Find a random pivot:
Call the quantum search subroutine to find a random pivot index j with $x_{\gamma} < x_j < x_{\delta}$. If no value is found, go to Step 2(c).
 - (b) Refine the search:
Call the AND-OR tree subroutine to check if $\text{value}(\mathcal{T}) < x_j$. If so, let $\delta \leftarrow j$; otherwise, let $\gamma \leftarrow j$.
 - (c) Backtrack if out of range:
Call the AND-OR subroutine to check if $x_{\gamma} \leq \text{value}(\mathcal{T}) < x_{\delta}$. If so, push (γ, δ) onto the stack. Otherwise, pop (γ, δ) off the stack. (If the stack is empty, let $\gamma \leftarrow \perp$ and $\delta \leftarrow \top$.)
3. Return γ as an index corresponding to the value of the MIN-MAX tree.

Clearly, the algorithm makes $O(W(N) \log(N))$ queries. We claim the following.

Theorem 1 *The algorithm returns the value of the MIN-MAX tree with probability at least $\frac{2}{3}$.*

To prove Theorem 1, we must consider the progress made by the random choices of pivots as well as the error probabilities of the subroutines for AND-OR and the searches (each errs with constant probability).

To begin with, assume that the subroutines for AND-OR and search never err (thus, $x_\gamma \leq \text{value}(\mathcal{T}) < x_\delta$ at all times). Under this assumption, the progress of the algorithm is determined by how quickly the subinterval converges. Once no value in Step 2(a) is found the algorithm has *converged* (with $x_\gamma = \text{value}(\mathcal{T})$) and can go to Step 3 and terminate (however it is harmless to perform more iterations before doing this).

Let $C(m)$ denote the expected number of iterations of the algorithm until it converges, assuming that m of its inputs are within its current range.

Then, for $m > 1$, $C(m)$ satisfies the recurrence

$$C(m) \leq \frac{2}{m} \left(\sum_{k=\lfloor m/2 \rfloor}^{m-1} C(k) \right) + 1. \quad (1)$$

This can be seen by assuming that the pivot is uniformly selected among all m possible positions within the subinterval and that $\text{value}(\mathcal{T})$ always lies in the larger side of the pivot. It is straightforward to verify that the recurrence implies $C(m) \in O(\log(m))$. Therefore, the expected number of iterations of Step 2 made by the algorithm before $x_\gamma = \text{value}(\mathcal{T})$, under the assumption that the subroutines never err, is $O(\log(N))$. By the Markov bound, $O(\log(N))$ iterations suffice to obtain error probability less than any particular constant.

We now consider the fact that the subroutines for AND-OR and searching can fail. First, note that, by incurring a multiplicative factor of only $O(\log \log(N))$, each call to the AND-OR and search algorithm can be amplified so that its error probability is $O(1/\log(N))$. This results in an $O(W(N) \log(N) \log \log(N))$ algorithm for MIN-MAX.

These amplification costs are not necessary in our algorithm, since it can cope with a constant fraction of errors in subroutine calls. To see why this is so, let ε be the probability that one or more subroutines err during one iteration of Step 2 of the algorithm. The algorithm begins some $O(\log(N))$ steps away from reaching a *good* state—of the form (γ', δ) such that $x_{\gamma'} = \text{value}(\mathcal{T})$. Before reaching a good state, an “incorrect” step for the algorithm places $\text{value}(\mathcal{T})$ outside the search interval, and a “correct” step either narrows the search interval or backtracks from a previous error. After reaching a good state, a “correct” step pushes a pair of the form (γ', δ) onto the stack and an “incorrect” step pops it off. In each iteration, the algorithm takes a correct step with probability at least $1 - \varepsilon$ and an incorrect step with probability at most ε . Therefore, with all but exponentially small probability, the number of correct steps minus the number of incorrect ones after $c \log(N)$ iterations is at least $\frac{\varepsilon}{2} \log(N)$. For suitably large c this means that, with constant probability, when the algorithm terminates, $x_\gamma = \text{value}(\mathcal{T})$ (typically with many copies of pairs of the form (γ', δ) on the top of its stack).

Finally, we note that, in game-playing contexts, it is useful to determine optimal moves. This corresponds to finding the subtree of a MIN-MAX tree that attains its value. If the leaf values x_1, \dots, x_N are distinct, this is easily deduced from $\text{value}(\mathcal{T})$. Otherwise, one can use a slightly modified version of the minimum/maximum finding algorithm in Ref. [7] to find the appropriate subtree.

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