

Entangled simultaneity versus classical interactivity in communication complexity

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Abstract

In 1999 Raz demonstrated a partial function that had an efficient *quantum two-way* communication protocol but no efficient *classical two-way* protocol and asked, whether there existed a function with an efficient *quantum one-way* protocol, but still no efficient classical two-way protocol. In 2010 Klartag and Regev demonstrated such a function and asked, whether there existed a function with an efficient *quantum simultaneous-messages* protocol, but still no efficient classical two-way protocol.

In this work we answer the latter question affirmatively and present a partial function *Shape*, which can be computed by a protocol sending entangled simultaneous messages of poly-logarithmic size, and whose classical two-way complexity is lower bounded by a polynomial.

1 Introduction

The setting of communication complexity is one of the strongest computational models, as of today, where we have tools to prove “hardness” – that is, where we know problems that don’t have efficient solutions.¹ In particular, we can compare the computational power of different communication complexity regimes (*classes*) via demonstrating that certain problem has an efficient solution in one regime, but not in the other.

There are three main types of the problems used for separating communication complexity classes: *total functions*, *partial functions* and *relations*. They form a “hierarchy” in the following sense: separations via total functions are the strongest (“most convincing”), separations via relations are the weakest (and thus the easiest to obtain) and separations via partial functions are “in between”. There are known cases where a quantum communication complexity class can be separated from a classical one via a relation, while a functional separation is provably impossible (cf. [GRdW08]).

The history of (exponential) separations in communication complexity that have demonstrated the advantage of quantum communication can be briefly outlined as follows.

- In 1999 Raz [Raz99] demonstrated a *partial function* that had an efficient *quantum two-way* communication protocol, but no efficient *classical two-way* protocol.

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¹In this work by calling a problem *hard* we always mean that *no efficient solution exists* (this is somewhat different from the usual notion of hardness in the context of computational complexity).

- In 2001 Buhrman, Cleve, Watrous and de Wolf [BCWdW01] showed that a *total function* (namely, the equality function) had an efficient *quantum simultaneous-messages* protocol, but no efficient *classical simultaneous-messages* protocol (without shared randomness).
- In 2004 Bar-Yossef, Jayram and Kerenidis [BYJK04] demonstrated a *relation* that had an efficient *quantum one-way* communication protocol, but no efficient *classical one-way* protocol.
- In 2008 Kempe, Kerenidis, Raz, de Wolf and the current author demonstrated ([GKK⁺08]) the same separation via a *partial function*.
- In 2008 the current author demonstrated ([Gav08]) a *relation* with an efficient *quantum one-way* protocol, but no efficient *classical two-way* protocol; the next year the result was strengthened ([Gav09]), to admit an efficient *entangled simultaneous-messages* protocol.
- In 2010 Klartag and Regev [KR11] demonstrated a *partial function* with an efficient *quantum one-way* protocol, but no efficient *classical two-way* protocol.

In this work we demonstrate a *partial function* that can be solved by a *quantum simultaneous-messages* protocol that sends $O(\log^2 n)$ entangled qubits; the classical two-way communication complexity of the same function is $\Omega(\sqrt{n})$. In particular, this answers an open question stated by Klartag and Regev.

A number of researchers believe that strong separations of this type are not possible for *total functions* – that is, unless the model is “too weak” (like *simultaneous message passing without shared randomness*, used in [BCWdW01]), it *might* be able to efficiently “emulate” classically any quantum protocol that computes a total function. Proving or disproving this hypothesis is a major open problem. However, if it is true, then the result of this work is very close to an “as strong as possible” demonstration of qualitative advantage of quantum communication over the classical one.

We note that our communication problem also has a *quantum one-way* protocol of cost $O(\log^2 n)$ (as follows from the existence of a simultaneous-messages protocol of cost $O(\log^2 n)$ that uses $O(\log^2 n)$ bits of entanglement). Till now, the only known example of super-polynomial advantage of quantum one-way over classical two-way communication in solving a functional problem has been the one demonstrated by Klartag and Regev. Their communication problem is called *Vector in Subspace Problem (VSP)* (originally proposed by Kremer [Kre95]), and it is “complete” for functions² in quantum one-way – i.e., the input to any function that admits an efficient quantum one-way protocol can be mapped *locally* to an instance of *VSP*, which *has the same answer as the original problem* and *solving which is efficient in terms of the original input size*. Accordingly, the present work gives an alternative proof of *qualitative* hardness of *VSP* for classical two-way communication³: if there were an efficient protocol for *VSP*, there would be one for our problem as well, and we prove the opposite.

A definition of our communication problem and an overview of our approach are given in Sections 2.1 and 2.1.1, respectively.

²both total and partial, though not for relations!

³Quantitatively, the lower bound given in [KR11] for the classical two-way complexity of *VSP* is much stronger: it is $\Omega(n^{1/3})$, while the following argument gives only $2^{\Omega(\sqrt{\log n})}$.

2 Preliminaries

For $x \in \{0, 1\}^n$ and $1 \leq i \leq n$, we will use both x_i and $x(i)$ to address the i 'th bit of x . Similarly, for $S \subseteq \{1, \dots, n\}$, both x_S and $x(S)$ will denote the $|S|$ -bit string, consisting of naturally-ordered bits of x , whose indices are in S . For $y \in \{0, 1\}^n$, let $x_y \stackrel{\text{def}}{=} x_{\{i|y_i=1\}}$. Let $|x|$ denote the Hamming weight of x . Let $x \oplus y$, $x \wedge y$, $x \vee y$ and $\neg x$ denote, respectively, the bit-wise XOR, AND, OR and NOT operations.

For any discrete set A , let \mathcal{U}_A denote the uniform distribution on A . Sometimes (e.g., in subscripts) we will write “ $\subseteq A$ ” instead of “ $\sim \mathcal{U}_A$ ”. We will sometimes emphasise that a distribution on $\{0, 1\}^{2n}$ is “viewed as bipartite” (i.e., assumed to be the joint distribution of two random variables, containing n bits each) by calling it a *distribution on* $\{0, 1\}^{n+n}$; similarly, we will write “ $(X, Y) \in \{0, 1\}^{n+n}$ ”, etc.

For a random variable $X \sim \mu$, we will use both $H(X)$ and $H(\mu)$ to denote the corresponding binary entropy, and similarly for the (binary) min-entropy $H_{\min}(X) = H_{\min}(\mu) \stackrel{\text{def}}{=} \min_{x_0} \{\log(1/\mu(x_0))\}$ and their conditional versions⁴. We need the following “weak chain rule with strong tail bound” for min-entropy:⁵

Claim 1. *Let ν be a bipartite distribution on $A \times B$. Then*

$$\begin{aligned} \mathbf{E}_{(Y_1, Y_2) \sim \nu} \left[H_{\min}(X_2 | X_1 = Y_1) \right] &\geq H_{\min}(\nu) - H_{(X_1, X_2) \sim \nu}(X_1) \\ &\geq H_{\min}(\nu) - \log |A| \end{aligned}$$

and for any $\Delta \geq 0$:

$$\Pr_{(Y_1, Y_2) \sim \nu} \left[H_{\min}(X_2 | X_1 = Y_1) \leq H_{\min}(\nu) - \log |A| - \Delta \right] \leq 2^{-\Delta}.$$

Proof. For all $x_1 \in A$ and $x_2 \in B$,

$$\begin{aligned} H_{\min}(\nu) &\leq \log \frac{1}{\Pr_{\nu}[X_1 = x_1, X_2 = x_2]} = \log \frac{1}{\Pr_{\nu}[X_1 = x_1] \cdot \Pr_{\nu}[X_2 = x_2 | X_1 = x_1]} \\ &= \log \frac{1}{\Pr_{\nu}[X_1 = x_1]} + \log \frac{1}{\Pr_{\nu}[X_2 = x_2 | X_1 = x_1]}. \end{aligned}$$

Taking x_2 that minimises the second term,

$$H_{\min}(\nu) \leq \log \frac{1}{\Pr_{\nu}[X_1 = x_1]} + H_{\min}^{\nu}(X_2 | X_1 = x_1). \quad (1)$$

⁴In the case of min-entropy we only allow *conditioning on events*, defined as the min-entropy of the corresponding distribution.

⁵In writing this paper we have been looking for more natural and sometimes more general formulations. E.g., Claim 1 allows phrasing Lemma 3 in terms of min-entropy; a more ad hoc formulation would get rid of the claim at the expense of – subjectively – less naturally-sounding lemma statement.

Averaging it over all x_1 -s gives the first desired inequality:

$$\begin{aligned} H_{\min}(\nu) &\leq \mathbf{E}_{(Y_1, Y_2) \sim \nu} \left[\log \frac{1}{\Pr_{(X_1, X_2) \sim \nu} [X_1 = Y_1]} + H_{\min} (X_2 | X_1 = Y_1) \right] \\ &= H_{(X_1, X_2) \sim \nu} (X_1) + \mathbf{E}_{(Y_1, Y_2) \sim \nu} \left[H_{\min} (X_2 | X_1 = Y_1) \right]. \end{aligned}$$

Averaging (1) over

$$A_\Delta \stackrel{\text{def}}{=} \left\{ x_1 \in A \mid H_{\nu} (X_2 | X_1 = x_1) \leq H_{\min}(\nu) - \log |A| - \Delta \right\}$$

gives the tail bound:

$$\begin{aligned} H_{\min}(\nu) &\leq \sum_{x_1 \in A_\Delta} \frac{\Pr_\nu [X_1 = x_1]}{\Pr_\nu [X_1 \in A_\Delta]} \cdot \left(\log \frac{1}{\Pr_\nu [X_1 = x_1]} + H_{\nu} (X_2 | X_1 = x_1) \right) \\ &= H_{(X_1, X_2) \sim \nu} (X_1 | X_1 \in A_\Delta) + \log \frac{1}{\Pr_\nu [X_1 \in A_\Delta]} \\ &\quad + \sum_{x_1 \in A_\Delta} \frac{\Pr_\nu [X_1 = x_1]}{\Pr_\nu [X_1 \in A_\Delta]} \cdot H_{\min} (X_2 | X_1 = x_1) \\ &\leq \log |A| + \log \frac{1}{\Pr_\nu [X_1 \in A_\Delta]} + H_{\min}(\nu) - \log |A| - \Delta, \end{aligned}$$

and so, $\log \frac{1}{\Pr_\nu [X_1 \in A_\Delta]} \geq \Delta$, as required. ■

We will use the following bound on the l_1 -distance between two distributions:

Claim 2. Let ν_1 and ν_2 be distributions on $\{0, 1\}^n$. Then

$$\|\nu_1 - \nu_2\|_1^2 \leq 8 \ln 2 \cdot \left(n - \min \left\{ H(\nu_1), H(\nu_2) \right\} \right).$$

Proof. Let u be the uniform distribution on $\{0, 1\}^n$, then

$$H(\mu) = \sum_x \mu(x) \log \frac{1}{\mu(x)} = - \sum_x \mu(x) \log \frac{\mu(x)}{2^{-n}} + \sum_x \mu(x) \log \frac{1}{2^{-n}} = n - d_{KL}(\mu \| u).$$

From the triangle and Pinsker's inequalities,

$$\begin{aligned} \|\nu_1 - \nu_2\|_1^2 &\leq 4 \max_{j \in \{1, 2\}} \left\{ \|\nu_j - u\|_1^2 \right\} \\ &\leq 8 \ln 2 \cdot \max_{j \in \{1, 2\}} \left\{ d_{KL}(\nu_j \| u) \right\} = 8 \ln 2 \cdot \left(n - \min \left\{ H(\nu_1), H(\nu_2) \right\} \right). \end{aligned}$$
■

Let S_n denote the group of permutations of the set $\{1, \dots, n\}$, and let $\sigma_i \in S_n$ be the i 'th cyclic permutation (i.e., $\sigma_i(j) = i + j$ if $i + j \leq n$ and $i + j - n$ otherwise). For $x \in \{0, 1\}^n$ and $\tau \in S_n$, denote by $\tau(x)$ the element of $\{0, 1\}^n$, whose $\tau(i)$ 'th position contains x_i for each i – in particular, $\sigma_j(x)$ is the j -bit cyclic shift of x .

2.1 Communication complexity

Please see [KN97] for an extensive overview of classical communication complexity. The quantum counterparts differ from the classical communication models in two aspects: the players are allowed to send quantum messages (accordingly, the complexity is measured in *qubits*) and to perform arbitrary quantum operations locally. We say that a communication model allows *prior entanglement* if the players can share any (input-independent) quantum state and use it in the protocol (in the case of simultaneous message passing, entanglement is only allowed between Alice and Bob).

The communication problem that we use for our separation is the following partial function.

Definition 1 (*Shape – Shifted Approximate Equality*). *Let Alice receive $(x_1, x_2) \in \{0, 1\}^{n+n}$ and Bob receive $(y_1, y_2) \in \{0, 1\}^{n+n}$. Then*

$$\text{Shape}(x_1, x_2, y_1, y_2) = \begin{cases} 1 & \text{if } \exists i : |\sigma_i(x_1) \oplus x_2 \oplus \sigma_i(y_1) \oplus y_2| \leq \frac{2n}{5}; \\ 0 & \text{if } \forall i : \frac{7n}{15} \leq |\sigma_i(x_1) \oplus x_2 \oplus \sigma_i(y_1) \oplus y_2| \leq \frac{8n}{15}; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

That is, $\text{Shape}(x_1, x_2, y_1, y_2)$ “asks” whether there exists a cyclic shift σ_i , such that $\sigma_i(x_1) \oplus x_2$ is close to $\sigma_i(y_1) \oplus y_2$. Note that both the “meaning of closeness” and the promised gap in the definition of Shape are $\Omega(n)$ – as we will see, the former is crucial for the lower bound argument, while the latter is (apparently) essential in order to admit an efficient quantum protocol.

2.1.1 Our techniques

To design a simultaneous-messages protocol for Shape , we will first build a protocol for the following “sub-problem”:

$$\text{Shape}_i(x_1, x_2, y_1, y_2) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } |\sigma_i(x_1) \oplus x_2 \oplus \sigma_i(y_1) \oplus y_2| \leq \frac{2n}{5}; \\ 0 & \text{if } \frac{7n}{15} \leq |\sigma_i(x_1) \oplus x_2 \oplus \sigma_i(y_1) \oplus y_2| \leq \frac{8n}{15}; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Our protocol for Shape_i will be such, that the quantum messages sent by the players will be independent of i , and only the referee will need to know i in order to measure the messages and produce the answer. Accordingly, reducing the error of solving Shape_i to sufficiently small (inverse-polynomial) value and sequentially applying the referee’s measurements corresponding to all $0 \leq i < n$ to the *same* quantum messages received from the players will produce answers to all n instances of Shape_i (with respect to the actual input), and with high probability all these answers will be correct. This gives a protocol for Shape .

To show hardness of Shape for classical two-way communication, we start by using relatively standard entropy-based arguments to say – towards contradiction – that if there is a short protocol for Shape , then there exists a large subset $A \subseteq \{0, 1\}^{n+n}$, such that *when* $(X, Y) \sim \mathcal{U}_A$, “something” is known about $\sigma_i(X) \oplus Y$ for every $0 \leq i < n$ – namely, $\Omega(1)$ bits of entropy are “missing”, on average, in each of the corresponding distributions. One possible way for A to have this property would be to “fix” certain bits of all its elements, and the number of fixed bit-positions should be sufficient in order to have them “overlap” in every

$\sigma_i(x) \oplus y$ when $(x, y) \in A$ – that is, roughly $\Omega(\sqrt{n})$ positions must be fixed (according to the “birthday paradox”), which means that A can have size at most $2^{2n-\Omega(\sqrt{n})}$, which, in turn, is sufficient for our lower bound.

However, there is another possibility for A to have the same property – namely, to fix only the bit-parity of all its elements and have size 2^{2n-1} , leading to no meaningful lower bound. This is the reason why *Shape* has been defined “with margins” (the problem asks whether the bit-wise XOR of strings is *close to* $\bar{0}$ – not necessarily equals it): this way we can choose the input distribution to be “noisy” and draw a stronger conclusion from the existence of a short protocol, namely that “*something*” is known about every $\sigma_i(X) \oplus \tilde{Y}$ when $(X, Y) \sim \mathcal{U}_A$, where \tilde{Y} is a noisy version of the random variable Y . Noise is known to “damage” high-degree Fourier coefficients (as quantified by the famous hypercontractive inequality) – in particular, a set A that only restricts the bit-parity of its elements would fail miserably with respect to the “noisy condition”.

We would like to use the hypercontractivity to prove an upper bound on the size of A ; indeed, the “noisy condition” is very similar to what has been analysed in [GKK⁺08] and led to a good bound on $|A|$. There is an interesting distinction between the two cases: in [GKK⁺08] “something” had to be known about $\tau(X) \oplus \tilde{Y}$ for every $\tau \in S_n$, whereas in this work we can only require that it holds for the σ_i s.⁶ The argument in [GKK⁺08] relies strongly on the symmetry resulting from allowing all $n!$ permutation – our modestly-permuted *Shape* seems to ask for different treatment.

There are at least two naturally-looking approaches to analyse a set A that satisfies the “noisy condition” with respect to a small family of permutations. First, we can use the fact that any such family corresponds to a rather small (of size roughly $2^n \cdot \langle$ the number of permutations \rangle) subset of all 2^{2n} Fourier coefficients in the characteristic function of A , which are “responsible” for the entire entropy loss – we can try to investigate the Fourier structure of Boolean functions that are “heavily supported” on the corresponding subset. Second, we can use entropy-inspired arguments and try to show that staying distinguishable after the action of one of polynomially-many allowed permutations followed by noise is, essentially, as hard as staying distinguishable after the action of a uniformly-random (disclosed) permutation from S_n . The proof will use the second approach.⁷

3 Solving *Shape* with simultaneous entangled messages

Here we give a protocol for solving *Shape* in $\mathcal{Q}^{\parallel, ent}$ – the model of quantum simultaneous message passing with entanglement (between Alice and Bob).

Protocol for *Shape_i*: In the beginning Alice and Bob share two states: $|A_0\rangle = \frac{1}{\sqrt{n}} \cdot \sum_{k=1}^n |k\rangle |k\rangle$ and $|B_0\rangle = \frac{1}{\sqrt{n}} \cdot \sum_{j=1}^n |j\rangle |j\rangle$, where Alice holds the first “register” and Bob

⁶Instead of $\{\sigma_i | 0 \leq i < n\}$ we could have fixed any other family, containing up to quasi-polynomial number of permutations; it may not be larger than that, as our quantum protocol would no longer be able to solve the problem efficiently.

⁷We still use the hypercontractivity to conclude the proof, in a way somewhat similar to [KKL88] (cf. Claim 3); in the language of the above informal description, that is done *after* using the presence of noise in the input distribution in order to limit (via entropy-inspired arguments) the potential role of high-degree Fourier coefficients.

holds the second “register” of each state. Upon receiving the input, the players apply local conditional phase-negations to transform the shared states into the form

$$|A_1\rangle = \frac{1}{\sqrt{n}} \cdot \sum_{k=1}^n (-1)^{x_1(k)+y_1(k)} |k\rangle |k\rangle, \quad |B_1\rangle = \frac{1}{\sqrt{n}} \cdot \sum_{j=1}^n (-1)^{x_2(j)+y_2(j)} |j\rangle |j\rangle.$$

Then the players send the parts of the shared states to the referee, who applies σ_{-i} to both registers of $|A_1\rangle$, resulting in

$$|A_2\rangle = \frac{1}{\sqrt{n}} \cdot \sum_{k=1}^n (-1)^{x_1(k)+y_1(k)} |\sigma_{-i}(k)\rangle |\sigma_{-i}(k)\rangle = \frac{1}{\sqrt{n}} \cdot \sum_{k=1}^n (-1)^{x_1(\sigma_i(k))+y_1(\sigma_i(k))} |k\rangle |k\rangle.$$

At this point,

$$\langle A_2 | B_1 \rangle = \frac{1}{n} \cdot \sum_{k=1}^n (-1)^{x_1(\sigma_i(k))+x_2(k)+y_1(\sigma_i(k))+y_2(k)} = 1 - \frac{2}{n} \cdot |\sigma_i(x_1) \oplus x_2 \oplus \sigma_i(y_1) \oplus y_2|$$

– in particular,

$$|\langle A_2 | B_1 \rangle| \begin{cases} \leq \frac{1}{15} & \text{if } \text{Shape}_i(x_1, x_2, y_1, y_2) = 0, \\ \geq \frac{1}{5} & \text{if } \text{Shape}_i(x_1, x_2, y_1, y_2) = 1. \end{cases}$$

The referee can distinguish the two cases with confidence $1/2 + \Omega(1)$ by performing the swap-test – a two-outcome measurement that “accepts” with probability $\frac{1+|\langle \phi_1 | \phi_2 \rangle|^2}{2}$ when performed over the pair of (pure) quantum states $|\phi_1\rangle$ and $|\phi_2\rangle$.

The above protocol can be repeated $O(\log \frac{1}{\varepsilon})$ times in parallel to bring the error down to any $\varepsilon > 0$ – let $\mathcal{P}_{i,\varepsilon}$ denote the resulting protocol. The total communication cost of $\mathcal{P}_{i,\varepsilon}$ is $O(\log n \cdot \log \frac{1}{\varepsilon})$ and it uses $O(\log n \cdot \log \frac{1}{\varepsilon})$ bits of entanglement.

Let $(\Pi_{i,\varepsilon}, I - \Pi_{i,\varepsilon})$ be the 2-outcome projective measurement that the referee applies in $\mathcal{P}_{i,\varepsilon}$ to the messages received from the players in order to determine the answer (with outcome $\Pi_{i,\varepsilon}$ corresponding to answering “ $\text{Shape}_i(x_1, x_2, y_1, y_2) = 1$ ”), and let this be the only step performed by the referee.⁸ Note that running $\mathcal{P}_{i,\varepsilon}$ didn’t require either Alice or Bob to know the actual value of i – only the referee had to know it in order to apply $(\Pi_{i,\varepsilon}, I - \Pi_{i,\varepsilon})$. This makes $\mathcal{P}_{i,\varepsilon}$ a perfect “building block” for solving the original problem.

Protocol for Shape : Let Alice and Bob send to the referee their messages, as prescribed by $\mathcal{P}_{1,\varepsilon'}$, for some ε' to be fixed soon. The referee sequentially measures the received messages with $(\Pi_{i,\varepsilon'}, I - \Pi_{i,\varepsilon'})$ for all $0 \leq i < n$. If at least one outcome $\Pi_{i,\varepsilon'}$ has been obtained, the referee answers “ $\text{Shape}(x_1, x_2, y_1, y_2) = 1$ ”; otherwise, “ $\text{Shape}(x_1, x_2, y_1, y_2) = 0$ ”.

Call the above protocol \mathcal{P} . Assume without loss of generality that $\text{Shape}(x_1, x_2, y_1, y_2) \in \{0, 1\}$ (i.e., the input fulfils the promise). To analyse the error of \mathcal{P} , note that the protocol can return the wrong answer only if at some round i the outcome of the measurement $(\Pi_{i,\varepsilon'}, I - \Pi_{i,\varepsilon'})$ was *wrong* – that is, the outcome was $\Pi_{i,\varepsilon'}$ while $\text{Shape}_i(x_1, x_2, y_1, y_2) = 0$, or vice versa. Note that while the probability of the outcome of the first performed measurement

⁸That is, the measurement $(\Pi_{i,\varepsilon}, I - \Pi_{i,\varepsilon})$ “contains” all the steps taken by the referee according to $\mathcal{P}_{i,\varepsilon}$. Alternatively, we can say that the referee “uncomputes” his previous (recursive) steps upon having performed the (only) measurement in the end of $\mathcal{P}_{i,\varepsilon}$.

being wrong is bounded above by ε' (as follows trivially from the error bound of $\mathcal{P}_{1,\varepsilon'}$), at the subsequent rounds the state being measured may have been “distorted” by the earlier measurements, which, in turn, may increase the error probability.

We analyse⁹ the probability that *the first wrong outcome during execution of \mathcal{P} has occurred at round j* – denote it by ε_j . For all $0 \leq i \leq j$, let $\Pi'_i \in \{\Pi_{i,\varepsilon'}, I - \Pi_{i,\varepsilon'}\}$ be the *right* outcome for round i . Let v be a unit vector representing the (pure) quantum state of the original messages received by the referee from the players (before any measurement has been applied). Denote:

$$\Pi'_i v = v + u_i.$$

From correctness of $\mathcal{P}_{i,\varepsilon'}$ s it follows that $\|u_i\|_2 \leq \sqrt{\varepsilon'}$.

Let v_j be the input state for the j 'th measurement. As we are assuming that all the previous measurements produced the right answers,

$$v_j = \frac{\Pi'_{j-1} \cdots \Pi'_1 v}{\left\| \Pi'_{j-1} \cdots \Pi'_1 v \right\|_2} = \frac{v + \Pi'_{j-1} \cdots \Pi'_2 u_1 + \cdots + \Pi'_{j-1} u_{j-2}}{\left\| \Pi'_{j-1} \cdots \Pi'_1 v \right\|_2} = \alpha(v + v'_j),$$

where $\alpha = 1/\|\Pi'_{j-1} \cdots \Pi'_1 v\|_2 \geq 1$ and $\|v'_j\|_2 \leq \sum_{i=1}^{j-1} \|u_i\|_2 < n\sqrt{\varepsilon'}$. Then

$$\varepsilon_j = 1 - \|\Pi'_j v_j\|_2^2 = 1 - \alpha^2 \|\Pi'_j v + \Pi'_j v'_j\|_2^2 \leq 1 - \left\| v \cdot (1 - \|u_j\|_2 - \|v'_j\|_2) \right\|_2^2 < 4n\sqrt{\varepsilon'}.$$

The error probability of \mathcal{P} is at most $\sum_{j=1}^n \varepsilon_j < 4n^2\sqrt{\varepsilon'}$ – choosing $\varepsilon' = \frac{\varepsilon^2}{16n^4}$ makes it less than ε . The resulting communication cost and entanglement requirements of \mathcal{P} are $O(\log n \cdot \log \frac{1}{\varepsilon'}) = O(\log^2 n + \log n \cdot \log \frac{1}{\varepsilon})$.

Theorem 1. *The complexity of Shape in $\mathcal{Q}^{\parallel,ent}$ is $O(\log^2 n)$.*

4 Solving Shape with classical interaction

In this and the next sections we prove a lower bound on the complexity of Shape in \mathcal{R} – the model of interactive classical communication:

Theorem 2. *The complexity of Shape in \mathcal{R} is $\Omega(\sqrt{n})$.*

The above bound is nearly-tight.¹⁰

We start by introducing several distributions.

Definition 2 (*Useful distributions* – T_δ , μ_0 , μ_1 and μ).

⁹Below we give a simple proof of a sub-optimal bound, sufficient for us. A more delicate treatment of similar setting is implicit in [ANTSV02] and explicit in Lemma 2 of [Aar04].

¹⁰Due to the “birthday paradox”, a single message from Alice that contains the values of $\Theta(\sqrt{n \log n})$ randomly-chosen bits of (x_1, x_2) allows Bob to answer $Shape(x_1, x_2, y_1, y_2)$ with polynomially-small error probability – using shared randomness to choose the positions, this can be implemented by a protocol of cost $O(\sqrt{n \log n})$. A similar approach can be used to get an $O(\sqrt{n \log n})$ -bit protocol in $\mathcal{R}^{\parallel,pub}$ – the model of classical simultaneous message passing with shared randomness.

- For $\delta \in [0, 1/2]$, let T_δ be the distribution on $\{0, 1\}^n$, where each position is 1 with probability δ , independently from the other positions. For a random variable $X \sim \nu$ taking values from $\{0, 1\}^n$, let $T_\delta(X)$ be the variable $X \oplus Z$ and $T_\delta(\nu)$ be the distribution of $T_\delta(X)$, and for $x \in \{0, 1\}^n$, let $T_\delta(x)$ be the variable $x \oplus Z$, where $Z \sim T_\delta$ in all cases.
- Let μ_0 be the uniform distribution on $\{0, 1\}^{4n}$.
- For $0 \leq i < n$, let $\mu_1^{(i)}$ be the uniform distribution of (X_1, X_2, Y_1, Y_2) taking values from $\{0, 1\}^{4n}$, modulo the condition $\sigma_i(X_1) \oplus X_2 \oplus \sigma_i(Y_1) \oplus Y_2 = \bar{0}$, and let $\tilde{\mu}_1^{(i)}$ be the distribution of $(T_{3/8}(X_1), X_2, Y_1, Y_2)$ when $(X_1, X_2, Y_1, Y_2) \sim \mu_1^{(i)}$. Let μ_1 be the distribution of sampling from $\tilde{\mu}_1^{(i)}$ for uniformly random i .
- Let μ be the distribution of sampling $(X_1, X_2, Y_1, Y_2) \sim \mu_Z$ when $Z \sim \mathcal{U}_{\{0,1\}}$.

Note that $\Pr_{\mu_j}[\text{Shape}(X_1, X_2, Y_1, Y_2) = j] \in 1 - o(1)$ for $j \in \{0, 1\}$, and therefore,

$$\Pr_{\mu}[\text{Shape}(X_1, X_2, Y_1, Y_2) = j] \in 1/2 \pm o(1).$$

We will show that solving *Shape* under μ with small constant error¹¹ requires $\Omega(\sqrt{n})$ bits of communication.

Proof outline

In Step 1 we say that if a short protocol were solving *Shape*, then there would exist a large input rectangle $A \times B$, strongly biased towards “ $\text{Shape}(x_1, x_2, y_1, y_2) = 0$ ” under μ . In Step 2 we say that if the above were true, then either A or B would “tell something” about each possible “noisy shifted convolution” of its elements (i.e., $\Omega(1)$ bits would be known about every $\sigma_i(x_1) \oplus T_{1/4}(x_2)$, $0 \leq i \leq n - 1$, when (x_1, x_2) is a uniformly-random element of the set). In Step 3 we say that if the above were true, then $\Omega(\sqrt{n})$ bits would be known about a uniformly-random element of either A or B . In Step 4 we conclude that Theorem 2 holds.

Step 1

If there were a short protocol solving *Shape* with small error, then there would exist a large input rectangle, strongly biased towards “ $\text{Shape}(x_1, x_2, y_1, y_2) = 0$ ” under μ . Formally:

Lemma 1. *Assume that a protocol of cost c solves *Shape* in \mathcal{R} with error at most $\varepsilon \leq 1/9$. Then for n large enough, there exists a rectangle $A \times B \subseteq \{0, 1\}^{2n} \times \{0, 1\}^{2n}$ of size at least 2^{4n-c-3} , such that*

$$\mu_1(A \times B) \leq 4\varepsilon \cdot \mu_0(A \times B).$$

Proof. If there is a randomised protocol of cost c that solves *Shape* with error at most ε , then some value of protocol’s random string achieves at most the same error under the input distribution μ – by fixing that value we obtain a *deterministic* protocol that solves *Shape* under μ with error at most ε . Any such protocol partitions the set of possible input pairs $((x_1, x_2), (y_1, y_2))$ into (at most) 2^c combinatorial rectangles.

¹¹The support of μ is $\{0, 1\}^{4n}$ – in particular, sometimes *Shape* is undefined under μ ; however, this happens with probability $o(1)$ and a good protocol must return *the* right answer from $\{0, 1\}$ for the given input always, except with arbitrarily small constant probability.

From the correctness of the protocol and the definition of μ it follows that with probability at least $1/2 - \varepsilon - o(1)$ the input belongs to a rectangle labelled with “0”. As there are at most 2^c rectangles, with probability at least $7/8$ the input belongs to a rectangle of size at least 2^{-c-3} . As the total error is at most ε , with probability at least $3/4$ the input belongs to a rectangle with relative error at most 4ε .

Assuming $\varepsilon \leq 1/9$, the probability of all three listed events taking place simultaneously is at least $1/72 - o(1)$, and the result follows. \blacksquare

Step 2

If input rectangle $A \times B$ is strongly biased towards “ $Shape(X_1, X_2, Y_1, Y_2) = 0$ ”, then in expectation with respect to uniformly-random i , $\Omega(1)$ bits are “known” either about $\sigma_i(X_1) \oplus T_{1/4}(X_2)$ when $(X_1, X_2) \sim \mathcal{U}_A$ or about $\sigma_i(Y_1) \oplus T_{1/4}(Y_2)$ when $(Y_1, Y_2) \sim \mathcal{U}_B$. Formally:

Lemma 2. *If $A \times B \subseteq \{0, 1\}^{2n} \times \{0, 1\}^{2n}$ satisfies $\mu_1(A \times B) \leq 4\varepsilon \cdot \mu_0(A \times B)$, then*

$$\min \left\{ \mathbf{E}_i \left[\mathbf{H}_{(X_1, X_2) \in A} (\sigma_i(X_1) \oplus T_{1/4}(X_2)) \right], \mathbf{E}_i \left[\mathbf{H}_{(Y_1, Y_2) \in B} (\sigma_i(Y_1) \oplus T_{1/4}(Y_2)) \right] \right\} \leq n - \frac{1}{92} + \varepsilon,$$

where $i \sim \mathcal{U}_{\{0, \dots, n-1\}}$.

Proof. By definition, $\mu_1 = \frac{1}{n} \sum_{i=0}^{n-1} \tilde{\mu}_1^{(i)}$, and therefore,

$$\frac{1}{n} \cdot \sum_{i=0}^{n-1} \tilde{\mu}_1^{(i)}(A \times B) \leq 4\varepsilon \cdot \mu_0(A \times B).$$

In particular, for some $I_0 \subseteq \{0, \dots, n-1\}$, $|I_0| \geq n/2$:

$$\forall i \in I_0 : \tilde{\mu}_1^{(i)}(A \times B) \leq 8\varepsilon \cdot \mu_0(A \times B).$$

Let $i_0 \in I_0$. Note that the noise operator is symmetric: $\Pr [T_\delta(a) = b] = \Pr [T_\delta(b) = a]$ for every a and b . Furthermore, $T_\delta(a \oplus b) = T_\delta(a) \oplus b$ and $T_{\delta_1}(T_{\delta_2}(a)) = T_{\delta_1(1-\delta_2)+\delta_2(1-\delta_1)}(a)$. Therefore,

$$\tilde{\mu}_1^{(i_0)}(x_1, x_2, y_1, y_2) \equiv \frac{\Pr [\sigma_{i_0}(x_1) \oplus T_{1/4}(x_2) = \sigma_{i_0}(y_1) \oplus T_{1/4}(y_2)]}{2^{3n}}.$$

Accordingly,

$$\begin{aligned} \frac{8\varepsilon \cdot |A \times B|}{2^{4n}} &= 8\varepsilon \cdot \mu_0(A \times B) \\ &\geq \tilde{\mu}_1^{(i_0)}(A \times B) = \sum_{\substack{(x_1, x_2) \in A \\ (y_1, y_2) \in B}} \frac{\Pr [\sigma_{i_0}(x_1) \oplus T_{1/4}(x_2) = \sigma_{i_0}(y_1) \oplus T_{1/4}(y_2)]}{2^{3n}} \end{aligned}$$

and so,

$$\mathbf{E}_{\substack{(X_1, X_2) \sim \mathcal{U}_A \\ (Y_1, Y_2) \sim \mathcal{U}_B}} \left[\Pr [\sigma_{i_0}(X_1) \oplus T_{1/4}(X_2) = \sigma_{i_0}(Y_1) \oplus T_{1/4}(Y_2)] \right] \leq \frac{8\varepsilon}{2^n}.$$

Let Z_1 be $\sigma_{i_0}(X_1) \oplus T_{1/4}(X_2)$ when $(X_1, X_2) \sim \mathcal{U}_A$ and Z_2 be $\sigma_{i_0}(Y_1) \oplus T_{1/4}(Y_2)$ when $(Y_1, Y_2) \sim \mathcal{U}_B$. Denote by ν_1 and ν_2 the distributions of Z_1 and Z_2 , respectively. Then

$$\frac{8\varepsilon}{2^n} \geq \Pr_{\substack{Z_1 \sim \nu_1 \\ Z_2 \sim \nu_2}} [Z_1 = Z_2] = \mathbf{E}_{Z \sim \nu_1} [\nu_2(Z)].$$

Therefore, $\Pr_{Z \sim \nu_1} [\nu_2(Z) \leq 16\varepsilon/2^n] \geq 1/2$; let $\mathcal{Z}_0 \stackrel{\text{def}}{=} \{z \in \{0, 1\}^n \mid \nu_2(Z) \leq 16\varepsilon/2^n\}$. Then

$$\|\nu_1 - \nu_2\|_1 \geq \sum_{z \in \mathcal{Z}_0} |\nu_1(z) - \nu_2(z)| \geq \nu_1(\mathcal{Z}_0) - \nu_2(\mathcal{Z}_0) \geq \frac{1}{2} - 16\varepsilon.$$

By Claim 2,

$$n - \min \left\{ H(\nu_1), H(\nu_2) \right\} \geq \frac{(\frac{1}{2} - 16\varepsilon)^2}{8 \ln 2} > \frac{1}{23} - 3\varepsilon.$$

Finally, since $H(\sigma_i(X_1) \oplus T_{1/4}(X_2)), H(\sigma_i(Y_1) \oplus T_{1/4}(Y_2)) \leq n$ always,

$$\begin{aligned} & \sum_{i=0}^{n-1} \left(n - \min \left\{ H_{\mathcal{U}_A}(\sigma_i(X_1) \oplus T_{1/4}(X_2)), H_{\mathcal{U}_B}(\sigma_i(Y_1) \oplus T_{1/4}(Y_2)) \right\} \right) \\ & \geq \sum_{i \in I_0} \left(n - \min \left\{ H_{\mathcal{U}_A}(\sigma_i(X_1) \oplus T_{1/4}(X_2)), H_{\mathcal{U}_B}(\sigma_i(Y_1) \oplus T_{1/4}(Y_2)) \right\} \right) \\ & > \frac{n}{2} \cdot \left(\frac{1}{23} - 3\varepsilon \right) \end{aligned}$$

and so,

$$\min \left\{ \mathbf{E}_i \left[H_{\mathcal{U}_A}(\sigma_i(X_1) \oplus T_{1/4}(X_2)) \right], \mathbf{E}_i \left[H_{\mathcal{U}_B}(\sigma_i(Y_1) \oplus T_{1/4}(Y_2)) \right] \right\} < n - \frac{1}{92} + \varepsilon,$$

as required. ■ Lemma 2

Step 3

If $\Omega(1)$ bits are “known” on average about each of $\sigma_i(X_1) \oplus T_{1/4}(X_2)$ when $(X_1, X_2) \sim \nu$, then $\Omega(\sqrt{n})$ bits are “known” about (X_1, X_2) itself. Formally:

Lemma 3. *If ν is a distribution on $\{0, 1\}^{n+n}$ for large enough n , such that*

$$\mathbf{E}_{i \in \{0, \dots, n-1\}} \left[H_{(X_1, X_2) \sim \nu}(\sigma_i(X_1) \oplus T_{1/4}(X_2)) \right] \leq n - \delta$$

for $\delta > 0$, then

$$H_{\min}(\nu) \leq 2n - \frac{\sqrt{\delta n}}{29}.$$

A proof of the lemma is given in Section 5. Note that the noise operator T is crucial for the statement to hold; without it, ν being the uniform distribution over all strings of even parity would provide a counterexample (this has been discussed in more detail in Section 2.1.1). Note also that the bound in the conclusion is asymptotically tight, as “entropy deficit” of $O(\sqrt{n})$ bits is sufficient for ν to satisfy the condition (cf. Section 2.1.1). On the other hand, the proof will not rely on any specific properties of the σ_i -s, and the condition can be somewhat relaxed by assuming sufficient entropy loss on the edge-wise XORs when averaging over a sufficiently large family of pairwise-disjoint perfect matchings between the bits of X_1 and $T_{\Omega(1)}(X_2)$.

Step 4

If a protocol of cost c solves *Shape* in \mathcal{R} with error at most $\varepsilon \leq 1/93$, then Lemma 1 guarantees that the conditions of Lemma 2 are satisfied by some rectangle $A \times B$ of size at least 2^{4n-c-3} . Lemma 2 guarantees that either $\nu = \mathcal{U}_A$ or $\nu = \mathcal{U}_B$ satisfies the condition of Lemma 3 for $\delta = \frac{1}{92} - \varepsilon$, which, in turn, implies that $\min\{|A|, |B|\} \leq 2^{2n-\Omega(\sqrt{n})}$. Theorem 2 follows.

5 Chasing the lost entropy

To prove Lemma 3, we will use some tools related to Boolean Fourier transform and hypercontractivity.¹²

Note that even though the noise operator T is a crucial part of the lemma condition (as discussed earlier), the proof will “handle” T with information-theoretic methods (cf. Lemma 4), while hypercontractivity (a technique, closely associated with the noise operator) will be used as an analytic tool for analysing the Fourier spectrum of certain “noiseless” probability distribution (cf. Lemma 5).

For $f, g : \{0, 1\}^n \rightarrow \mathbb{R}$ and $p > 0$, let $\|f\|_p \stackrel{\text{def}}{=} (\mathbf{E}_{x \in \{0,1\}^n} [|f(x)|^p])^{1/p}$, $\|f\|_\infty \stackrel{\text{def}}{=} \max_x \{|f(x)|\}$ and $\langle f, g \rangle \stackrel{\text{def}}{=} \mathbf{E}_x [f(x) \cdot g(x)]$, so that $\|f\|_2^2 = \langle f, f \rangle$. For $s \subseteq \{1, \dots, n\}$ and $x \in \{0, 1\}^n$, let $\chi_s(x) \stackrel{\text{def}}{=} (-1)^{|x_s|}$ and $\hat{f}(s) \stackrel{\text{def}}{=} \langle f, \chi_s \rangle$. The linear mapping $f \rightarrow \hat{f}$ is *norm-preserving* in the following sense: $\|f\|_2^2 = \sum_s \hat{f}(s)^2$ (*Parseval’s identity*).

Fact 1 (*Hypercontractive inequality [Bon70, Bec75]*). *If $1 \leq p \leq q$, then*

$$\left\| \sum_{s \subseteq \{1, \dots, n\}} \left(\frac{p-1}{q-1} \right)^{|s|/2} \cdot \hat{f}(s) \cdot \chi_s \right\|_q \leq \|f\|_p.$$

From it we derive the following variations on the “KKL theme” (cf. [KKL88]).

Claim 3. *Let $f : \{0, 1\}^n \rightarrow \mathbb{R}$, $\alpha \stackrel{\text{def}}{=} \|f(x)\|_\infty$ and $\beta \stackrel{\text{def}}{=} \|f(x)\|_1$. Then for any $\delta \in [0, 1]$:*

$$\sum_{s \subseteq \{1, \dots, n\}} \delta^{|s|} \cdot \hat{f}(s)^2 \leq \alpha^2 \cdot \left(\frac{\beta}{\alpha} \right)^{\frac{2}{1+\delta}},$$

¹²Note that we are changing the meaning of the notation $\|v\|_p$: in Sections 3 and 4 it stood for $(\sum_x |v(x)|^p)^{1/p}$; now we define it to be $(\mathbf{E}_x [|v(x)|^p])^{1/p}$, which is more common in the context of Fourier transform.

and for any $t \leq 2 \ln \frac{\alpha}{\beta}$:

$$\sum_{|s| \leq t} \hat{f}(s)^2 \leq \beta^2 \cdot \left(\frac{2e \cdot \ln \frac{\alpha}{\beta}}{t} \right)^t.$$

Proof. Choosing $q = 2$ and $p = 1 + \delta$ in the hypercontractive inequality (Fact 1), we get the first desired bound:

$$\begin{aligned} \sum_s \delta^{|s|} \cdot \hat{f}(s)^2 &= \left\| \sum_s \delta^{|s|/2} \cdot \hat{f}(s) \cdot \chi_s \right\|_2^2 \\ &\leq \|f\|_{1+\delta}^2 = \left(\mathbf{E}_x [|f(x)|^{1+\delta}] \right)^{\frac{2}{1+\delta}} = \left(\alpha^{1+\delta} \cdot \mathbf{E}_x [|f(x)/\alpha|^{1+\delta}] \right)^{\frac{2}{1+\delta}} \\ &\leq \left(\alpha^{1+\delta} \cdot \mathbf{E}_x [|f(x)/\alpha|] \right)^{\frac{2}{1+\delta}} = \alpha^{\frac{2\delta}{1+\delta}} \cdot \beta^{\frac{2}{1+\delta}}, \end{aligned}$$

where the first equality follows from Parseval's identity.

Now observe that

$$\sum_{|s| \leq t} \delta^{|s|} \cdot \hat{f}(s)^2 \leq \sum_{s \subseteq \{1, \dots, n\}} \delta^{|s|} \cdot \hat{f}(s)^2 \leq \alpha^2 \cdot \left(\frac{\beta}{\alpha} \right)^{\frac{2}{1+\delta}} \leq \alpha^2 \cdot \left(\frac{\beta}{\alpha} \right)^{2-2\delta} = \beta^2 \cdot \left(\frac{\alpha}{\beta} \right)^{2\delta},$$

where the second inequality is what we have just proved. Let $\delta \stackrel{\text{def}}{=} \frac{t}{2 \ln(\alpha/\beta)}$, then

$$\sum_{|s| \leq t} \hat{f}(s)^2 \leq \frac{\beta^2 \cdot (\alpha/\beta)^{2\delta}}{\delta^t} = \beta^2 \cdot \frac{(\alpha/\beta)^{t/\ln(\alpha/\beta)}}{(t/2 \ln(\alpha/\beta))^t} = \beta^2 \cdot \left(\frac{2e \cdot \ln \frac{\alpha}{\beta}}{t} \right)^t,$$

as required. ■ *Claim 3*

In the rest of this section we prove Lemma 3.

Proof outline

Let us say, informally, that a distribution of bits is *noise-resistant* if its “noisy version” is distinguishable from the uniform distribution. In Step 1 we observe that a “projection” of a noise-resistant distribution to a randomly-selected subset of bits can (also) be distinguished from the uniform – that is, it cannot have the full entropy. This argument is applied to the distribution of $\sigma_i(X_1) \oplus X_2$, which is noise-resistant by assumption, letting us conclude that a “typical” $(\sigma_i(X_1) \oplus X_2)_S$ has entropy $|S| - \Omega(\delta)$.

In Step 2 we use chain decomposition over $H((\sigma_i(X_1) \oplus X_2)_S)$ in order to “pinpoint” the entropy loss in $\sigma_i(X_1) \oplus X_2$. That is, as a result of the first two steps we “trade the noise-resistance off” for localising the entropy loss in $\sigma_i(X_1) \oplus X_2$.

In Step 3 we apply hypercontractivity to conclude that ν cannot have the full entropy, quod erat demonstrandum.

Step 1

If a distribution is noise-resistant, then its “projection” to a random $2/3$ -fraction of bits is also distinguishable from the uniform – this will let us conclude that under the assumptions of Lemma 3, entropy loss must occur in a “typical” $(\sigma_i(X_1) \oplus X_2)_S$ for $|S| = \lceil 2n/3 \rceil$. Formally:

Lemma 4. *If ν is a distribution on $\{0, 1\}^n$, such that $H(T_{1/4}(\nu)) \leq n - \delta$ for $\delta > 0$, then*

$$\mathbf{E}_{|S|=\lceil 2n/3 \rceil} \left[H_{X \sim \nu} (X_S | S) \right] \leq \left\lceil \frac{2n}{3} \right\rceil - \delta + 2^{\log n - \Omega(n)},$$

where S is uniformly distributed over the subsets of $\{1, \dots, n\}$ of size $\lceil 2n/3 \rceil$.

Note that the distributions of $\sigma_i(X_1) \oplus T_{1/4}(X_2)$ and of $T_{1/4}(\sigma_i(X_1) \oplus X_2)$ are the same. Therefore, the above lemma implies that

$$\mathbf{E}_{|S|=\lceil 2n/3 \rceil} \left[H((\sigma_i(X_1) \oplus X_2)_S | S) \right] \leq \left\lceil \frac{2n}{3} \right\rceil - (n - H(\sigma_i(X_1) \oplus T_{1/4}(X_2))) + 2^{\log n - \Omega(n)}.$$

From the assumption of Lemma 3 it follows that

$$\mathbf{E}_{\substack{i \in \{0, \dots, n-1\} \\ |S|=\lceil 2n/3 \rceil}} \left[H_{(X_1, X_2) \sim \nu} ((\sigma_i(X_1) \oplus X_2)_S | S) \right] \leq \left\lceil \frac{2n}{3} \right\rceil - \delta + 2^{\log n - \Omega(n)}. \quad (2)$$

Proof of Lemma 4. For any $x_0 \in \{0, 1\}^n$, $Y \sim T_{1/4}(x_0)$ can be sampled via the following procedure.¹³ Pick independently $W, Z \sim \mathcal{U}_{\{0,1\}^n}$; on those coordinates j where $W_j = 1$, let $Y_j = x_0(j)$; otherwise, let $Y_j = Z_j$. The same method can be used to sample $Y \sim T_{1/4}(\nu)$:

$$Y = (W \wedge X) \vee (\neg W \wedge Z),$$

where $X \sim \nu$ and $W, Z \sim \mathcal{U}_{\{0,1\}^n}$ independently.

By non-increasing of conditional entropy,

$$n - \delta \geq H(Y) \geq H(Y|W) = H(X_W|W) + H(Z_{\neg W}|W) = H(X_W|W) + \mathbf{E}[n - |W|].$$

Note that

$$H(X_W|W) + \mathbf{E}[n - |W|] = \sum_{k=0}^n \Pr[|W| = k] \cdot \left(\mathbf{E}_{|S|=k} [H(X_S|S)] + n - k \right)$$

and that $\mathbf{E}_{|S|=k} [H(X_S|S)] + n - k$ is non-increasing in k , as

$$\mathbf{E}_{|S|=k+1} [H(X_S|S)] - \mathbf{E}_{|S|=k} [H(X_S|S)] \leq 1.$$

Accordingly,

$$\begin{aligned} n - \delta &\geq \left(\mathbf{E}_{|S|=\lceil 2n/3 \rceil} [H(X_S|S)] + n - \left\lceil \frac{2n}{3} \right\rceil \right) \cdot \left(1 - \Pr \left[|W| > \left\lceil \frac{2n}{3} \right\rceil \right] \right) \\ &\geq \mathbf{E}_{|S|=\lceil 2n/3 \rceil} [H(X_S|S)] + n - \left\lceil \frac{2n}{3} \right\rceil - n \cdot 2^{-\Omega(n)}, \end{aligned}$$

where the last inequality follows from the Chernoff bound. ■ Lemma 4

¹³This intuitive and convenient way of looking at the noisy distribution is used in [Sam15] for very similar purpose – to analyse the noise-resistance of a Boolean function (cf. Remark 1.10 there).

Step 2

Here we will use chain decomposition and non-increasing of conditional entropy in order to “pinpoint” the entropy loss in $\sigma_i(X_1) \oplus X_2$.

Let us look at $\mathbf{E}_{|S|=\lceil 2n/3 \rceil} [H(Y_S|S)]$ for some Y , taking values from $\{0, 1\}^n$:

$$\begin{aligned}
\mathbf{E}_{|S|=\lceil 2n/3 \rceil} [H(Y_S|S)] &= \mathbf{E}_{\tau \in S_n} \left[\sum_{i=1}^{\lceil 2n/3 \rceil} H(Y_{\tau(i)} | \tau, Y_{\tau(1)}, \dots, Y_{\tau(i-1)}) \right] \\
&= \sum_{i=1}^{\lceil 2n/3 \rceil} \mathbf{E}_{\tau \in S_n} \left[H(Y_{\tau(i)} | \tau, Y_{\tau(1)}, \dots, Y_{\tau(i-1)}) \right] \\
&= \sum_{i=0}^{\lceil 2n/3 \rceil - 1} \mathbf{E}_{\substack{j \in \{1, \dots, n\} \\ S \subseteq \{1, \dots, n\} \setminus \{j\} \\ |S|=i}} [H(Y_j | S, Y_S)] \\
&\geq \left\lfloor \frac{2n}{3} \right\rfloor \cdot \mathbf{E}_{\substack{j \in \{1, \dots, n\} \\ S \subseteq \{1, \dots, n\} \setminus \{j\} \\ |S|=\lceil 2n/3 \rceil}} [H(Y_j | S, Y_S)],
\end{aligned}$$

where the inequality follows from non-increasing of conditional entropy.

Combining the above with (2), we get that for large enough n ,

$$\mathbf{E}_{\substack{i \in \{0, \dots, n-1\} \\ j \in \{1, \dots, n\} \\ S \subseteq \{1, \dots, n\} \setminus \{j\} \\ |S|=\lceil 2n/3 \rceil}} \left[\frac{H}{(X_1, X_2) \sim \nu} \left((\sigma_i(X_1) \oplus X_2)_j | S, (\sigma_i(X_1) \oplus X_2)_S \right) \right] \leq 1 - \frac{\delta}{n}.$$

From non-increasing of conditional entropy it further follows that

$$\begin{aligned}
1 - \frac{\delta}{n} &\geq \mathbf{E}_{\substack{i \in \{0, \dots, n-1\} \\ j \in \{1, \dots, n\} \\ S \subseteq \{1, \dots, n\} \setminus \{j\} \\ |S|=\lceil 2n/3 \rceil}} \left[\frac{H}{(X_1, X_2) \sim \nu} \left((\sigma_i(X_1) \oplus X_2)_j | S, (\sigma_i(X_1) \oplus X_2)_S \right) \right] \\
&\geq \mathbf{E}_{i, j, S} \left[\frac{H}{\nu} \left((\sigma_i(X_1) \oplus X_2)_j | S, (\sigma_i(X_1))_S, X_2(S) \right) \right] \\
&= \mathbf{E}_{i, j, S} \left[\frac{H}{\nu} \left(X_1(\sigma_{-i}(j)) \oplus X_2(j) | S, (\sigma_i(X_1))_S, X_2(S) \right) \right]. \tag{3}
\end{aligned}$$

As for any i_0 , at most $\lfloor n/3 \rfloor$ pairs of the form $(X_1(\sigma_{-i_0}(j)), X_2(j))$ can be outside of $X_1(S_1) \times X_2(S_2)$, the expectation (3) is at least

$$\mathbf{E}_{\substack{i \in \{0, \dots, n-1\} \\ j \in \{1, \dots, n\} \\ S_1 \subseteq \{1, \dots, n\} \setminus \{\sigma_{-i}(j)\} \\ S_2 \subseteq \{1, \dots, n\} \setminus \{j\} \\ |S_1|=|S_2|=\lceil 5n/6 \rceil}} \left[\frac{H}{\nu} \left(X_1(\sigma_{-i}(j)) \oplus X_2(j) | S_1, S_2, X_1(S_1), X_2(S_2) \right) \right]$$

$$= \mathbf{E}_{\substack{S_1, S_2 \subseteq \{1, \dots, n\} \\ |S_1| = |S_2| = \lceil \frac{5n}{6} \rceil}} \left[\mathbf{E}_{\substack{j_1 \in \{1, \dots, n\} \setminus S_1 \\ j_2 \in \{1, \dots, n\} \setminus S_2}} \left[H_{\nu} (X_1(j_1) \oplus X_2(j_2) | S_1, S_2, X_1(S_1), X_2(S_2)) \right] \right].$$

In particular, there exist subsets $T_1, T_2 \subseteq \{1, \dots, n\}$ of size $\lfloor n/6 \rfloor$, such that

$$\mathbf{E}_{\substack{j_1 \in T_1 \\ j_2 \in T_2}} \left[H_{(X_1, X_2) \sim \nu} (X_1(j_1) \oplus X_2(j_2) | X_1(\overline{T_1}), X_2(\overline{T_2})) \right] \leq 1 - \frac{\delta}{n},$$

where $\overline{T_i}$ stands for $\{1, \dots, n\} \setminus T_i$.

Let $I_1 \stackrel{\text{def}}{=} T_1$, $I_2 \stackrel{\text{def}}{=} n + T_2$, $I \stackrel{\text{def}}{=} I_1 \cup I_2$ and $\overline{I} \stackrel{\text{def}}{=} \{1, \dots, 2n\} \setminus I$. Then

$$\frac{\delta}{n} \leq \mathbf{E}_{\substack{j_1 \in I_1 \\ j_2 \in I_2}} \left[1 - H_{Y \sim \nu} (Y_{j_1} \oplus Y_{j_2} | Y_{\overline{I}}) \right] \leq 2 \cdot \mathbf{E}_{j_1 \neq j_2 \in I} \left[1 - H_{Y \sim \nu} (Y_{j_1} \oplus Y_{j_2} | Y_{\overline{I}}) \right], \quad (4)$$

where $|I| \geq n/3 - 2$.

Step 3

Finally, we apply hypercontractivity to argue that if (4) holds then ν cannot have the full entropy. Formally:

Lemma 5. *If ρ is a distribution on $\{0, 1\}^n$ for large enough n , then*

$$\mathbf{E}_{j_1 \neq j_2 \in \{1, \dots, n\}} \left[1 - H_{X \sim \rho} (X_{j_1} \oplus X_{j_2}) \right] \leq \frac{45}{n^2} \cdot \left(n - H_{\min}(\rho) \right)^2.$$

Using it, we can write (assuming large enough n):

$$\begin{aligned} & \mathbf{E}_{W \sim \nu} \left[\left(|I| - H_{Y \sim \nu} (Y_I | Y_{\overline{I}} = W_{\overline{I}}) \right)^2 \right] \\ & \geq \frac{|I|^2}{45} \cdot \mathbf{E}_{W \sim \nu} \left[\mathbf{E}_{j_1 \neq j_2 \in I} \left[1 - H_{Y \sim \rho} (Y_{j_1} \oplus Y_{j_2} | Y_{\overline{I}} = W_{\overline{I}}) \right] \right] \\ & \geq \frac{n^2}{406} \cdot \mathbf{E}_{j_1 \neq j_2 \in I} \left[1 - H_{Y \sim \rho} (Y_{j_1} \oplus Y_{j_2} | Y_{\overline{I}}) \right] \\ & \geq \frac{\delta n}{812}, \end{aligned}$$

where the first inequality is an application of Lemma 5 with respect to Y_I taking values from $\{0, 1\}^{|I|}$, conditioned on the given value of $Y_{\overline{I}}$; the second one uses the fact that $|I| \geq n/3 - 2$; the last one follows from (4) and the fact that H_{\min} is never greater than H . Using Claim 1 with A being the support of $Y_{\overline{I}}$, B being the support of Y_I and $\Delta = 2 \log n + 2$, we get

$$\begin{aligned} \frac{1}{4n^2} & \geq \mathbf{Pr}_{W \sim \nu} \left[H_{Y \sim \nu} (Y_I | Y_{\overline{I}} = W_{\overline{I}}) \leq H_{\min}(\nu) - (2n - |I|) - 2 \log n - 2 \right] \\ & = \mathbf{Pr}_{W \sim \nu} \left[|I| - H_{Y \sim \nu} (Y_I | Y_{\overline{I}} = W_{\overline{I}}) \geq 2n - H_{\min}(\nu) + 2 \log n + 2 \right], \end{aligned}$$

and so,

$$\frac{\delta n}{812} \leq \mathbf{E}_{W \sim \nu} \left[\left(|I| - H_{\min} (Y_I | Y_{\bar{I}} = W_{\bar{I}}) \right)^2 \right] \leq \left(2n - H_{\min}(\nu) + 2 \log n + 2 \right)^2 + 1.$$

Lemma 3 follows.

Proof of Lemma 5. For $j_1 \neq j_2 \in \{1, \dots, n\}$, let $\Delta_{j_1, j_2} \stackrel{\text{def}}{=} |\mathbf{Pr}_\rho [X_{j_1} = X_{j_2}] - 1/2|$. Then

$$\mathbf{E}_{j_1 \neq j_2} \left[1 - \frac{H}{\rho} (X_{j_1} \oplus X_{j_2}) \right] < \mathbf{Pr}_{j_1 \neq j_2} [\Delta_{j_1, j_2} > 1/4] + 6 \mathbf{E}_{j_1 \neq j_2} [\Delta_{j_1, j_2}^2] < 22 \mathbf{E}_{j_1 \neq j_2} [\Delta_{j_1, j_2}^2], \quad (5)$$

where the first inequality follows from the Taylor expansion

$$\begin{aligned} 1 - \frac{H}{\rho} (X_{j_1} \oplus X_{j_2}) &= \frac{1}{2 \ln 2} \cdot \sum_{i=1}^{\infty} \frac{(2\Delta_{j_1, j_2})^{2i}}{i(2i-1)} \leq \frac{2\Delta_{j_1, j_2}^2}{\ln 2} \cdot \sum_{i=0}^{\infty} (2\Delta_{j_1, j_2})^{2i} \\ &= \frac{2\Delta_{j_1, j_2}^2}{\ln 2 \cdot (1 - 4\Delta_{j_1, j_2}^2)} \leq \frac{2\Delta_{j_1, j_2}^2}{\ln 2 \cdot (1 - 2\Delta_{j_1, j_2})}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \Delta_{j_1, j_2} &= \frac{1}{2} \cdot \left| 2 \mathbf{Pr}_\rho [X_{j_1} = X_{j_2}] - 1 \right| = \frac{1}{2} \cdot \left| \mathbf{E}_\rho [\chi_{\{j_1, j_2\}}(x)] \right| \\ &= \frac{1}{2} \cdot \left| \sum_x \rho(x) \chi_{\{j_1, j_2\}}(x) \right| = 2^{n-1} \cdot |\hat{\rho}(\{j_1, j_2\})|, \end{aligned}$$

and from (5),

$$\mathbf{E}_{j_1 \neq j_2} \left[1 - \frac{H}{\rho} (X_{j_1} \oplus X_{j_2}) \right] < 22 \mathbf{E}_{j_1 \neq j_2} [\Delta_{j_1, j_2}^2] < \frac{6 \cdot 2^{2n}}{\binom{n}{2}} \cdot \sum_{|s|=2} \hat{\rho}(s)^2.$$

From Claim 3 with $t = 2$,

$$\sum_{|s|=2} \hat{\rho}(s)^2 \leq \|\rho\|_1^2 \cdot \left(e \cdot \ln \frac{\|\rho\|_\infty}{\|\rho\|_1} \right)^2 = 2^{-2n} \cdot \left(\frac{e}{\log e} \right)^2 \cdot \left(n - H_{\min}(\rho) \right)^2,$$

and the result follows. ■ Lemma 5

6 Discussion

We have shown that the *partial function Shape* is easy for $\mathcal{Q}^{\parallel, \text{ent}}$ but hard for \mathcal{R} , which may be viewed as improving earlier understanding of *when quantum communication can outperform classical communication*. Nevertheless, prior to this work there have been a number of results that have emphasised the advantage of quantum communication: [Raz99, BCWdW01, BYJK04, GKK⁺08, Gav08, KR11] and many others.

On the other hand, current knowledge of *when classical communication can “replace” quantum communication* – i.e., when the advantage of using quantum communication can be at

most (quasi-) polynomial in terms of complexity – is surprisingly limited (cf. [Aar04, GRdW08]). We do not want to speculate now whether or not it is possible, say, to find *a partial function, easy for \mathcal{Q}^{\parallel} ¹⁴ but hard for \mathcal{R}* – finding that out is, obviously, an important open problem; however, our intuition strongly suggests that *no total function can have that property*. Proving or refuting it would be a breakthrough.

On a different note, several questions regarding the complexity of *Shape* in other models of communication are still open. Answering any of the following will necessarily lead to a yet-unknown (as of now) separation of communication complexity classes.

- What is the \mathcal{Q}^{\parallel} -complexity of *Shape*? If it has an efficient protocol, that would strengthen qualitatively the result of this work. If it is hard, that would give the first super-polynomial separation between $\mathcal{Q}^{\parallel,ent}$ and \mathcal{Q}^{\parallel} via a partial function (a relational separation is known).
- What is the $\mathcal{Q}^{\parallel,pub}$ -complexity¹⁴ of *Shape*? The consequences would be similar.
- What is the $\mathcal{R}^{\parallel,ent}$ -complexity¹⁵ of *Shape*? The consequences would be similar, and not even a relational separation is currently known between $\mathcal{Q}^{\parallel,ent}$ and $\mathcal{R}^{\parallel,ent}$.

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¹⁴ \mathcal{Q}^{\parallel} is the model of (unentangled) quantum simultaneous message passing; closely related to it is $\mathcal{Q}^{\parallel,pub}$ – the model of quantum simultaneous message passing with shared randomness.

¹⁵ $\mathcal{R}^{\parallel,ent}$ is the model of classical simultaneous message passing with entanglement.

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